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Abstract

The concept of M-convex functions is generalized for functions defined on constant-parity jump systems. Such function arises from minimum weight perfect b-matchings and from a separable convex function (sum of univariate convex functions) on the degree sequences of an undirected graph. As a generalization of a recent result of Apollonio and Sebő for the minsquare factor problem, a local optimality criterion is given for minimization of an M-convex function subject to a component sum constraint.

Keywords: jump system, degree sequence, graph factor, discrete convex function, local optimality
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1 Introduction

A recent paper of Apollonio and Sebő [2] has shown that the minsquare factor problem on a graph can be solved in polynomial time. The problem is, given an undirected graph possibly containing loops and parallel edges, to find a subgraph with a specified number of edges that minimizes the sum of squares of the degrees (= numbers of incident edges) of vertices. The key observation in [2] is that global optimality is guaranteed by local optimality in the neighborhood of ℓ_1-distance four in the space of degree sequences. It has also been observed in [2] that this local optimality criterion remains valid when the objective function is generalized to a separable convex function (= sum of univariate convex functions) of the degree sequence.

The objective of this paper is to put the above results in a more general context of discrete convex analysis [21] by introducing the concept of M-convex functions on constant-parity jump systems. A separable convex function of the degree sequences of a graph is an M-convex function in this sense.

A jump system [4] is a set of integer points with an exchange property (to be described in Section 2); see also [14], [16]. It is a generalization of a matroid [6], [15], a delta-matroid [3], [5], [7], and a base polyhedron of an integral polymatroid (or a submodular system) [11]. Minimization of a separable convex function over a jump system has been studied in [1], where a local criterion for optimality as well as a greedy algorithm is given.

Study of nonseparable nonlinear functions on matroidal structures was started with valuated matroids [8], [9], which have come to be accepted as discrete concave functions; see [18], [20]. This concept has been generalized to M-convex functions on base polyhedra [19], which play a central role in discrete convex analysis [21]. Valuated delta-matroids [10] afford another generalization of valuated matroids; see also [10], [17], [23], [24]. In all these generalizations global optimality is equivalent to local optimality defined in an appropriate manner. In addition, discrete duality such as discrete separation and min-max formula holds for valuated matroids and M-convex functions on base polyhedra, whereas it fails for valuated delta-matroids. M-convex functions on constant-parity jump systems, to be introduced in this paper, are a common generalization of valuated delta-matroids and M-convex functions on base polyhedra.

In this paper, we investigate into minimization of an M-convex function on a constant-parity jump system. It is shown, in particular, that (i) global optimality for unconstrained minimization is equivalent to local optimality in the neighborhood of ℓ_1-distance two (Theorem 3.3), and (ii) global optimality for constrained minimization on a hyperplane of a constant component sum is equivalent to local optimality in the neighborhood of ℓ_1-distance four (Theorem 4.1). The former generalizes the optimality criterion in [1] for separable convex function minimization over a jump system, and the latter the optimality criterion in [2] for the minsquare factor problem. Theorem 4.3 reveals convexity of the optimal values with respect to the component sum, on the basis of which algorithms are constructed for the constrained minimization in Section 5.

2 Exchange Axioms

Let V be a finite set. For $u \in V$ we denote by χ_u the characteristic vector of u, with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for $v \neq u$. For $x = (x(v)), y = (y(v)) \in \mathbf{Z}^V$ define

$$\begin{aligned} x(V) &= \sum_{v \in V} x(v), \\ ||x||_1 &= \sum_{v \in V} |x(v)|, \\ \mathrm{supp}\,(x) &= \{v \in V \mid x(v) \neq 0\}, \\ \mathrm{supp}^+(x) &= \{v \in V \mid x(v) > 0\}, \\ \mathrm{supp}^-(x) &= \{v \in V \mid x(v) < 0\}, \end{aligned}$$

$$[x, y] = \{z \in \mathbf{Z}^V \mid \min(x(v), y(v)) \leq z(v) \leq \max(x(v), y(v)), \forall v \in V\}.$$

A vector $s \in \mathbf{Z}^V$ is called an (x, y)*-increment* if $s = \chi_u$ or $s = -\chi_u$ for some $u \in V$ and $x+s \in [x, y]$. An (x, y)*-increment pair* will mean a pair of vectors (s, t) such that s is an (x, y)-increment and t is an $(x + s, y)$-increment.

A nonempty set $J \subseteq \mathbf{Z}^V$ is said to be a *jump system* if satisfies an exchange axiom, called the *2-step axiom*: for any $x, y \in J$ and for any (x, y)-increment s with $x + s \notin J$, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$. A set $J \subseteq \mathbf{Z}^V$ is a *constant-sum system* if $x(V) = y(V)$ for any $x, y \in J$, and a *constant-parity system* if $x(V) - y(V)$ is even for any $x, y \in J$.

We introduce a stronger exchange axiom:

(J-EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$ and $y - s - t \in J$.

This property characterizes a constant-parity jump system, a fact communicated to the author by J. Geelen (see Section 6.1 for a proof).

Lemma 2.1 (Geelen [12]) *A nonempty set J is a constant-parity jump system if and only if it satisfies* (J-EXC).

It turns out (see Section 6.2 for a proof) that (J-EXC) can be replaced by a weaker axiom:

(J-EXC$_\mathrm{w}$) For any distinct $x, y \in J$ there exists an (x, y)-increment pair (s, t) such that $x + s + t \in J$ and $y - s - t \in J$.

Lemma 2.2 *A set J satisfies* (J-EXC) *if and only if it satisfies* (J-EXC$_\mathrm{w}$)*.*

We call $f : J \to \mathbf{R}$ an *M-convex function* if it satisfies the following exchange axiom:

(M-EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$, $y - s - t \in J$, and

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t).$$

We adopt the convention that $f(x) = +\infty$ for $x \notin J$.

It turns out that the exchange axiom (M-EXC) is equivalent to a local exchange axiom:

(M-EXC$_\mathrm{loc}$) For any $x, y \in J$ with $||x - y||_1 = 4$ there exists an (x, y)-increment pair (s, t) such that $x + s + t \in J$, $y - s - t \in J$, and

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t).$$

Theorem 2.3 *A function $f : J \to \mathbf{R}$ defined on a constant-parity jump system J satisfies* (M-EXC) *if and only if it satisfies* (M-EXC$_\mathrm{loc}$)*.*

(Proof) The proof is technical and given in Section 6.3. ■

This implies that (M-EXC) can be replaced by a weaker axiom:

(M-EXC$_\mathrm{w}$) For any distinct $x, y \in J$ there exists an (x, y)-increment pair (s, t) such that $x + s + t \in J$, $y - s - t \in J$, and

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t).$$

Theorem 2.4 *A function $f : J \to \mathbf{R}$ satisfies* (M-EXC) *if and only if it satisfies* (M-EXC$_\mathrm{w}$)*.*

(Proof) It suffices to prove the "if" part. (M-EXC$_\mathrm{w}$) implies (J-EXC$_\mathrm{w}$) for J, and hence J is a constant-parity jump by Lemma 2.2. Then the claim follows from Theorem 2.3. ■

Note that addition of a linear function preserves M-convexity. That is, for an M-convex function f and a vector $p = (p(v)) \in \mathbf{R}^V$, the function $f[-p]$ defined by $f[-p](x) = f(x) - \langle p, x\rangle$ with $\langle p, x\rangle = \sum_{v \in V} p(v)x(v)$ is M-convex.

Remark 2.1 Our definition of an M-convex function is consistent with the previously considered special cases where (i) J is a constant-sum jump system, and (ii) J is a constant-parity jump system contained in $\{0, 1\}^V$. Case (i) is equivalent to J being the set of integer points in the base polyhedron of an integral submodular system [11], and then our M-convex function is the same as the M-convex function investigated in [19], [21]. Case (ii) is equivalent to J being an even delta-matroid [23], [24], and then f is M-convex in our sense if and only if $-f$ is a valuated delta-matroid in the sense of [10]. ■

Examples of M-convex functions follow.

Example 2.1 A *separable convex function* on a constant-parity jump system J, i.e., a function $f : J \to \mathbf{R}$ of the form $f(x) = \sum_{v \in V} \varphi_v(x(v))$ with univariate (one-dimensional) convex functions φ_v, is M-convex. In particular, the sum of squares $f(x) = \sum_{v \in V}(x(v))^2$ is M-convex. Such functions have been investigated in [1], [2]. ■

Example 2.2 Minimum weight factors in a graph yield an M-convex function. Let $G = (V, E)$ be an undirected graph that may contain loops and parallel edges. For a subgraph $H = (V, F)$, denote its *degree sequence* by $\deg_H = \sum\{\chi_u + \chi_v \mid (u, v) \in F\} \in \mathbf{Z}^V$. It is well known [4], [16] that

$$J = \{\deg_H \mid H \text{ is a subgraph of } G\}$$

forms a constant-parity jump system, called the *degree system* of G. Given edge weighting $w : E \to \mathbf{R}$, define a function $f : J \to \mathbf{R}$ by

$$f(x) = \min\{w(F) \mid H = (V, F) \text{ is a subgraph of } G \text{ with } \deg_H = x\}$$

with notation $w(F) = \sum_{e \in F} w(e)$, where $f(x)$ represents the minimum weight of a subgraph with degree sequence x.

This f is an M-convex function. In fact, (M-EXC) can be verified by the alternating path argument as follows. For distinct $x, y \in J$ let F_x and F_y be subsets of edges such that $f(x) = w(F_x)$ and $f(y) = w(F_y)$ with $x = \sum\{\chi_u + \chi_v \mid (u, v) \in F_x\}$ and $y = \sum\{\chi_u + \chi_v \mid (u, v) \in F_y\}$. Let s be an (x, y)-increment, and put $u_* = \mathrm{supp}\,(s)$. We may assume, without loss of generality, that $s = \chi_{u_*}$. Starting with an edge in $F_y \setminus F_x$ incident to u_* we construct an alternating path P by adding an edge in $F_x \setminus F_y$ and an edge in $F_y \setminus F_x$ alternately. The path P is not necessarily simple so that it may contain the same vertex more than once, whereas it consists of distinct edges. We assume that P is maximal in the sense that it cannot be extended further beyond the end vertex, say, v_*. Then there exists an $(x + \chi_{u_*}, y)$-increment t with $\mathrm{supp}\,(t) = v_*$; more specifically, $t = \chi_{v_*}$ or $-\chi_{v_*}$ according to whether P consists of odd or even number of edges. Denote by $F_x \Delta P$ the symmetric difference of F_x and P, and by $F_y \Delta P$ that of F_y and P. Since $x+s+t = \sum\{\chi_u+\chi_v \mid (u, v) \in F_x \Delta P\}$ and $y-s-t = \sum\{\chi_u+\chi_v \mid (u, v) \in F_y \Delta P\}$, we have $f(x + s + t) \leq w(F_x \Delta P)$ and $f(y - s - t) \leq w(F_y \Delta P)$, whereas $w(F_x \Delta P) + w(F_y \Delta P) = w(F_x) + w(F_y) = f(x) + f(y)$. Hence (M-EXC). Note that the alternating path argument above serves also as a proof of (J-EXC) for J. ■

Example 2.3 As a variant of the construction from the degree system in Example 2.2, an M-convex function arises from minimum weight perfect b-matchings; see [13], [22] for b-matchings. Let $G = (V, E)$ be an undirected graph that may have loops, but no parallel edges, and $w : E \to \mathbf{R}$ be an edge weighting. Let $J \subseteq \mathbf{Z}^V$ be the set of vectors $x \in \mathbf{Z}^V$ such that a perfect x-matching exists in G, and define a function $f : J \to \mathbf{R}$ by setting $f(x)$ to be the minimum weight of a perfect x-matching:

$$\begin{aligned} f(x) \quad = \quad & \min\{\sum_{e \in E} \lambda(e) w(e) \mid \\ & \sum_{e \in \delta(v)} \lambda(e) = x(v) \ (\forall v \in V); \lambda(e) \in \mathbf{Z}_+ \ (\forall e \in E)\}, \end{aligned} \tag{2.1}$$

where $\delta(v)$ denotes the set of edges incident to vertex $v \in V$, and $\mathbf{Z}_+$ the set of nonnegative integers. This function is M-convex as in Example 2.2. ■

Example 2.4 Let $A(t)$ be a skew-symmetric polynomial matrix in variable t. The degree in t of the principal minors of $A(t)$ yields a valuated delta-matroid, as is pointed out in [10], [23], and hence the negative of an M-convex function. ■

Remark 2.2 Unlike in the previously studied special cases where J is a base polyhedron or an even delta-matroid, an M-convex function on a jump system is not always extensible to a convex function. Nevertheless, our results will provide convincing evidences to indicate its discrete convexity. ■

3 Unconstrained Minimization

We consider minimization of an M-convex function $f : J \to \mathbf{R}$ defined on a constant-parity jump system $J \subseteq \mathbf{Z}^V$.

First we note a property of an M-convex function that indicates its discrete convexity. Given $f : J \to \mathbf{R}$ and $x, y \in J$, a sequence of points in J, say, $x_0, x_1, \ldots, x_m$, is called a *steepest-descent chain* connecting x to y if $x_0 = x$, $x_m = y$, and for $i = 1, \ldots, m$ we have $x_i = x_{i-1} + s_i + t_i$ for some (x_{i-1}, y)-increment pair (s_i, t_i) such that $f(x_{i-1}+s_i+t_i) \leq f(x_{i-1}+s+t)$ for every (x_{i-1}, y)-increment pair (s, t); we have $m = ||x-y||_1/2$. An M-convex function turns out to be convex along a steepest-descent chain, as follows.

Proposition 3.1 *Let $f : J \to \mathbf{R}$ be an M-convex function, and $x_0, x_1, \ldots, x_m$ be a steepest-descent chain connecting $x \in J$ to $y \in J$. Then*

$$f(x_{i-1}) + f(x_{i+1}) \geq 2f(x_i) \qquad (i = 1, \ldots, m-1). \tag{3.1}$$

(Proof) Put $x_i = x_{i-1} + s + t$ and $x_{i+1} = x_i + s' + t'$. By (M-EXC) we have

$$
\begin{aligned}
& f(x_{i-1}) + f(x_{i+1}) \\
& \geq \min[f(x_{i-1} + s + t) + f(x_{i-1} + s' + t'), \\
& \qquad f(x_{i-1} + s + t') + f(x_{i-1} + s' + t), \\
& \qquad f(x_{i-1} + s + s') + f(x_{i-1} + t + t')] \ \geq 2f(x_i).
\end{aligned}
$$

■

As an immediate corollary we see that a nonoptimal point can be improved with a suitable increment pair.

Proposition 3.2

(1) *If $x, y \in J$ and $f(x) > f(y)$, there exists an (x, y)-increment pair (s, t) such that $f(x) > f(x + s + t)$.*

(2) *If $x, y \in J$ and $f(x) \geq f(y)$, there exists an (x, y)-increment pair (s, t) such that $f(x) \geq f(x + s + t)$.*

This implies, in turn, that global optimality (minimality) of an M-convex function is guaranteed by local optimality in the neighborhood of ℓ_1-distance two.

Theorem 3.3 *Let $f : J \to \mathbf{R}$ be an M-convex function on a constant-parity jump system J, and let $x \in J$. Then $f(x) \leq f(y)$ for all $y \in J$ if and only if $f(x) \leq f(y)$ for all $y \in J$ with $||x - y||_1 \leq 2$.*

(Proof) The "only if" part is obvious, and the "if" part follows from Proposition 3.2. ■

The minimizers of an M-convex function form a constant-parity jump system, as follows. We denote by $\arg\min f[-p]$ the set of minimizers of function $f[-p]$.

Proposition 3.4 *For any $p \in \mathbf{R}^V$, $\arg\min f[-p]$ is a constant-parity jump system, if it is nonempty.*

(Proof) Let β denote the minimum value of $f[-p]$, and let $x, y \in \arg\min f[-p]$. Then, in (M-EXC) we have $2\beta = f[-p](x) + f[-p](y) \geq f[-p](x + s + t) + f[-p](y - s - t) \geq 2\beta$, which implies $x + s + t, y - s - t \in \arg\min f[-p]$. ■

Remark 3.1 The local optimality criterion for M-convex functions on jump systems in Theorem 3.3 contains a number of previous results as special cases. In the case of constant-sum jump systems, case (i) in Remark 2.1, the present theorem reduces to the optimality criterion for M-convex functions on base polyhedra established in [19] (see Theorem 6.26 of [21]), and, moreover, Proposition 3.2 (1) above coincides with Proposition 6.23 of [21]. In the case of constant-parity jump systems contained in $\{0,1\}^V$, case (ii) in Remark 2.1, Theorem 3.3 reduces to the optimality criterion for valuated delta-matroids established in [10]. Both of these are generalizations, in different directions, of the optimality criterion for valuated matroids given in [8], [9]. It is noted that the optimality criterion for valuated matroids given in [8], [9] is the origin of this type of optimality criteria for nonseparable nonlinear objective functions, and the above two special cases are generalizations in different directions thereof. Separable convex functions on jump systems have been considered in [1]. ■

4 Minimization under Sum Constraint

In this section we investigate into minimization of an M-convex function $f(x)$ when the sum of the components of x is specified. Recalling the notation $x(V)$ for the sum of components of a vector x, we introduce some other notations concerning the feasible regions of our optimization problem:

$$\begin{aligned}
k_{\min} &= \min\{x(V) \mid x \in J\}, \\
k_{\max} &= \max\{x(V) \mid x \in J\}, \\
\Lambda &= \{k \mid k_{\min} \leq k \leq k_{\max},\ k \equiv k_{\min} (\mathrm{mod}\ 2)\}, \\
J_k &= \{x \in J \mid x(V) = k\} \qquad (k \in \Lambda),
\end{aligned}$$

where $J_k \neq \emptyset$ for each $k \in \Lambda$ by (J-EXC) and it may be that $k_{\min} = -\infty$ and/or $k_{\max} = +\infty$.

Our problem is to minimize $f(x)$ subject to $x \in J_k$, where $k \in \Lambda$ is a parameter. Denote by f_k and M_k the minimum value and the set of minimizers, respectively, i.e.,

$$\begin{aligned}
f_k &= \min\{f(x) \mid x \in J_k\} \qquad (k \in \Lambda), \\
M_k &= \{x \in J_k \mid f(x) = f_k\} \qquad (k \in \Lambda),
\end{aligned}$$

where we assume that, for each $k \in \Lambda$, f_k is finite and M_k is nonempty. By convention we put $f_k = +\infty$ for $k \notin \Lambda$.

Global optimality (minimality) on J_k is guaranteed by local optimality in the neighborhood of ℓ_1-distance four. Compare this with the unconstrained optimization treated in Theorem 3.3, which refers to the neighborhood of ℓ_1-distance two. It is emphasized that J_k is not necessarily a jump system, and accordingly, Theorem 3.3 does not apply to minimization of f over J_k.

Theorem 4.1 *Let $f : J \to \mathbf{R}$ be an M-convex function on a constant-parity jump system J, and let $x \in J_k$ with $k \in \Lambda$. Then $f(x) \leq f(y)$ for all $y \in J_k$ if and only if $f(x) \leq f(y)$ for all $y \in J_k$ with $||x - y||_1 \leq 4$.*

(Proof) The "only if" part is obvious. To prove the "if" part by contradiction, assume that $f(x) > f(y)$ for some $y \in J_k$ and take such y with minimum $||y-x||_1$. Since $x(V) = y(V)$ and $x \neq y$, both $\mathrm{supp}^+(y-x)$ and $\mathrm{supp}^-(y-x)$ are nonempty.

Claim 1: If $u \in \mathrm{supp}^+(y-x)$ and $v \in \mathrm{supp}^-(y-x)$, then

$$f(x) + f(y) < f(x + \chi_u - \chi_v) + f(y - \chi_u + \chi_v).$$

Proof of Claim 1: We have $f(x) \leq f(x + \chi_u - \chi_v)$ by the assumed local optimality, and $f(x) \leq f(y - \chi_u + \chi_v)$ since $y - \chi_u + \chi_v$ is closer to x than y. Adding these two and $f(y) < f(x)$ yields the desired inequality.

By (M-EXC) for (x, y), together with Claim 1, there exist $u_1 \in \mathrm{supp}^+(y-x)$, $u_2 \in \mathrm{supp}^+(y-x-\chi_{u_1})$, $v_1 \in \mathrm{supp}^-(y-x)$, and $v_2 \in \mathrm{supp}^-(y-x-\chi_{v_1})$ such that

$$f(x) + f(y) \geq f(x + \chi_{u_1} + \chi_{u_2}) + f(y - \chi_{u_1} - \chi_{u_2}), \tag{4.1}$$

$$f(x) + f(y) \geq f(x - \chi_{v_1} - \chi_{v_2}) + f(y + \chi_{v_1} + \chi_{v_2}). \tag{4.2}$$

By (M-EXC) for $(x + \chi_{u_1} + \chi_{u_2}, x - \chi_{v_1} - \chi_{v_2})$ and the local optimality, we obtain

$$\begin{aligned}
& f(x + \chi_{u_1} + \chi_{u_2}) + f(x - \chi_{v_1} - \chi_{v_2}) \\
& \geq \min[f(x + \chi_{u_1} - \chi_{v_1}) + f(x + \chi_{u_2} - \chi_{v_2}), \\
& \qquad f(x + \chi_{u_1} - \chi_{v_2}) + f(x + \chi_{u_2} - \chi_{v_1}), \\
& \qquad f(x) + f(x + \chi_{u_1} + \chi_{u_2} - \chi_{v_1} - \chi_{v_2})] \\
& \geq 2f(x).
\end{aligned} \tag{4.3}$$

Similarly, by (M-EXC) for $(y - \chi_{u_1} - \chi_{u_2}, y + \chi_{v_1} + \chi_{v_2})$, we obtain

$$\begin{aligned}
& f(y - \chi_{u_1} - \chi_{u_2}) + f(y + \chi_{v_1} + \chi_{v_2}) \\
& \geq \min[f(y - \chi_{u_1} + \chi_{v_1}) + f(y - \chi_{u_2} + \chi_{v_2}), \\
& \qquad f(y - \chi_{u_1} + \chi_{v_2}) + f(y - \chi_{u_2} + \chi_{v_1}), \\
& \qquad f(y) + f(y - \chi_{u_1} - \chi_{u_2} + \chi_{v_1} + \chi_{v_2})] \\
& \geq f(x) + f(y),
\end{aligned} \tag{4.4}$$

since $f(y - \chi_{u_i} + \chi_{v_j}) \geq f(x)$ and $f(y - \chi_{u_1} - \chi_{u_2} + \chi_{v_1} + \chi_{v_2}) \geq f(x)$ by the choice of y. Adding (4.1), (4.2), (4.3) and (4.4) yields a contradiction. ■

The ℓ_1-distance of four in Theorem 4.1 cannot be replaced by ℓ_1-distance of two, as we see in the following example.

Example 4.1 ([2]) Let $J \subseteq \mathbf{Z}^6$ be the degree system (see Example 2.2) of an undirected graph consisting of vertex-disjoint two triangles, and $f : J \to \mathbf{R}$ be an M-convex function representing the sum of squares of the components (see Example 2.1). Let $k = 8$ and $x = (2, 2, 2, 1, 1, 0)$, for which $f(x) = 14$. For any point $y \in J_8$ with $||y - x||_1 = 2$ we have $f(y) = 14$, whereas for $x^* = (2, 1, 1, 2, 1, 1)$ we have $f(x^*) = 12$ and $||x^* - x||_1 = 4$. ■

Remark 4.1 Theorem 4.1 above is a generalization of Theorem 1 of [2], since the degree system of a graph is a constant-parity jump system (Example 2.2) and a separable convex function on a constant-parity jump system is an M-convex function (Example 2.1). In fact, the result of [2] was the primary motivation behind Theorem 4.1. ■

The following theorem reveals a kind of monotonicity of the minimizers of f on J_k.

Theorem 4.2 *For any* $x_k \in M_k$ *with* $k \in \Lambda$ *there exists* $(x_l \in M_l \mid l \in \Lambda \setminus \{k\})$ *such that* $x_{k_{\min}} \leq \cdots \leq x_{k-2} \leq x_k \leq x_{k+2} \leq \cdots \leq x_{k_{\max}}$.

(Proof) We show the existence of such x_{k-2}. Then x_{k+2} can be shown to exist in a similar manner, and the other x_l (with $l \leq k-4$ or $l \geq k+4$) are by induction.

Take $y \in M_{k-2}$ with minimum $||y - x_k||_1$. If $y \leq x_k$, we are done with $x_{k-2} = y$. Otherwise, take $u \in \mathrm{supp}^-(y - x_k)$ and apply (M-EXC) to obtain either

$$\exists v \in \mathrm{supp}^-(y - x_k) : \ f_{k-2} + f_k \geq f(y + \chi_u + \chi_v) + f(x_k - \chi_u - \chi_v)$$

or

$$\exists v \in \mathrm{supp}^+(y - x_k) : \ f_{k-2} + f_k \geq f(y + \chi_u - \chi_v) + f(x_k - \chi_u + \chi_v).$$

In the first case the right-hand side is lower bounded by $f_k + f_{k-2}$ and hence $y + \chi_u + \chi_v \in M_k$ and $x_k - \chi_u - \chi_v \in M_{k-2}$; then we can take $x_{k-2} = x_k - \chi_u - \chi_v$. The second case cannot occur, since the right-hand side is lower bounded by $f_{k-2} + f_k$, from which follows $y + \chi_u - \chi_v \in M_{k-2}$, whereas $||(y + \chi_u - \chi_v) - x_k||_1 = ||y - x_k||_1 - 2$; a contradiction to the choice of y. ■

Theorem 4.3 *Minimum values* f_k *form a convex sequence:*

$$f_{k-2} + f_{k+2} \geq 2f_k \qquad (k \in \Lambda \setminus \{k_{\min}, k_{\max}\}). \tag{4.5}$$

(Proof) By Theorem 4.2 we can take $x_{k-2} \in M_{k-2}$ and $x_{k+2} \in M_{k+2}$ with $x_{k-2} \leq x_{k+2}$, and also $u \in \mathrm{supp}^+(x_{k+2} - x_{k-2})$. By (M-EXC) there exists $v \in \mathrm{supp}^+(x_{k+2} - x_{k-2})$ such that

$$f_{k-2} + f_{k+2} \geq f(x_{k-2} + \chi_u + \chi_v) + f(x_{k+2} - \chi_u - \chi_v) \geq 2f_k.$$

■

Convexity of the minimum values motivates us to consider the subgradient. For $\alpha \in \mathbf{R}$ define $f^\alpha : J \to \mathbf{R}$ by

$$f^\alpha(x) \quad = \quad f(x) - \alpha x(V). \tag{4.6}$$

Then we have

$$\min_{x \in J} f^{\alpha}(x) = \min_{l \in \Lambda} \min_{x \in J_l} f^{\alpha}(x) = \min_{l \in \Lambda}(f_l - \alpha l). \tag{4.7}$$

By Theorem 4.3, the minimum of $f_l - \alpha l$ over $l \in \Lambda$ is attained by $l = k$ if

$$(f_k - f_{k-2})/2 \leq \alpha \leq (f_{k+2} - f_k)/2. \tag{4.8}$$

Hence

$$f_k = k\alpha + \min\{f^{\alpha}(x) \mid x \in J\} \tag{4.9}$$

for α in the range of (4.8). This shows that the optimal value f_k can be computed by solving an unconstrained minimization problem for another M-convex function f^{α}.

It is noted, however, that not every minimizer of f^{α} belongs to J_k. A point $x \in J$ minimizes $f^{\alpha}(x)$ if and only if $x \in M_k$ for some k with $k_-(\alpha) \leq k \leq k_+(\alpha)$, where

$$\begin{aligned} k_-(\alpha) &= \min\{k \mid \min_l (f_l - \alpha l) = f_k - \alpha k\}, && (4.10) \\ k_+(\alpha) &= \max\{k \mid \min_l (f_l - \alpha l) = f_k - \alpha k\}. && (4.11) \end{aligned}$$

Theorem 4.4 *For each $\alpha \in \mathbf{R}$, $\bigcup\{M_k \mid k_-(\alpha) \leq k \leq k_+(\alpha)\}$ is a constant-parity jump system. In particular, M_k is a base polyhedron if $k = k_{\min}$, or $k = k_{\max}$, or $f_{k-2} + f_{k+2} > 2f_k$ with $k \in \Lambda \setminus \{k_{\min}, k_{\max}\}$.*

(Proof) The first statement follows from Proposition 3.4, since, as observed above, $\bigcup\{M_k \mid k_-(\alpha) \leq k \leq k_+(\alpha)\}$ coincides with $\arg\min f^{\alpha}$. For the second statement it suffices to note that for such k we can choose an α with $k_-(\alpha) = k = k_+(\alpha)$ and that a constant-sum jump system is a base polyhedron. ■

Remark 4.2 Theorems 4.2, 4.3 and 4.4 above are natural generalizations of the similar results of [17] for valuated delta-matroids. ■

5 Algorithms

The local optimality criteria in Theorems 3.3 and 4.1 for unconstrained and constrained minimization, respectively, naturally suggest descent-type algorithms. At each feasible nonoptimal point, an improved point can be found with $O(|V|^2)$ function evaluations in unconstrained minimization and $O(|V|^4)$ function evaluations in constrained minimization. Although we do not enter into further technical details, the number of updates of the solution point may be bounded by the ℓ_1-distance from the initial point to the optimal point, or by the difference of the objective function values at the initial point and at the optimal point if the objective function is integer-valued.

Two other algorithms can be constructed for constrained minimization, to minimize $f(x)$ subject to $x \in J_k$, on the basis of Theorems 4.2 and 4.3. It is assumed that an algorithm is available for unconstrained minimization. For the convenience of descriptions it is also assumed that $k_{\min}$ and $k_{\max}$ are finite.

An increasing sequence of optimal solutions, the existence of which is guaranteed by Theorem 4.2, can be generated by the following algorithm. Once a global minimizer x^* is found, the algorithm computes the whole set of f_k $(k \in \Lambda)$ with $\mathrm{O}((k_{\max} - k_{\min})|V|^2)$ evaluations of f. Note that the algorithm works even if $k_{\min}$ and/or $k_{\max}$ are not known in advance.

Algorithm I
  Compute $x^* \in J$ that minimizes f;
  Set $k^* := x^*(V)$, $x_{k^*} := x^*$, $f_{k^*} := f(x_{k^*})$;
  for $k := k^* + 2, k^* + 4, \cdots, k_{\max}$ **do**
    Find $\{u, v\} \subseteq V$ that minimizes $f(x_{k-2} + \chi_u + \chi_v)$
    and put $x_k := x_{k-2} + \chi_u + \chi_v$ and $f_k := f(x_k)$;
  for $k := k^* - 2, k^* - 4, \cdots, k_{\min}$ **do**
    Find $\{u, v\} \subseteq V$ that minimizes $f(x_{k+2} - \chi_u - \chi_v)$
    and put $x_k := x_{k+2} - \chi_u - \chi_v$ and $f_k := f(x_k)$.

Convexity of the sequence f_k makes it possible to convert the constrained minimization to an unconstrained minimization of f^α with an appropriate value of α; see (4.9). Here f^α is M-convex and, by our assumption, the minimum of $f^\alpha(x)$ over $x \in J$ can be computed efficiently. We assume that we can find $k_+(\alpha)$ and $k_-(\alpha)$ of (4.11) and (4.10) by maximizing (resp. minimizing) $x(V)$ among the minimizers of $f^\alpha(x)$ by means of some variant of an unconstrained minimization algorithm.

The following algorithm computes $k_{\min}$, $k_{\max}$ and f_k $(k \in \Lambda)$ by searching for appropriate values of α. It requires $\mathrm{O}((k_{\max} - k_{\min})|V|^2)$ evaluations of f.

Algorithm II
  Let α be sufficiently large;
  Minimize f^α to find $k_{\min} = k_+(\alpha) = k_-(\alpha)$ and $f_{k_{\min}}$;
  Let α be sufficiently small
    (α is a negative number with a large absolute value);
  Minimize f^α to find $k_{\max} = k_+(\alpha) = k_-(\alpha)$ and $f_{k_{\max}}$;
  if $k_{\max} - k_{\min} \geq 4$ **then** search$(k_{\min}, k_{\max})$.

Here the procedure "search(k_1, k_2)" is defined when $k_1 + 4 \leq k_2$ as follows.

procedure search(k_1, k_2)
  $\alpha := (f_{k_2} - f_{k_1})/(k_2 - k_1)$;

Minimize f^α to find $k_+ = k_+(\alpha)$, $k_- = k_-(\alpha)$, $f_+ = f_{k_+}$ and $f_- = f_{k_-}$;
for $k := k_- + 2, k_- + 4, \cdots, k_+ - 2$ **do**
    $f_k := ((k - k_-)f_+ + (k_+ - k)f_-)/(k_+ - k_-)$;
if $k_1 + 4 \leq k_-$ **then** search(k_1, k_-);
if $k_+ + 4 \leq k_2$ **then** search(k_+, k_2).

The second algorithm, as it stands, computes the values of f_k, and not the optimal solutions x_k. If x_k's are wanted, they can be computed easily in procedure "search" by generating a sequence of points $x_k \in J_k \cap \arg\min f^\alpha$ by applying (J-EXC) to the pair of the optimal solutions x_{k_-} and x_{k_+}.

6 Proofs

6.1 Proof for (J-EXC)

A proof of Lemma 2.1, different from Geelen's [12], is provided here. This proof can be extended for Theorem 2.3.

For a constant-parity system J, the 2-step axiom of a jump system is simplified to:

(J-EXC$_+$) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$.

It suffices to prove (J-EXC$_+$) $\Rightarrow$ (J-EXC), since (J-EXC) $\Rightarrow$ (J-EXC$_+$) is obvious and (J-EXC) implies J being a constant-parity system.

We first note the following fact.

Lemma 6.1 *Assume* (J-EXC$_+$)*, let $y \in J$, and let z be a point at ℓ_1-distance four from y, represented as $z = y - s_1 - s_2 - s_3 - s_4$ with $s_i \in \mathbf{Z}^V$ and $||s_i||_1 = 1$ for $i = 1, 2, 3, 4$. If $z \in J$, then $y - s_i - s_j \in J$ and $y - s_k - s_l \in J$ for some $i, j, k, l \in \{1, 2, 3, 4\}$ with $\{i, j, k, l\} = \{1, 2, 3, 4\}$.*

(Proof) Consider an undirected graph G with vertex-set $\{1, 2, 3, 4\}$ and edge-set $\{(i, j) \mid y - s_i - s_j \in J\}$. It follows from (J-EXC$_+$) for (y, z) with $s = -s_i$ that, for each vertex i, there exists an edge incident to i. Similarly, it follows from (J-EXC$_+$) for (z, y) with $s = s_i$ that, for each vertex i, there exists an edge not incident to i. Such a graph has a perfect matching consisting of two edges, say, (i, j) and (k, l) with $\{i, j, k, l\} = \{1, 2, 3, 4\}$. This means that $y - s_i - s_j \in J$ and $y - s_k - s_l \in J$. ■

To prove (J-EXC$_+$) $\Rightarrow$ (J-EXC) by contradiction, we assume that there exists a pair (x, y) for which (J-EXC) fails. That is, we assume that the set of such pairs

$$\begin{aligned} \mathcal{D} \;=\; \{(x, y) \mid x, y \in J,\ \exists s_* : (x, y)\text{-increment such that} \\ \forall t : (x + s_*, y)\text{-increment} : x + s_* + t \not\in J \text{ or } y - s_* - t \not\in J\} \end{aligned}$$

is nonempty.

Take a pair $(x, y) \in \mathcal{D}$ with minimum $||x-y||_1$, where $||x-y||_1 \geq 4$, and fix s_* satisfying the condition above, and put $u_* = \text{supp}\,(s_*)$. Denoting the set of $(x+s_*, y)$-increments by I, we have

$$x + s_* + t \notin J \quad \text{or} \quad y - s_* - t \notin J \qquad (t \in I). \tag{6.1}$$

Put $U = \text{supp}\,(y-x)$ and, for $v \in U$, let t_v denote the (uniquely determined) (x, y)-increment such that $\text{supp}\,(t_v) = v$; we have $t_v = \sigma(v)\chi_v$ using the notation σ defined by: $\sigma(v) = 1$ for $v \in \text{supp}^+(y-x)$ and $\sigma(v) = -1$ for $v \in \text{supp}^-(y-x)$. Define $\alpha \in \mathbf{R}$ by

$$\alpha = \begin{cases} 1/2 & (s_* \in I,\ x + 2s_* \notin J,\ y - 2s_* \in J), \\ 0 & (\text{otherwise}), \end{cases}$$

and $p \in \mathbf{R}^V$ by

$$\sigma(v)p(v) = \begin{cases} \alpha & (v = u_*), \\ -\alpha & (v \in U \setminus \{u_*\}, x + s_* + t_v \in J), \\ -\alpha + 1 & (v \in U \setminus \{u_*\}, x + s_* + t_v \notin J, y - s_* - t_v \in J), \\ 0 & (\text{otherwise}). \end{cases}$$

Claim 1:

$$\langle p, s_* + t \rangle = 0 \qquad \text{if } t \in I,\ x + s_* + t \in J, \tag{6.2}$$

$$\langle p, s_* + t \rangle = 1 \qquad \text{if } t \in I,\ y - s_* - t \in J. \tag{6.3}$$

The equality (6.2) is easy to see, whereas (6.3) can be shown as follows. By (6.1) we have $x + s_* + t \notin J$, and hence

$$\langle p, s_* + t \rangle = \begin{cases} 2\alpha = 1 & \text{if } t = s_*, \\ \alpha + (-\alpha + 1) = 1 & \text{if } t \neq s_*. \end{cases}$$

Next, let P denote the set of $(x+s_*, y)$-increment pairs.

Claim 2: There exists $(s_0, t_0) \in P$ such that $y - s_0 - t_0 \in J$ and

$$\langle p, s_0 + t_0 \rangle \leq \langle p, s + t \rangle \quad \text{if } (s, t) \in P,\ y - s - t \in J. \tag{6.4}$$

Since s_* is an (x, y)-increment and J satisfies (J-EXC$_+$), there exists $t_* \in I$ such that $x + s_* + t_* \in J$, where t_* may possibly be identical with s_*. We see that $x + s_* + t_*$ is distinct from y since $||x - y||_1 \geq 4$. By (J-EXC$_+$) for $(y, x + s_* + t_*)$ there exists an $(x + s_* + t_*, y)$-increment pair (s, t) such that $y - s - t \in J$. This shows the existence of $(s, t) \in P$ with $y - s - t \in J$. Then (6.4) is satisfied by the pair $(s, t) = (s_0, t_0)$ that minimizes $\langle p, s + t \rangle$ over $(s, t) \in P$ subject to the condition $y - s - t \in J$.

Claim 3: $(x, y') \in \mathcal{D}$ with $y' = y - s_0 - t_0$.

To show this, first note that s_* is an (x, y')-increment, and let t be an $(x + s_*, y')$-increment. We have $t \in I$, $(s_0, t) \in P$, and $(t_0, t) \in P$. Hence, by (6.4), we have

$$\langle p, s_0 + t_0 \rangle \leq \langle p, s_0 + t \rangle \quad \text{if} \quad y - s_0 - t \in J, \tag{6.5}$$

$$\langle p, s_0 + t_0 \rangle \leq \langle p, t_0 + t \rangle \quad \text{if} \quad y - t_0 - t \in J. \tag{6.6}$$

We assume $y' - s_* - t \in J$ and derive $x + s_* + t \notin J$. By Lemma 6.1 with $z = y - s_0 - t_0 - s_* - t$, at least one of the following three cases occur: (i) $y - s_0 - t_0 \in J$ and $y - s_* - t \in J$, (ii) $y - s_0 - t \in J$ and $y - s_* - t_0 \in J$, (iii) $y - t_0 - t \in J$ and $y - s_* - s_0 \in J$. In either case we have

$$\langle p, s_* + t \rangle \geq 1, \tag{6.7}$$

since, in case (ii), for example, we have

$$\langle p, s_0 + t_0 + s_* + t \rangle = \langle p, s_0 + t \rangle + \langle p, s_* + t_0 \rangle \geq \langle p, s_0 + t_0 \rangle + 1$$

by (6.3) and (6.5). By (6.7) and (6.2) we see $x + s_* + t \notin J$. Hence $(x, y') \in \mathcal{D}$.

Finally, since $||x - y'||_1 = ||x - y||_1 - 2$, Claim 3 contradicts our choice of $(x, y) \in \mathcal{D}$. Therefore we conclude $\mathcal{D} = \emptyset$, completing the proof of Lemma 2.1.

6.2 Proof for (J-EXC) $\Leftrightarrow$ (J-EXC$_{\rm w}$)

A proof of Lemma 2.2 is provided here. By the argument in Section 6.1, it suffices to show (J-EXC$_{\rm w}$) $\Rightarrow$ (J-EXC$_+$), which we prove by induction on $||x - y||_1$. Take distinct $x, y \in J$ and an (x, y)-increment s. By (J-EXC$_{\rm w}$) there exists an (x, y)-increment pair (s_1, t_1) such that $x + s_1 + t_1 \in J$ and $y - s_1 - t_1 \in J$. If $s \in \{s_1, t_1\}$, we are done. Otherwise, put $y' = y - s_1 - t_1$. We have $||x - y'||_1 = ||x - y||_1 - 2$ and s is an (x, y')-increment. By the induction hypothesis, (J-EXC$_+$) with (x, y') and s implies $x + s + t \in J$ for some (x, y')-increment t, which is also an (x, y)-increment.

6.3 Proof for (M-EXC) $\Leftrightarrow$ (M-EXC$_{\rm loc}$)

A proof of Theorem 2.3 is provided here. It suffices to prove (M-EXC$_{\rm loc}$) $\Rightarrow$ (M-EXC). For $x \in J$, $d \in \mathbf{Z}^V$, and $p \in \mathbf{R}^V$, define $f(x, d) = f(x + d) - f(x)$ and $f_p(x, d) = f(x + d) - f(x) - \langle p, d \rangle$. We then have

$$f_p(x, d) + f_p(y, -d) = f(x, d) + f(y, -d). \tag{6.8}$$

We use an abbreviation f_p for $f[-p]$.

Lemma 6.2 *Assume $x, y \in J$, $||x - y||_1 = 4$, and $p \in \mathbf{R}^V$. If* (M-EXC$_{\rm loc}$) *is satisfied, then*

$$f_p(y) - f_p(x) \geq \min(\pi_{12} + \pi_{34}, \pi_{13} + \pi_{24}, \pi_{14} + \pi_{23}), \tag{6.9}$$

where $\pi_{ij} = f_p(x, s_i + s_j)$ *for* $i, j \in \{1, 2, 3, 4\}$ *with reference to the representation* $y = x + s_1 + s_2 + s_3 + s_4$ *with* $s_i \in \mathbf{Z}^V$ *and* $||s_i||_1 = 1$ *for* $i = 1, 2, 3, 4$.

(Proof) Note that $x + s_i + s_j = y - s_k - s_l$ if $\{i, j, k, l\} = \{1, 2, 3, 4\}$. (M-EXC$_{\rm loc}$) for f is equivalent to that for $f[-p]$, which implies

$$\begin{aligned} f_p(y) - f_p(x) \quad \geq \quad & \min[f_p(x, s_1 + s_2) + f_p(x, s_3 + s_4), \\ & f_p(x, s_1 + s_3) + f_p(x, s_2 + s_4), \\ & f_p(x, s_1 + s_4) + f_p(x, s_2 + s_3)]. \end{aligned}$$

■

To prove by contradiction, we assume that there exists a pair (x, y) for which (M-EXC) fails. That is, we assume that the set of such pairs

$$\begin{aligned} \mathcal{D} \quad = \quad & \{(x, y) \mid x, y \in J,\ \exists s_* : (x, y)\text{-increment such that} \\ & \forall t : (x + s_*, y)\text{-increment} : f(x, s_* + t) + f(y, -s_* - t) > 0\} \end{aligned}$$

is nonempty. Take a pair $(x, y) \in \mathcal{D}$ with minimum $||x - y||_1$; we have $||x - y||_1 > 4$ by (M-EXC$_{\rm loc}$). Let s_* be an (x, y)-increment satisfying the condition above, and put $u_* = \mathrm{supp}\,(s_*)$. Denoting the set of $(x + s_*, y)$-increments by I, we have

$$f(x, s_* + t) + f(y, -s_* - t) > 0 \qquad (t \in I). \tag{6.10}$$

Put $U = \mathrm{supp}\,(y - x)$ and, for $v \in U$, let t_v denote the (uniquely determined) (x, y)-increment such that $\mathrm{supp}\,(t_v) = v$; we have $t_v = \sigma(v)\chi_v$ using the notation σ defined by: $\sigma(v) = 1$ for $v \in \mathrm{supp}^+(y - x)$ and $\sigma(v) = -1$ for $v \in \mathrm{supp}^-(y - x)$. Using this convention, define $\alpha \in \mathbf{R}$ by

$$\alpha = \begin{cases} f(x, 2s_*)/2 & (s_* \in I,\ x + 2s_* \in J), \\ (-f(y, -2s_*) + \varepsilon)/2 & (s_* \in I,\ x + 2s_* \not\in J,\ y - 2s_* \in J), \\ 0 & (\text{otherwise}), \end{cases}$$

and $p \in \mathbf{R}^V$ by

$$\sigma(v)p(v) = \begin{cases} \alpha & (v = u_*), \\ f(x, s_* + t_v) - \alpha & (v \in U \setminus \{u_*\}, x + s_* + t_v \in J), \\ -f(y, -s_* - t_v) - \alpha + \varepsilon & (v \in U \setminus \{u_*\}, x + s_* + t_v \not\in J, \\ & \quad y - s_* - t_v \in J), \\ 0 & (\text{otherwise}) \end{cases}$$

with some $\varepsilon > 0$.

Claim 1:

$$\begin{aligned} f_p(x, s_* + t) \quad &= \quad 0 \qquad \text{if } t \in I, x + s_* + t \in J, & (6.11) \\ f_p(y, -s_* - t) \quad &> \quad 0 \qquad \text{if } t \in I. & (6.12) \end{aligned}$$

The equality (6.11) follows from

$$f_p(x, s_* + t) = f(x, s_* + t) - \langle p, s_* \rangle - \langle p, t \rangle$$
$$= \begin{cases} f(x, 2s_*) - 2\alpha = 0 & \text{if } t = s_*, \\ f(x, s_* + t) - \alpha - [f(x, s_* + t) - \alpha] = 0 & \text{if } t \neq s_*. \end{cases}$$

The inequality (6.12) can be shown as follows. We may assume $y - s_* - t \in J$, since otherwise $f_p(y, -s_* - t) = +\infty$. If $x + s_* + t \in J$, we have $f_p(x, s_* + t) = 0$ by (6.11) and

$$f_p(x, s_* + t) + f_p(y, -s_* - t) = f(x, s_* + t) + f(y, -s_* - t) > 0$$

by (6.8) and (6.10). Otherwise ($y - s_* - t \in J$ and $x + s_* + t \notin J$), we have

$$f_p(y, -s_* - t) = f(y, -s_* - t) + \langle p, s_* \rangle + \langle p, t \rangle$$
$$= \begin{cases} f(y, -2s_*) + 2\alpha = \varepsilon & \text{if } t = s_*, \\ f(y, -s_* - t) + \alpha + [-f(y, -s_* - t) - \alpha + \varepsilon] = \varepsilon & \text{if } t \neq s_*. \end{cases}$$

Next, let P denote the set of $(x + s_*, y)$-increment pairs.

Claim 2: There exists $(s_0, t_0) \in P$ such that $y - s_0 - t_0 \in J$ and

$$f_p(y, -s_0 - t_0) \leq f_p(y, -s - t) \qquad (\forall (s, t) \in P). \tag{6.13}$$

Since s_* is an (x, y)-increment and J satisfies (J-EXC), there exists $t_* \in I$ such that $x + s_* + t_* \in J$, where t_* may possibly be identical with s_*. We see that $x + s_* + t_*$ is distinct from y since $||x - y||_1 > 4$. By (J-EXC) there exists an $(x + s_* + t_*, y)$-increment pair (s, t) such that $y - s - t \in J$. This shows the existence of $(s, t) \in P$ with $y - s - t \in J$. Then (6.13) is satisfied by the pair $(s, t) = (s_0, t_0)$ that minimizes $f_p(y, -s - t)$ over $(s, t) \in P$.

Claim 3: $(x, y') \in \mathcal{D}$ with $y' = y - s_0 - t_0$.

To show this, first note that s_* is an (x, y')-increment, and let t be an $(x + s_*, y')$-increment. We have $t \in I$, $(s_0, t) \in P$, and $(t_0, t) \in P$. Hence, by (6.13), we have

$$f_p(y, -s_0 - t_0) \leq f_p(y, -s_0 - t), \quad f_p(y, -s_0 - t_0) \leq f_p(y, -t_0 - t). \tag{6.14}$$

Suppose that $x + s_* + t \in J$ and $y' - s_* - t \in J$. From (6.8), (6.11), Lemma 6.2, (6.12) and (6.14) we obtain

$$\begin{aligned} & f(x, s_* + t) + f(y', -s_* - t) \\ &= f_p(x, s_* + t) + f_p(y', -s_* - t) \\ &= f_p(y', -s_* - t) \\ &= f_p(y - s_0 - t_0 - s_* - t) - f_p(y - s_0 - t_0) \\ &\geq \min[f_p(y, -s_0 - t_0) + f_p(y, -s_* - t), \\ &\qquad f_p(y, -s_0 - t) + f_p(y, -s_* - t_0), \end{aligned}$$

$$
\begin{aligned}
& f_p(y, -t_0 - t) + f_p(y, -s_* - s_0)] \\
& \quad -f_p(y, -s_0 - t_0) \\
& > \min[f_p(y, -s_0 - t_0), f_p(y, -s_0 - t), f_p(y, -t_0 - t)] \\
& \quad -f_p(y, -s_0 - t_0) \\
& = 0.
\end{aligned}
$$

This shows $(x, y') \in \mathcal{D}$.

Finally, since $||x - y'||_1 = ||x - y||_1 - 2$, Claim 3 contradicts our choice of $(x, y) \in \mathcal{D}$. Therefore we conclude $\mathcal{D} = \emptyset$, completing the proof of Theorem 2.3.

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