MATHEMATICAL ENGINEERING TECHNICAL REPORTS

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(Communicated by Kazuo Murota)

METR 2004–34 June 2004

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June 9, 2004

Abstract

In this paper, we consider convergence and stability analysis for a class of bimodal piecewise linear systems. We first provide two necessary and sufficient conditions for the planar bimodal piecewise linear systems to be stable. These two conditions are given in terms of eigenvalue loci of subsystems and coefficients of characteristic polynomials, respectively. Then, an extension to the higher-order case is addressed. We discuss some properties of trajectories of the higher-order bimodal piecewise linear systems, and we derive a necessary condition and a sufficient condition for the stability. The conditions are given in terms of the eigenvalue loci and the observability of subsystems. The sufficient condition for stability allows existence of unstable dynamics, while the conventional Lyapunov approach requires that all subsystems are stable. Furthermore, we discuss a stabilizing controller design based on the derived sufficient condition.

1 Introduction

Hybrid control has been paid much attention in the area of control system design, because we have many practical control applications which contain both continuous-time dynamical systems and logical or switching elements. There have been a lot of mathematical models proposed to represent behavior of hybrid control systems. One of the typical models is the piecewise linear system (PLS); The system consists of some pairs of linear time-invariant dynamics and a cell which is a piece of a partition of the state space, and the state evolves along the dynamics corresponding to the cell in which the state exists. The class of PLSs is one of the fundamental classes of hybrid dynamical systems, because the continuous dynamics is linear in each cell and the discrete dynamics is the simplest one. Therefore, study on PLSs is important as a first step to establish hybrid control theory.

In spite of recent progress in hybrid control theory, there still remain fundamental issues to be clarified. The most fundamental issue is stability. Recently, many results based on Lyapunov functions have been obtained on stability for several classes of hybrid dynamical systems (see [3, 5, 14] and the references therein), where we need to show the existence of a Lyapunov function which guarantees the stability. On the other hand, no converse theorem[1] has been derived due to its hybrid nature. In other words, no necessary and sufficient stability condition based on Lyapunov functions has been derived for any classes of hybrid dynamical systems.

In fact, we can not completely check the stability even for the class of PLSs. In general, direct applications of the Lyapunov methods to the class of PLSs lead to not any necessary and sufficient conditions but only sufficient conditions for the stability. In addition, we must restrict the available class of Lyapunov functions within a class of piecewise quadratic functions to give a systematic way of finding the Lyapunov functions [16, 17]. This also causes the conservativeness of the stability conditions. Therefore, we need a new approach to get a less conservative stability condition or hopefully to derive a necessary and sufficient condition for the stability.

To this end, we here try to investigate the stability problem for bimodal PLSs (BPLSs) from a different perspective. Instead of Lyapunov methods, we focus on eigenvalue loci of subsystems to investigate the stability. We discuss several properties of trajectories of BPLSs. In particular, a necessary condition and a sufficient condition for stability are derived. The conditions are given in terms of the eigenvalue loci and the observability of subsystems. The sufficient condition for stability allows existence of unstable dynamics, while the piecewise quadratic Lyapunov approach for BPLSs requires that all subsystems are stable [19]. Furthermore, we can derive two necessary and sufficient conditions for the planar BPLSs. The first one is characterized by the eigenvalue loci of subsystems. The second one consists of a condition for each subsystem and a coupling condition, and they are given in terms of coefficients of characteristic polynomials of subsystems.

This paper is organized as follows. Section 2 describes basic setup for representing a class of BPLSs. Section 3 is devoted to stability analysis of the planar BPLS. We give a necessary and sufficient condition for the planar BPLS to be stable in terms of eigenvalue loci of the subsystems. Moreover, another necessary and sufficient stability condition is provided in terms of coefficients of characteristic polynomials in Section 4. We give some extensions in Section 5. A necessary condition and a sufficient condition for stability are derived in Sections 5.1 and 5.2, respectively. In Section 6, our

[1] For smooth dynamical systems, there are theorems such that the given conditions are necessary for stability. Such theorems are usually called converse theorems [13].

results are applied to the stabilization problem, i.e. we propose a method of designing a stabilizing controller based on the derived sufficient condition for stability.

In this paper, we will use the following notation. The symbols $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}_+$ represent the set of integers, the set of real numbers, and the set of positive real numbers, respectively. The symbols $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ stand for the set of all n-dimensional real column vectors and the set of all $n \times m$ real matrices, respectively. Let an n-dimensional vector $x = [x_1, x_2, \ldots, x_n]$ be given. If $x_i > 0$ and $x_1 = \ldots = x_{i-1} = 0$ for some i, we denote it by $x \succ 0$. Furthermore, if $x = 0$ or $x \succ 0$, we denote it by $x \succeq 0$. Also, $x \prec 0$ and $x \preceq 0$ mean that $-x \succ 0$ and $-x \succeq 0$, respectively.

2 Bimodal piecewise linear systems

We consider a class of bimodal piecewise linear systems (BPLSs) represented by

$$\dot{x} = \begin{cases} A_1 x, & \text{if } y \geq 0, \\ A_2 x, & \text{if } y \leq 0, \end{cases} \tag{1}$$

$$y = cx, \tag{2}$$

where $A_1, A_2 \in \mathbb{R}^{n\times n}$, and $c\ (\neq 0) \in \mathbb{R}^{1\times n}$. See Figure 1.

Throughout this paper, we assume that the BPLS (1)–(2) is well-posed, i.e. the BPLS has a unique solution for each initial state. Definition of well-posedness can be chosen from well-posedness in the sense of Carathéodory [8], C-well-posedness, H-well-posedness [6], and well-posedness under the switch-driven rule [7]. The choice does not influence our results shown below, although the sliding mode phenomenon [4, 18] may cause some problems.

We will use the following notation under the well-posedness assumption. The solution from a given initial state x_0 is denoted by $x(t, x_0)$ where the initial time is always set 0. Furthermore, the corresponding variable y is denoted by $y(t, x_0)$.

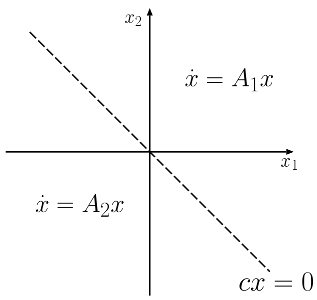

Figure 1: Bimodal piecewise linear system (1) –(2) with $n = 2$.

We here define two special subsets of the state space as follows:

$$\begin{aligned}\mathbb{S}_1 &= \{x \in \mathbb{R}^n \mid U_1 x \succeq 0\},\\ \mathbb{S}_2 &= \{x \in \mathbb{R}^n \mid U_2 x \preceq 0\},\end{aligned}$$

where U_i represents the observable matrix of the pair (c, A_i), i.e.

$$U_i = \left[c^\top, (cA_i)^\top, \ldots, (cA_i^{n-1})^\top\right]^\top, \tag{3}$$

for each i $(= 1, 2)$. Let an initial state $x_0 \in \mathbb{S}_1$ be given. Then no mode transition takes place for any $t \geq 0$, if and only if $x(t, x_0) \in \mathbb{S}_1$ holds for all $t \geq 0$. The property also holds when the set $\mathbb{S}_1$ is replaced by $\mathbb{S}_2$.

We then describe several mathematical definitions of stability that will be investigated in this paper. The origin is called *stable* if, for each $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that

$$\|x_0\| < \delta(\varepsilon) \Rightarrow \|x(t, x_0)\| < \varepsilon, \quad \forall t \geq 0 \tag{4}$$

holds, that is, we use the term 'stable' in the sense of Lyapunov. The origin is called *attractive* if $\lim_{t\to\infty} x(t, x_0) = 0$ for any initial state x_0. The origin is called *globally asymptotically stable* if it is stable and attractive.

A conventional systematic way to check global asymptotic stability of a class of PLSs is to look for piecewise quadratic Lyapunov functions with S-procedure [16, 17]. Let us begin with an investigation of validity of the method in the class of BPLSs.

Proposition 1 ([19]) *Consider the BPLS (1)–(2). Then there exists a quadratic Lyapunov function with S-procedure for the system, only if both A_1 and A_2 are Hurwitz.*

Proposition 1 implies that no piecewise quadratic Lyapunov functions exist for systems with unstable dynamics, even if the origin is stable as illustrated in an example shown below. In other words, the piecewise quadratic Lyapunov method can not lead to any necessary and sufficient conditions. This motivates us to propose a new approach for stability analysis which is different from the quadratic Lyapunov method.

Example 2 Let us consider a BPLS with

$$\begin{aligned}A_1 &= \begin{bmatrix} -1, & 1 \\ -1, & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1, & 3 \\ -3, & 1 \end{bmatrix},\\ c &= \begin{bmatrix} 1, & 0 \end{bmatrix}.\end{aligned}$$

Figure 2 illustrates a trajectory of the system. Clearly the system is globally asymptotically stable, although $\dot{x} = A_2 x$ is unstable.

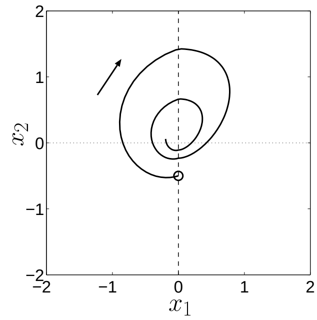

Figure 2: Trajectory of Example 2.

3 Stability analysis for the planar bimodal piecewise linear system

Our focus in this section is restricted to the planar system, i.e. $n = 2$. Fortunately, we can obtain a necessary and sufficient condition for stability in this case.

We first show the following two lemmas related to stability of the system. The lemmas will be extended to the higher-order case in Section 5.

Lemma 3 *Consider the system (1)–(2) with* $n = 2$*. The following two statements (a) and (b) are equivalent for each* i $(= 1, 2)$*.*

(a) *There exists an initial state* x_0 *satisfying the following two conditions:*

- **(a-1)** $\forall t \geq 0, \ \ x(t, x_0) \in \mathbb{S}_i$,
- **(a-2)** $\lim_{t\to\infty} x(t, x_0) \neq 0$.

(b) *The matrix* A_i *has a non-negative real eigenvalue.*

PROOF. 1 It can be proved by Lemma 10 shown in Section 5 which is a generalization of Lemma 3. See several remarks followed by Lemma 10 for the detail. □

Applying Lemma 3 to the system (1)–(2), we see that the origin is not asymptotically stable, if either A_1 or A_2 has a non-negative real eigenvalue. Then assume that the condition (b) holds. As mentioned before, there exists an initial state such that no mode transitions occur in this case. Conversely, suppose that the condition (b) does not hold. Then, we see that a mode transition necessarily takes place, if the trajectory does not converge to 0 as time goes to infinity.

Lemma 4 *Consider the system (1)–(2) with* $n = 2$*. The following statement holds true for each* i $(= 1, 2)$*: If all the eigenvalues of* A_i *are negative real, then*

(c-1) *$\forall t \geq 0,\ \ x(t, x_0) \in \mathbb{S}_i$, and*

(c-2) $\lim_{t\to\infty} x(t, x_0) = 0$,

hold for any x_0 satisfying

$$cx_0 = 0 \ \text{ and } \ x_0 \in \mathbb{S}_i. \tag{5}$$

PROOF. 2 It can be proved by Lemma 13 derived in Section 5 which is a generalization of Lemma 4. See several remarks followed by Lemma 13 for the detail. □

The conditions (c-1) and (c-2) mean that every trajectory starting from the switching plane (5) tends to zero as time goes to infinity without any transitions. In other words, once the state reaches at the switching plane (5), it converges to zero if all the eigenvalues of A_i are negative real.

Using Lemmas 3 and 4, we obtain a necessary and sufficient condition for stability of the planar BPLS.

Theorem 5 *For the BPLS (1)–(2) with $n = 2$, the following statements hold true.*

(i) *Suppose that either A_1 or A_2 has a real eigenvalue. Then the origin is globally asymptotically stable, if and only if all the real eigenvalues of A_1 and A_2 are negative.*

(ii) *Suppose that A_1 and A_2 have complex eigenvalues of the forms $\sigma_1 \pm \mathrm{j}\omega_1$ and $\sigma_2 \pm \mathrm{j}\omega_2$, respectively, where $\sigma_1, \sigma_2 \in \mathbb{R}$ and $\omega_1, \omega_2 \in \mathbb{R}_+$. Then the origin is globally asymptotically stable, if and only if*

$$\frac{\sigma_1}{\omega_1} + \frac{\sigma_2}{\omega_2} < 0 \tag{6}$$

holds.

PROOF. 3 Proof of (i): It follows from Lemmas 3 and 4 that the origin is attractive if and only if all the real eigenvalues are negative. The proof of stability in the Lyapunov sense is similar to Proof of (ii) shown next.

Proof of (ii): Under the dynamics $\dot{x} = A_i x$, we have

$$y(t, x_0) = \chi \mathrm{e}^{\sigma_i t} \sin(\omega_i t + \theta), \tag{7}$$

where χ and θ are constant values depending on the given initial state x_0. Note that the pairs (c, A_1) and (c, A_2) are observable, because $n = 2$ and all the eigenvalues are complex. Thus $x_0 \neq 0$ yields $\chi \neq 0$. Consequently, at least one transition always takes place for every non-zero initial state. In addition, noting that

$$\mathrm{e}^{A_i \pi/\omega_i} = \mathrm{e}^{\sigma_i \pi/\omega_i} I_2, \quad i = 1, 2,$$

we have

$$x(n\hat{t}, \bar{x}_0) = \mathrm{e}^{(\frac{\sigma_1}{\omega_1}+\frac{\sigma_2}{\omega_2})n\pi}\bar{x}_0,$$

for each initial state $\bar{x}_0$ satisfying $c\bar{x}_0 = 0$, where $n \in \mathbb{Z}$ and $\hat{t} := (1/b_1 + 1/b_2)\pi$. Hence, $\lim_{t\to\infty} x(t, x_0) = 0$, $\forall x_0$, if and only if the condition (6) holds.

Finally, we discuss the stability problem of the origin in the Lyapunov sense. Suppose that the condition (6) holds. Define

$$k_i := \sup_{0\le t\le \frac{\pi}{b_i}} \|\mathrm{e}^{A_i t}\|, \quad i = 1, 2.$$

Note that $1 \le k_i < \infty$. Every initial state x_0 satisfying $\|x_0\| \le \varepsilon/(k_1k_2)$ yields $x(t, x_0) \le \varepsilon$, $\forall t \ge 0$. Therefore, the origin is stable in the Lyapunov sense. □

Let us now investigate the conditions in Theorem 5. Let A_2 be given. If A_2 has a non-negative real eigenvalue, then the system is not asymptotically stable for any A_1. If A_2 has complex eigenvalues of the form $\sigma_2 \pm \mathrm{j}\omega_2$, then the stability condition for the eigenvalues of A_1 is characterized by the shaded portion in Figure 3. It is seen that the matrix A_1 does not need to be Hurwitz, when $\sigma_2 < 0$. Suppose that all the eigenvalues of A_2 are negative real. Then, the origin is globally asymptotically stable when A_1 does not have any non-negative real eigenvalues. Finally, consider a case in which $A = A_1 = A_2$. In such case, the conditions in Theorem 5 imply that all the eigenvalues of A are in the open left half complex plane, which is the same stability condition for linear time invariant systems.

Theorem 5 includes simple stability tests for the two special cases.

Corollary 6 *For the BPLS (1)–(2) with $n = 2$, the following statements hold true.*

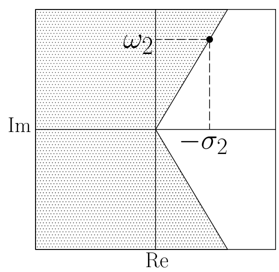

Figure 3: Stability condition for the eigenvalues of A_1, where $\sigma_2 \pm \mathrm{j}\omega_2$ stand for the complex eigenvalues of A_2

(i) *If both A_1 and A_2 are Hurwitz, then the origin is globally asymptotically stable.*

(ii) *If neither A_1 nor A_2 is Hurwitz, then the origin is not globally asymptotically stable.*

Remark 7 *A version of Theorem 5 with continuity of vector fields has been obtained by Çamlıbel et al. [1]. They have derived the same stability condition in Theorem 5 for a subclass of planar BPLSs represented by*

$$\dot{x} = \begin{cases} Ax, & \text{if } cx \geq 0, \\ Ax - bcx, & \text{if } cx \leq 0, \end{cases} \tag{8}$$

where $A \in \mathbb{R}^{2\times 2}$, $b \in \mathbb{R}^2$, and $c \in \mathbb{R}^{1\times 2}$. The research is independent of our work (see [9] and [11] for the Japanese and conference versions, respectively), and their focus was different from ours in the sense that they obtained the result through an investigation of the stability for the class of planar bimodal linear complementarity systems represented by

$$\begin{aligned} \dot{x} &= Ax + bz, \\ w &= cx + z, \\ z &\geq 0,\ w \geq 0,\ zw = 0, \end{aligned}$$

which can be rewritten as (8).

Example 8 Let us consider a planar BPLS with

$$\begin{aligned} A_1 &= \begin{bmatrix} \zeta, & 1 \\ -1, & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1, & 3 \\ -3, & 1 \end{bmatrix}, \\ c &= \begin{bmatrix} 1, & 0 \end{bmatrix}, \end{aligned}$$

where ζ is a constant value. The matrix A_2 has complex eigenvalues $1 \pm \mathrm{j}3$. We here examine the relationship between stability of the origin and the value of ζ.

(i) When $\zeta \geq 2$, A_1 has a non-negative real eigenvalue. Thus, the origin is not globally asymptotically stable from Theorem 5–(i). Furthermore, from Lemma 3, there exists a trajectory which does not tend to zero as $t \to \infty$ as illustrated in Figure 4–(i).

(ii) When $\zeta \leq -2$, all the eigenvalues of A_1 are negative real. Thus, the origin is globally asymptotically stable from Theorem 1–(i). In addition, from Lemma 4, all trajectories tend to zero as $t \to \infty$, once the state reaches at the switching plane (5). See Figure 4–(ii).

(iii) When $-2/\sqrt{10} < \zeta < 2$, A_1 has complex eigenvalues which do not satisfy the condition (6). Therefore, the origin is not globally asymptotically stable from Theorem 5–(ii) as shown in Figure 4–(iii).

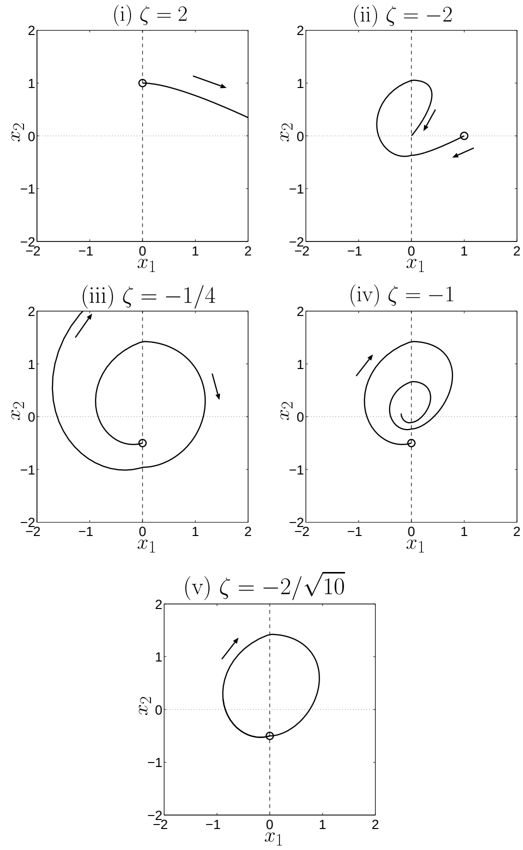

Figure 4: Trajectories in Example 8. (i) A_1 has a non-negative real eigenvalue. (ii) All the eigenvalues of A_1 are negative real. (iii) The condition (6) is not satisfied. (iv) The condition (6) is satisfied. (v) The condition (9) is satisfied.

(iv) When $-2 < \zeta < -2/\sqrt{10}$, A_1 has complex eigenvalues satisfying the condition (6). Therefore, the origin is globally asymptotically stable from Theorem 5–(ii). See Figure 4–(iv).

(v) When $\zeta = -2/\sqrt{10}$, A_1 has complex eigenvalues satisfying

$$\frac{\sigma_1}{\omega_1} + \frac{\sigma_2}{\omega_2} = 0. \tag{9}$$

Thus, each trajectory is a closed orbit corresponding to the given initial state as illustrated in Figure 4–(v).

Recall that we can not determine the stability in the case (ii) or (iv) by using the piecewise quadratic Lyapunov approach (see Proposition 1).

4 A stability test based on coefficients of characteristic polynomials

In this section, another necessary and sufficient stability condition for the planar BPLS is derived. The condition is characterized by coefficients of characteristic polynomials.

Theorem 9 *Consider the BPLS (1)–(2) with $n = 2$, and let*

$$\det(sI - A_i) = s^2 + \alpha_i s + \beta_i, \quad i = 1, 2.$$

Then, the origin is globally asymptotically stable, if and only if all the following inequalities hold;

$$\beta_i > \max(0, -\frac{|\alpha_i|\alpha_i}{4}), \qquad i = 1, 2, \tag{10}$$

$$\frac{|\alpha_1|\alpha_1}{\beta_1} + \frac{|\alpha_2|\alpha_2}{\beta_2} > 0. \tag{11}$$

Before moving to the proof of the theorem, let us now investigate the conditions in Theorem 9. On one hand, the condition (10) is imposed on each subsystem. It is illustrated as the shaded portion in Figure 5. Clearly, the condition (10) yields $\beta_i > 0$. In addition, the inequality (10) implies that A_i does not have any non-negative real eigenvalues. Roughly speaking, the inequality (10) corresponds to Theorem 5–(i). On the other hand, the condition (11) is a coupling condition of two subsystems. The inequality (11) corresponds to Theorem 5–(ii). Let A_2 be given. Then, the shaded portions in Figure 6 represent the stability condition for A_1.

PROOF. 4 (⇐): It follows from (10) and (11) that at least one of α_1 and α_2 is positive.

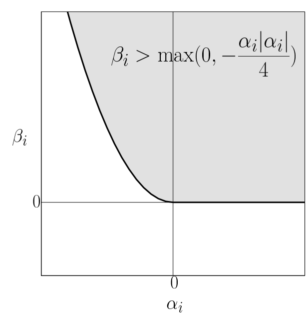

Figure 5: The shaded portion represents the region in which the inequality (10) holds.

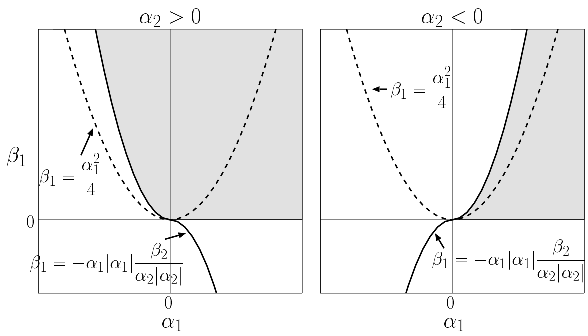

Figure 6: Stability condition for α_1 and β_1. The shaded portions imply the stability conditions.

First, suppose both α_1 and α_2 are positive. Recall that both β_1 and β_2 are positive, which implies both A_1 and A_2 are Hurwitz. From Corollary 6, we conclude that the origin is globally asymptotic stable.

Next, we consider the remainder part. Suppose that $\alpha_1 > 0$ and $\alpha_2 \leq 0$ without loss of generality. Then, A_2 has complex eigenvalues in the closed right half complex plane. We divide our proof into the following two parts: (I) $\beta_1 > 0$ and $\beta_1 \leq \alpha_1^2/4$, and (II) $\beta_1 > \alpha_1^2/4$. Note that the inequality (10) does not hold when neither the case (I) nor (II) is satisfied.

Case (I): Assume $\beta_1 > 0$ and $\beta_1 \leq \alpha_1^2/4$. Then, the two eigenvalues of A_1 are negative real. It follows from Theorem 5 that the origin is globally asymptotic stable.

Case (II): Assume $\beta_1 > \alpha_1^2/4$. Substituting $\alpha_1 > 0$ and $\alpha_2 \leq 0$ into (11) yields

$$\alpha_1^2\beta_2 > \alpha_2^2\beta_1.$$

By multiplying by 4 and then adding $-\alpha_1^2\alpha_2^2$ to the above inequality, we have

$$\alpha_1^2(4\beta_2 - \alpha_2^2) > \alpha_2^2(4\beta_1 - \alpha_1^2),$$

or equivalently

$$\frac{-\alpha_1}{\sqrt{4\beta_1 - \alpha_1^2}} + \frac{-\alpha_2}{\sqrt{4\beta_2 - \alpha_2^2}} < 0, \tag{12}$$

which implies that the eigenvalue condition (6) holds. Thus, the origin is globally asymptotically stable.
($\Rightarrow$): It follows from the asymptotic stability that neither A_1 nor A_2 has any non-negative real eigenvalues, which implies that the inequality (10) is required. Then, we divide our proof into the following two parts to show

that the inequality (11) holds, : (Ⅲ) Both A_1 and A_2 are Hurwitz. (Ⅳ) One of A_1 and A_2 is Hurwitz and the other is not Hurwitz.

Case (Ⅲ): Suppose that both A_1 and A_2 are Hurwitz. Then all the coefficients of the characteristic polynomials are positive real, which concludes that the inequality (11) holds.

Case (Ⅳ): Suppose that A_1 is Hurwitz and A_2 is not Hurwitz without loss of generality. It follows from the asymptotic stability that the eigenvalues of A_2 are complex numbers in the closed right half complex plane, which yields

$$\alpha_2 \leq 0, \;\; \beta_2 > 0, \;\; \text{and } \beta_2 > \alpha_2^2/4. \tag{13}$$

In this case, one of the following statements is satisfied: (Ⅳ–1) All the eigenvalues of A_1 are negative real. (Ⅳ–2) A_1 has complex eigenvalues which satisfy the condition (6).

Case (Ⅳ–1): Suppose that all the eigenvalues of A_1 are negative real, which implies

$$\alpha_1 > 0, \;\; \beta_1 > 0, \;\; \text{and } \beta_1 \leq \alpha_1^2/4. \tag{14}$$

Manipulating (13) and (14) yields (11).

Case (Ⅳ–2): Suppose that A_1 has complex eigenvalues which satisfy the condition (6). Then, the matrix A_i has complex eigenvalues of the form

$$\frac{-\alpha_i \pm \mathrm{j}\sqrt{4\beta_i - \alpha_i^2}}{2}, \quad i = 1, 2. \tag{15}$$

By substituting the complex eigenvalues of the form (15) into the condition (6), we have (12) which is equivalent to (11), because $\alpha_1 > 0$ and $\alpha_2 \leq 0$. □

5 Stability analysis of the higher-order bimodal piecewise linear system

5.1 A necessary condition for asymptotic stability

Let us begin with the following lemma in order to derive a necessary condition for the asymptotic stability of the higher-order BPLS. It is a generalization of Lemma 3 provided in Section 3.

Lemma 10 *Consider the system (1)–(2). Then the following statements (a), (b) and (b') are equivalent for each* i $(= 1, 2)$.

(a) *There exists an initial state* x_0 *such that the following two properties hold:*

(a-1) $\forall t \geq 0, \;\; x(t, x_0) \in \mathbb{S}_i$.

(a-2) $\lim_{t\to\infty} x(t, x_0) \neq 0$.

(b) *At least one of the following properties holds:*

(b-1) *The pair (c, A_i) is not detectable.*

(b-2) *The matrix A_i has a non-negative real eigenvalue.*

(b') *At least one of the following properties holds:*

(b'-1) *The pair (c, A_i) is not detectable.*

(b'-2) *The pair (c, A_i) has a non-negative real observable mode.*

The proof will be provided in Appendix A.2.

Let us now show that the condition (b) in Lemma 10 with $n = 2$ is equivalent to the condition (b) in Lemma 4 provided in Section 3. To this end, suppose that $n = 2$ for a while. Also, suppose that the condition (b-1) in Lemma 10 holds. Then A_i has a non-negative real eigenvalue, because $n = 2$ and $c \neq 0$. Consequently, the conditions (b-1) and (b-2) in Lemma 10 are equivalent when $n = 2$. This completes the desired statement.

Applying Lemma 10 to the system (1)–(2) yields the following theorem which provides a necessary condition for asymptotic stability of the BPLS (1)–(2).

Theorem 11 *Consider the BPLS (1)–(2). Then the origin is asymptotically stable, only if the following two conditions hold:*

(i) *Both the pairs (c, A_1) and (c, A_2) are detectable.*

(ii) *Neither the matrix A_1 nor A_2 has any non-negative real eigenvalues.*

Remark 12 We can replace the condition (ii) by the following condition (ii') in Theorem 11.

(ii') Neither the pair (c, A_1) nor (c, A_2) has any non-negative real observable modes.

5.2 A sufficient condition for attractiveness

We here give the following lemma which leads to a sufficient condition for stability of the higher-order BPLS (1)–(2). It is a generalization of Lemma 4 derived in Section 3.

Lemma 13 *For the system (1)–(2), the following statements (c) and (d) are equivalent for each i $(= 1, 2)$.*

(c) *Two conditions*

(c-1) $\forall t \geq 0$, $x(t, x_0) \in \mathbb{S}_i$, *and*

(c-2) $\lim_{t\to\infty} x(t, x_0) = 0$,

hold for any x_0 *satisfying*

$$cx_0 = 0 \quad \text{and} \quad x_0 \in \mathbb{S}_i. \tag{16}$$

(d) *The pair* (c, A_i) *is detectable and at least one of the following two conditions holds:*

(d-1) *The observable index of the pair* (c, A_i) *is equal to* 1*.*

(d-2) *The observable index of the pair* (c, A_i) *is equal to* 2 *and all the observable modes of the pair* (c, A_i) *are negative real.*

The proof will be provided in Appendix A.3.

Let us now investigate the relationship between Lemmas 4 and 13 when $n = 2$. To this end, suppose that $n = 2$ for a while. It is seen that the condition (d) in Lemma 13 holds, if all the eigenvalues of the matrix A_i are negative real. On the other hand, we can show that all the eigenvalues of the matrix A_i are negative real, if the necessary condition for asymptotic stability and the condition (d) in Lemma 13 hold. To show it, suppose that the pair (c, A_i) satisfies the conditions in Theorem 11 holds. In addition, first, suppose that the condition (d-1) in Lemma 13 holds. Then the matrix A_i has two real eigenvalues, because the observable index is 1. Moreover, from the assumption, both the observable mode and the unobservable mode are negative real, which means that all the eigenvalues of A_i are negative real. Next, suppose that the condition (d-2) in Lemma 13 holds. This implies that all the eigenvalues of A_i are negative real, which completes the desired statement.

Applying Lemmas 10 and 13 to the BPLS (1)–(2), we obtain the following theorem which provides a sufficient condition for attractiveness.

Theorem 14 *Suppose that the system (1)–(2) satisfies the conditions in Theorem 11. Then the origin is attractive, if the following two conditions hold for at least one of the pairs* (c, A_1) *and* (c, A_2)*:*

(i) *The observable index is less than or equal to* 2*.*

(ii) *All the observable modes are negative real.*

Theorem 14 has the following two features.

1. Theorem 14 is equivalent to Theorem 5–(i) derived in Section 3 when $n = 2$. In other words, Theorem 14 is a generalization of Theorem 5–(i).

2. The sufficient condition for attractiveness does not imply that both A_1 and A_2 are Hurwitz, while the piecewise quadratic Lyapunov approach requires the stability of subsystems (see Proposition 1).

6 Stabilization for bimodal piecewise linear systems

In this section, we are interested in the BPLS with control inputs of the form

$$\dot{x} = \begin{cases} A_1 x + B_1 u_1, & \text{if } cx \geq 0, \\ A_2 x + B_2 u_2, & \text{if } cx \leq 0, \end{cases} \tag{17}$$

where $x \in \mathbb{R}^n$ is the state, and $u_1 \in \mathbb{R}^{m_1}$ and $u_2 \in \mathbb{R}^{m_2}$ are the inputs. The objective here is to find feedback gains K_1 and K_2 that stabilize the system (17). In particular, we investigate a controller design method based on the stability condition derived in Section 5.2. To this end, we must make the observability index of one of the pairs $(c, A_i + B_i K_i)$ $(i = 1, 2)$ less than or equal to two. The following proposition provides a sufficient condition for the existence of such a feedback gain.

Proposition 15 *Consider the system (17).*

(i) *There exists a feedback gain K_i such that the pair $(c, A_i + B_i K_i)$ satisfies the condition (d) in Lemma 13, if the following three conditions hold:*

- **(i-1)** $\min\{\rho \mid CA_i^{\rho-1} B_i \neq 0\} \leq 2$.
- **(i-2)** *The pair (A_i, B_i) is controllable.*
- **(i-3)** *All invariant zeros*[2] *of (c, A_i, B_i) are in the open left half complex plane.*

(ii) *Suppose that the conditions in (i) hold for i, and let K_i be given such that $(c, A_i + B_i K_i)$ is detectable and satisfies conditions (i) and (ii) in Theorem 14. Then, the origin is attractive, if there exists a feedback gain K_j $(j \neq i)$ such that the closed loop system is well-posed and satisfies the conditions in Theorem 11.*

PROOF. 5 Proof of (i): The result follows from [15, Theorem 1].

Proof of (ii): It is seen from Theorem 14 that the statement holds true. □

Example 16 Consider an illustrative example of three water-tanks as depicted in Figure 7, where x_i is the water level of tank i $(= 1, 2, 3)$, and the input u is the volume of water discharged into tank 1. The valve at tank 2

[2] A complex number s is called an *invariant zero* for the triple (C, A, B) where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{\ell \times n}$, if

$$\operatorname{rank} \begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix} < n + \min(\ell, m)$$

holds.

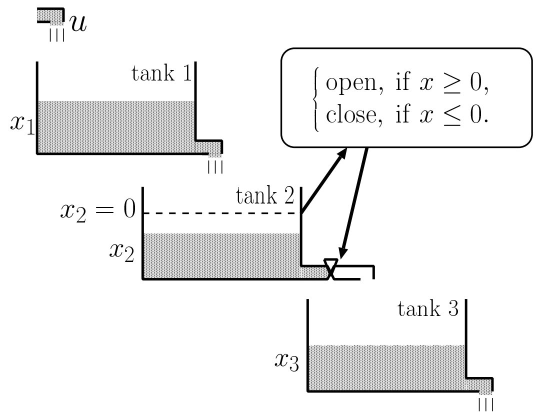

Figure 7: Three water-tanks.

is open if $x_2 \geq 0$, and it is closed if $x_2 \leq 0$ as illustrated in Figure 7. For simplicity, all coefficients are normalized to 1. Then, equations of motion of the system at the neighborhood of the origin are given by

$$\dot{x} = \begin{cases} A_1 + Bu, & \text{if } cx \geq 0, \\ A_2 + Bu, & \text{if } cx \leq 0, \end{cases} \tag{18}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Suppose that $u = 0$. Noting that the matrix A_2 has a non-negative real eigenvalue, we see that the origin is not asymptotically stable. Figure 8–(i) shows trajectories of the system (18) with $u = 0$.

Let us now design a controller for the origin to be attractive. The triple (c, A_1, B) satisfies the condition (i) in Proposition 15; (i-1) $\min\{\rho \mid cA_1^{\rho-1}B \neq 0\} = 2$, (i-2) the pair (A_1, B) is controllable, and (i-3)the invariant zero is $s = -1$. Indeed, the pair $(c, A_1 + BK_1)$ satisfies the conditions in Lemma 13 for $K_1 = [-1, -2, 0]$. See [15] for a design method of the feedback gain K_1. Furthermore, the closed system is attractive after we choose $K_2 = [0, -2, 0]$, because the closed loop system is well-posed in the sense of Carathéodory [8] and the pair $(C, A_2 + BK_2)$ does not satisfy the condition (b) in Lemma 10. Figure 8-(ii) shows trajectories of the system (18) with $K_1 = [-1, -2, 0]$ and $K_2 = [0, -2, 0]$, where the initial state is set $[-1, -1, 1]^\top$. A transition takes place at $t = 1.85$ in such case.

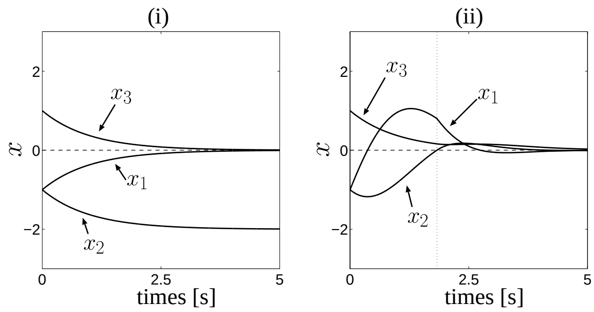

Figure 8: Trajectories of the tank system (18). (Left) Uncontrolled. (Right) Controlled.

7 Conclusion

In this paper, we have investigated the stability problem for a class of bimodal piecewise linear systems (BPLSs). We have provided two necessary and sufficient conditions for the planar BPLSs to be stable. These two conditions are given in terms of eigenvalue loci of subsystems and coefficients of characteristic polynomials, respectively. Then, we have discussed some properties of trajectories of the higher-order BPLSs and derived a necessary condition and a sufficient condition for the stability. The conditions are given in terms of the eigenvalue loci and the observability of subsystems. Furthermore, we have discussed a stabilizing controller design based on the derived sufficient condition for BPLSs.

There still remain several open problems on stability of piecewise linear systems to be addressed in the future, although we have established some basic tools for stability analysis in this paper. We need to discuss stability analysis for the higher-order case in detail to get the tighter stability conditions. In addition, extensions to the multi-modal case and the piecewise affine case should be addressed. An extension to the multi-modal case appears in [10].

Acknowledgements

The authors gratefully acknowledge helpful discussions with M. Yamakita and J. Imura at Tokyo Institute of Technology, and M. Ishikawa at Kyoto University. This work was supported in part by the Grants-in-Aid for JSPS fellows No.15003305 and Scientific Research (C) (2) No.14550438 of the Ministry of Education, Culture, Sports, Science and Technology, Japan.

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A Proofs of Lemmas 10 and 13

A.1 Preliminaries

In this subsection, we will summarize some properties which will play important roles in the proofs of Lemmas 10 and 13. Symbols defined in this subsection will be used in the subsequent proofs.

We will prove the lemmas only when $i = 1$, because we can prove them similarly when $i = 2$. For the sake of briefly, we omit the index i of A_i in the equations (1)–(2), i.e. we consider the system of the form

$$\dot{x} = Ax, \quad y = cx, \tag{19}$$

where $A \in \mathbb{R}^{n\times n}$, and $c\ (\neq 0) \in \mathbb{R}^{1\times n}$.

We first discuss properties on observability of the pair (c, A). Assume the matrix A and the vector c are expressed as

$$A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad c = \begin{bmatrix} c_0 & 0 \end{bmatrix}, \tag{20}$$

where the pair $(c_0,\ A_{11})$ is observable without loss of generality [12]. We denote the size of A_{11} by m, i.e., m denotes the observability index of the pair (c, A). In addition, partition x as $[x_o^\top, x_u^\top]^\top := x$ where $x_o \in \mathbb{R}^m$ and $x_u \in \mathbb{R}^{n-m}$.

We next describe some features of the eigenvalues of A_{11}. Let p denote the number of the real and distinct eigenvalues of A_{11}, and let $2q$ denote the number of the complex and distinct eigenvalues. Note that there exists a unique linearly independent eigenvector associating with each distinct eigenvalue, because the pair (c_0, A_{11}) is observable. Let λ_i $(i = 1, \ldots, p)$ stand for the real and distinct eigenvalues of A_{11}. Furthermore, let us denote the multiplicity of λ_i by μ_i. Then, there exist non-zero vectors $v_{ij} \in \mathbb{R}^m$ $(i = 1, \ldots, p,\ j = 2, \ldots, \mu_i)$ for each λ_i such that

$$\begin{aligned} (\lambda_i I - A_{11}) v_{i1} &= 0, && (21) \\ (\lambda_i I - A_{11}) v_{ij} &= -v_{i(j-1)}, && (22) \end{aligned}$$

hold. Similarly, we denote the complex and distinct eigenvalues by $\sigma_i \pm \mathrm{j}\omega_i$ $(i = 1, \ldots, q)$ where $\sigma_i \in \mathbb{R}$ and $\omega_i \in \mathbb{R}_+$. Also, let ν_i represent the multiplicity of $\sigma_i \pm \mathrm{j}\omega_i$. Then, there exist non-zero vectors $g_{ij} \in \mathbb{R}^m$ and $h_{ij} \in \mathbb{R}^m$ $(i = 1, \ldots, q,\ j = 2, \ldots, \nu_i)$ for each $\sigma_i \pm \mathrm{j}\omega_i$ such that

$$\begin{aligned} &\{(\sigma_i \pm \mathrm{j}\omega_i) I - A_{11}\}(g_{i1} \pm \mathrm{j}h_{i1}) = 0, && (23) \\ &\{(\sigma_i \pm \mathrm{j}\omega_i) I - A_{11}\}(g_{ij} \pm \mathrm{j}h_{ij}) \\ &\qquad = -(g_{i(j-1)} \pm \mathrm{j}h_{i(j-1)}), && (24) \end{aligned}$$

hold. Here, v_{ij} and $r_{ij} := [g_{ij}, h_{ij}]$ are called elements of a basis of the generalized eigenspace. We will use the following three facts where they follow from the observability of the pair (c_0, A_{11}).

Fact 1:

$$\begin{aligned} c_0 v_{i1} &\neq 0, \quad i = 1, \ldots, p, && (25) \\ c_0 r_{i1} &\neq 0, \quad i = 1, \ldots, q. && (26) \end{aligned}$$

Fact 2:

$$\left(\sum_{i=1}^{p} \mu_i\right) + 2\left(\sum_{i=1}^{q} \nu_i\right) = m.$$

Fact 3: Let $T \in \mathbb{R}^{m \times m}$ denote a matrix which consists of all elements of a basis of the generalized eigenspace. Then, rank $T = m$ holds.

A.2 Proof of Lemma 10

We will prove Lemma 10 only when $i = 1$. Suppose that the assumptions made in Appendix A.1 hold without loss of generality. Also, we will use the symbols defined in Appendix A.1.

(b $\Rightarrow$ a): We first prove that (b-1) $\Rightarrow$ (a). Suppose that the pair (c, A) is not detectable. Partition an initial state x_0 as $[x_{o0}^\top, x_{u0}^\top]^\top$ where $x_{o0} \in \mathbb{R}^m$

and $x_{u0} \in \mathbb{R}^{n-m}$. Suppose that $x_{o0} = 0$, which yields (a-1). Also, it follows that (a-2) holds for some x_{u0}.

We then show that (b-2) $\Rightarrow$ (a). Let λ_1 be a non-negative real eigenvalue of A_{11}. Define v_{11} by (21), and let $T_1 \in \mathbb{R}^{m\times(m-1)}$ represent a matrix which consists of all elements of a basis of the generalized eigenspace except v_{11}. In addition, let $T := [v_{11}, T_1]$ and

$$\begin{bmatrix} c_1 & \dots & c_m \end{bmatrix} \;:=\; c_0 T. \tag{27}$$

Furthermore, let $x_{o0} := T[c_1, 0, \dots, 0]^\top$. Choosing $x_{u0} \in \mathbb{R}^{n-m}$ arbitrarily, we have $y(t, x_0) = c_1^2 \mathrm{e}^{\lambda_1 t} > 0$, $\forall t \geq 0$ where $x_0 = [x_{o0}^\top, x_{u0}^\top]^\top$, which leads to (a).

(a $\Rightarrow$ b): We prove the contrapositive of the desired statement. Define $\mathbb{K} := \{i \in \{1, \dots, q\} \mid \sigma_i \geq 0\}$. Let an initial state x_0 be given.

First, assume that its elements corresponding to modes of $\sigma_i \pm \mathrm{j}\omega_i$ are zero for all $i \in \mathbb{K}$. Then $\lim_{t\to\infty} x(t, x_0) = 0$ holds, if no transition takes place. Consequently, the condition (a) does not hold.

Then, assume that x_0 has some non-zero elements which correspond to modes of $\sigma_i \pm \mathrm{j}\omega_i$ for some $i \in \mathbb{K}$. We will show that some transitions take place under the assumption. Let us choose a positive real number σ_{up} which is larger than all σ_i ($i = 1, \dots, q$). Consider a function represented by

$$f(t) = y(t, x_0)\mathrm{e}^{-\sigma_{\mathrm{up}} t}.$$

It is sufficient to show that there exists a time $\bar{t}$ (≥ 0) such that $f(\bar{t}) < 0$, because sign of $y(t, x_0)$ is coincident with that of $f(t)$ for all t. Computing the Laplace transformation of the signal $f(t)$, we have a stable, rational and strictly proper transfer function whose dominant pole is complex (see [2] for the definition of the dominant pole). It follows from [2, Proposition 1] that the desired statement holds true. $\square$

A.3 Proof of Lemma 13

We will prove it only when $i = 1$. Suppose that the assumptions made in Appendix A.1 hold without loss of generality. Also, we will use the symbols defined in Appendix A.1.

(d $\Rightarrow$ c): We first prove (d-1) $\Rightarrow$ (c). Suppose that the condition (d-1) holds, and let an initial value $x_0 = [x_{o0}, x_{u0}^\top]^\top$ satisfying the equation (16) be given where $x_{o0} \in \mathbb{R}$ and $x_{u0} \in \mathbb{R}^{n-1}$. The value of x_{o0} must be zero, because $c_0 \neq 0$. Then we obtain $y(t, x_0) = 0$ for all t, which yields the condition (c-1). Moreover, $x(t, x_0)$ tends to zero as t goes to infinity, since A_{22} is Hurwitz.

We then prove (d-2) $\Rightarrow$ (c). We divide our proof into the following two parts: (I) A_{11} is a simple matrix, i.e. A_{11} is similar to a diagonal matrix, (II) A_{11} is similar to a second order Jordan block.

(I): Let λ_1 and λ_2 be negative real eigenvalues of A_{11}. We choose x_0 satisfying the equation (16). Using the modal expansion, we obtain

$$y(t, x_0) = \chi_1 \mathrm{e}^{\lambda_1 t} + \chi_2 \mathrm{e}^{\lambda_2 t},$$

where χ_1 and χ_2 satisfy

$$\begin{aligned} y(0, x_0) &= \chi_1 + \chi_2 = 0, \\ \dot{y}(0, x_0) &= \chi_1 \lambda_1 + \chi_2 \lambda_2 \geq 0. \end{aligned}$$

In view of above equations, we see $y(t, x_0) \geq 0$, $\forall t \geq 0$, which yields (c-1). In addition, (c-2) holds because A is Hurwitz.

(II): Let λ_1 be the negative real eigenvalue of A_{11}. We choose x_0 satisfying (16). Using the modal expansion, we obtain

$$y(t, x_0) = \chi t \mathrm{e}^{\lambda_1 t}, \tag{28}$$

where χ is a positive constant of which the value depends on x_0. The equation (28) leads to (c-1) and (c-2).

(c $\Rightarrow$ d). First, $c \neq 0$ yields $m \geq 1$. Next, we will prove the contrapositive of the remainder part. We divide our proof into the following five parts: (III) The pair (c, A) is not detectable, (IV) A_{11} has some complex eigenvalues, (V) A_{11} has a non-negative real eigenvalue and another real eigenvalue, (VI) A_{11} has a non-negative real eigenvalue whose multiplicity is two, and (VII) $m \geq 3$ and all eigenvalues of A_{11} are negative real.

(III): We have already proved it in the proof of (a) $\Rightarrow$ (b) shown in Appendix A.2.

(IV): Let $\sigma_1 \pm \mathrm{j}\omega_1$ be complex eigenvalues of A_{11}. Define r_{11} by (23), and let $T_1 \in \mathbb{R}^{m \times (m-2)}$ stand for a matrix which consist of all elements of a basis of the generalized eigenspace except r_{11}. In addition, let $T := [r_{11}, T_1]$. We define c_i by (27). Note that $[c_1,\ c_2] \neq 0$ from Fact 1. Let $x_{o0} := T[c_1, c_2, 0, \ldots, 0]^\top$. Choosing $x_{u0} \in \mathbb{R}^{n-m}$ arbitrarily, we have

$$y(t, x_0) = (c_1^2 + c_2^2) \mathrm{e}^{\sigma_1 t} \sin \omega_1 t,$$

where $x_0 = [x_{o0}^\top, x_{u0}^\top]^\top$. Thus $y(0, x_0) = 0$ and $\dot{y}(0, x_0) > 0$ hold. This implies that the equation (16) holds. Moreover, the condition (c-1) is not satisfied at $t = \pi/\omega_1$.

(V): Let λ_1 denote a non-negative real eigenvalue of A_{11}. Also, let λ_2 represent a real eigenvalue of A_{11} satisfying $\lambda_2 < \lambda_1$. Define v_{11} and v_{21} by (21), and let $T_1 \in \mathbb{R}^{m \times (m-2)}$ denote a matrix which consists of all elements of a basis of the generalized eigenspace except v_{11} and v_{21}. In addition, let $T := [v_{11},\ v_{21},\ T_1]$. We define c_i by (27). Note that $c_1 \neq 0$ and $c_2 \neq 0$. Let $x_{o0} := T[c_1, -c_1^2/c_2, 0, \ldots, 0]^\top$. Choosing $x_{u0} \in \mathbb{R}^{n-m}$ arbitrarily, we have

$$y(t, x_0) = c_1^2 (\mathrm{e}^{\lambda_1 t} - \mathrm{e}^{\lambda_2 t}).$$

where $x_0 = [x_{o0}^\top, x_{u0}^\top]^\top$. We see $y(0, x_0) = 0$ and $\dot{y}(0, x_0) > 0$, which means that the equation (16) is satisfied. It follows from $\lambda_1 > \lambda_2$ that $\lim_{t\to\infty} y(t, x_0) \neq 0$ holds, and hence we have $\lim_{t\to\infty} x(t, x_0) \neq 0$.

(VI): Let λ_1 denote an eigenvalue of A_{11} whose multiplicity is 2. Define v_{11} and v_{12} by (21)–(22), and let $T_1 \in \mathbb{R}^{m\times(m-2)}$ stand for a matrix which consists of all elements of a basis of the generalized eigenspace except v_{11} and v_{12}. In addition, let $T := [v_{11},\ v_{12},\ T_1]$. We define c_i by (27). Note that $c_1 \neq 0$. Let $x_{o0} = T[-c_2, c_1, 0, \dots, 0]^\top$. Choosing $x_{u0} \in \mathbb{R}^{n-m}$ arbitrarily, we have

$$y(t, x_0) = c_1^2 t \mathrm{e}^{\lambda_1 t}, \tag{29}$$

where $x_0 = [x_{o0}^\top, x_{u0}^\top]^\top$. We see that $y(0, x_0) = 0$ and $\dot{y}(0, x_0) > 0$ hold, which means (16) is satisfied. It follows from $\lambda_1 \geq 0$ that $\lim_{t\to\infty} y(t, x_0) \neq 0$ holds, which implies $\lim_{t\to\infty} x(t, x_0) \neq 0$.

(VII): We divide our proof into the following three parts: The number of distinct and negative real eigenvalues of A_{11} is (VII–1) one, (VII–2) two, and (VII–3) greater than or equal to three.

(VII–1): Let λ_1 denote the negative real eigenvalue of A_{11}. Note that the multiplicity of λ_1 is greater than or equal to three because $m \geq 3$. Define v_{11}, v_{12} and v_{13} by (21)–(22), and let $T_1 \in \mathbb{R}^{m\times(m-3)}$ represent a matrix which consists of all elements of a basis of the generalized eigenspace except v_{11}, v_{12} and v_{13}. Let $T := [v_{11},\ v_{12},\ v_{13},\ T_1]$. We define c_i by (27). Let us choose z_1, z_2, and z_3 satisfying

$$\begin{aligned} &\chi_1 = 0, \\ &\chi_2 > 0, \\ &\chi_2 + \chi_3 \frac{1}{2} < 0, \end{aligned}$$

where

$$\begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix} := \begin{bmatrix} c_1 & c_2 & c_3 \\ 0 & c_1 & c_2 \\ 0 & 0 & c_1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}.$$

Define $x_{o0} = T[z_1, z_2, z_3, 0, \dots, 0]^\top$. Choosing $x_{u0} \in \mathbb{R}^{n-m}$ arbitrarily, let $x_0 = [x_{o0}^\top, x_{u0}^\top]^\top$. Computing $y(t, x_0)$, we have

$$y(t, x_0) = \mathrm{e}^{\lambda_1 t}(\chi_2 t + \chi_3 \frac{t^2}{2}).$$

It follows that $y(0, x_0) = 0$ and $\dot{y}(0, x_0) > 0$, which implies the equation (16) holds at the initial time. Moreover, we see that the condition (c-1) does not hold, because $y(1, x_0) < 0$ if no transition takes place.

(VII–2): The matrix A_{11} has a negative real eigenvalue λ_1 whose multiplicity is greater than or equal to two, because the number of the distinct and negative real eigenvalues is two and $m \geq 3$. Let λ_2 be a negative real eigenvalue which differs from λ_1.

We first show that there exist χ_i and $\bar{t}$ (> 0) satisfying

$$\chi_1 + \chi_3 = 0, \tag{30}$$
$$\chi_1\lambda_1 + \chi_2 + \chi_3\lambda_2 > 0, \tag{31}$$
$$\chi_1 e^{\lambda_1 \bar{t}} + \chi_2 \bar{t} e^{\lambda_1 \bar{t}} + \chi_3 e^{\lambda_2 \bar{t}} < 0. \tag{32}$$

Now let

$$\begin{aligned} \bar{\chi}_3 &:= (\lambda_2 - \lambda_1)\chi_3, \\ \phi(t) &:= \frac{1}{t e^{\lambda_1 t}} \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}. \end{aligned}$$

Note that $\phi(t) > 0$ for all $t > 0$. We see that there exists $\bar{t} > 0$ such that $\phi(\bar{t}) \neq 1$ due to linearly independence of $e^{\lambda_1 t}$, $t e^{\lambda_1 t}$ and $e^{\lambda_2 t}$. We choose χ_2 and χ_3 satisfying

$$\begin{cases} \chi_2 > 0, \ -\chi_2/\phi(\bar{t}) > \bar{\chi}_3 > -\chi_2, & \text{if } \ \phi(\bar{t}) > 1, \\ \bar{\chi}_3 > 0, \ -\phi(\bar{t})\bar{\chi}_3 > \chi_2 > -\bar{\chi}_3, & \text{if } \ 0 < \phi(\bar{t}) < 1. \end{cases}$$

Then, all the equations (30)–(32) are satisfied, when $\chi_1 = -\chi_3$. We are now ready to provide the desired statement.

Define v_{11}, v_{12} and v_{21} by (21)–(22), and let $T_1 \in \mathbb{R}^{m \times (m-3)}$ denote a matrix which consists of all elements of a basis of the generalized eigenspace except v_{11}, v_{12} and v_{21}. In addition, let $T := [v_{11}, \ v_{12}, \ v_{21}, \ T_1]$. We define c_i by (27). Choose χ_i and $\bar{t}$ satisfying (30)–(32). Let

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := \begin{bmatrix} c_1 & c_2 \\ 0 & c_1 \end{bmatrix}^{-1} \begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix}, \quad z_3 := \chi_3/c_3.$$

In addition, let $x_{o0} := T[z_1, z_2, z_3, 0, \ldots, 0]^\top$. Choosing $x_{u0} \in \mathbb{R}^{m-3}$ arbitrarily, we have

$$y(0, x_0) = \chi_1 + \chi_3 = 0, \tag{33}$$
$$\dot{y}(0, x_0) = \chi_1\lambda_1 + \chi_2 + \chi_3\lambda_2 > 0, \tag{34}$$
$$y(\bar{t}, x_0) = \chi_1 e^{\lambda_1 \bar{t}} + \chi_2 \bar{t} e^{\lambda_1 \bar{t}} + \chi_3 e^{\lambda_2 \bar{t}} < 0, \tag{35}$$

where $x_0 = [x_{o0}^\top, x_{u0}^\top]^\top$, which leads to the conclusion that the condition (c-1) does not hold.

(VII–3): Let λ_1, λ_2 and λ_3 be negative real eigenvalues. Suppose that $0 > \lambda_1 > \lambda_2 > \lambda_3$ without loss of generality.

We first show that there exist χ_1, χ_2, χ_3 and $\bar{t}$ (> 0) satisfying

$$\chi_1 + \chi_2 + \chi_3 = 0, \tag{36}$$
$$\chi_1\lambda_1 + \chi_2\lambda_2 + \chi_3\lambda_3 > 0, \tag{37}$$
$$\chi_1 e^{\lambda_1 \bar{t}} + \chi_2 e^{\lambda_2 \bar{t}} + \chi_3 e^{\lambda_3 \bar{t}} < 0. \tag{38}$$

Let

$$\bar{\chi}_3 := \frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1}\chi_3,$$
$$\phi(t) := \frac{e^{\lambda_3 t} - e^{\lambda_1 t}}{e^{\lambda_2 t} - e^{\lambda_1 t}}\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}.$$

Note that $\phi(t) > 0$ for all $t > 0$. There exists $\bar{t} > 0$ such that $\phi(\bar{t}) \neq 1$ due to linearly independence of $e^{\lambda_1 t}$, $e^{\lambda_2 t}$ and $e^{\lambda_3 t}$. We choose χ_2 and χ_3 satisfying

$$\begin{cases} \bar{\chi}_3 > 0, -\phi(\bar{t})\bar{\chi}_3 < \chi_2 < -\bar{\chi}_3, & \text{if } \phi(\bar{t}) > 1, \\ \chi_2 > 0, -\chi_2/\phi(\bar{t}) < \bar{\chi}_3 < -\chi_2, & \text{if } 0 < \phi(\bar{t}) < 1. \end{cases}$$

Then the equations (36)-(38) are satisfied, when $\chi_1 = -\chi_2 - \chi_3$. We are now ready to complete the proof.

Define v_{11}, v_{21} and v_{31} by (21)–(22), and let $T_1 \in \mathbb{R}^{m\times(m-3)}$ denote a matrix which consists of all elements of a basis of the generalized eigenspace except v_{11}, v_{21} and v_{31}. In addition, let $T := [v_{11},\ v_{21},\ v_{31},\ T_1]$. We define c_i by (27). Choose χ_i and $\bar{t}$ satisfying (36)–(38). Let $z_i := \chi_i/c_i$ $(i = 1, 2, 3)$. In addition, let $x_{o0} := T[z_1, z_2, z_3, 0, \ldots, 0]^\top$. Choosing $x_{u0} \in \mathbb{R}^{m-3}$ arbitrarily, let $x_0 = [x_{o0}^\top, x_{u0}^\top]^\top$. Then, we have

$$y(0, x_0) = \chi_1 + \chi_2 + \chi_3 = 0, \quad (39)$$
$$\dot{y}(0, x_0) = \chi_1\lambda_1 + \chi_2\lambda_2 + \chi_3\lambda_3 > 0, \quad (40)$$
$$y(\bar{t}, x_0) = \chi_1 e^{\lambda_1\bar{t}} + \chi_2 e^{\lambda_2\bar{t}} + \chi_3 e^{\lambda_3\bar{t}} < 0, \quad (41)$$

which leads to the conclusion that the condition (c-1) does not hold. □