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**Abstract**

In this paper, we consider stability analysis of planar and multi-modal piecewise linear systems. Necessary and sufficient stability conditions for the systems are derived. The conditions are given in terms of poles and zeros of subsystems, and they are computationally tractable. We then show three numerical examples which provide typical trajectories of piecewise linear systems in order to clarify differences between a class of linear time-invariant systems and a class of piecewise linear systems from the view point of stability.

## 1 Introduction

A great variety of engineering systems contain both continuous-time dynamical systems and logical or switching elements. Such systems are called hybrid dynamical systems. In the last decade, a lot of techniques have been developed for analysis and controller synthesis for hybrid dynamical systems. Each result depends on the mathematical model which represents behavior of hybrid dynamical systems. One of the typical models is the piecewise linear system (PLS). The system consists of some pairs of linear time-invariant dynamics and a switching rule given by a linear function of the continuous state. Study on PLSs is important as a first step to establish hybrid control theory, because the hybrid dynamics is the simplest in all classes of hybrid dynamical systems.

Unlike linear time-invariant systems, checking stability of a given PLS is a very hard problem due to its hybrid nature. In fact, there exist no systematic ways to exactly check stability of the class of PLSs, although many results have been obtained on stability for several classes of hybrid dynamical systems which include the class of PLSs (see [3, 11] and the references therein). Most of the results on stability are extensions of Lyapunov's theorem, where we need to show the existence of a Lyapunov function to guarantee the stability. The Lyapunov methods provide not only sufficient conditions but also necessary conditions for stability under hybrid natures. Actually, the converse theorems ensure the existence of a Lyapunov function when the system is asymptotically stable [12, 19]. On the other hand, we must restrict available classes of Lyapunov functions within a class of piecewise quadratic functions [5, 13, 15] or a class of sums of squares [14, 16] to give systematic ways of finding the Lyapunov functions. This causes the conservativeness of the stability conditions. Therefore, we need a new approach to get an explicit necessary and sufficient stability condition for the class of PLSs.

In recent years, direct analysis of behavior of hybrid states has led to exact stability tests for several classes of hybrid dynamical systems. Xu and Antsaklis [18] derived necessary and sufficient conditions for stabilizability of a class of planar and linear switched systems through an investigation of behavior of the systems. In addition, they proposed stabilizing control laws for the switched systems. Çamlıbel et al. [2] provided a necessary and sufficient stability condition for a class of planar and linear complementarity systems. Recently, two necessary and sufficient stability conditions were obtained for a class of planar and bimodal PLSs [10] which includes the class of planar and linear complementarity systems as a special case. The two conditions are given in terms of eigenvalue loci of subsystems and coefficients of characteristic polynomials, respectively. All the three papers do not treat any extensions of Lyapunov's theorem.

In this paper, we consider stability analysis of planar and multi-modal PLSs through an investigation of behavior of the hybrid state in order to derive necessary and sufficient conditions for stability. To this end, we define two concepts, namely transitive mode and weak transitive mode, which characterize behavior of the hybrid state. The two concepts also characterize a necessary and sufficient stability condition for the systems. The stability condition may be given in terms of poles and zeros of subsystems, and it is computationally tractable. Note that this paper does not treat any Lyapunov approaches. Finally, three numerical examples are addressed. They illustrate typical trajectories of PLSs and clarify differences between a class of linear time-invariant systems and a class of PLSs from a view point of stability.

This paper is organized as follows. Section 2 describes a basic setup for representing a class of planar and multi-modal PLSs. In Section 3, two concepts of transitive mode and weak transitive mode are defined, and a necessary and sufficient stability condition is given in terms of the two concepts. Section 4 is devoted to definitions on poles and zeros of subsystems. Section 5 provides two necessary and sufficient conditions for stability in terms of poles and zeros of subsystems. Finally, three numerical examples are illustrated in Section 6.

In this paper, we will use the following notation. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ represent the set of real numbers and the set of positive real numbers, respectively. The symbols $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ stand for the set of all $n$-dimensional real column vectors and the set of all $n \times m$ real matrices, respectively. For a set $\mathbb{A}$, $\mathrm{int}(\mathbb{A})$ and $\partial\mathbb{A}$ mean the interior of $\mathbb{A}$ and the boundary of $\mathbb{A}$, respectively. For two sets $\mathbb{A}$ and $\mathbb{B}$, $\mathbb{A}\backslash\mathbb{B}$ denotes difference of the two sets.

## 2 Planar multi-modal piecewise linear systems

We consider a class of planar and multi-modal piecewise linear systems represented by

$$\dot{x} = \begin{cases} A_1 x, & \text{if } x \in \mathbb{S}_1 \\ A_2 x, & \text{if } x \in \mathbb{S}_2 \\ \quad\vdots & \quad\vdots \\ A_m x, & \text{if } x \in \mathbb{S}_m. \end{cases} \tag{1}$$

Here $A_i \in \mathbb{R}^{2\times 2}$, and each $\mathbb{S}_i$ is a convex cone of the form

$$\mathbb{S}_i := \{x \in \mathbb{R}^2 \mid C_i x \geq 0\}, \tag{2}$$

where $C_i = [c_{i1}^\top, c_{i2}^\top]^\top \in \mathbb{R}^{2\times 2}$ $(i = 1, \ldots, m)$. A constituent matrix $A_i$ may be equal to another constituent matrix $A_j$ as seen in Figure 1.

We here introduce two notions, called the proper state space and well-posedness in order to clarify the class of systems treated in this paper. Actually, the two notions exclude the cases which are out of our interests.

First, the state space of the system (1) is said to be *proper*, if all the

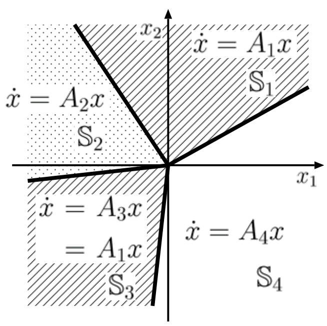


Figure 1: Planar multi-modal piecewise linear model.

following statements hold:

$$
\begin{aligned}
&\mathbb{S}_i \neq \mathbb{R}^2, && \forall i, && (3)\\
&\mathrm{int}\mathbb{S}_i \neq \emptyset, && \forall i, && (4)\\
&\cup_{i=1}^{m} \mathbb{S}_i = \mathbb{R}^2, && && (5)\\
&\mathrm{int}(\mathbb{S}_i \cap \mathbb{S}_j) = \emptyset, && \forall (i,j),\ (i \neq j). && (6)
\end{aligned}
$$

It is clear that (3)–(6) are quite natural and hence they are not restrictive under memoryless nonlinearities. Note that both (3) and (4) hold, if $\det C_i \neq 0$ for all $i$.

Second, the system (1) is said to be *well-posed*, if the system has a unique solution for each initial state. Definition of solutions can be chosen from the concepts of well-posedness in the sense of Carathéodory [8], C-well-posedness, H-well-posedness [6] and well-posedness under the switch-driven rule [7]. The choice does not influence our results shown below. On the other hand, we do not treat systems with sliding modes in the sense of Filippov [4, 17]. In fact, Lemma 22 provided in Appendix A does not hold under existence of sliding modes.

The solution from a given initial state $x_0$ is denoted by $x(t, x_0)$ where the initial time is always set 0.

Finally, we make a mild assumption. We suppose $\det C_i \neq 0$ for all $i$ in Sections 4 and 5.1 to avoid the notational complexity. Note that the assumption does not make the resulting stability condition conservative, because each system whose state space is proper satisfies the assumption after additional partition with the $x_1$-axis and the $x_2$-axis as illustrated in Example 1 below. Also, the assumption shall be removed in Section 5.2, though the notational complexity grows.

**Example 1** *Define* $\mathbb{S}_1$, $\mathbb{S}_2$ *and* $\bar{\mathbb{S}}_3$ *by (2) with*

$$C_1 = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -1, & 0 \\ 0, & 1 \end{bmatrix}, \quad \bar{C}_3 = \begin{bmatrix} 0, & 0 \\ 0, & -1 \end{bmatrix}.$$

*Clearly,* $\det \bar{C}_3 = 0$. *Let us partition* $\bar{\mathbb{S}}_3$ *into* $\mathbb{S}_3$ *and* $\mathbb{S}_4$ *with*

$$C_3 = \begin{bmatrix} -1, & 0 \\ 0, & -1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1, & 0 \\ 0, & -1 \end{bmatrix},$$

*respectively (see Figure 2). Then* $\det C_i \neq 0$ *for all* $i$.

# 3 Stability analysis

In this section, we summarize the basic concepts of stability analysis for the system (1). We first discuss behavior of trajectories of the system (1)

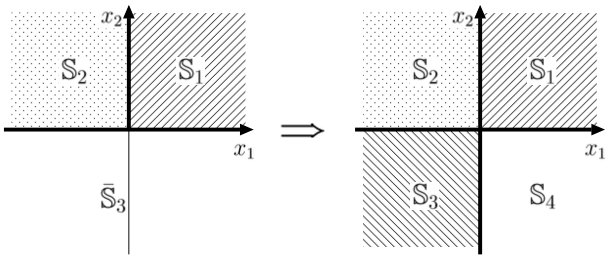


Figure 2: Additional partition of the state space.

by introducing two notions, namely transitive mode and weak transitive mode. We then give a necessary and sufficient stability condition in terms of transitive modes and weak transitive modes. The stability condition will be characterized by poles and zeros of the system (1) in Section 5.

We first investigate trajectories of the system (1). Suppose that the system is well-posed and the state space is proper. Then the system has one of the following two properties: **(T-i)** Infinitely many events occur on each trajectory as illustrated in Figure 3-(i). **(T-ii)** The number of events which take place on each trajectory is finite as depicted in Figure 3-(ii). The two properties are closely connected with transitive modes and weak transitive modes defined as follows.

**Definition 2** (i) *A mode $i$ is said to be* transitive*, if*
$\forall x_0 \in \mathbb{S}_i \backslash \{0\},\ \exists t > 0,\ x(t, x_0) \notin \mathbb{S}_i.$

(ii) *A mode $i$ is said to be* weakly transitive*, if one of the following two statements holds for all $x_0 \in \mathbb{S}_i \backslash \{0\}$:*
(a) $\exists t > 0$, $x(t, x_0) \notin \mathbb{S}_i$.
(b) $(\forall\ t \geq 0,\ x(t, x_0) \in \mathbb{S}_i)$ *and* $\lim_{t\to\infty} x(t, x_0) = 0$.

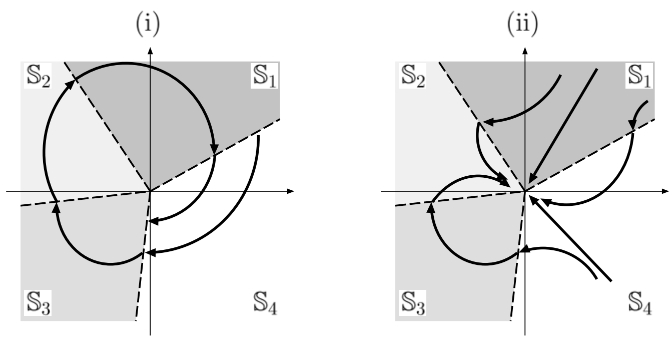


Figure 3: Trajectories of the system (1)

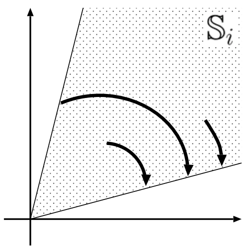


Figure 4: Trajectories in a transitive mode.

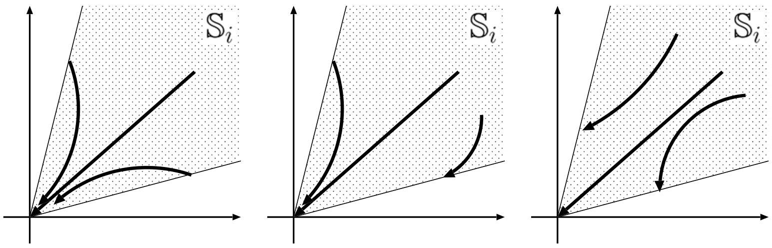


Figure 5: Trajectories in weak transitive modes.

Trajectories in a transitive mode and three possible weak transitive modes can be seen in Figures 4 and 5, respectively, in which thick curves represent the trajectories.

We first focus on the weak transitive modes. Each weak transitive mode has the following three features which are immediate from Definition 2: (i) A mode $i$ is weakly transitive, if $i$ is transitive. (ii) The origin is not stable, if there exists a mode $i$ which is not weakly transitive. (iii) There exists at least one weak transitive mode, if the origin is asymptotically stable and (T-ii) holds.

We then investigate the relationship between transitive modes and the property (T-i). To this end, we define a set of vectors on the boundary $\partial\mathbb{S}_i$ as follows.

**Definition 3** *The set of vectors defined by*

$$\mathbb{B}_i := \{x_0 \in \partial\mathbb{S}_i \backslash \{0\} \mid \exists\varepsilon > 0, \forall t \in [0, \varepsilon), x(t, x_0) \in \mathbb{S}_i\},$$

*is called the* inward boundary *of* $\mathbb{S}_i$. *(See Figure 6-(i).)*

Each transitive mode possesses the following property which will be proved in Section 5 as a part of the proof of Lemma 8.

**Lemma 4** *The following two statements are equivalent for each* $i$.

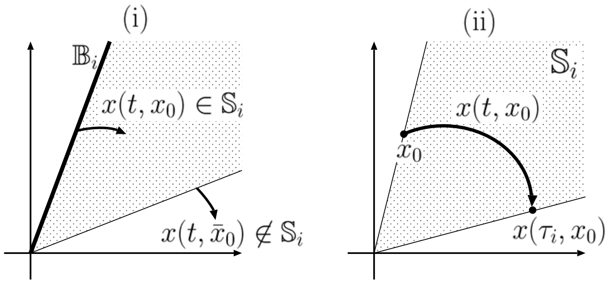


Figure 6: (i) The thick line represents the inward boundary $\mathbb{B}_i$. (ii) An illustration of (7).

(i) *The mode $i$ is transitive.*

(ii) $\mathbb{B}_i \neq \emptyset$ *and there exists a unique* $\tau_i > 0$ *such that*

$$\forall x_0 \in \mathbb{B}_i,\ \{x(\tau_i,x_0) \in \partial\mathbb{S}_i \text{ and } \forall t \in (0,\tau_i),\ x(t,x_0) \in \text{int}\mathbb{S}_i\}. \tag{7}$$

An illustration of (7) can be seen in Figure 6-(ii).

By Lemma 4, (T-i) holds if and only if all modes are transitive. Suppose that all modes are transitive. Then

$$\forall x_0 \in \mathbb{R}^2,\ \exists a \in \mathbb{R},\ ax_0 = x\left(\sum_{i=1}^{m} \tau_i, x_0\right),$$

where each $\tau_i$ is defined according to (ii) in Lemma 4. Clearly, the origin is globally asymptotically stable if and only if $a < 1$ for all $x_0$.

Using the terminologies of transitive modes and weak transitive modes, we can show the following theorem which provides a necessary and sufficient condition for the system (1) to be stable.

**Theorem 5** *Consider the system (1). Suppose that the system is well-posed and the state space is proper. Then the following statements hold.*

(i) *Suppose that all modes are transitive. Then the origin is globally asymptotically stable, if and only if* $x(\sum_{i=1}^{m} \tau_i, x_0) < x_0$ *holds for all* $x_0 \in \mathbb{R}^2 \backslash \{0\}$.

(ii) *Suppose that there exists a mode which is not transitive. Then the origin is globally asymptotically stable, if and only if all modes are weakly transitive.*

We will rewrite Theorem 5 in terms of poles and zeros of subsystems in Section 5 where the complete proof is provided. Note that the stability conditions in Theorems 11 and 16 shown there are computationally tractable as seen in algorithms attached below the theorems.

# 4 Transfer function representation

This section introduces a $2 \times 2$ matrix transfer function for each mode. Each function is the Laplace transform of initial value responses for two initial states on $\partial\mathbb{S}_i$. The poles and zeros of the functions characterize necessary and sufficient stability conditions in the next section. Throughout in this section, we assume $\det C_i \neq 0$.

We first partition $C_i^{-1}$ as

$$[z_{i1}, z_{i2}] := C_i^{-1}. \tag{8}$$

Each vector $z_{ij}$ is on $\partial\mathbb{S}_i$ (see Figure 7).

We then consider a transfer function of the form

$$\begin{aligned} T_i(s) &= C_i(sI - A_i)^{-1}C_i^{-1} \\ &= \frac{1}{s^2 + \alpha_i s + \beta_i}\begin{bmatrix} s + \gamma_{i11}, & \gamma_{i12} \\ \gamma_{i21}, & s + \gamma_{i22} \end{bmatrix}. \end{aligned} \tag{9}$$

Clearly, $T_i(s)$ represents the Laplace transform of initial value responses whose initial values are on $\partial\mathbb{S}_i$.

We can easily check if $z_{ij}$ is on the inward boundary $\mathbb{B}_i$ as follows.

**Proposition 6** *Consider the system (1). Assume* $\det C_i \neq 0$. *Then* $z_{ij} \in \mathbb{B}_i$ *holds, if and only if* $\gamma_{i\bar{j}j} \geq 0$ *where* $\bar{j} \in \{k \in \{1, 2\} \mid k \neq j\}$.

**PROOF. 1** *Simple calculations show* $c_{i\bar{j}}\dot{x}(0, z_{ij}) = \gamma_{i\bar{j}j}$, *which leads to the result.* □

It is important to analyze behavior of trajectories whose initial values are on inward boundaries $\mathbb{B}_i$ as shown in Lemma 4. The following definition chooses two special zeros associated with an initial state on $\mathbb{B}_i$ from $\gamma_{ijk}$ $(j = 1, 2,\ k = 1, 2)$ in $T_i(s)$, if it exists.

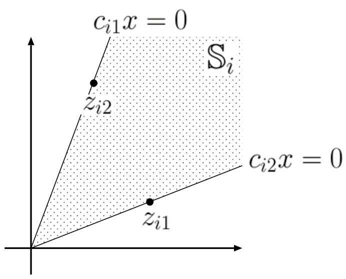


Figure 7: An illustration of $z_{i1}$ and $z_{i2}$.

**Definition 7** *Define*

$$p_i := \begin{cases} 1, & \text{if } \gamma_{i21} \geq 0, \\ 2, & \text{otherwise.} \end{cases} \quad (10)$$

*In addition,*

$$q_i \in \{ j \in \{1,2\} \mid j \neq p_i \} . \quad (11)$$

*Furthermore, define*

$$\gamma_i := \gamma_{ip_ip_i}, \quad \delta_i := \gamma_{iq_ip_i} \quad (12)$$

*to simplify the notation.*

Note that $z_{ip_i}$ is on the inward boundary $\mathbb{B}_i$, if $\mathbb{B}_i \neq \emptyset$. Conversely, $\mathbb{B}_i = \emptyset$, if $\delta_i < 0$.

# 5 Necessary and sufficient stability conditions

In this section, we rewrite Theorem 5 and provide necessary and sufficient stability conditions for the system (1) in terms of poles and zeros of $T_i(s)$. In Section 5.1, we give a stability condition under the assumption such that $\det C_i = 0$ for all $i$. Section 5.2 is devoted to the elimination of the non-singular assumption from the stability condition.

In this section, we often omit the index $i$ which expresses a mode from symbols to simplify the notation. All the proofs of lemmas and theorems in this section are found in Appendices B and C.

## 5.1 Non-singular case

The following lemma provides a necessary and sufficient condition for a mode $i$ to be transitive.

**Lemma 8** *Consider the system (1). Assume* $\det C_i \neq 0$. *Then the following three statements are equivalent.*

(i) *The mode $i$ is transitive.*

(ii) *There exists a unique $\tau > 0$ such that*
$$c_p x(\tau, z_p) = 0 \text{ and } \forall t \in [0, \tau],\ x(t, z_p) \in \mathbb{S}_i.$$

(iii) *It holds that*

$$\beta_i > \begin{cases} \frac{\alpha^2}{4}, & \text{if } \alpha \leq 2\gamma, \\ \gamma\alpha - \gamma^2, & \text{if } \alpha \geq 2\gamma. \end{cases} \quad (13)$$

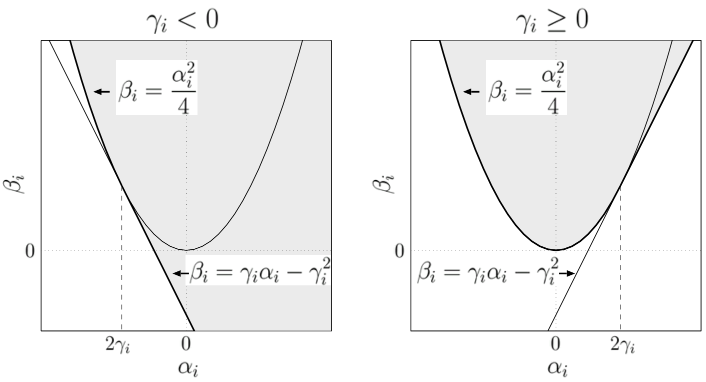


Figure 8: Each shaded portion represents the region satisfying (13).

The statement (ii) in Lemma 8 is equivalent to (ii) in Lemma 4.

The regions satisfying (13) are illustrated in Figure 8. In general, the region with an unstable zero ($\gamma_i < 0$) is larger than the region with a stable zero ($\gamma_i > 0$).

The following lemma describes the relationship between $z_p$ and $z_q$ under (13).

**Lemma 9** *Consider the system (1). Assume* $\det C_i \neq 0$*. In addition, (13) holds for* $i$*. Define* $\tau$ *according to (ii) in Lemma 8. Then, it holds that*

$$x(\tau, z_p) = \eta z_q, \tag{14}$$

*where*[1]

$$\eta := \begin{cases} \frac{\delta}{\sqrt{\beta-\alpha\gamma+\gamma^2}} \exp\left(\frac{-\alpha\theta}{\sqrt{4\beta-\alpha^2}}\right), & \text{if } \alpha^2 < 4\beta, \\ \frac{2\delta}{\alpha-2\gamma} \exp\left(\frac{-\alpha}{\alpha-2\gamma}\right), & \text{if } \alpha^2 = 4\beta, \\ \delta \exp\left(\frac{\lambda_2 \log|\lambda_2+\gamma| - \lambda_1 \log|\lambda_1+\gamma|}{\lambda_1-\lambda_2}\right), & \text{if } \alpha^2 > 4\beta, \end{cases} \tag{15}$$

$$\theta = \mathrm{Arccos}\left(\frac{\alpha - 2\gamma}{2\sqrt{\beta - \alpha\gamma + \gamma^2}}\right), \tag{16}$$

$$\lambda_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2}, \tag{17}$$

$$\lambda_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2}. \tag{18}$$

Note that $\eta$ is well-defined under (13).

[1] For $x \in [-1, 1]$, $\mathrm{Arccos}(x)$ expresses the principal value of the inverse cosine of $x$, i.e. $\mathrm{Arccos}(x) \in [0, \pi]$.

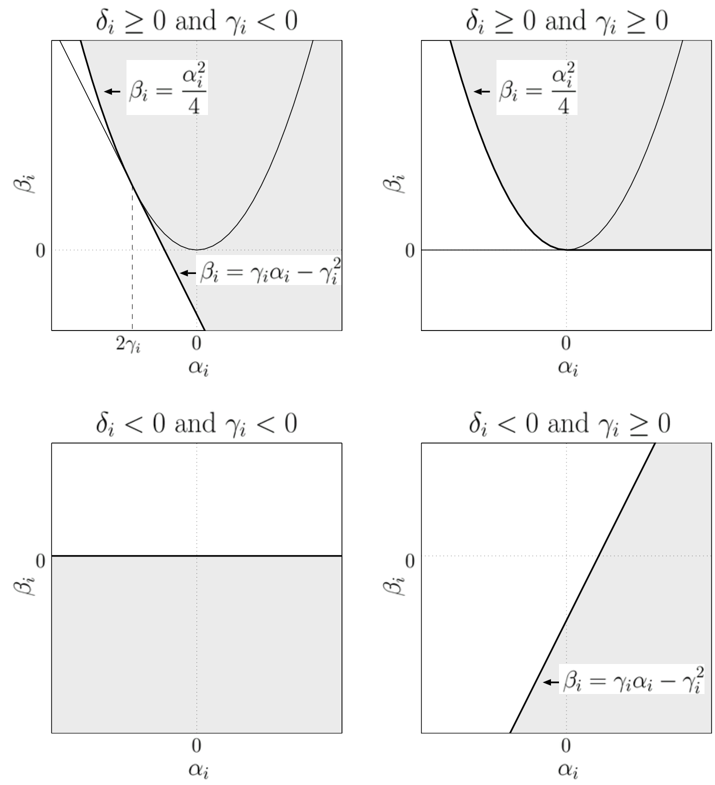


Figure 9: Each shaded portion implies the region satisfying (19).

The following lemma provides a necessary and sufficient condition for a mode $i$ to be weakly transitive.

**Lemma 10** *Consider the system (1). Assume* $\det C_i \neq 0$. *The following two statements are equivalent.*

(i) *The mode $i$ is weakly transitive.*

(ii) *It holds that*

$$\begin{cases} \beta > \frac{\alpha^2}{4}, & \text{if } \delta \geq 0 \text{ and } \alpha \leq 2\underline{\gamma}, \\ \beta > \underline{\gamma}\alpha - \underline{\gamma}^2, & \text{if } \delta \geq 0 \text{ and } \alpha \geq 2\underline{\gamma}, \\ \beta < \overline{\gamma}\alpha - \overline{\gamma}^2, & \text{if } \delta < 0, \end{cases} \tag{19}$$

*where* $\overline{\gamma} := \max(0, \gamma)$ *and* $\underline{\gamma} := \min(0, \gamma)$.

The regions satisfying (19) are depicted in Figure 9. Let us concentrate on the case $\delta_i \geq 0$. The region defined by (19) is larger than the region defined by (13), when $\gamma_i > 0$ and $\delta_i \geq 0$. On the other hand, the two regions defined by (13) and (19) are equivalent, when $\gamma_i \leq 0$ and $\delta_i \geq 0$.

We are now ready to give a necessary and sufficient stability condition in terms of poles and zeros of $T_i(s)$.

**Theorem 11** *Consider the system (1). Suppose that the system is well-posed, the state space is proper and* $\det C_i \neq 0$ *for all* $i$*. Then the following two statements hold true.*

(i) *Suppose (13) holds for all* $i$*. Then the origin is globally asymptotically stable, if and only if it holds that*

$$\prod_{i=1}^{m} (\eta_i \| z_{iq_i} \|) < \prod_{i=1}^{m} \| z_{ip_i} \|. \tag{20}$$

(ii) *Suppose that there exists a mode* $i$ *such that (13) does not hold. Then the origin is globally asymptotically stable, if and only if (19) holds for all* $i$*.*

We can determine the stability of the system (1) by using the following algorithm based on Theorem 11.

**Algorithm 1**

**Step 1)**

**(i)** Check if the state space of a given system is proper. If yes, then go to (ii). If not, the algorithm ends.

**(ii)** Check if the system is well-posed (see [6, 7, 8] for details). If yes, then go to Step 2. If not, the algorithm ends.

**Step 2)**

**(i)** Check if $\det C_i \neq 0$ for all $i$. If not, partition the state space with the $x_1$-axis and the $x_2$-axis.

**(ii)** Compute $z_{i1}$, $z_{i2}$, $\alpha_i$, $\beta_i$, $\gamma_i$, $\delta_i$, $p_i$ and $q_i$ ($i = 1, \ldots, m$) according to (8)–(12).

**Step 3)** Check if (13) holds for all $i$. If yes, then go to Step 4-i. If not, then go to Step 4-ii.

**Step 4)**

**(i)** **(a)** Compute $\eta_i$ ($i = 1, \ldots, m$) according to (15).

**(b)** Check if (20) is satisfied. If yes, the origin is globally asymptotically stable. If not, the origin is not globally asymptotically stable.

**(ii)** Check if (19) holds for all $i$. If yes, the origin is globally asymptotically stable. If not, the origin is not globally asymptotically stable.

## 5.2 General case

In this section, we consider the planar and multi-modal piecewise linear system (1) with some cells $\mathbb{S}_i$ satisfying $\det C_i = 0$ and provide yet another algorithm to check the stability of the system (1). The new algorithm requires no further partition of the state space as seen in Step 2-i of Algorithm 1. To this end, we investigate properties on transition modes and weak transition modes whose cells $\mathbb{S}_i$ satisfy $\det C_i = 0$.

We first define $\alpha$ and $\beta$ in the same way as the non-singular case, i.e.

$$\det(sI - A_i) = s^2 + \alpha_i s + \beta_i. \tag{21}$$

We then present versions of Lemmas 8, 9 and 10 with singular $C_i$.

**Lemma 12** *Consider the system (1). Assume the state space is proper and* $\det C_i = 0$*. Then the following three statements are equivalent.*

(i) *The mode $i$ is transitive.*

(ii) *The statement (ii) in Lemma 4 holds.*

(iii) $4\beta > \alpha^2$.

**Lemma 13** *Consider the system (1). Assume the state space is proper and* $\det C_i = 0$*. In addition, suppose* $4\beta > \alpha^2$*. Define $\tau$ according to (ii) in Lemma 4. Then*

$$x(\tau, x_0) = -\exp\left(\frac{-\alpha\pi}{\sqrt{4\beta - \alpha^2}}\right) x_0 \tag{22}$$

*holds for all* $x_0 \in \mathbb{B}_i$.

**Lemma 14** *Consider the system (1). Assume the state space is proper and* $\det C_i = 0$*. Then, the following statements are equivalent.*

(i) *The mode $i$ is weakly transitive.*

(ii) $\beta > \begin{cases} \frac{\alpha^2}{4}, & \text{if } \alpha \leq 0, \\ 0, & \text{if } \alpha \geq 0. \end{cases}$

Let us now define $\gamma$, $\delta$, $p$, $q$, $z_p$ and $z_q$ in the singular case as shown below. The definition unifies Lemmas 8 and 12, and Lemmas 10 and 14. Moreover, the definition enables us to give a stability condition for the system (1) with singular $C_i$ using a similar statement to Theorem 11.

**Definition 15** *Define*

$$\gamma_i = \infty, \quad \delta_i = p_i = q_i = 1, \quad z_{i1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{23}$$

$$\eta_i := \exp\left( \frac{-\alpha_i \pi}{\sqrt{4\beta_i - \alpha_i^2}} \right). \tag{24}$$

We are now ready to remove the non-singular assumption from Theorem 11. We can prove the following theorem in a similar way to the proof of Theorem 11.

**Theorem 16** *Consider the system (1). Suppose that the system is well-posed and the state space is proper. Define*

$$\mathbb{I}_{\mathrm{r}} := \{i \in \{1, 2, \ldots, \} \mid \det C_i \neq 0\}, \tag{25}$$

$$\mathbb{I}_{\mathrm{s}} := \{i \in \{1, 2, \ldots, \} \mid \det C_i = 0\}. \tag{26}$$

*Moreover, define $z_{i1}$, $z_{i2}$, $\alpha_i$, $\beta_i$, $\gamma_i$, $\delta_i$ $p_i$, $q_i$ and $\eta_i$ according to (8)–(12) and (15) for all $i \in \mathbb{I}_{\mathrm{r}}$. Also, define them according to (21), (23) and (24) for all $i \in \mathbb{I}_{\mathrm{s}}$. Then, the statements (i) and (ii) in Theorem 11 hold true.*

We establish a new algorithm for a stability test based on Theorem 16 as follows.

**Algorithm 2** Algorithm 2 has the same procedures as Algorithm 1 except Steps 2 and 4-i-a. Steps 2 and 4-i-a are replaced by the following 2' and 4-i-a', respectively.

**Step 2')**

- **(i)** Classify all indices $1, 2, \ldots, m$ into the two sets $\mathbb{I}_{\mathrm{r}}$ and $\mathbb{I}_{\mathrm{s}}$ defined by (25) and (26), respectively.
- **(ii)** Compute $z_{i1}$, $z_{i2}$, $\alpha_i$, $\beta_i$, $\gamma_i$, $\delta_i$, $p_i$ and $q_i$ according to (8)–(12) for all $i \in \mathbb{I}_{\mathrm{r}}$.
- **(iii)** Compute $z_{i1}$, $\alpha_i$, $\beta_i$, $\gamma_i$, $\delta_i$, $p_i$ and $q_i$ according to (21) and (23) for all $i \in \mathbb{I}_{\mathrm{s}}$.

**Step 4-i-a')** Compute $\eta_i$ according to (15) for all $i \in \mathbb{I}_{\mathrm{r}}$. Also, compute $\eta_i$ according to (24) for all $i \in \mathbb{I}_{\mathrm{s}}$.

## 6 Numerical examples

This section illustrates three typical numerical examples in order to clarify differences between trajectories of linear time invariant systems and piecewise linear systems from a view point of stability.

**Example 17** Consider a four-modal piecewise linear system represented by

$$\dot{x} = A_i x, \quad \text{if } x \in \mathbb{S}_i, \tag{27}$$

where

$$\begin{aligned}
A_1 &= A_3 = \begin{bmatrix} \sigma, & 1 \\ -\omega^2, & \sigma \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 1, & \pi \\ -\pi, & 1 \end{bmatrix}, && (28)\\
\mathbb{S}_i &:= \{x \in \mathbb{R}^2 \mid C_i x \geq 0\}, \quad i = 1,2,3,4, && (29)\\
C_1 &= \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -1, & 0 \\ 0, & 1 \end{bmatrix}, && (30)\\
C_3 &= \begin{bmatrix} -1, & 0 \\ 0, & -1 \end{bmatrix}, \quad C_4 = \begin{bmatrix} 1, & 0 \\ 0, & -1 \end{bmatrix}. && (31)
\end{aligned}$$

Each set $\mathbb{S}_i$ expresses the $i$-th quadrant.

The aim here is to derive a necessary and sufficient condition for asymptotic stability of the system in terms of $\sigma$ ($\in \mathbb{R}$) and $\omega$ ($\in \mathbb{R}_+$). To this end, we investigate the stability condition via Algorithm 1.

**Step 1)**

- **(i)** (3)–(6) hold.
- **(ii)** It is seen that the system (27) is well-posed in the sense of Carathéodory from the condition in [8].

**Step 2)**

- **(i)** Clearly, $\det C_i \neq 0$ for all $i$.
- **(ii)** We have

$$\begin{array}{llll}
\alpha_1 = -2\sigma, & \beta_1 = \sigma^2 + \omega^2, & \gamma_1 = -\sigma, & \delta_1 = 1, \\
\alpha_2 = -2, & \beta_2 = 1 + \pi^2, & \gamma_2 = -1, & \delta_2 = \pi, \\
\alpha_3 = -2\sigma, & \beta_3 = \sigma^2 + \omega^2, & \gamma_3 = -\sigma, & \delta_3 = 1, \\
\alpha_4 = -2, & \beta_4 = 1 + \pi^2, & \gamma_4 = -1, & \delta_4 = \pi.
\end{array}$$

**Step 3)** The inequality (13) holds for all $i$.

**Step 4)** We obtain

$$\eta_1 = \eta_3 = \frac{\mathrm{e}^{\left(\frac{\sigma\pi}{2\omega}\right)}}{\omega}, \qquad \eta_2 = \eta_4 = \mathrm{e}^{\frac{1}{2}}. \tag{32}$$

Substituting (32) into (20) yields

$$\mathrm{e}^{1+\frac{\sigma\pi}{\omega}} < \omega^2. \tag{33}$$

This is the necessary and sufficient stability condition. Bellow we illustrate typical trajectories of the system for three pairs of $\omega$ and $\sigma$.

**(i)** $\omega = 5$ and $\sigma = 2$: The condition (33) is satisfied. Therefore, the origin is asymptotically stable, while all constituent matrices $A_i$ are unstable: This phenomenon has been pointed out in [1]. A trajectory of the system with $\omega = 5$ and $\sigma = 2$ can be seen in Figure 10–(i).

**(ii)** $\omega = 5$ and $\sigma = 5$: The origin is unstable because the condition (33) is not satisfied. See Figure 10-(ii).

**(iii)** $\omega = 5$ and $\sigma = \frac{\omega}{\pi}(-1 + 2\log\omega)$: Then

$$e^{1+\frac{\sigma\pi}{\omega}} = \omega^2$$

holds. Each trajectory is a closed orbit as depicted in Figure 10–(iii) in this case. This confirms the proposed stability test is exact.

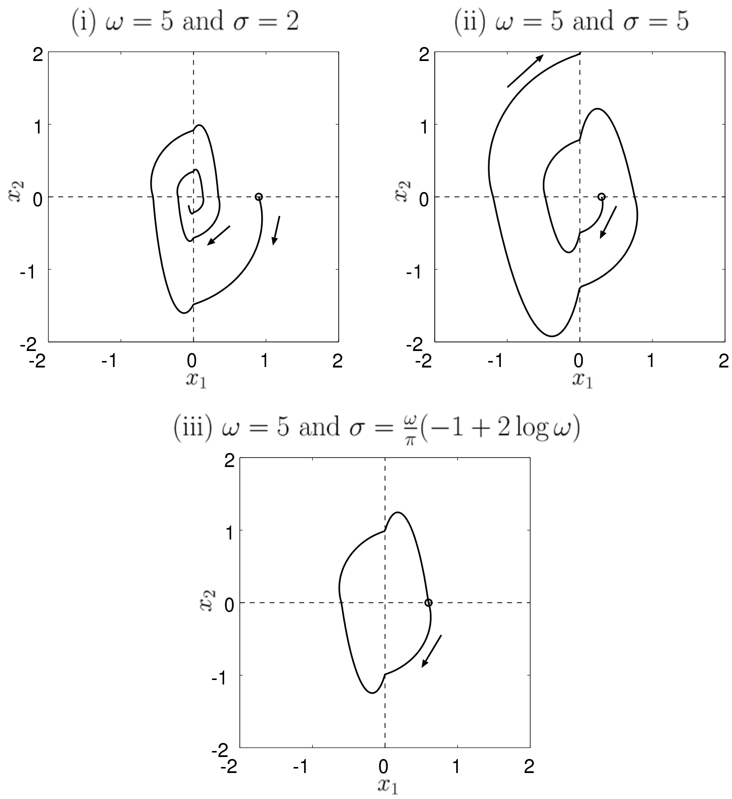


Figure 10: Trajectories of Example 17.

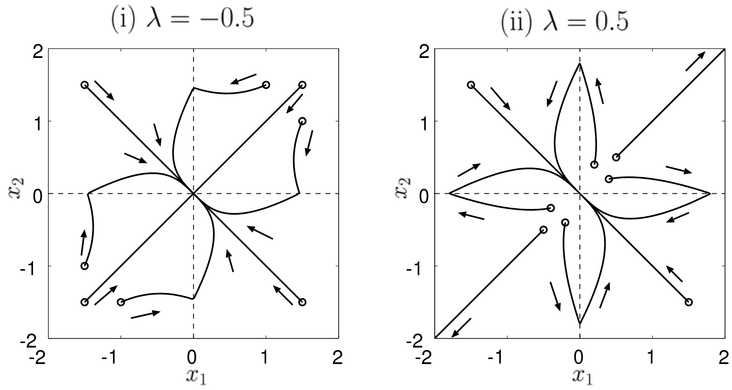


Figure 11: Trajectories of Example 18.

**Example 18** Consider a four-modal system with

$$\begin{aligned} A_1 &= A_3 = \frac{1}{2}\begin{bmatrix} \lambda+1, & \lambda-1 \\ \lambda-1, & \lambda+1 \end{bmatrix}, \\ A_2 &= A_4 = \begin{bmatrix} -2, & -1 \\ -1, & -2 \end{bmatrix}, \\ \mathbb{S}_i &:= \text{the } i\text{−th quadrant}, \end{aligned}$$

i.e., $C_i$ (i=1,2,3,4) are given by (30) and (31).

We here assume $\lambda < 1$, which guarantees well-posedness of the system in the sense of Carathéodory. We then see that (13) does not hold for any $i$. Thus, let us investigate (19). Consequently, we see that the origin is asymptotically stable, if and only if $\lambda < 0$. Typical trajectories can be seen in Figure 11.

We have a remark for the unstable case, i.e. $0 \le \lambda < 1$. Lemma 22 shown in Appendix A ensures that every trajectory converges to the origin, if the initial state is not on the line through $[1,1]^\top$ and $-[1,1]^\top$. In other words,

$$\lim_{t\to\infty} x(t, x_0) = 0, \quad \text{a.e. } x_0 \in \mathbb{R}^2$$

holds, even if the origin is unstable as illustrated in Figure 11–(ii).

**Example 19** Consider a three-modal system with

$$\begin{aligned} A_1 &= \frac{1}{2}\begin{bmatrix} \lambda+1, & \lambda-1 \\ \lambda-1, & \lambda+1 \end{bmatrix}, \ A_2 = A_3 = \begin{bmatrix} -1, & \pi \\ -\pi, & -1 \end{bmatrix}, \\ C_1 &= \begin{bmatrix} 1, & 0 \\ -\sqrt{3}, & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1, & 0 \\ \sqrt{3}, & -1 \end{bmatrix}, \qquad C_3 = \begin{bmatrix} -1, & 0 \\ 0, & 0 \end{bmatrix}. \end{aligned}$$

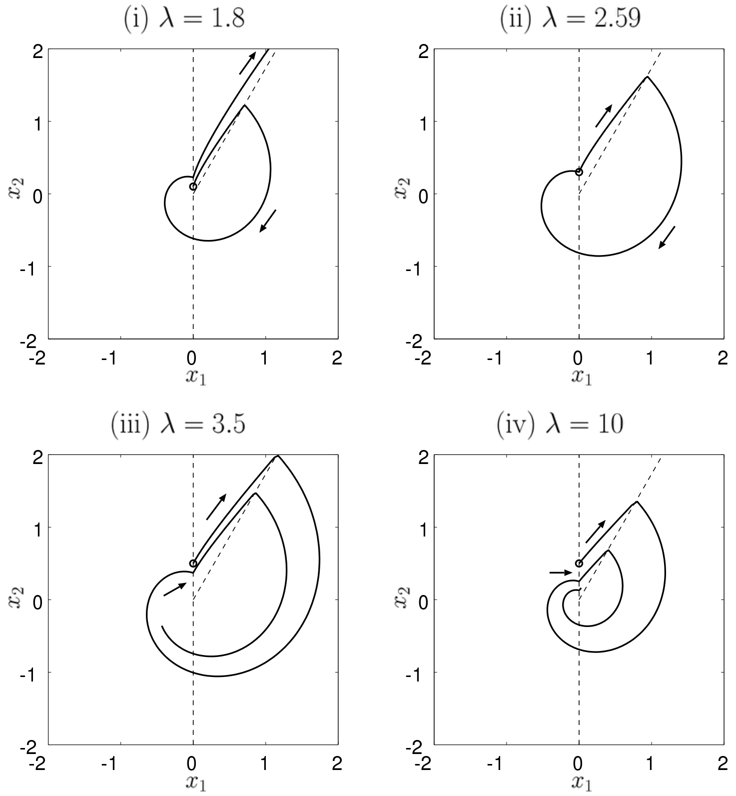


Figure 12: Trajectories of Example 19.

Clearly, $\det C_3 = 0$. Assume $\lambda > 1$ to guarantee well-posedness of the system. Let us investigate the stability of the system via Algorithm 2. As a result, we see that the origin is asymptotically stable, if and only if it holds that

$$\lambda > \frac{\frac{11}{6} + \log \frac{\sqrt{3}+1}{2}}{\frac{11}{6} + \log \frac{\sqrt{3}-1}{2}} \simeq 2.59. \tag{34}$$

Note that $\lambda$ is one of the eigenvalues of $A_1$. Therefore, (34) implies that an eigenvalue of $A_1$ must be greater than the value defined by the right hand side of (34). Moreover, the greater the value of $\lambda$ is, the faster the state converges to the origin as illustrated in Figure 12. Roughly speaking, the more unstable the subsystem is, the more stable the hybrid system is, in this case.

# 7 Conclusion

In this paper, we have derived necessary and sufficient conditions for planar and multi-modal piecewise linear systems to be stable. The conditions are given in terms of poles and zeros of subsystems, and they are computationally tractable. Also, we have shown three numerical examples which provide typical trajectories of piecewise linear systems. They clarify differences between a class of linear time invariant systems and a class of piecewise linear systems from a view point of stability.

There still remain several open problems on stability of piecewise linear systems to be addressed in the future, although we have established some basic tools for stability analysis in this paper. In particular, we need to discuss stability analysis for the higher-order case. A necessary condition and a sufficient condition for stability of higher-order and bimodal systems are addressed in [9, 10]. The conditions are given in terms of eigenvalue loci of subsystems and observability.

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# A Preliminaries for the proof of Theorem 11

In this section, we analyze behavior of trajectories for each subsystem of the system (1), which will play important roles to show the proof of Theorem 11. We here often omit the index $i$ which expresses a mode from symbols to simplify the notation.

We first classify the indices $1, 2, \ldots, m$ into three groups according to the values of $\gamma_{11}$ and $\gamma_{22}$.

**Definition 20** *Consider the system (1). Assume* $\det C_i \neq 0$ *for all* $i$. *Then define*

$$\begin{aligned} \mathbb{I}_1 &:= \{i \in \mathbb{I} \mid \gamma_{i12} \geq 0 \text{ and } \gamma_{i21} \geq 0\}, \\ \mathbb{I}_2 &:= \{i \in \mathbb{I} \mid \gamma_{i12} < 0 \text{ and } \gamma_{i21} < 0\}, \\ \mathbb{I}_3 &:= \{i \in \mathbb{I} \mid i \notin \mathbb{I}_1 \text{ and } i \notin \mathbb{I}_2\}, \end{aligned}$$

*where* $\mathbb{I} := \{1, 2, \ldots m\}$.

Each of the index sets $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ has the following features.

**Proposition 21** *Consider the system (1). Assume* $\det C_i \neq 0$. *Then the following statements hold true.*

(i) $\beta \leq \gamma\alpha - \gamma^2$, if $i \in \mathbb{I}_1$.

(ii) $\beta < \gamma\alpha - \gamma^2$, if $i \in \mathbb{I}_2$.

(iii) $\beta \geq \gamma\alpha - \gamma^2$, if $i \in \mathbb{I}_3$.

**PROOF. 2** *It is straightforward to verify that*

$$\beta - (\gamma\alpha - \gamma^2) = -\gamma_{11}\gamma_{22},$$

*which leads to the desired statements.* □

We then summarize properties on the index set $\mathbb{I}_2$.

**Lemma 22** *Consider the system (1). Assume the system is well-posed, the state space is proper and* $\det C_i \neq 0$ *for all* $i$. *Then the following statements hold true.*

(i) $i \in \mathbb{I}_2 \Leftrightarrow \delta_i < 0$.

(ii) *If $\mathbb{I}_2 \neq \emptyset$, then $\mathbb{I}_1 \neq \emptyset$.*

(iii) *If $i \in \mathbb{I}_2$, then $A_i$ has distinct and real eigenvalues.*

(iv) *Suppose $i \in \mathbb{I}_2$. Let $\lambda_{\min}$ denote the minimal eigenvalue of $A_i$. Then there exists an eigenvector $v_{\min}$ associating with $\lambda_{\min}$ such that $x(t, v_{\min}) \in \mathbb{S}_i$ holds for all $t \geq 0$.*

(v) *Suppose $i \in \mathbb{I}_2$. Define $v_{\min}$ according to (iv). In addition, define*

$$\mathbb{V}_i := \{x \in \mathbb{R}^2 \mid x = av_{\min},\ a \in \mathbb{R}\}.$$

*Then*

$$\forall x_0 \in \mathbb{S}_i \backslash \mathbb{V}_i, \quad \exists t > 0, \quad x(t, x_0) \notin \mathbb{S}_i.$$

**PROOF. 3** *Proof of (i): The result follows from the definitions of $\delta_i$ and $\mathbb{I}_2$.*

*Proof of (ii): Suppose that $\mathbb{I}_2 \neq \emptyset$ and $\mathbb{I}_1 = \emptyset$. Then there exists a sliding mode, which contradicts the well-posedness assumption.*

*Proof of (iii): The result follows from Proposition 21.*

*Proof of (iv): Define $\bar{A}_i := C_i A_i C_i^{-1}$. Let $\lambda_1$ and $\lambda_2$ denote the eigenvalues of $\bar{A}_i$. Assume $\lambda_1 > \lambda_2$ without loss of generality. Let $v_j = [v_{j1}, v_{j2}]^\top$ be an eigenvector of $\bar{A}_i$ associating with $\lambda_j$ for each $j$ $(= 1, 2)$. Note that $\gamma_{12} < 0$ implies $v_{21} \neq 0$. Thus, we assume $v_{21} = 1$ and*

$$\det[v_1, v_2] = v_{11}v_{22} - v_{12} = 1, \tag{35}$$

*without loss of generality. Then,*

$$\begin{aligned} c_2\dot{x}(0, z_1) &= v_{12}v_{22}(\lambda_1 - \lambda_2) < 0, & (36)\\ c_1\dot{x}(0, z_2) &= -v_{11}(\lambda_1 - \lambda_2) < 0. & (37) \end{aligned}$$

*Thanks to (35), (36) and (37), it holds that*

$$\begin{aligned} &v_{11} > 0,\\ &v_{22}(v_{11}v_{22} - 1) < 0. \end{aligned}$$

*Then, we obtain $v_{22} > 0$. Hence we see $x(t, C_i^{-1}v_2) \in \mathbb{S}_i$ holds for all $t \geq 0$, because $C_i^{-1}v_2$ is an eigenvector of $A_i$.*

*Proof of (v): Define $\lambda_1$, $\lambda_2$, $v_1$ and $v_2$ according to the definitions in the proof of (iv). We consider any initial state $C_i^{-1}x_0$ $(\in \mathbb{S}_i)$ satisfying $x_2 < v_{22}x_1$. It is straightforward to verify that*

$$\begin{aligned} c_2 x(t, C_i^{-1}x_0) &= v_{12}(v_{22}x_1 - x_2)\mathrm{e}^{\lambda_1 t}\\ &\quad + v_{22}(v_{11}x_2 - v_{12}x_1)\mathrm{e}^{\lambda_2 t}. \end{aligned}$$

*It is seen from (36) that $v_{12} < 0$, which leads to the desired statement. The case such that $x_2 > v_{22}x_1$ can be shown similarly.* □

We finally show the following corollary which is immediate from Definitions 2 and 20, and Lemma 8.

**Corollary 23** *Consider the system (1). Assume* $\det C_i \neq 0$. *The following statements hold true.*

(i) *Suppose* $i \in \mathbb{I}_1$ *and (19) holds for* $i$. *Then, (b) in Definition 2 holds true for two initial states* $x_0 = z_{i1}$ *and* $x_0 = z_{i2}$.

(ii) *Suppose* $i \in \mathbb{I}_3$. *In addition, suppose that (13) does not hold for* $i$ *and (19) holds for* $i$. *Then, (b) in Definition 2 holds true for* $x_0 = z_{ip_i}$.

# B Proofs of Lemmas 8, 9 and 10, and Theorem 11

**Proof of Lemma 8:** Proof of (ii) $\Leftrightarrow$ (iii): Suppose that (ii) holds. Then Proposition 6 yields $i \in \mathbb{I}_3$.

We then suppose that $\beta = \gamma\alpha - \gamma^2$. In this case, one of the poles of $T_i(s)$ is $-\gamma$, and we have

$$c_p x(t, z_p) = \mathrm{e}^{(-\alpha+\gamma)t}.$$

Hence (ii) does not hold.

Therefore, it suffices to show (ii) $\Leftrightarrow$ (iii) under $\beta > \gamma\alpha - \gamma^2$. We divide our proof into the following three parts: (I) $\alpha^2 < 4\beta$, (II) $\alpha^2 = 4\beta$, and (III) $\alpha^2 > 4\beta$.

Case $\alpha^2 < 4\beta$: We obtain

$$c_p x(t, z_p) = \mathrm{e}^{-\frac{\alpha}{2}t} \left\{ \cos \omega t + \frac{1}{\omega} \left( \gamma - \frac{\alpha}{2} \right) \sin \omega t \right\}, \tag{38}$$

where $\omega = \sqrt{\beta - \frac{\alpha^2}{4}}$. Thus, we see that (ii) $\Leftrightarrow$ (iii) always holds true in this case.

Case $\alpha^2 = 4\beta$: We have

$$c_p x(t, z_p) = \mathrm{e}^{-\frac{\alpha}{2}t} \left\{ t(\gamma - \frac{\alpha}{2}) + 1 \right\}. \tag{39}$$

It is seen that there exists $\tau_i > 0$ satisfying (ii), if and only if $\alpha > 2\gamma$ holds.

Case $\alpha^2 > 4\beta$: We get

$$c_p x(t, z_p) = \frac{\lambda_1 + \gamma}{(\lambda_1 - \lambda_2)} \mathrm{e}^{\lambda_1 t} - \frac{\lambda_2 + \gamma}{(\lambda_1 - \lambda_2)} \mathrm{e}^{\lambda_2 t}, \tag{40}$$

where $\lambda_1$ and $\lambda_2$ are defined by (17) and (18), respectively. There exists $\tau_i > 0$ satisfying (ii), if and only if $\alpha - 2\gamma > \sqrt{\alpha^2 - 4\beta}$ holds.

Proof of (i) $\Leftrightarrow$ (iii): It also suffices to prove (i) $\Leftrightarrow$ (iii) under $\beta > \gamma\alpha - \gamma^2$ for the same reason in the proof of (ii) $\Leftrightarrow$ (iii).

Define

$$\mathbb{A}_i := \{x \in \mathbb{R}^2 \mid [\bar{x}_1, \bar{x}_2]^\top = C_i^{-1}x, \ \ \bar{x}_1 \geq 0, \ \bar{x}_2 \geq 0, \ \text{and } \bar{x}_1 + \bar{x}_2 > 0\}.$$

Note that the two sets $\mathbb{S}_i \backslash \{0\}$ and $\mathbb{A}_i$ are equivalent. Note also that $\gamma_{pq} < 0$ is satisfied from $i \in \mathbb{I}_3$. Using these two facts, we can prove (i) $\Leftrightarrow$ (iii) in a similar way to the proof of (ii) $\Leftrightarrow$ (iii). □

**Proof of Lemma 9:** We can get the explicit form of $\tau$ from (38), (39) and (40). Let $y(s)$ denote the $(q,p)$ element of $T_i(s)$. In addition, let $y(t)$ stand for the inverse Laplace transform of $y(s)$. Then we have

$$x(\tau, z_p) = y(\tau) z_q. \tag{41}$$

Substituting $\tau$ into (41) yields (14). □

**Proof of Lemma 10:** We divide our proof into the following three parts: (I) $i \in \mathbb{I}_1$, (II) $i \in \mathbb{I}_2$, and (III) $i \in \mathbb{I}_3$.

Case $i \in \mathbb{I}_1$: (i) $\Rightarrow$ (ii); Note that $x(t, z_p) \in \mathbb{S}_i$ holds for all $t \geq 0$, because $i \in \mathbb{I}_1$. Therefore $x(t, z_p)$ must converge to the origin, which implies that $A_i$ is Hurwitz. Hence (ii) holds.

(ii) $\Rightarrow$ (i); The condition (ii) means that $A_i$ is Hurwitz in this case. Therefore, every trajectory converges to the origin, if no events take place. This leads to (i).

Case $i \in \mathbb{I}_2$: The condition (ii) implies that the minimal eigenvalue of $A_i$ is negative in this case. It follows from (iv) and (v) in Lemma 22 that (i) $\Leftrightarrow$ (ii).

Case $i \in \mathbb{I}_3$: The condition (ii) implies that $A_i$ has complex eigenvalues or that $A_i$ is Hurwitz.

(i) $\Rightarrow$ (ii); We will prove the contrapositive of the desired statements. Suppose (ii) does not hold. Then, (iii) in Lemma 8 is not satisfied. Therefore, (a) in Definition 2 does not hold for $x_0 = z_p$. Moreover, (b) in Definition 2 does not hold for $x_0 = z_p$, because $A_i$ has a non-negative real eigenvalue.

(ii) $\Rightarrow$ (i); If $A_i$ has complex eigenvalues, then (a) in Definition 2 holds for all initial states. Otherwise, or if $A_i$ is Hurwitz, then we can prove that (i) holds in the same way as the proof of (ii) $\Rightarrow$ (i) in Case $i \in \mathbb{I}_1$. □

**Proof of Theorem 11:** Proof of (i): We see that $x(\sum_{i=1}^{m} \tau_i, x_0) < x_0$ holds for all $x_0 \in \mathbb{S}_i \backslash \{0\}$, if and only if (20) holds. In other words, $\lim_{t\to\infty} x(t, x_0) = 0$, $\forall x_0$, if and only if (20) holds.

Stability of the origin in the sense of Lyapunov follows from linearity, if (20) holds for all $i$.

Proof of (ii): ($\Rightarrow$); It is seen from Lemma 10 that there exists an initial state $x_0$ such that $\lim_{t\to\infty} x(t, x_0) \neq 0$, if there exists an $i$ such that (19) does not hold. We have proved the contrapositive of the desired statement.

($\Leftarrow$); It is seen from Lemma 22 (ii) and Corollary 23 that there exists a $z_{ij}$ such that (b) in Definition 2 holds for $x_0 = z_{ij}$. Therefore, $\lim_{t\to\infty} x(t, x_0) = 0$ holds for all $x_0$.

Stability of the origin in the sense of Lyapunov follows from linearity, if (19) holds for all $i$. $\square$

## C Proofs of Lemmas 12, 13 and 14

**Proof of Lemma 12:** Define

$$r := \begin{cases} 1, & \text{if } c_1 \neq 0, \\ 2, & \text{otherwise.} \end{cases}$$

(i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Suppose that (iii) does not hold. Then $A_i$ has a real eigenvalue. We get

$$c_r x(t, x_0) = \begin{cases} \zeta t \mathrm{e}^{-\frac{\alpha t}{2}}, & \text{if } 4\beta = \alpha^2, \\ \zeta \left( \mathrm{e}^{\lambda_1 t} - \mathrm{e}^{\lambda_2 t} \right), & \text{if } 4\beta < \alpha^2, \end{cases} \tag{42}$$

where $x_0 \in \mathbb{B}_i$ and $\zeta (\neq 0)$ is a constant value depending on $x_0$, and $\lambda_1$ and $\lambda_2$ are defined by (17) and (18), respectively. In view of (42), we see that (ii) does not hold.

(iii) $\Rightarrow$ (i): We have

$$c_r x(t, x_0) = \zeta \mathrm{e}^{-\frac{\alpha t}{2}} \sin(\omega t + \theta), \tag{43}$$

where $x_0 \in \mathbb{S}_i$, $\omega = \sqrt{\beta - \frac{\alpha^2}{4}}$, and $\zeta (\neq 0)$ and $\theta$ are constant values depending on $x_0$. The equation (43) leads to (i). $\square$

**Proof of Lemma 13:** Substituting $x_0 \in \mathbb{B}_i$ into (43) yields $\theta = 0$, which leads to (22). $\square$

**Proof of Lemma 14:** (i) $\Rightarrow$ (ii): Suppose (ii) does not hold. Then, there exists an eigenvector $v_i \in \mathbb{S}_i$ associating with a non-negative real eigenvalue of $A_i$. Thus, $x(t, v_i) \in \mathbb{S}_i, \forall t \geq 0$ holds, which implies (i) does not hold.

(ii) $\Rightarrow$ (i): The statement (ii) means that $A_i$ has complex eigenvalues or that $A_i$ is Hurwitz. If $A_i$ has complex eigenvalues, then (a) in Definition 2 is immediate from Lemma 12. If $A_i$ is Hurwitz, then we see that (i) holds in the same way as the proof of Lemma 10. $\square$