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Abstract

We study a polytope which arises from a mixed integer programming formulation of the quadratic semi-assignment problem. We introduce an isomorphic projection in order to transform the polytope to another essentially equivalent and tractable polytope. As a result, some basic polyhedral properties, such as the dimension, the affine hull, and the trivial facets, are obtained in quite a simple way. We further present valid inequalities called clique- and cut-inequalities and give complete characterizations for them to be facet-defining. We also discuss a simultaneous lifting of the clique-type facets. Finally, we show an application of the quadratic semi-assignment problem to hub location problems with some computational experiences.

Keywords: Quadratic semi-assignment problem, Polytope, Facet, Hub location

1 Introduction

The *quadratic semi-assignment problem* is a linearly constrained 0-1 quadratic programming problem. Let M and N be mutually disjoint sets with $|M| = m$ and $|N| = n$. Given weights a_{ik} $(\forall i \in M, \forall k \in N)$ and b_{ikjl} $(\forall i, \forall j \in M, i < j, \forall k, \forall l \in N)$, the quadratic semi-assignment problem is formulated as follows:

$$\begin{array}{lll} \text{QSAP} \quad \text{min.} & \displaystyle\sum_{i \in M} \sum_{k \in N} a_{ik} x_{ik} + \sum_{\substack{i,j \in M \\ i<j}} \sum_{k \in N} \sum_{l \in N} b_{ikjl} x_{ik} x_{jl} & \\ \quad\quad\quad\;\; \text{s. t.} & \displaystyle\sum_{k \in N} x_{ik} = 1 & (\forall i \in M), \\ & x_{ik} \in \{0, 1\} & (\forall i \in M, \forall k \in N). \end{array}$$

The quadratic semi-assignment problem is NP-hard. An application of the problem can be found in the area of scheduling [13, 20, 21]. Skutella dealt with the problem of

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scheduling unrelated parallel machines which is formulated as a special case of QSAP and proposed an approximation algorithm [18]. A *hub location problem* is also an application of the quadratic semi-assignment problem [16] (see Appendex B).

We consider a mixed integer programming (MIP) formulation of the problem and introduce an associated polytope which we call *quadratic semi-assignment polytope* (QSAP-polytope). The aim of this paper is to investigate theoretical aspects of the QSAP-polytope. A mixed integer programming formulation of QSAP is given independently by Billionnet and Elloumi [3] and by Saito, Matuura, and Matsui [16] in the context of a hub location problem (see Appendix B). The computational experience in [16] shows that for 'most' of the generated instances, the linear programming relaxation of the MIP is integral. Our additional computational experience in Appendix B of this paper shows that 'all' of the generated instances (including that in [16]) can be solved to optimality by adding *triangle inequalities* to the linear relaxation problems. This motivates us to study a polytope associated with the MIP formulation in view of the quadratic semi-assignment problem.

We define the quadratic semi-assignment polytope by the convex hull of all the feasible solutions of the MIP. First, we show the dimension of the QSAP-polytope and then prove that the affine hull of the polytope is composed of the equality system of the MIP formulation (Theorem 8). Similarly, we prove that the inequality constraints of the MIP formulation are facet-defining (Theorem 10). Next, we discuss *clique-inequalities* and *cut-inequalities*, originally introduced for unconstrained 0-1 quadratic programming problems by Padberg [14]. We show necessary and sufficient conditions that the inequalities define facets of the polytope (Theorems 13–16). We also obtain another class of facets by simultaneous lifting of the clique-type facets (Theorem 18). In Section 2, we discuss connections between the QSAP-polytope and other polytopes: the *Boolean quadric polytope* [14], the *quadratic assignment polytope* [8, 15], and the *stable set polytope in a higher dimension* [6]. Our results are obtained by employing the *star-transformation* developed for the quadratic assignment polytope by Jünger and Kaibel [8]. By modifying the transformation for the QSAP-polytope, we obtain a tractable full-dimensional polytope, which is essentially equivalent to the original one.

The rest of this paper is organized as follows. In Section 2, we define the quadratic semi-assignment polytope and present the mixed integer programming reformulation of QSAP. In Section 3, we introduce the star-transformation to the QSAP-polytope, and obtain some basic polyhedral properties. In Sections 4.1 and 4.2, we give necessary and sufficient conditions that the clique- and the cut-inequalities define facets. In Section 4.3, we discuss simultaneous lifting of clique-inequalities. In Appendix A, we show some facet defining inequalities which are neither clique- nor cut-inequalities. Finally, we present some computational experiences in Appendix B.

2 QSAP-Polytope and MIP Formulation

In this section, we first introduce the quadratic semi-assignment polytope. Second, we show a MIP formulation of QSAP in [3, 16]. We also mention some polytopes related to the QSAP-polytope.

To linearize the objective function of QSAP, we introduce auxiliary variables $y = (y_{ikjl})$ satisfying $y_{ikjl} = x_{ik}x_{jl}$. Then the objective function of QSAP is written by

$$\sum_{i \in M} \sum_{k \in N} a_{ik} x_{ik} + \sum_{\substack{i,j \in M \\ i<j}} \sum_{k \in N} \sum_{l \in N} b_{ikjl} y_{ikjl},$$

which is called the *linearization technique* in [1].

We explain the above linearization from the graph theoretical point of view. To this end, we define the graph $G = (V, E)$ by the node set $V := M \times N$ and the edge set $E := \left\{\{(i,k),(j,l)\} \in \binom{V}{2} \mid i \neq j\right\}$, where $\binom{V}{2}$ is the set of all the 2-subsets of V. We associate the weights a_{ik} and b_{ikjl} to the node $(i,k) \in V$ and the edge $\{(i,k),(j,l)\} \in E$, respectively. It is easy to see that the characteristic vector $\chi_T \in \mathbb{R}^V$ of a node subset $T \subseteq V$ is a feasible solution of QSAP if and only if T forms an m-clique of G (see Figure 1). Hence, QSAP turns into a problem to find an m-clique which minimizes the sum of its node and edge weights.

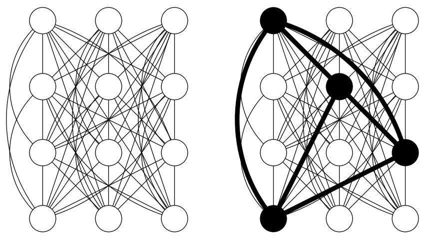

Figure 1: The graph G with $m = 4$ and $n = 3$, and an m-clique.

We define the quadratic semi-assignment polytope $\mathrm{QSAP}_{m,n}$ by the convex hull of all the feasible solutions, where $E(T) = \{\{u,v\} \in E \mid u, v \in T\}$ for any node subset $T \subseteq V$.

Definition 1. $\mathrm{QSAP}_{m,n} = \mathrm{conv}\{(\chi_C, \chi_{E(C)}) \in \mathbb{R}^V \times \mathbb{R}^E \mid C \textit{ is an } m\textit{-clique of } G\}$.

Next, we present a MIP formulation of QSAP discussed in [3, 16]. For $T \subseteq V$, we define $x(T) := \sum_{v \in T} x_v$ and $y(T) := y(E(T))$. For mutually disjoint subsets $S, T \subseteq V$, we denote the cut set by $E(S : T) := \{\{u,v\} \in E \mid u \in S,\ v \in T\}$ and we abbreviate $y(E(S : T))$ to $y(S : T)$. We omit brackets for singletons, e.g., we use χ_u instead of $\chi_{\{u\}}$. We also use χ_{uv} instead of $\chi_{\{\{u,v\}\}}$. For all $i \in M$, we define a subset of V by $\mathrm{row}_i = \{(i,k) \mid k \in N\}$. Then we have a MIP formulation of QSAP.

$$\begin{aligned}
\text{MIP}\quad \text{min.}\quad & \sum_{v \in V} a_v x_v + \sum_{e \in E} b_e y_e \\
\text{s. t.}\quad & x(\mathrm{row}_i) = 1 && (\forall i \in M), && (1)\\
& -x_{ik} + y((i,k) : \mathrm{row}_j) = 0 && (\forall (i,k) \in V, \forall j \in M, i \neq j), && (2)\\
& y_e \geq 0 && (\forall e \in E), && (3)\\
& x_v \in \{0,1\} && (\forall v \in V). && (4)
\end{aligned}$$

We note that this MIP formulation is based on the proposition below. The role of the linear constraints (1)–(3) for the polytope $\mathrm{QSAP}_{m,n}$ will be discussed in Section 3.

Proposition 2. *A vector* $(x, y) \in \mathbb{R}^V \times \mathbb{R}^E$ *is a vertex of* $\mathrm{QSAP}_{m,n}$ *if and only if it satisfies* (1), (2), (3), *and* (4).

Finally, we remark on some polytopes related to $\mathrm{QSAP}_{m,n}$. Padberg dealt with a quadratic programming problem with 0-1 constraints and introduced the Boolean quadric polytope defined by

$$\mathrm{BQP}_n = \mathrm{conv}\{(\chi_C, \chi_{\mathcal{E}_n(C)}) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid C \text{ is a clique of } K_n\},$$

where $K_n = (\mathcal{V}_n, \mathcal{E}_n)$ is a complete graph with n vertices [14]. Jünger and Kaibel studied the quadratic assignment polytope QAP_n which arises from the quadratic assignment problem [8]. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with $\mathcal{V} = N \times N$ and $\mathcal{E} = \{\{(i,j),(k,l)\} \in \binom{\mathcal{V}}{2} \mid i \neq k, j \neq l\}$. Then the quadratic assignment polytope is defined by

$$\mathrm{QAP}_n = \mathrm{conv}\{(\chi_C, \chi_{\mathcal{E}(C)}) \in \mathbb{R}^{\mathcal{V}} \times \mathbb{R}^{\mathcal{E}} \mid C \text{ is an } n\text{-clique of } \mathcal{G}\}.$$

We can show the following property easily.

Proposition 3. *The polytope* $\mathrm{QSAP}_{m,n}$ *is isomorphic to a face of the Boolean quadric polytope* BQP_{mn}*, and the quadratic assignment polytope* QAP_n *is isomorphic to a face of the polytope* $\mathrm{QSAP}_{n,n}$.

As a generalization of BQP_n, Fujie et al. [6] studied the polytope

$$\mathrm{ESTAB}(H) = \mathrm{conv}\{(\chi_C, \chi_{F(C)}) \in \mathbb{R}^W \times \mathbb{R}^F \mid C \text{ is a clique of } \overline{H}\}$$

for any graph $H = (W, F)$, where $\overline{H}$ is a complement of H. An isomorphic image of $\mathrm{QSAP}_{m,n}$, to be denoted by $\mathrm{QSAP}^\star_{m,n^\star}$ in Section 3, becomes a special case of ESTAB. More precisely, we have that $\mathrm{QSAP}^\star_{m,n^\star} = \mathrm{ESTAB}(G^\star)$ (for the definition of the graph $G^\star$, see the next section). We also note that $\mathrm{QSAP}^\star_{m,2-1} = \mathrm{BQP}_m$.

3 Star Transformation and Basic Facial Structures

In this section, we introduce an isomorphic projection, which we call *star-transformation*, for $\mathrm{QSAP}_{m,n}$ in order to obtain a full-dimensional polytope. The basic polyhedral properties of $\mathrm{QSAP}_{m,n}$ are derived from those of the full-dimensional polytope. This is a straightforward modification of the star-transformation developed for the QAP-polytope by Jünger and Kaibel [8].

We define the star-transformation in graph-theoretical terms. Let $G^\star = (V^\star, E^\star)$ be a graph with node set $V^\star := M \times (N \setminus \{n\})$ and edge set $E^\star := \{\{(i,k),(j,l)\} \in \binom{V^\star}{2} \mid i \neq j\}$. It is easy to see that a map $\kappa : 2^V \to 2^{V^\star}$ defined by $\kappa(T) = T \cap V^*$ is a bijection between the set of m-cliques in G and the set of cliques in $G^\star$, where empty set is also a clique (see Figure 2).

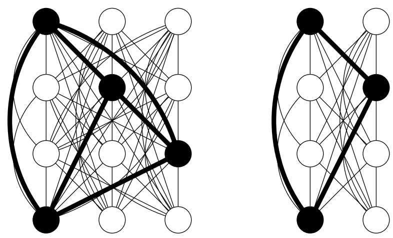

Figure 2: The effect of the star-transformation.

This leads us to introduce a polytope defined on $G^\star$, which is denoted by $\mathrm{QSAP}^\star_{m,n^\star}$. For $G^\star$ and $\mathrm{QSAP}^\star_{m,n^\star}$, we use similar notations to those of G and $\mathrm{QSAP}_{m,n}$, respectively.

Definition 4. $\mathrm{QSAP}^\star_{m,n^\star} = \mathrm{conv}\{(\chi_C, \chi_{E^\star(C)}) \in \mathbb{R}^{V^\star} \times \mathbb{R}^{E^\star} \mid C \textit{ is a clique of } G^\star\}$.

Proposition 5. *The polytope* $\mathrm{QSAP}^\star_{m,n^\star}$ *is full-dimensional, i.e.,* $\dim(\mathrm{QSAP}^\star_{m,n^\star}) = |V^\star| + |E^\star|$.

Proof. The vectors $(0,0)$, $(\chi_u, 0)$ $(\forall u \in V^\star)$, $(\chi_u + \chi_v, \chi_{uv})$ $(\forall \{u,v\} \in E^\star)$ are affinely independent in $\mathrm{QSAP}^\star_{m,n^\star}$. □

We show some trivial facet defining inequalities of $\mathrm{QSAP}^\star_{m,n^\star}$. Since $\mathrm{QSAP}^\star_{m,n^\star}$ is full-dimensional, these inequalities are essentially unique up to positive multiples. At the last of this section, we will see that they correspond to facets of $\mathrm{QSAP}_{m,n}$.

Proposition 6. *The inequalities*

$$y_e \geq 0 \qquad (\forall e \in E^\star), \quad (5)$$
$$-x_{ik} + y((i,k) : \mathrm{row}^\star_j) \leq 0 \qquad (\forall (i,k) \in V^\star, \forall j \in M, i \neq j), \quad (6)$$
$$x(\mathrm{row}^\star_i \cup \mathrm{row}^\star_j) - y(\mathrm{row}^\star_i \cup \mathrm{row}^\star_j) \leq 1 \qquad (\forall i, \forall j \in M, i < j) \quad (7)$$

define facets of $\mathrm{QSAP}^\star_{m,n^\star}$.

Proof. It is easy to show the validity of (5)–(7). To prove that they define facets, we explicitly provide with $\dim(\mathrm{QSAP}^\star_{m,n^\star})$ affinely independent vectors in the corresponding faces.

For (5), the vectors

$$(0,0),\ (\chi_u, 0)\ (\forall u \in V^\star),\ (\chi_u + \chi_v, \chi_{uv})\ (\forall \{u,v\} \in E^\star \setminus \{e\})$$

are affinely independent in the face $\{(x,y) \in \mathrm{QSAP}^\star_{m,n^\star} \mid y_e = 0\}$.

For (6), we put $v = (i,k)$. Then the vectors

$$(0,0),\ (\chi_u, 0)\ (\forall u \in V^\star \setminus \{v\}),\ (\chi_v + \chi_w, \chi_{vw})\ (\forall w \in \mathrm{row}^\star_j),$$
$$(\chi_{v_1} + \chi_{v_2}, \chi_{v_1 v_2})\ (\forall \{v_1, v_2\} \in E^\star(V^\star \setminus \{v\}),$$
$$(\chi_u + \chi_v + \chi_w, \chi_{uv} + \chi_{uw} + \chi_{vw})\ (\forall u \notin \mathrm{row}^\star_j \cup \{v\}, \exists w \in \mathrm{row}^\star_j)$$

are affinely independent in the face $\{(x,y) \in \mathrm{QSAP}^\star_{m,n^\star} \mid -x_v + y(v : \mathrm{row}^\star_j) = 0\}$.

Finally, for (7), we put $T = \mathrm{row}^\star_i \cup \mathrm{row}^\star_j$. Then the vectors

$$
\begin{aligned}
&(\chi_u, 0)\ (\forall u \in T),\\
&(\chi_u + \chi_w, \chi_{uw})\ (\forall w \in V^\star \setminus T, \exists u \in T),\\
&(\chi_u + \chi_v, \chi_{uv})\ (\forall \{u,v\} \in E^\star(T)),\\
&(\chi_u + \chi_v + \chi_w, \chi_{uv} + \chi_{uw} + \chi_{vw})\ (\forall \{u,v\} \in E^\star(T : V^\star \setminus T), \exists v \in T),\\
&(\chi_u + \chi_{w_1} + \chi_{w_2}, \chi_{uw_1} + \chi_{uw_2} + \chi_{w_1w_2})\ (\forall \{w_1,w_2\} \in E^\star(V^\star \setminus T), \exists u \in T)
\end{aligned}
$$

are affinely independent in the face $\{(x,y) \in \mathrm{QSAP}^\star_{m,n^\star} \mid x(T) - y(T) = 1\}$. □

To derive some basic polyhedral structures of $\mathrm{QSAP}_{m,n}$, we describe the star-transformation from geometric point of view. More precisely, we give an explicit description of an affine map such that the image of $\mathrm{QSAP}_{m,n}$ is isomorphic to $\mathrm{QSAP}^\star_{m,n^\star}$. For this purpose, we introduce an affine map $\phi : \mathbb{R}^V \times \mathbb{R}^E \to \mathbb{R}^V \times \mathbb{R}^E$ defined by

$$
(\phi(x,y))_v := \begin{cases} x_v & (\forall v \in V^\star),\\ x_v - (1 - x(\mathrm{row}^\star(v))) & (\forall v \in V \setminus V^\star),\end{cases}
$$

$$
(\phi(x,y))_e := \begin{cases} y_e & (\forall e \in E^\star),\\ y_e - (x_u - y(u : \mathrm{row}^\star(v))) & (\forall e = \{u,v\} \in E(V^\star : V \setminus V^\star)),\\ y_e - (1 - x(\mathrm{row}^\star(u) \cup \mathrm{row}^\star(v)) & \\ \quad + y(\mathrm{row}^\star(u) \cup \mathrm{row}^\star(v))) & (\forall e = \{u,v\} \in E(V \setminus V^\star)),\end{cases}
$$

where $\mathrm{row}(v) = \mathrm{row}_i$ and $\mathrm{row}^\star(v) = \mathrm{row}^\star_i$ for any $v \in \mathrm{row}_i$. Let $\mathcal{A}$ and $\mathcal{U}$ be affine subspaces defined by

$$
\mathcal{A} := \left\{ (x,y) \in \mathbb{R}^V \times \mathbb{R}^E \,\middle|\, \begin{array}{ll} x(\mathrm{row}_i) = 1 & (\forall i \in M),\\ -x_{ik} + y((i,k) : \mathrm{row}_j) = 0 & (\forall (i,k) \in V, \forall j \in M, i \neq j) \end{array} \right\}
$$

and

$$
\mathcal{U} := \{(x,y) \in \mathbb{R}^V \times \mathbb{R}^E \mid x_v = 0\ (\forall v \in V \setminus V^\star),\ y_e = 0\ (\forall e \in E \setminus E^\star)\}.
$$

It is obvious that $\mathrm{QSAP}_{m,n} \subseteq \mathcal{A}$ and $\mathrm{QSAP}^\star_{m,n^\star} \subseteq \mathbb{R}^{V^\star} \times \mathbb{R}^{E^\star} \cong \mathcal{U}$. The definition of ϕ directly implies that $\phi(\mathcal{A}) \subseteq \mathcal{U}$. Furthermore, the following lemma holds.

Lemma 7. *The map $\phi : \mathcal{A} \to \mathcal{U}$ is an affine isomorphism.*

Proof. It is easy to see that the matrix corresponding to the affine map ϕ is a lower triangular matrix whose diagonal elements are all ones. Thus, the map ϕ is an affine isomorphism. □

Let $\pi : \mathbb{R}^V \times \mathbb{R}^E \to \mathbb{R}^{V^\star} \times \mathbb{R}^{E^\star}$ be a linear map defined by $\pi(x, \bar{x}, y, \bar{y}) = (x, y)$, where $x \in \mathbb{R}^{V^\star}$, $\bar{x} \in \mathbb{R}^{V \setminus V^\star}$, $y \in \mathbb{R}^{E^\star}$, and $\bar{y} \in \mathbb{R}^{E \setminus E^\star}$. Then the star-transformation is described as

$$
\mathrm{QSAP}^\star_{m,n^\star} = \pi \circ \phi(\mathrm{QSAP}_{m,n}).
$$

We define a linear map $\iota : \mathbb{R}^{V^\star} \times \mathbb{R}^{E^\star} \to \mathbb{R}^V \times \mathbb{R}^E$ by

$$(\iota(x,y))_v := \begin{cases} x_v & (\forall v \in V^\star), \\ 0 & (\forall v \in V \setminus V^\star), \end{cases} \quad (\iota(x,y))_e := \begin{cases} y_e & (\forall e \in E^\star), \\ 0 & (\forall e \in E \setminus E^\star). \end{cases}$$

Since affine independency is invariant under ι, we have $\dim(\mathrm{QSAP}^\star_{m,n^\star}) \leq \dim(\phi(\mathrm{QSAP}_{m,n}))$. Thus, Lemma 7 implies the first result of this paper.

Theorem 8.

1. *The dimension of* $\mathrm{QSAP}_{m,n}$ *is*

$$\dim(\mathrm{QSAP}_{m,n}) = \dim(\mathbb{R}^V \times \mathbb{R}^E) - (m + \frac{1}{2}m(m-1)(2n-1)).$$

2. *The affine subspace* $\mathcal{A}$ *is the affine hull of* $\mathrm{QSAP}_{m,n}$.

Proof. Since $\phi(\mathcal{A}) = \mathcal{U}$, we have $\dim(\mathcal{A}) = \dim(\mathcal{U})$. It is easy to show that $\iota(\mathrm{QSAP}^\star_{m,n^\star}) \subseteq \phi(\mathrm{QSAP}_{m,n})$ and $\dim(\mathrm{QSAP}^\star_{m,n^\star}) \leq \dim(\phi(\mathrm{QSAP}_{m,n}))$. Hence,

$$\begin{aligned} \dim(\mathcal{A}) = \dim(\mathcal{U}) = \dim(\mathbb{R}^V \times \mathbb{R}^E) - (m + \frac{1}{2}m(m-1)(2n-1)) = \dim(\mathbb{R}^{V^\star} \times \mathbb{R}^{E^\star}) \\ = \dim(\mathrm{QSAP}^\star_{m,n^\star}) \leq \dim(\phi(\mathrm{QSAP}_{m,n})) = \dim(\mathrm{QSAP}_{m,n}) \leq \dim(\mathcal{A}). \end{aligned}$$

The equality $\dim(\mathrm{QSAP}_{m,n}) = \dim(\mathcal{A})$ implies the second statement. □

Finally, we consider a lifting of facets of $\mathrm{QSAP}^\star_{m,n^\star}$ to those of $\mathrm{QSAP}_{m,n}$. The following lemma shows that a simple lifting, called *zero-lifting*, is applicable.

Lemma 9. *If* $\sum_{v \in V^\star} a_v x_v + \sum_{e \in E^\star} b_e y_e \leq c$ *defines a facet of* $\mathrm{QSAP}^\star_{m,n^\star}$*, then the zero-lifting of the inequality, i.e.,* $\sum_{v \in V^\star} a_v x_v + \sum_{v \in V \setminus V^\star} 0 \cdot x_v + \sum_{e \in E^\star} b_e y_e + \sum_{e \in E \setminus E^\star} 0 \cdot y_e \leq c$ *defines a facet of* $\mathrm{QSAP}_{m,n}$.

Proof. The validity is clear. Since $\sum_{v \in V^\star} a_v x_v + \sum_{e \in E^\star} b_e y_e \leq c$ defines a facet of $\mathrm{QSAP}^\star_{m,n^\star}$, there exists a set $P^\star$ of $\dim(\mathrm{QSAP}^\star_{m,n^\star})$ affinely independent vectors on the facet. Lemma 7 shows that $P := \phi^{-1} \circ \iota(P^\star)$ is also a set of affinely independent vectors. It is easy to show that P is contained in the face of $\mathrm{QSAP}_{m,n}$ defined by the corresponding zero-lifting. Since $\dim(\mathrm{QSAP}_{m,n}) = \dim(\mathrm{QSAP}^\star_{m,n^\star})$, the zero-lifting defines a facet of $\mathrm{QSAP}_{m,n}$. □

By Lemma 9, we can show that the nonnegativity constraints (3) in MIP define facets of $\mathrm{QSAP}_{m,n}$

Theorem 10. *For any* $e \in E$*, the inequality* $y_e \geq 0$ *defines a facet of* $\mathrm{QSAP}_{m,n}$.

Proof. By Lemma 9, the zero-liftings of (5), (6), and (7) of $\mathrm{QSAP}^\star_{m,n^\star}$ are facet defining for $\mathrm{QSAP}_{m,n}$. For each inequality in (6) and (7), we can obtain an inequality $y_e \geq 0$ by adding some of equalities in (1) and (2). □

We remark that $0 \le x \le 1$ and $y \le 1$ are redundant constraints for MIP. Hence, they do not define facets of $\mathrm{QSAP}_{m,n}$.

Proposition 11. *For any $v \in V$ and any $e \in E$, the inequalities $0 \le x_v \le 1$ and $y_e \le 1$ are implied by* (1), (2), *and* (3).

Proof. The inequality $0 \le x_v \le 1$ holds because $x_{ik} = y((i,k) : \mathrm{row}_j))$, $y_e \ge 0$, and $x(\mathrm{row}_i) = 1$. Since $y_e \ge 0$ and $y((i,k) : \mathrm{row}_j)) = x_{ik} \le 1$, we have $y_e \le 1$. □

4 Families of Facets of $\mathrm{QSAP}_{m,n}$

In this section, we present two families of valid inequalities of $\mathrm{QSAP}_{m,n}$, called *clique-inequalities* and *cut-inequalities*, and give complete characterizations for them to be facet-defining. Since Lemma 9 ensures that it is enough to deal with the full-dimensional polytope $\mathrm{QSAP}^\star_{m,n^\star}$, we concentrate on $\mathrm{QSAP}^\star_{m,n^\star}$ throughout this section.

Padberg proposed families of valid inequalities for the Boolean quadric polytope called *clique-inequalities* and *cut-inequalities* and showed the conditions for them to be facet-defining [14]. Jünger and Kaibel introduced a family of valid inequalities for the QAP-polytope called *ST-inequalities* [9], which are also valid for the Boolean quadric polytope. Since $\mathrm{QSAP}^\star_{m,n^\star}$ is isomorphic to a face of $\mathrm{BQP}_{m\times n^\star}$, ST-inequalities of $\mathrm{BQP}_{m\times n^\star}$ are also valid for $\mathrm{QSAP}^\star_{m,n^\star}$. We note that clique- and cut-inequalities are special cases of ST-inequalities.

Proposition 12. *For any pair of subsets $S, T \subseteq V^\star$ with $S \cap T = \emptyset$ and $\beta \in \mathbb{Z}$, the ST-inequality*

$$-\beta x(S) + (\beta - 1)x(T) - y(S) - y(T) + y(S : T) \le \frac{\beta(\beta-1)}{2} \tag{8}$$

is valid for $\mathrm{QSAP}^\star_{m,n^\star}$.

Proof. The inequality $(x(T) - x(S) - \beta)(x(T) - x(S) - (\beta - 1)) \ge 0$ holds for any integer β and vector $x \in \{0,1\}^{V^\star}$. By substituting $y_{uv} = x_u x_v$, we have (8). □

We introduce *clique-inequalities*

$$(\beta - 1)x(T) - y(T) \le \frac{\beta(\beta-1)}{2} \quad (T \subseteq V^\star, \beta \in \mathbb{Z}) \tag{9}$$

and *cut-inequalities*

$$-x(S) - y(S) - y(T) + y(S : T) \le 0 \quad (S, T \subseteq V^\star, S \cap T = \emptyset). \tag{10}$$

Note that the inequalities (9) and (10) are ST-inequalities with $S = \emptyset$ and with $\beta = 0$, respectively. The clique- and cut-inequalities constitute a fairly large class of valid inequalities. For example, the facet defining inequality (5) is obtained by setting $T = \{u, v\}$ and $\beta = 1$ in (9). If we take $S = \{v\}$ and $T = \emptyset$ in (10), however, we have

$x_v \geq 0$ which does not define a facet. Thus our issue is to decide the condition for parameters S, T, and β of (9) and (10) to be facet-defining.

In Sections 4.1 and 4.2, we completely determine the conditions for the clique- and cut-inequalities to be facet-defining. In Section 4.3, we present another family of facets by simultaneous lifting.

4.1 Clique-inequalities

It is easy to see that (9) does not define a facet for any $\beta \leq 0$ and for any $T \subseteq V^\star$ with $|M(T)| = 1$, where $M(T) = \{i \in M \mid \mathrm{row}_i \cap T \neq \emptyset\}$. Hence, it is necessary that $|M(T)| \geq 2$ and $\beta \geq 1$ for (9) to be facet-defining.

Theorem 13. *Let $T \subseteq V^\star$ be a node subset with $|M(T)| = 2$. Then the clique-inequality (9) defines a facet of $\mathrm{QSAP}^\star_{m,n^\star}$ if and only if one of the following conditions holds:*

1. *$\beta = 1$ and $T = \{u, v\} \in E^\star$,*
2. *$\beta = 2$ and $T = \mathrm{row}^\star_i \cup \mathrm{row}^\star_j$ with $i, j \in M$ $(i \neq j)$.*

Proof. Assume that $\beta = 1$. If $T = \{u, v\} \in E^\star$, we can see that (9) defines a facet by Proposition 6. Conversely, suppose that $|T| \geq 3$, then (9) is the summation of $-y_e \leq 0$ $(e \in E^\star(T))$. Let $\beta = 2$. If $T = \mathrm{row}^\star_i \cup \mathrm{row}^\star_j$ with $i \neq j$, we can see that (9) defines a facet by Proposition 6. For any $T \subset \mathrm{row}^\star_i \cup \mathrm{row}^\star_j$, (9) is the summation of inequalities (5) and (6), which proves the converse. Finally, we show that (9) does not define a facet if $\beta \geq 3$. We may assume that $T = \mathrm{row}^\star(T)$. Since $x(T) - y(T) \leq 1$ defines a facet and $x(T) \leq 2$ is valid for $\mathrm{QSAP}^\star_{m,n^\star}$, we have

$$(\beta - 1)x(T) - y(T) \leq 1 + 2(\beta - 2) \leq \frac{\beta(\beta - 1)}{2}.$$

□

Theorem 14. *Let $T \subseteq V^\star$ be a node subset with $|M(T)| \geq 3$. Then the clique-inequality (9) defines a facet of $\mathrm{QSAP}^\star_{m,n^\star}$ if and only if $2 \leq \beta \leq |M(T)| - 1$.*

Proof. We first prove that if $2 \leq \beta \leq |M(T)| - 1$ then (9) defines a facet of $\mathrm{QSAP}^\star_{m,n^\star}$. Suppose that there is an inequality $a^\top x + b^\top y \leq c$ valid for $\mathrm{QSAP}^\star_{m,n^\star}$ such that

$$\begin{aligned} F := &\left\{ (x, y) \in \mathrm{QSAP}^\star_{m,n^\star} \;\middle|\; (\beta - 1)x(T) - y(T) = \frac{\beta(\beta - 1)}{2} \right\} \\ \subseteq &\{(x, y) \in \mathrm{QSAP}^\star_{m,n^\star} \mid a^\top x + b^\top y = c\}. \end{aligned}$$

(a) For any $\{u, v\}$, $\{u, w\} \in E^\star(T)$, by the condition, there exists $R \subseteq T \backslash \mathrm{row}^\star(\{u, v, w\})$ such that each of $R \cup \{v\}$ and $R \cup \{w\}$ is a $(\beta - 1)$-clique and each of $R \cup \{u, v\}$ and $R \cup \{u, w\}$ is a β-clique. Since their incidence vectors lie on F, it holds that

$$a(R \cup \{v\}) + b(R \cup \{v\}) = c, \tag{11}$$
$$a(R \cup \{u, v\}) + b(R \cup \{u, v\}) = c. \tag{12}$$

Subtracting (11) from (12), we have

$$a_u + b(u : (R \cup \{v\})) = 0. \tag{13}$$

Similarly, we have

$$a_u + b(u : (R \cup \{w\})) = 0. \tag{14}$$

From (13) and (14), $b_{uv} = b_{uw}$ holds. Thus, we can conclude that there is a number μ such that $b_e = \mu$ for any $e \in E^\star(T)$. From (13) and (11), we have

$$a_u = -(\beta - 1)\mu \quad (\forall u \in T),\ c = -\frac{\beta(\beta-1)}{2}\mu.$$

(b) For any $\{u, v\} \in E^\star(T : V^\star \setminus T)$ such that $u \in T$ and $v \in V^\star \setminus T$, there exists a $(\beta - 1)$-clique $R \subseteq T$ such that $R \cup \{u\}$ is a β-clique. Then we have

$$a(R) + b(R) = c, \tag{15}$$
$$a(R \cup \{u\}) + b(R \cup \{u\}) = c, \tag{16}$$
$$a(R \cup \{v\}) + b(R \cup \{v\}) = c, \tag{17}$$
$$a(R \cup \{u, v\}) + b(R \cup \{u, v\}) = c. \tag{18}$$

By considering $(18) - (17) - (16) + (15)$, we have $b_{uv} = 0$. Subtracting (15) from (17) yields $a_v + b(v : R) = 0$. Since $b_{uv} = 0$, we can conclude that $a_v = 0$ for any $v \in V^\star \setminus T$.

(c) For any $\{u, v\} \in E^\star(V^\star \setminus T)$, there exists a $(\beta - 1)$-clique $R \subseteq T$. Then we have

$$a(R \cup \{u, v\}) + b(R \cup \{u, v\}) = c. \tag{19}$$

From (a), (b), and (19), we can conclude that $b_{uv} = 0$.

(d) Since $(0, 0) \in \mathrm{QSAP}^\star_{m,n^\star}$ and $a^\top x + b^\top y \leq c$ is valid for $\mathrm{QSAP}^\star_{m,n^\star}$, we have $\mu \leq 0$.

From (a)–(d), $a^\top x + b^\top y \leq c$ is a nonnegative multiple of (9). Hence, (9) defines a facet of $\mathrm{QSAP}^\star_{m,n^\star}$.

Next, we prove the converse by showing that the clique-inequality (9) does not define a facet of $\mathrm{QSAP}^\star_{m,n^\star}$ if $|M(T)| \leq \beta$.

(i) Suppose $|M(T)| \leq \beta - 2$. Then it is clear that (9) does not define a facet.

(ii) Suppose $|M(T)| = \beta - 1$. We define $B := \mathrm{row}^\star(T)$. It is easy to see that

$$(\beta - 2)x(B) - y(B) \leq \frac{(\beta-1)(\beta-2)}{2}$$

is the summation of inequalities in (7). By considering a cut-inequality $-x_v+y(v:U) \leq 0$ with a proper choice of $U \subset V^\star$ for each $v \in B \setminus T$, we have

$$(\beta-2)x(T) - y(T) \leq \frac{(\beta-1)(\beta-2)}{2}.$$

Together with $x(T) \leq |M(T)| = \beta - 1$, we have

$$(\beta-1)x(T) - y(T) \leq \frac{(\beta-1)(\beta-2)}{2} + \beta - 1 = \frac{\beta(\beta-1)}{2}.$$

(iii) Suppose $|M(T)| = \beta$. Then, the proof is similar to that of $|M(T)| = \beta - 1$.

□

4.2 Cut-inequalities

Note that the cut-inequality (10) cannot be facet defining if $|M(S)| \geq 2$ and $|M(T)| = 1$. Throughout this subsection, we define $\overline{T} := V^\star \setminus T$ for any $T \subseteq V^\star$.

Let $S, T \subseteq V^\star$ be node subsets such that $S \cap T = \emptyset$ and $|M(S)| = |M(T)| = 1$. Then the cut-inequality (10) becomes

$$-x(S) + y(S:T) \leq 0 \tag{20}$$

because $E^\star(S) = E^\star(T) = \emptyset$.

Theorem 15. *Let $S, T \subseteq V^\star$ be node subsets such that $S \cap T = \emptyset$ and $|M(S)| = |M(T)| = 1$. Then the cut-inequality (20) defines a facet of $\mathrm{QSAP}^\star_{m,n^\star}$ if and only if*

1. *$M(S) \neq M(T)$,*
2. *$|S| = 1$, i.e., $S = \{u\}$ for some $u \in V^\star$, and*
3. *$T = \mathrm{row}^\star(T)$.*

Proof. Since sufficiency is proved in Proposition 6, we prove the converse. If the first condition is not satisfied, i.e., $M(S) = M(T)$, then (20) is the summation of $-x_v \leq 0$ for each $v \in S$. If the second condition is not satisfied, i.e., $|S| \geq 2$, then (20) is the summation of $-x_v + y(v : \mathrm{row}^\star(T)) \leq 0$ for each $v \in S$ and $-y(S : T \setminus \mathrm{row}^\star(T)) \leq 0$. If the third condition is not satisfied, i.e., $T \subsetneq \mathrm{row}^\star(T)$, then (20) is the summation of $-x_u + y(u : \mathrm{row}^\star(T)) \leq 0$ and $-y(S : T \setminus \mathrm{row}^\star(T)) \leq 0$. □

Theorem 16. *Let $S, T \subseteq V^\star$ be node subsets such that $S \cap T = \emptyset$, $|M(S)| \geq 1$, and $|M(T)| \geq 2$. Then the cut-inequality (10) defines a facet of $\mathrm{QSAP}^\star_{m,n^\star}$ if and only if*

1. *$\forall \{w_1, w_2\} \in E^\star(T)$, $\exists u \in S$ such that $\{u, w_1, w_2\}$ is a 3-clique, and*
2. *$\forall \{u_1, u_2\} \in E^\star(S)$, $\exists \{w_1, w_2\} \in E^\star(T)$ such that $\{u_1, u_2, w_1, w_2\}$ is a 4-clique.*

Proof. First, we prove the sufficiency. Suppose that there is an inequality $a^\top x + b^\top y \leq c$ valid for $\mathrm{QSAP}^\star_{m,n^\star}$ such that

$$\begin{aligned} F :=& \{(x,y) \in \mathrm{QSAP}^\star_{m,n^\star} \mid -x(S) - y(S) - y(T) + y(S:T) = 0\} \\ \subseteq& \{(x,y) \in \mathrm{QSAP}^\star_{m,n^\star} \mid a^\top x + b^\top y = c\}. \end{aligned}$$

(a) Since $(0,0) \in F$, we have $c = 0$.

(b) For any $v \notin S$, $(\chi_v, 0) \in F$ implies $a_v = 0$.

(c) For any $\{v_1, v_2\} \in E^\star(\overline{S \cup T})$, $(\chi_{v_1} + \chi_{v_2}, \chi_{v_1 v_2}) \in F$ implies $b_{v_1 v_2} = 0$.

(d) For any $\{v, w\} \in E^\star(\overline{S \cup T} : T)$, $(\chi_v + \chi_w, \chi_{vw}) \in F$ implies $b_{vw} = 0$.

(e) For any $\{u, w\} \in E^\star(S : T)$, $(\chi_u + \chi_w, \chi_{uw}) \in F$ implies $a_u + b_{uw} = 0$.

(f) For any $\{u, v\} \in E^\star(S : \overline{S \cup T})$ such that $u \in S$ and $v \in \overline{S \cup T}$, by the assumption $|M(T)| \geq 2$, there exists $w \in T$ such that $\{u, w\} \in E^\star(S : T)$. Hence, $b_{uv} = 0$.

(g) For any $\{w_1, w_2\} \in E^\star(T)$, by the first condition, there exists $u \in S$ such that $\{u, w_1, w_2\}$ is a 3-clique. Each 3-clique contains two 2-cliques $\{u, w_1\}$ and $\{u, w_2\}$. Thus $b_{uw_1} = b_{uw_2} = -a_u$ and $b_{w_1 w_2} = a_u$. Furthermore, for any $u_1, u_2 \in S$ with $M(u_1) = M(u_2)$, there exists $\{w_1, w_2\} \in E^\star(T)$ such that both of $\{u_1, w_1, w_2\}$ and $\{u_2, w_1, w_2\}$ are 3-cliques, since otherwise we must have $|M(T)| = 2$ and $M(u_1) = M(u_2) \subseteq M(T)$, contradicting the first condition. Thus $a_{u_1} = a_{u_2} = b_{w_1 w_2}$. Hence, for each $i \in M(S)$, there exists a number μ_i such that $\forall u \in S \cap \mathrm{row}^\star_i$, $\forall \{u, w\} \in E^\star(S \cap \mathrm{row}^\star_i : T)$, and $\forall \{w_1, w_2\} \in E^\star(T)$, $a_u = -b_{uw} = b_{w_1 w_2} = \mu_i$.

(h) For any $\{u_1, u_2\} \in E^\star(S)$, by the condition 2, there exist $w_1, w_2 \in T$ such that $\{u_1, u_2, w_1, w_2\}$ is a 4-clique. Each 4-clique contains two 3-cliques $\{u_1, w_1, w_2\}$ and $\{u_2, w_1, w_2\}$. It follows that $a_{u_1} = a_{u_2}$. Thus we can conclude that $\mu_i = \mu_j$ for any $i, j \in M(S)$. Hence, there exists a number μ such that $\forall u \in S$, $\forall \{u, w\} \in E^\star(S : T)$, $\forall \{w_1, w_2\} \in E^\star(T)$, and $\forall \{u_1, u_2\} \in E^\star(S)$, $b_{uw} = -a_u = -b_{w_1 w_2} = -b_{u_1 u_2} = \mu$.

Since $(\chi_u, 0) \in \mathrm{QSAP}^\star_{m,n^\star}$ for any $u \in S$, we have $\mu \geq 0$.

Second, we prove the converse. Suppose that the first condition does not hold, i.e., there exists $e = \{w_1, w_2\} \in E^\star(T)$ such that $\{u, w_1, w_2\}$ is not a 3-clique for any $u \in S$. It is sufficient to show the validity of the inequality

$$-x(S) - y(S) - y(T) + y(S:T) + y_e \leq 0 \tag{21}$$

because (10) is the summation of $-y_e \leq 0$ and (21). For any clique C of the graph $G^\star$, we put $(x, y) = (\chi_C, \chi_{E^\star(C)})$. If $e \notin E^\star(C)$, it is clear that (x, y) satisfies (21). Otherwise, it is easy to see that $S \cap C = \emptyset$ holds because the first condition does not hold. Thus, we have $x(S) = y(S) = y(S : T) = 0$. Hence, the left hand side of (21) is equal to $-y(T) + y_e = -y(T \setminus \{w_1, w_2\}) \leq 0$. Next, suppose that the second condition does not hold, i.e., there exists $e = \{u_1, u_2\} \in E^\star(S)$ such that $\{u_1, u_2, w_1, w_2\}$ is not a 4-clique for any $\{w_1, w_2\} \in E^\star(T)$. It is sufficient to show that $(x, y) = (\chi_C, \chi_{E^\star(C)})$ satisfies (21) for any clique C of $G^\star$. If $e \notin E^\star(C)$, it is clear that (x, y) satisfies (21). Otherwise, it is easy to see that $|T \cap C| \leq 1$ holds because the second condition does not hold. Since $|T \cap C| \leq 1$, we have $y(S : T) \leq x(S)$. Thus we have $-x(S)-y(S)-y(T)+y(S:T)+y_e \leq -y(S)-y(T)+y_e = -y(S\setminus\{u_1,u_2\})-y(T) \leq 0$. □

4.3 Simultaneous Lifting of Clique-type Facets

We discuss a simultaneous lifting of facets. We characterize the lifted facets by vertices of some polyhedron. Sherali et al. also dealt with a simultaneous lifting of facets for the Boolean quadric polytope [17] (see also [6]).

When $T \subseteq V^\star$ is a node subset such that $3 \leq |M(T)| \leq m-1$, Theorem 14 implies that the clique inequality

$$x(T) - y(T) \leq 1 \tag{22}$$

defines a facet of $\mathrm{QSAP}^\star_{|M(T)|,n^\star}$. For any $U \subseteq V^\star \setminus \mathrm{row}^\star(T)$ with $|M(U)| = 1$, we consider a lifting of (22) as follows:

$$x(T) - y(T) + px(U) + qy(U:T) \leq 1 \tag{23}$$

for some $p, q \in \mathbb{R}$.

We determine the parameters (p, q) for the inequality (23) to be facet-defining. Let $W \subseteq T$ be an l-clique, where $0 \leq l \leq |M(T)|$. Since $(\chi_W, \chi_{E^\star(W)}) \in \mathrm{QSAP}^\star_{m,n^\star}$, it is necessary that

$$l - \frac{l(l-1)}{2} + p + ql \leq 1 \tag{24}$$

for the inequality (23) to be valid. It is obvious that (p, q) satisfies (24) if and only if $p + lq \leq (l-2)(l-1)/2$. Thus it is necessary for the inequality (23) to be valid that (p, q) is in a polyhedron $Q(T)$ defined by

$$Q(T) = \{(p, q) \in \mathbb{R}^2 \mid p + lq \leq \frac{(l-2)(l-1)}{2} \ (0 \leq l \leq |M(T)|)\}.$$

It is easy to determine the vertices and extreme rays of $Q(T)$ (see Figure 3).

Lemma 17. *The set of vertices of $Q(T)$ is $\{(-(l-1)(l+2)/2, l-1) \mid 0 \leq l \leq |M(T)|-1\}$ and that of extreme rays of $Q(T)$ is $\{(0,-1), (-(|M(T)|-1), 1)\}$.*

Theorem 18. *Let $T, U \subseteq V^\star$ be a pair of node subsets satisfying $M(T) \cap M(U) = \emptyset$, $3 \leq |M(T)| \leq m-1$, and $|M(U)| = 1$. Then the inequality (23) defines a facet of $\mathrm{QSAP}^\star_{m,n^\star}$ if and only if (p, q) is a vertex of $Q(T)$.*

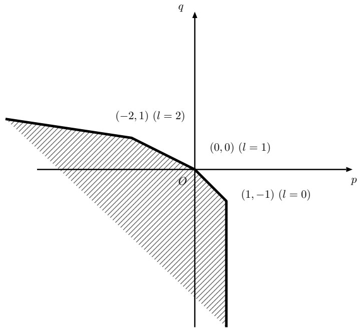

Figure 3: The polyhedron $Q(T)$ with $|M(T)| = 3$.

Proof. It is clear that the inequality (23) is valid to $\mathrm{QSAP}^{\star}_{m,n^{\star}}$ for any $(p, q) \in Q(T)$. If $(p, q) \in Q(T)$ is not a vertex of $Q(T)$, then (p, q) is a convex and conic combination of some vertices and extreme rays of $Q(T)$, which implies (23) is the summation of other inequalities.

Conversely, suppose that (p, q) is a vertex of $Q(T)$, Lemma 17 requires that $(p, q) = (-(l-1)(l+2)/2, l-1)$ for some l satisfying $0 \le l \le |M(T)| - 1$. To show that (23) defines a facet, suppose that there is an inequality $a^{\top} x + b^{\top} y \le c$ valid for $\mathrm{QSAP}^{\star}_{m,n^{\star}}$ such that

$$\begin{aligned} F := & \{(x, y) \in \mathrm{QSAP}^{\star}_{m,n^{\star}} \mid x(T) - y(T) - \frac{(l-1)(l+2)}{2} x(U) + (l-1) y(U : T) = 1\} \\ \subseteq & \{(x, y) \in \mathrm{QSAP}^{\star}_{m,n^{\star}} \mid a^{\top} x + b^{\top} y = c\}. \end{aligned}$$

(a) For any $u \in T$, $(\chi_u, 0) \in F$ implies $a_u = c$.

(b) For any $\{u, v\} \in E^{\star}(T)$, $(\chi_u + \chi_v, \chi_{uv}) \in F$ implies $b_{uv} = -c$.

(c) Let W be an l-clique in T. For any $v \in U$, the incidence vector of $W \cup \{v\}$ lies on F. Then we have

$$cl - c\frac{l(l-1)}{2} + a_v + b(v : W) = c. \tag{25}$$

(d) Let W be an l-clique in T, then there exists a node u in $T \setminus \mathrm{row}^{\star}(W)$ such that $W \cup \{u\}$ an $(l+1)$-clique in T. For any $v \in U$, the incidence vector of $W \cup \{u, v\}$ lies on F. Thus we have

$$c(l+1) - c\frac{l(l+1)}{2} + a_v + b(v : W) + b_{uv} = c. \tag{26}$$

By subtracting (25) from (26), we have $b_{uv} = c(l-1)$. Hence $a_v = -c(l-1)(l+2)/2$.

(e) For any $w \in \overline{T \cup U}$, there exists $u \in T$ such that $\{u, w\} \in E^\star(T : \overline{T \cup U})$ because $|M(T)| \geq 3$. Since the incidence vector of $\{u, w\}$ lies on F, we have $a_w + b_{uw} = 0$.

(f) For any $\{w_1, w_2\} \in E^\star(\overline{T \cup U})$, there exists $u \in T$ such that $\{u, w_1, w_2\}$ is a 3-clique because $|M(T)| \geq 3$. Since the incidence vector of $\{u, w_1, w_2\}$ lies on F, we have $b_{w_1 w_2} = 0$.

(g) For any $w \in \overline{T \cup U}$, there exists $\{u_1, u_2\} \in E^\star(T)$ such that $\{u_1, u_2, w\}$ is a 3-clique because $|M(T)| \geq 3$. Since the incidence vector of $\{u_1, u_2, w\}$ lies on F, we have $a_w = 0$, and thus $b_e = 0$ for any $e \in E^\star(T : \overline{T \cup U})$.

(h) For any $\{v, w\} \in E^\star(U : \overline{T \cup U})$ with $v \in U$ and $w \in \overline{T \cup U}$, take any l-clique W in T. Since the incidence vectors of $W \cup \{v\}$ and $W \cup \{v, w\}$ lie on F, we have $b_{vw} = 0$.

(i) Since $(0, 0) \in \mathrm{QSAP}^\star_{m,n^\star}$ and $a^\top x + b^\top y \leq c$ is valid for $\mathrm{QSAP}^\star_{m,n^\star}$, we have $c \geq 0$.

□

5 Conclusions

We investigated a polytope arising from the quadratic semi-assignment problem. By introducing the star-transformation, we obtained basic polyhedral properties of the dimension, the affine hull and trivial facets in quite a simple way. We also showed that the affine hull and the trivial facets correspond to the equality and the inequality constraints in the known MIP formulation, respectively. We obtained families of facets called clique- and cut-inequalities and gave complete characterizations for them to be facet-defining. We derived another class of facets from the clique-type facets by simultaneous lifting. Finally, we remark that there are some facets except the above ones (see Appendix A).

A Other Facets of $\mathrm{QSAP}^\star_{m,n^\star}$

In this appendix, we show some facet defining inequalities of $\mathrm{QSAP}^\star_{m,n^\star}$ which are neither clique- nor cut-inequalities.

Proposition 19. *Let $S, T \subseteq V^\star$ be node subsets such that $S \cap T = \emptyset$, $|M(S)| = 2$, $|M(T)| \geq 4$, and $|M(S) \cap M(T)| = 2$. Then the inequality*

$$-x(S) - y(S) - y(T) + y(S : T) + y(T_1) \leq 0 \tag{27}$$

defines a facet of $\mathrm{QSAP}^\star_{m,n^\star}$, where $T_1 = T \cap \mathrm{row}^\star(S)$.

Proof. First, we show the validity of the inequality. For any clique C of the graph $G^\star$, we define the vector $(x, y) = (\chi_C, \chi_{E^\star(C)})$. When $|C \cap T_1| \le 1$, we have $y(T_1) = 0$. When $|C \cap T_1| = 2$, we have $C \cap S = \emptyset$ and thus $y(S : T) = 0$. Hence, for each case, the vector (x, y) satisfies the inequality (27).

Next, suppose that there is an inequality $a^\top x + b^\top y \le c$ valid for $\mathrm{QSAP}^\star_{m,n^\star}$ such that

$$\begin{aligned} F :=& \{(x, y) \in \mathrm{QSAP}^\star_{m,n^\star} \mid -x(S) - y(S) - y(T) + y(T_1) + y(S : T) = 0\} \\ \subseteq& \{(x, y) \in \mathrm{QSAP}^\star_{m,n^\star} \mid a^\top x + b^\top y = c\}. \end{aligned}$$

Then, by the following observation, we have the desired result.

(a) Since $(0, 0) \in F$, we have $c = 0$.

(b) For any $v \notin S$, we have $a_v = 0$.

(c) For any $e \in E^\star(\overline{S \cup T})$, we have $b_e = 0$.

(d) For any $e \in E^\star(T_1)$, we have $b_e = 0$.

(e) For any $e \in E^\star(\overline{S \cup T} : T)$, we have $b_e = 0$.

(f) For any $\{u, w\} \in E^\star(S : T)$ such that $u \in S$ and $w \in T$, we have $a_u + b_{uw} = 0$.

(g) For any $\{u, v\} \in E^\star(S : \overline{S \cup T})$ such that $u \in S$ and $v \in \overline{S \cup T}$, there exists $w \in T$ such that $\{u, w\} \in E^\star(S : T)$ because $|M(T)| \ge 4$. Thus we have $b_{uv} = 0$.

(h) Let T_2 be $T_2 = T \setminus T_1$. For any $\{w_1, w_2\} \in E^\star(T_2)$ and for any $u \in S$, $\{u, w_1, w_2\}$ is a 3-clique because $|M(T)| \ge 4$ and $|M(S) \cap M(T)| = 2$. Thus, for any $u \in S$ and any $e \in E^\star(T_2)$, there exists a number μ such that $a_u = b_e = \mu$.

(i) For any $\{u_1, u_2\} \in E^\star(S)$, there exist $w_1, w_2 \in T$ such that $\{u_1, u_2, w_1, w_2\}$ is a 4-clique, because $|M(T)| \ge 4$. Thus, we have $b_e = \mu$ for any $e \in E^\star(S)$.

(j) For any $\{w_1, w_2\} \in E^\star(T_1 : T_2)$, there exists $u \in S$ such that $\{u, w_1, w_2\}$ is a 3-clique because $|M(S)| = 2$ and $M(S) \cap M(T_2) = \emptyset$. Thus, we have $b_e = \mu$ for any $e \in E^\star(T_1 : T_2)$.

Hence, the inequality (27) is a product of μ and $a^\top x + b^\top y \le c$. Since $(\chi_u, 0) \in \mathrm{QSAP}^\star_{m,n^\star}$ for any $u \in S$, we have $\mu \ge 0$.

□

Proposition 20. *Let $S, T \subseteq V^\star$ be node subsets such that $S \cap T = \emptyset$, $|M(S)| \ge 3$, $|M(T)| = 3$, and $2 \le |M(S) \cap M(T)| \le 3$. Then the inequality*

$$-x(S) - y(S) - y(T) + y(S : T) + y(S_1) \le 0 \tag{28}$$

defines a facet of $\mathrm{QSAP}^\star_{m,n^\star}$, where $S_1 = S \cap \mathrm{row}^\star(T)$.

Proof. First, we show the validity of the inequality. For any clique C of the graph $G^\star$, we define the vector $(x, y) = (\chi_C, \chi_{E^\star(C)})$. When $|C \cap S_1| \leq 1$, we have $y(S_1) = 0$. When $|C \cap S_1| = 2$, we have $x(S) = y(S : T)$. When $|C \cap S_1| = 3$, we have $C \cap T = \emptyset$ and thus $y(S : T) = 0$. Hence, for each case, the vector (x, y) satisfies the inequality (28).

Next, suppose that there is an inequality $a^\top x + b^\top y \leq c$ valid for $\mathrm{QSAP}^\star_{m,n^\star}$ such that

$$\begin{aligned} F :=&\{(x,y) \in \mathrm{QSAP}^\star_{m,n^\star} \mid -x(S) - y(S) + y(S_1) - y(T) + y(S:T) = 0\} \\ \subseteq&\{(x,y) \in \mathrm{QSAP}^\star_{m,n^\star} \mid a^\top x + b^\top y = c\}. \end{aligned}$$

Then, by the following observation, we have the desired result.

(a) Since $(0, 0) \in F$, we have $c = 0$.

(b) For any $v \notin S$, we have $a_v = 0$.

(c) For any $e \in E^\star(\overline{S \cup T})$, we have $b_e = 0$.

(d) For any $e \in E^\star(\overline{S \cup T} : T)$, we have $b_e = 0$.

(e) For any $\{u, w\} \in E^\star(S : T)$ such that $u \in S$ and $w \in T$, we have $a_u + b_{uw} = 0$.

(f) For any $\{u_1, u_2\} \in E^\star(S_1)$, there exists $w \in T$ such that $\{u_1, u_2, w\}$ is a 3-clique because $|M(T)| = 3$. Since $\{u_1, w\}, \{u_2, w\} \in E^\star(S : T)$, we have $b_{u_1u_2} = 0$.

(g) For any $\{u, v\} \in E^\star(S : \overline{S \cup T})$ such that $u \in S$ and $v \in \overline{S \cup T}$, there exists $w \in T$ such that $\{u, w\} \in E^\star(S : T)$. Thus, we have $b_{uv} = 0$.

(h) For any $u \in S_2$ and any $\{w_1, w_2\} \in E^\star(T)$, $\{u, w_1, w_2\}$ is a 3-clique. Thus there exists a number μ such that $b_e = a_u = \mu$ for any $u \in S_2$ and any $e \in E^\star(T)$.

(i) For any $u \in S_1$, there exists $\{w_1, w_2\} \in E^\star(T)$ $\{u, w_1, w_2\}$ is a 3-clique because $|M(T)| = 3$. Thus, we have $a_u = \mu$ for any $u \in S_1$.

(j) For any $\{u_1, u_2\} \in E^\star(S) \setminus E^\star(S_1)$, there exists $\{w_1, w_2\} \in E^\star(T)$ such that $\{u_1, u_2, w_1, w_2\}$ is a 4-clique because $|M(T)| = 3$. Thus, we have $b_{u_1u_2} = \mu$.

Hence, the inequality (28) is a product of a number μ and $a^\top x + b^\top y \leq c$. Since $(\chi_u, 0) \in \mathrm{QSAP}^\star_{m,n^\star}$ for any $u \in S$, we have $\mu \geq 0$.

□

B Computational Results

In this appendix, we apply our results on QSAP to the hub location problem (HLP) and present some computational results.

The hub location problem arose in the airline industry, telecommunications, and postal delivery systems (see a survey paper [4]). We consider a variant of HLP, which we call *hub network design problem* [16, 19]. The hub-and-spoke structure consists of two sets of nodes called *hubs* and *non-hubs*. We denote the set of hubs by H and non-hubs by NH, respectively. We assume that hubs are completely interconnected and any pair of non-hubs can interact only via hubs. For any ordered pair of nodes (i, j), $f_{ij} \geq 0$ denotes the amount of flow from i to j. Suppose further that the transportation cost $d_{ij} \geq 0$ per unit flow is defined for any pair of nodes except that of non-hubs because direct interactions between non-hubs are not allowed. The hub network design problem finds a connection of each non-hub to exactly one of the hubs which minimizes the total transportation cost. Here, each flow is assigned to a minimum cost route among the routes allowed by the connection. The problem is formulated as follows, where we drop the transportation cost associated with pairs of hubs because it is a constant.

$$\begin{aligned}
\text{HLP} \quad \text{min.} \quad & \sum_{i \in NH} \sum_{j \in NH} f_{ij} \left(\sum_{k \in H} d_{ik} x_{ik} + \sum_{k \in H} \sum_{l \in H} d_{kl} x_{ik} x_{jl} + \sum_{l \in H} d_{lj} x_{jl} \right) \\
& + \sum_{i \in NH} \sum_{l \in H} f_{il} \sum_{k \in H} (d_{ik} + d_{kl}) x_{ik} + \sum_{k \in H} \sum_{j \in NH} f_{kj} \sum_{l \in H} (d_{kl} + d_{lj}) x_{jl} \\
\text{s. t.} \quad & \sum_{k \in H} x_{ik} = 1 \qquad\qquad (\forall i \in NH), \\
& x_{ik} \in \{0, 1\} \qquad\qquad (\forall i \in NH, \forall k \in H).
\end{aligned}$$

The above formulation appears in the paper [16], for example. It is not hard to see that HLP is a special case of QSAP, where $M = NH$ and $N = H$. The problem HLP is known to be NP-hard even if $|H| = 3$ (see [19]). Some polyhedral results of hub network design problems and hub location problems appear in [7, 10, 11].

Next, we describe our computational experiments for HLP. Experiments were performed on PC with Celeron 1.2 GHz CPU, using glpk 3.2.3[1] to solve linear relaxations. We first solved a linear relaxation of MIP. If a non-integer optimal solution is found, we added facet defining inequalities as cutting planes. We have the following simple facet defining inequalities (29) and (30) as corollaries of the theorems in Section 4. The inequalities (29) and (30) are called *triangle inequalities* of BQP_n in [14].

Corollary 21. *For any* $T = \{u, v, w\} \subseteq V^\star$ *with* $|M(T)| = 3$, *the clique-inequality*

$$x_u + x_v + x_w - y_{uv} - y_{uw} - y_{vw} \leq 1 \tag{29}$$

defines a facet of $\text{QSAP}^\star_{m,n^\star}$.

Corollary 22. *For any* $S = \{u\}$ *and* $T = \{v, w\} \subseteq V^\star$ *with* $|M(S) \cup M(T)| = 3$, *the cut-inequality*

$$-x_u + y_{uv} + y_{uw} - y_{vw} \leq 0 \tag{30}$$

defines a facet of $\text{QSAP}^\star_{m,n^\star}$.

[1] http://www.gnu.org/software/glpk/glpk.html

The computation was done with two kinds of data sets called CAB data [12] and AP data [5]. Both data sets are available at the web site [2]. CAB data is based on the actual airline data between 25 major US cities. The instances were generated as follows. We dealt with the case where the number of hubs is four, i.e., $h = 4$. We chose 10, 15, 20, and 25 cities out of the total 25 cities. For each case, since hubs were not specified in the data set, we generated instances according to the every combination of hubs. In addition, we introduced a parameter $0 \leq \alpha \leq 1$ called discount factor by O'Kelly [12]. The parameter shows the discount of costs among hubs and replaces d_{kl} by αd_{kl}. We dealt with the case where α=1.0, 0.8, 0.6, and 0.4, respectively. Tables 1 and 2 show that integer optimal solutions are obtained for almost all instances only by solving linear relaxation of MIP. For the remaining instances, we successfully obtained integer optimal solutions by adding our cuts. Computational times took at most only a few seconds for each instance. AP data is derived from the mail flows in Australia [5]. Since hubs were not specified in the data set, we chose an optimal location of hubs, which is already known [5]. Table 3 shows that integer optimal solutions are obtained for all instances without adding any cuts. Computational times took at most only a few minutes for each instance.

Table 1: Computational results for CAB data with $n = 4$ (1)

n	$m+n$	number of instances	α	non-integer instances	non-integer instances with cuts		time [s] max and average
4	10	210	1.0	0	0	max	0.010
						average	0.003
		210	0.8	0	0	max	0.010
						average	0.003
		210	0.6	0	0	max	0.010
						average	0.003
		210	0.4	0	0	max	0.010
						average	0.003
	15	1365	1.0	3	0	max	0.100
						average	0.014
		1365	0.8	3	0	max	0.080
						average	0.014
		1365	0.6	0	0	max	0.070
						average	0.011
		1365	0.4	0	0	max	0.070
						average	0.011

Table 2: Computational results for CAB data with $n = 4$ (2)

n	$m+n$	number of instances	α	non-integer instances	non-integer instances with cuts		time [s] max and average
4	20	4845	1.0	3	0	max	0.560
						average	0.099
		4845	0.8	1	0	max	0.460
						average	0.081
		4845	0.6	0	0	max	0.570
						average	0.079
		4845	0.4	0	0	max	0.460
						average	0.071
	25	12650	1.0	5	0	max	2.820
						average	0.404
		12650	0.8	9	0	max	2.970
						average	0.394
		12650	0.6	1	0	max	2.580
						average	0.342
		12650	0.4	1	0	max	2.560
						average	0.294

Table 3: Computational results for AP data

$n = 2$	$m+n$	10	20	25	40	50
	time [s]	0.000	0.120	0.380	5.220	17.500
3		10	20	25	40	50
		0.010	0.270	1.030	18.010	53.090
4		10	20	25	40	50
		0.010	0.490	2.250	37.290	110.980
5		10	20	25	40	50
		0.010	0.810	3.860	72.080	208.230

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