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Abstract

Multimodular functions and L-convex functions have been investigated almost independently, but they are, in fact, equivalent objects that can be related through a simple coordinate transformation. Some facts known for L-convex functions can be translated to new results for multimodular functions, and vice versa. In particular, the local optimality condition for global optimality found in the literature of multimodular functions should be rectified, and a discrete separation theorem holds for multimodular functions.

Keywords: discrete convex function; multimodular function; L-convex function; local optimality
AMS Classifications: 90C10; 90C25, 90C35, 90C27.

1 Introduction

Discrete convex functions have long been attracting research interest in operations research and related disciplines [2, 7, 11, 13]. Among others, submodular functions, defined on subsets or integer vectors, certainly form a central class of discrete convex functions and the relationship between submodularity and convexity is now fully understood [3, 5, 9, 11, 14]. Ramifications of submodular functions have been investigated, quite independently, in various contexts; multimodular functions in discrete-event control [1, 2, 8], submodular integrally convex functions in nonlinear discrete optimization [4], and L-convex functions in discrete convex analysis [10, 11].

The objective of this short note is to indicate a close relationship, equivalence through a simple coordinate transformation, between multimodular functions and L-convex functions, and to derive some new results for multimodular functions from facts known about L-convex functions, and vice versa. Specifically, it is pointed out that (i) the local optimality condition for

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global optimality found in the literature of multimodular functions should be rectified (Example 1, Theorem 1), (ii) a discrete separation theorem holds for multimodular functions (Theorem 2), and (iii) an L-convex function can be characterized in terms of the convexity of its piecewise linear extension (Theorem 3).

2 Definitions

We consider functions defined on integer lattice points, $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$, that may possibly take $+\infty$, where the effective domain of f is denoted by $\mathrm{dom}\, f = \{x \in \mathbf{Z}^n \mid f(x) < +\infty\}$. For a subset Y of $V = \{1, \ldots, n\}$, we denote by χ_Y the characteristic vector of Y; the ith component of χ_Y equals one or zero according to whether i belongs to Y or not. For $i \in V$, χ_i is a short-hand notation for $\chi_{\{i\}}$. For vectors $x, y \in \mathbf{Z}^n$ we denote by $x \vee y$ and $x \wedge y$ the vectors of componentwise maximum and minimum of x and y, i.e.,

$$(x \vee y)_i = \max(x_i, y_i), \quad (x \wedge y)_i = \min(x_i, y_i) \qquad (i \in V).$$

A function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be *submodular* if $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$ for all $x, y \in \mathbf{Z}^n$. A function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is said to be *multimodular* if the function $\tilde{f} : \mathbf{Z}^{n+1} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{f}(x_0, x) = f(x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}) \qquad (x_0 \in \mathbf{Z}, x \in \mathbf{Z}^n) \tag{1}$$

is submodular in $n+1$ variables.

A function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, g \neq \emptyset$ is said to be *L-convex* if (i) it is submodular and (ii) there exists $r \in \mathbf{R}$ such that $g(p + \mathbf{1}) = g(p) + r$ for all $p \in \mathbf{Z}^n$, where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{Z}^n$. A function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is called *L$^\natural$-convex*[1)] if the function $\tilde{g} : \mathbf{Z}^{n+1} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{g}(p_0, p) = g(p - p_0\mathbf{1}) \qquad (p_0 \in \mathbf{Z}, p \in \mathbf{Z}^n) \tag{2}$$

is L-convex in $n+1$ variables. Whereas L$^\natural$-convex functions are conceptually equivalent to L-convex functions by the relation (2), the class of L$^\natural$-convex functions in n variables is strictly larger than that of L-convex functions in n variables. It is shown in [6] that L$^\natural$-convex functions are the same as submodular integrally convex functions introduced in [4].

Multimodularity and L$^\natural$-convexity have the following relationship.

Lemma 1 ([11, p.183]) *A function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is multimodular if and only if it can be represented as*

$$f(x) = g(x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + \cdots + x_n) \qquad (x \in \mathbf{Z}^n) \tag{3}$$

[1)] "L$^\natural$-convex" should be read "L-natural-convex."

for some $L^\natural$*-convex function* g*. Conversely, a function* $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ *is* $L^\natural$*-convex if and only if it can be represented as*

$$g(p) = f(p_1, p_2 - p_1, p_3 - p_2, \ldots, p_n - p_{n-1}) \qquad (p \in \mathbf{Z}^n) \tag{4}$$

for some multimodular function f*.*

3 Multimodular Function Minimization

Minimization is most fundamental for discrete convex functions, and global optimality (minimality) is often characterized by local optimality with a suitable definition of locality.

For multimodular functions the following statement has been made in [1, Cor. 2.2] and [2, p. 19, Cor. 2]: For a multimodular function f and $x \in \mathrm{dom}\, f$, we have

$$f(x) \leq f(y) \quad (\forall\, y \in \mathbf{Z}^n) \iff f(x) \leq f(x \pm \chi_i) \quad (\forall\, i \subseteq V).$$

However, this is not correct, as the following example shows.[2)]

Example 1 For a multimodular function $f(x_1, x_2) = 2x_1^2 + 2(x_1+x_2)^2 - 3x_1$, we have

$$\begin{array}{lll} f(-1,\ \ 1) = 5, & f(0,\ \ 1) = 2, & f(1,\ \ 1) = 7. \\ f(-1,\ \ 0) = 7, & f(0,\ \ 0) = 0, & f(1,\ \ 0) = 1, \\ f(-1, -1) = 13, & f(0, -1) = 2, & f(1, -1) = -1. \end{array}$$

The origin $x = (0, 0)$ is a local minimum, satisfying $f(x) \leq f(x \pm \chi_i)$ for $i = 1, 2$, but it is not the global minimum since $f(0, 0) > f(1, -1)$.

The correct optimality criterion can be obtained readily from the following corresponding result for $\mathrm{L}^\natural$-convex functions.

Lemma 2 ([11, Theorem 7.14]) *For an* $L^\natural$*-convex function* g *and* $p \in \mathrm{dom}\, g$*, we have*

$$g(p) \leq g(q) \quad (\forall\, q \in \mathbf{Z}^n) \iff g(p) \leq g(p \pm \chi_Y) \quad (\forall\, Y \subseteq V).$$

A combination of the above lemma with the relationship shown in Lemma 1 yields the following optimality criterion for multimodular functions.

Theorem 1 *For a multimodular function* f *and* $x \in \mathrm{dom}\, f$*, we have*

$$f(x) \leq f(y) \quad (\forall\, y \in \mathbf{Z}^n) \iff f(x) \leq f(x \pm y) \quad (\forall\, y \in \mathcal{T}),$$

where $\mathcal{T}$ *is the set of vectors of the form* $\chi_{i_1} - \chi_{i_2} + \cdots + (-1)^{k-1}\chi_{i_k}$ *for some increasing sequence of indices* $i_1 < i_2 < \cdots < i_k$*.*

[2)] This serves also as a counterexample to [2, p. 253, Cor. 11].

(Proof) By Lemma 1 we have $f(x) = g(p)$ if $x = Dp$, where $D = (d_{ij} \mid 1 \leq i,j \leq n)$ is a bidiagonal matrix with $d_{ii} = 1$ for $i = 1,\ldots,n$ and $d_{i+1,i} = -1$ for $i = 1,\ldots,n-1$. The set $\mathcal{T}$ is obtained as $\mathcal{T} = \{Dp \mid p = \chi_Y$ for some $Y \subseteq V\}$. Then the claim follows from Lemma 2. ■

When $n = 2$, for example, we have $\mathcal{T} = \{\chi_1, \chi_2, \chi_1 - \chi_2\}$. The added direction $\chi_1 - \chi_2$ detects the nonoptimality of $x = (0,0)$ in Example 1.

Here is a remark on minimization algorithms. For L$^\natural$-convex functions the algorithm of Favati–Tardella [4] is the first polynomial-time algorithm; it is based on the ellipsoid method. A steepest descent-type polynomial-time algorithm is given by [12] (see also [11, Section 10.3]). The algorithm is based on the recently developed combinatorial algorithms for submodular set-function minimization as well as on a scaling technique. By the relationship stated in Lemma 1 these algorithms can be adapted for multimodular function minimization.

4 Other Implications

Two other implications of the relationship given in Lemma 1 are indicated.

4.1 Discrete separation theorem

In general terms, a discrete separation theorem is a statement that, for $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ and $h : \mathbf{Z}^n \to \mathbf{R} \cup \{-\infty\}$ belonging to certain classes of discrete convex/concave functions, if $f(x) \geq h(x)$ for all $x \in \mathbf{Z}^n$, then there exist $\alpha^* \in \mathbf{R}$ and $\xi^* \in \mathbf{R}^n$ such that

$$f(x) \geq \alpha^* + \langle \xi^*, x \rangle \geq h(x) \qquad (\forall x \in \mathbf{Z}^n),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product (pairing), and we can take $\alpha^* \in \mathbf{Z}$ and $\xi^* \in \mathbf{Z}^n$ if f and h are integer-valued. A discrete separation theorem often captures deep combinatorial properties beyond mere convex-extensibility (see, e.g., [5, Section 4.2], [11, Section 1.4.4]).

It is known [11, Theorem 8.16] that a discrete separation theorem holds for L$^\natural$-convex functions, from which we can derive the following theorem for multimodular functions.

Theorem 2 *Let $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ and $h : \mathbf{Z}^n \to \mathbf{R} \cup \{-\infty\}$ be functions such that f and $-h$ are multimodular, and assume*[3)] *that $f(x_0)$ and $h(x_0)$ are finite for some $x_0 \in \mathbf{Z}^n$. If $f(x) \geq h(x)$ $(\forall x \in \mathbf{Z}^n)$, there exist $\alpha^* \in \mathbf{R}$ and $\xi^* \in \mathbf{R}^n$ such that*

$$f(x) \geq \alpha^* + \langle \xi^*, x \rangle \geq h(x) \qquad (\forall x \in \mathbf{Z}^n). \tag{5}$$

[3)] This assumption can be weakened by referring to their conjugate functions, as in [11, Theorem 8.16].

Moreover, if f and h are integer-valued, there exist integer-valued $\alpha^* \in \mathbf{Z}$ and $\xi^* \in \mathbf{Z}^n$.

(Proof) Put $g(p) = f(Dp)$ and $k(p) = h(Dp)$ with the matrix D defined in the proof of Theorem 1. Then g and $-k$ are L$^\natural$-convex by Lemma 1 and the discrete separation theorem for L$^\natural$-convex functions [11, Theorem 8.16] yields $\alpha^* \in \mathbf{R}$ and $\eta^* \in \mathbf{R}^n$ such that

$$g(p) \geq \alpha^* + \langle \eta^*, p \rangle \geq k(p) \qquad (\forall p \in \mathbf{Z}^n).$$

This implies (5) with $\xi^* = D^{-\top}\eta^*$ since $\langle \eta^*, p \rangle = \langle \eta^*, D^{-1}x \rangle = \langle D^{-\top}\eta^*, x \rangle$. If f and h are integer-valued, so are g and k, and we have $\alpha^* \in \mathbf{Z}$ and $\eta^* \in \mathbf{Z}^n$ by the integrality assertion in the discrete separation theorem for L$^\natural$-convex functions. Finally, we note that D^{-1} is an integral matrix (with $(D^{-1})_{ij} = 1$ for $i \geq j$ and $(D^{-1})_{ij} = 0$ for $i < j$), which guarantees $\xi^* \in \mathbf{Z}^n$ if $\eta^* \in \mathbf{Z}^n$. ■

4.2 Characterization of L-convexity

It is known that an L$^\natural$-convex function g can be extended to a convex function, and that the convex extension can be constructed as a collection of the Lovász extensions of the submodular set-functions derived from g. A converse of this statement can be obtained through the translation of a fundamental fact known in the literature of multimodular functions.

Given a real vector $q \in \mathbf{R}^n$, we denote by $\hat{q}_1 > \hat{q}_2 > \cdots > \hat{q}_m$ the distinct values of its components and put

$$U_k = U_k(q) = \{i \in V \mid q_i \geq \hat{q}_k\} \qquad (k = 1, \ldots, m).$$

For a function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$, we define a function $\overline{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ according to the expression

$$\overline{g}(p+q) = (1 - \hat{q}_1)g(p) + \sum_{k=1}^{m-1} (\hat{q}_k - \hat{q}_{k+1}) g(p + \chi_{U_k}) + \hat{q}_m g(p + \chi_{U_m}) \quad (6)$$

for $p \in \mathbf{Z}^n$ and $q \in \mathbf{R}^n$ with $0 \leq q_i < 1$ for every $i \in V$. By construction we have $\overline{g}(p) = g(p)$ for $p \in \mathbf{Z}^n$, which means that $\overline{g}$ is a piecewise linear extension of g.

It is known [11, Theorem 7.20] that $\overline{g}$ is convex if g is L$^\natural$-convex. A fundamental theorem ([1, Cor. 2.1], [2, p. 16, Theorem 1]) in the theory of multimodular functions implies that the converse is also true, as follows.

Theorem 3 *g is L$^\natural$-convex if and only if $\overline{g}$ is convex.*

(Proof) This is an immediate consequence of the combination of Lemma 1 with [1, Cor. 2.1] or [2, p. 16, Theorem 1]. It is noted that the “only if”-part is given in [11, Theorem 7.20], and for the “if”-part an alternative proof using results in [11] is also possible as follows. Suppose that $\overline{g}$ is convex. Then g is submodular in each unit hypercube by the fundamental theorem of Lovász [9] (see [11, Theorem 4.16]). This local submodularity implies global submodularity. On the other hand, g is integrally convex by the construction of $\overline{g}$ and the coincidence of $\overline{g}$ with g on integer vectors. Thus, g is submodular integrally convex, and hence $\mathrm{L}^\natural$-convex by [11, Theorem 7.21]. ■

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