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Dual Subdivision

a new class of subdivision schemes using projective duality

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Abstract

This paper proposes a new class of subdivision schemes. Previous subdivision processes are described by the movement and generation of vertices, and the faces are specified indirectly as polygons defined by those vertices. In the proposed scheme, on the other hand, the subdivision process is described by the generation of faces, and the vertices are specified indirectly as the intersections of these faces. In this sense, this paper gives a framework for a wide class of new subdivision methods. Moreover, the new subdivision is a dual framework of an ordinary subdivision based on the principle of duality in projective geometry. So, the new subdivision scheme inherits various properties of the ordinary subdivision schemes.

In this paper, we define the dual subdivision and derive its basic properties. Next, we derive properties on surfaces in general based on the duality independently of the subdivision. In particular, we derive the duality between the star-shapedness of a surface and the boundedness of its dual. Then, we define an "inflection plane" in order to represent surfaces which have inflection points in the dual space.

Moreover, we derive two main theorems. The first theorem gives a sufficient condition and a necessary condition for the dual subdivision surfaces to be C^k-continuous. Second, using an idea called the "subdivision kernel", we derive a sufficient condition for the dual subdivision surfaces to be bounded.

1 Introduction

Subdivision [11, 13, 3, 20, 18] is a well-known method for geometric design and for computer graphics, because the subdivision makes smooth surfaces with arbitrary topology. A subdivision scheme is defined by subdivision matrices and a rule of change of connectivity. So, many researchers study the condition of continuity of subdivision surfaces depending on subdivision matrices [20, 22, 16, 15, 1, 7, 21, 17, 4]. Moreover, multiresolution analysis [14, 5, 19, 10, 9, 8] derived by subdivision theory is extremely useful on mesh editing.

Subdivisions on rectangular or triangular meshes were studied extensively. For example, the most popular subdivisions are the Catmull-Clark subdivision and the Loop subdivision [13]. These subdivisions are designed for irregular rectangular or triangular meshes. Most subdivision methods are for triangular or rectangular meshes; there are only a few methods for the other type of meshes. Some subdivisions on hexangular meshes were developed [2, 6]. However, faces generated by these subdivisions are not "flat".

In this paper, we derive a new subdivision scheme. This is the dual framework of an ordinary subdivision based on the principle of duality in projective geometry. The proposed dual subdivision can generate meshes, composed of non-triangular "flat" faces.

In an ordinary subdivision, we compute new positions of the vertices using old positions of vertices and matrices called subdivision matrices. On the other hand, in the dual subdivision, we compute new equations of faces using old equations of faces and subdivision matrices. Moreover, a mesh which has the subdivision connectivity is the dual of a mesh which has the dual subdivision connectivity. In short, a mesh made by the dual subdivision is the dual mesh of a mesh made by ordinary subdivision. In this sense, the dual subdivision method can be made by an ordinary subdivision method. Conversely, an ordinary subdivision method can be made by the dual subdivision method.

The dual subdivision is a wide class of new subdivisions. The dual subdivision generates faces by subdivision matrices. The subdivision matrices are shared between an ordinary subdivision and the dual subdivision. So, the dual subdivision has properties similar to the ordinary subdivision. In this paper, we explain such properties.

Levin and Wartenbarg [12] already proposed some dual schemes. However, their schemes are convexity-preserving interpolations. Dual subdivision can naturally represent surfaces with arbitrary topology. We achieved that using "Inflection plane".

2 Polar Transformation

In this section, we explain a well-known duality, called polar transformation.

A hyperplane in a d dimensional space is represented as $a_1x_1 + a_2x_2 + \cdots + a_dx_d + a_{d+1} = 0$. A quadratic surface in the d dimensional space is represented as $\mathbf{x}A\mathbf{x}^\top = 0$, where A is a $(d+1) \times (d+1)$ symmetric matrix, $\mathbf{x} = (x_1, x_2, \cdots, x_d, 1)$ is a $d+1$ dimensional vector.

We associate a point $p = (p_1, p_2, \cdots, p_d)$ with a hyperplane $D(p) :\ \mathbf{x}A(p, 1)^\top = 0$, where $(p, 1) = (p_1, p_2, \cdots, p_d, 1)$. Conversely, we associate a hyperplane $h :\ \mathbf{x}A(p, 1)^\top = 0$ with a point $D(h) = p$. Here, $D(D(p)) = p,\ D(D(h)) = h$. Therefore, transformation D is a dual transformation based on the quadratic surface $\mathbf{x}A\mathbf{x}^\top = 0$, called polar transformation. The point p and the hyperplane $D(p)$ (the point $D(h)$ and the hyperplane h) are a pole and a polar surface of $\mathbf{x}A\mathbf{x}^\top = 0$. Usually, we use a parabola or a sphere for the quadratic surface $\mathbf{x}A\mathbf{x}^\top = 0$.

In this paper, we use a unit sphere for the quadratic surface. Then, we have following properties:

- When a point p is on a hyperplane h, and only then, a point $D(h)$ is on a hyperplane $D(p)$.
- When a point p exists in upper (lower) half-space partitioned by a hyperplane h, the point $D(h)$ exists in upper (lower) half-space partitioned by the hyperplane $D(p)$. Here, a lower half-space means the half-space which has the origin and the upper half-space means the other.

Here, the polar transformation using a unit sphere $(a, b, c) \leftrightarrow ax+by+cz-1 = 0$ is a projective duality $(p_x, p_y, p_z, p_w) \leftrightarrow p_x x + p_y y + p_z z - p_w w = 0$. The projective duality satisfies above first property.

3 Ordinary Subdivision

In this section, we review ordinary subdivisions in general.

3.1 Subdivision Matrix

A subdivision scheme is defined by subdivision matrices and a rule of connectivity change. The subdivision scheme, when it is applied to 2-manifold irregular meshes, generates smooth surfaces at the limit. Fig. 1 is an example of the Loop subdivision. In this figure, (a) is an original mesh; subdividing (a), we get (b); subdividing (b) once more, we get (c); subdividing infinite times, we get the smooth surface (d). We call (d) the subdivision surface. Here, a face is divided into four new faces. This is a change of connectivity. In this paper, the change of connectivity is fixed to this type, but other types of connectivity change can be argued similarly.

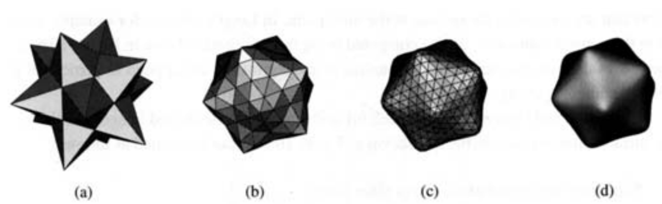

Figure 1: Loop subdivision [19].

Next, let us consider how to change the positions of the old vertices, and how to decide the positions of the new vertices. They are specified by matrices called "subdivision matrices". The subdivision matrices are defined at vertices and they depend on degree k of the vertex (the degree is the number of edges connected to the vertex). For example, Fig. 2 denotes a vertex v_0^j which has five edges. Let $v_1^j, v_2^j, \cdots, v_5^j$ be the vertices at the other terminal of the five edges. Then, subdivision matrix S_5 is defined as follows:

$$\begin{pmatrix} v_0^{j+1} \\ v_1^{j+1} \\ \vdots \\ v_5^{j+1} \end{pmatrix} = S_5^j \begin{pmatrix} v_0^{j} \\ v_1^{j} \\ \vdots \\ v_5^{j} \end{pmatrix}.$$

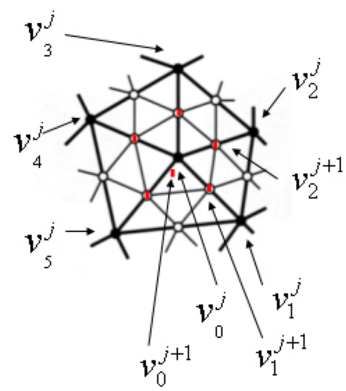

Figure 2: subdivision matrix.

Here, the subdivision matrix S_5^j is a square matrix. j means j-th step of the subdivision. Here, neighbor vertices of a vertex v are called vertices on the 1-disc of v. The subdivision matrix is generally defined not only on vertices in the 1-disc, but also on other vertices around, $\cdots$. Here, we argue only subdivision matrices that depend on vertices in the 1-disc. However, we can argue other subdivision matrices, similarly. In this paper, we assume that the subdivision matrix is independent on j. A subdivision scheme of this type is called "stationary".

In this way, subdivision matrix is written for a vertex. However, since a newly generated vertex is computed by two subdivision matrices at the ends of the edge, the two subdivision matrices must generate the same location of the vertex. So, the subdivision matrices have this kind of restriction.

For example, subdivision matrices S_k $(k \geq 3)$ for the Loop subdivision are

$$S_k = \begin{pmatrix} 1-k\beta & \beta & \beta & \beta & \beta & \beta & \cdots & \beta \\ \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & \cdots & 0 & \frac{1}{8} \\ \frac{3}{8} & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 & \cdots & 0 \\ \frac{3}{8} & 0 & \frac{1}{8} & \frac{3}{8} & \frac{1}{8} & 0 & \cdots & 0 \\ \vdots & & & & \ddots & & & \\ \frac{3}{8} & \frac{1}{8} & 0 & 0 & \cdots & 0 & \frac{1}{8} & \frac{3}{8} \end{pmatrix},$$

where k is the degree of the associated vertex, and

$$\beta = \frac{1}{k}\left(\frac{5}{8} - \left(\frac{3}{8} + \frac{1}{4}\cos\left(\frac{2\pi}{k}\right)\right)^2\right).$$

The degree k of a vertex is at least two. A vertex whose degree is two is a boundary vertex. The degree of a vertex of 2-manifold meshes is at least three. In this paper, we do not argue boundaries of meshes. So, we assume that the degree is at least three.

As seen above, a stationary subdivision scheme is defined by subdivision matrices S_k $(k \geq 3)$. Then, by the theorem 2.1 in [1], the limit surface of subdivision $f : |K| \to \mathbf{R}^3$ is the following parametric surface:

$$f[p](y) = \sum_i v_i \phi_i(y),$$
$$v_i \in \mathbf{R}^3,\ \phi_i(y) \in \mathbf{R},\ y \in |K|, p = (v_0, v_1, \cdots),$$

where K is a complex, $|K|$ is a topological space, that is, the mesh, i is an index of a vertex, v_i is the position of the i-th vertex, $p = (v_0, v_1, \cdots)$, $\phi_i(y)$ is the weight function with the i-th vertex. Moreover, the weight function $\phi_i(y)$ is dependent only on the subdivision matrices. If the sum of each row of the subdivision matrix is 1, vertices at each stage of subdivision is affine combinations of original vertices. Therefore,

$$\forall y \in |K|,\ \sum_i \phi_i(y) = 1.$$

So, weight functions make affine combinations, too. If the combination is not affine, it is not invariant under the translation of the coordinates systems, and hence we usually consider only affine combinations. Therefore, in what follows we assume that the sum of element in each row of the subdivision matrix is equal to 1.

Here, we denote $\phi(y) = (\phi_0(y), \phi_1(y), \cdots)$. Then, $\phi(y)$ decides a set of representable surfaces. Then, the set is spanned by $\phi(y)$. So, we call the weight functions basis functions. The limit surface of the subdivision is a point in such a functional space.

3.2 Nested Functional Spaces

In this subsection, we explain nested functional spaces. This is an essence of the subdivision. The limit surface of subdivision is

$$f[p](y) = (p, \phi(y)),$$

where p is a column vector whose elements are v_i, $\phi(y)$ is a row vector whose elements are $\phi_i(y)$. If the subdivision surface converges,

$$\begin{aligned} f[p](y) &= (p, \phi(y/2)) \\ &= (Sp, \phi(y)) \\ &= (p, S^\top \phi(y)). \end{aligned}$$

So,

$$\phi(y/2) = S^\top \phi(y),$$

where S is a global subdivision matrix. A global subdivision matrix is made by the local subdivision matrices S_k $(k \geq 3)$. Here, we denote a space spanned by the basis functions $\phi^j(y)$ at the j-th step of subdivision as V^j. Then, this equation means

$$V^0 \subset V^1 \subset V^2 \subset \cdots .$$

This is a family of nested functional spaces. All sets of basis functions of subdivision have this structure.

Fig. 3 denotes a filter bank of a mesh. Using the nested functional spaces, we can apply wavelet transformations on meshes. We call this multiresolution analysis.

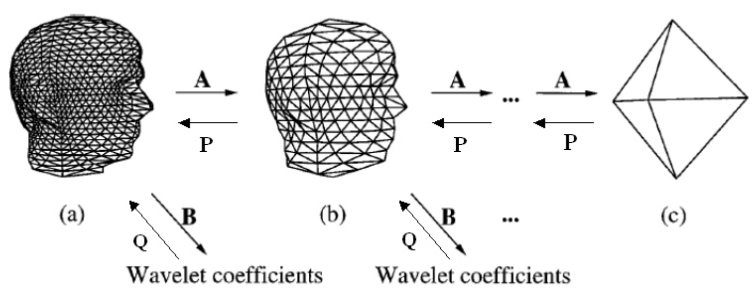

Figure 3: a filter bank of a mesh [19]

Multiresolution analysis greatly contributes to the mesh editing. So, many researchers study MRA and its applications.

4 Dual Subdivision

Here, we propose a dual subdivision method.

4.1 Definition of Dual Subdivision

The ordinary subdivision is specified by how the vertices are generated and located. On the other hand, the dual subdivision, which we will define here, is specified by how the faces are generated and located.

We assumed that the sum of each row of subdivision matrix is 1.

Here, p^j is a column vector of vertices at the j-th subdivision step. Using a subdivision matrix S, p^{j+1} is written as:

$$p^{j+1} = Sp^j,$$

where

$$p^j = \begin{pmatrix} p^j_{0x} & p^j_{0y} & p^j_{0z} \\ p^j_{1x} & p^j_{1y} & p^j_{1z} \\ & \vdots & \end{pmatrix}.$$

Therefore, if we denote

$$f^j = \begin{pmatrix} p^j_{0x} & p^j_{0y} & p^j_{0z} & -1 \\ p^j_{1x} & p^j_{1y} & p^j_{1z} & -1 \\ & \vdots & & \end{pmatrix},$$

we get

$$f^{j+1} = Sf^j,$$

where the elements of each row of f^j are coefficients of the equation $p^j_{ix}x + p^j_{iy}y + p^j_{iz}z - 1 = 0$. Therefore, the equations of planes are subdivided. These equations are the dual of vertices $(p^j_{ix}, p^j_{iy}, p^j_{iz})$. So, this subdivision is a dual framework of ordinary subdivision. Moreover, dual subdivision can be defined by a projective duality $(p_x, p_y, p_z, p_w) \leftrightarrow p_x x + p_y y + p_z z - p_w w = 0$.

Now, for any triangular mesh M, using the duality, we get a dual mesh $D(M)$. Here, if the degree of a vertex v of M is k, then v is the intersection of faces $f_i,\ i = 1, 2, \cdots, k$, so these vertices $D(f_i),\ i = 1, 2, \cdots, k$ is on the face $D(v)$. So, we get following proposition:

Proposition 4.1 (Dual mesh)

If the degree of the vertex v of M is k, the face $D(v)$ of $D(M)$ is a k-gon.

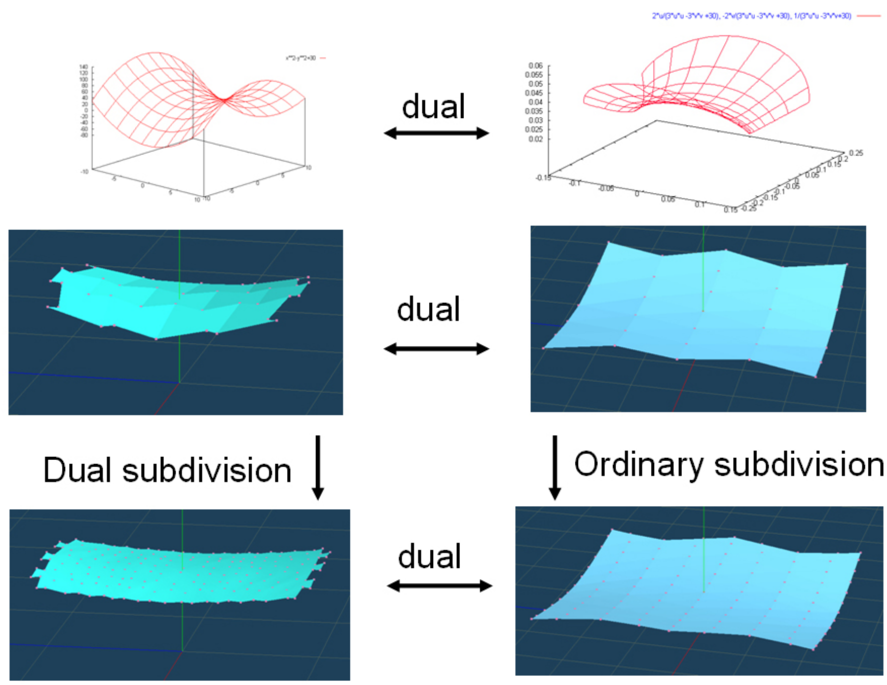

Figure 4: The duality between ordinary subdivision and dual subdivision. Upper left figure is a surface which has saddle points. Upper right figure is the dual surface. Middle right figure is a triangular mesh made by plotting points on upper right surface. Middle left mesh is the dual mesh of the middle right mesh. we get lower right mesh made by ordinary subdivision for the middle right mesh. On the other hand, we get lower left mesh made by dual subdivision for the middle left mesh. Then, the lower left mesh is the dual mesh of the lower right mesh. Dual subdivision is defined as such. Like this, dual subdivision can represent surfaces which have saddle points.

Here, we define the rule of connectivity change of dual subdivision. The connectivity change of dual subdivision is defined as dual of the connectivity change of ordinary subdivision. Fig. 4 denotes the definition of dual subdivision. In short, the mesh made by a dual subdivision in dual space is the dual mesh of a mesh made by a corresponding ordinary subdivision in primal space.

So, we get following proposition.

Proposition 4.2 (Dual subdivision)

Applying ordinary subdivision to meshes in the primal space is equivalent to applying dual subdivision to dual meshes in the dual space.

Now, if an ordinary subdivision makes "flat" faces, we get a dual subdivision based on the projective duality. Then, vertices in dual space is decided by the intersection of faces. So, we get a dual subdivision. Conversely, if we make a dual subdivision whose newly generated faces intersect a vertex, we get an ordinary subdivision which makes "flat" faces. However, an ordinary subdivision does not make "flat" faces, we do not get a dual subdivision. Because, the intersection of faces is not a vertex, in short, the faces do not have a concurrent. If un-flat faces divides into "flat" faces by adding edges, we get a dual subdivision based on this change of connectivity.

Here, it shows that if meshes made by ordinary subdivision approximate surfaces very well, dual meshes made by dual subdivision approximate dual surfaces very well, too. This shows that dual subdivision has a useful property that it can represent surfaces by "flat" polygons on non-triangular meshes, for example, hexagonal meshes.

4.2 Star-Shape and Boundedness

In this section, we explain important properties. These properties are based on star-shapedness and dual transformation $(a, b, c) \leftrightarrow ax + by + cz - 1 = 0$ and 2-manifold. So, these properties hold independently of the dual subdivision.

Here, we denote a 2-manifold in the primal space as S, the dual shape of S as $D(S)$. Points of $D(S)$ are the dual of tangent planes of S. Tangent planes of $D(S)$ is the dual of points of S. We can see $D(S)$ as an envelope surface defined by the dual of points of S.

First, we get following theorems.

Theorem 4.1 (Duality of 2-manifold)

If S is a bounded 2-manifold and if any subset of S is not flat, $D(S)$ is a 2-manifold.

Proof *First, we assume that S is smooth (C^1). Then any bounded 2-manifold S in primal space has an open covering which is composed of finite number of open sets whose topology is equal to that of a disc. Moreover, this dual transform is a continuous mapping. So, the open sets are mapped to open sets of tangent planes in dual space. Since any subset of S is not flat, an envelope surface made by an open set of tangent planes is an open set of points in dual space. Therefore, the dual surface of the open covering is represent by union of open sets. Since the open covering is composed of finite number of open sets, the dual surface of the open covering is an open set. Moreover, there exists an open covering whose union is equal to S. So, the dual surface of this open covering is equal to $D(S)$. Therefore, there exists open sets, whose topology is equal to that of a disc, at any point of $D(S)$. So, $D(S)$ is a 2-manifold.*

Second, we assume that S is not smooth (C^1). Here, we define tangent plane at point where tangent planes is discontinuous as convex combinations of tangent planes at neighborhood of the point. Then, since any subset of S is not flat, tangent planes on any open set of S is an open set. So, similarly, $D(S)$ is a 2-manifold. □

$D(S)$ is generally not a 2-manifold if S is a 2-manifold. For example, S is a plane, then $D(S)$ is a point.

Theorem 4.2 (Duality of smoothness)

We assume that S is a bounded 2-manifold and any subset of S and $D(S)$ is not flat. Then $D(S)$ is tangent plane continuous if and only if S is tangent plane continuous. Moreover, if S and $D(S)$ have no inflection points, then $D(S)$ being C^1-continuous is equivalent to S being C^1-continuous.

Proof *Since S is tangent plane continuous, points of $D(S)$ are continuous. Moreover, any subset of S is not flat. So, the dual of tangent planes of S is non-degenerate. Here, points of S are continuous. So, tangent planes of $D(S)$ are continuous. Therefore, $D(S)$ is tangent plane continuous. Similarly, only if part is clear.*

If S and $D(S)$ have inflection points, normals of S and $D(S)$ reverse at the dual inflection points. See the chapter "Inflection plane". □

If a subset of S is a plane, the dual of tangent planes of the subset of S is a point of $D(S)$. Since tangent plane of the point of $D(S)$ is not unique, $D(S)$ is not C^1-continuous.

Next we explain duality between star-shape and boundedness. Star-shape is a class of shapes (Fig. 5 shows examples of two dimensional star-shapes. However, in this paper, we consider three dimensional star-shape which is a 2-manifold in $\mathbf{R}^3$). Star-shape S has a point from which one can see all the surfaces of S.

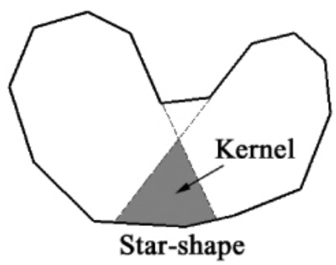

Figure 5: Examples of two dimensional star-shapes.

So, we define as follows.

Definition 4.1 (Kernel)

$K(S)$ is the intersection of all lower half-spaces which are divided by tangent plane of S.

Here, a lower half-space means the half-space which has the origin and the upper half-space means the other.

If $K(S) \neq \phi$, S is called a star-shape and $K(S)$ is called the kernel of S. For example, if the boundary of $K(S)$ is equal to S, then S is convex. Star-shape is generally not convex.

Then we get following theorem.

Theorem 4.3 (Kernel and duality)

If a point k is in the interior of $K(S)$, the plane $D(k)$ has no intersection with the dual shape $D(S)$. So, all planes which have no intersection with the convex hull of $D(S)$ form the dual of $K(S)$.

Proof *This is obvious, because all tangent planes of S does not intersect the point k.* □

So, we get following corollary.

Corollary 4.1 (Star-shape and boundedness)

If S is a star-shape and if the origin is in the interior of $K(S)$, $D(S)$ is bounded.

Proof *Since the interior of $K(S)$ exists, we can get the origin to be in interior of $K(S)$. Then, we can get an open sphere, which is in interior of $K(S)$, whose center is the origin of primal space. Here, we denote the radius of the sphere $r\ (> 0)$. Then dual shape of the sphere is exterior of closed sphere whose radius is $1/r$ and center is the origin of dual space. By the second property of the duality, $D(S)$ is in the closed sphere. Therefore, $D(S)$ is bounded.* □

The converse of this corollary is generally false. However, we get the following theorem.

Theorem 4.4

Assume that S is a bounded and boundaryless closed 2-manifold, and S is tangent plane continuous. Then, if a bounded $D(S)$ exists, some position of the origin exists such that S is a star-shape.

Proof *Consider the contraposition of this theorem. If S is not a star-shape, since S is tangent plane continuous and S is a bounded and boundaryless closed 2-manifold, any point in primal space intersects at least one tangent plane of S. So, wherever the origin in primal space is, $D(S)$ is not bounded.* □

Moreover, we get following theorems.

Theorem 4.5 (Bounded and smooth star-shape)

Assume that $D(S)$ is boundaryless and closed and tangent plane continuous, and any subset of S is not flat. If S is a bounded star-shape and if the origin is in the interior of $K(S)$, then $D(S)$ is a bounded star-shape.

Proof **Proof of theorem 4.5**

$D(S)$ is a 2-manifold from the theorem 4.1. Moreover, since the origin is in interior of $K(S)$, $D(S)$ is bounded. Here, if $D(S)$ is not a star-shape, the origin in dual space is not in the kernel of $D(S)$. Then, since $D(S)$ is closed and tangent plane continuous, S is not bounded. □

So, we can see that if we want to consider S and $D(S)$ are bounded and tangent plane continuous and boundaryless closed 2-manifolds, S and $D(S)$ are star-shapes.

4.3 Properties of dual subdivision

Proposition 4.3 (Expansion of kernel)

If global subdivision matrix S is a nonnegative matrix, in ordinary space, the mesh made by ordinary subdivision is in the convex hull of the original mesh, and in dual space, the kernel of the mesh made by dual subdivision includes the kernel of the dual original mesh.

Proof *If global subdivision matrix S is a nonnegative matrix, in primal space the convex hull of the original mesh includes that of the subdivided mesh. Since the dual of the kernel is all planes which have no intersection with the convex hull, the kernel of the dual subdivided mesh includes that of the dual original mesh.* □

Next, we derive an important theorem for smoothness of limit surfaces.

Theorem 4.6 (Duality of smoothness)

Assume that there exists local parameterizations, which have Jacobi matrix of maximal rank 2 except at extraordinary points, on basis functions of ordinary stationary subdivision, and there exists unique tangent planes at extraordinary points, and any subset of the limit surface is not flat, and the limit surfaces of ordinary and dual subdivision have no inflection points. Then, the limit surfaces of the dual subdivision are C^1-continuous if and only if the limit surfaces of the ordinary subdivision are C^1-continuous:

$$C^1_{\mathbf{ordinary}} \Leftrightarrow C^1_{\mathbf{dual}}.$$

Proof *For any basis function generated by ordinary stationary subdivision, if Jacobi matrix is degenerate at a point on the basis function except the extraordinary point, then Jacobi matrix is degenerate at any point of the basis function. Then, Jacobi matrix is degenerate on the part, which corresponds to the basis function, of the subdivision surface except a finite number of extraordinary points. So, Jacobi matrix of maximal rank 2 except at extraordinary points means that Jacobi matrix is non-degenerate at any point of the surface generated by ordinary subdivision except extraordinary points. Therefore there exists unique tangent planes of the limit surface of ordinary subdivision except extraordinary points. Here, there exists unique tangent planes at extraordinary points. So, there exists unique tangent planes of the surface generated by ordinary subdivision. Thus, any subset of $D(S)$ is not flat. Moreover, any subset of the limit surface is not flat. Therefore, by theorem 4.2, we get this theorem.* □

As is the case with the theorem 4.2, generally if the limit surface of ordinary stationary subdivision which has unique tangent planes at any point is C^1-continuous, the limit surface of the dual subdivision is not C^1-continuous.

Following proposition is very important from a practical viewpoint.

Proposition 4.4 (MRA for dual subdivision)

Taking the dual meshes of meshes in primal spaces at each stage of multiresolution analysis, we get multiresolution analysis for dual subdivision.

Proof *The j-th dual mesh in dual space is dual of the j-th mesh in primal space. So, dual subdivision derives the dual multiresolution analysis, similarly. However, when we use the dual multiresolution analysis, we need the dual subdivision connectivity.* □

5 Inflection plane

We talked about the smoothness of the limit surfaces of dual subdivision. However, even if a dual surface in the dual space is smooth and if any subset of the surface is not flat, the associated surface in the primal space is not necessarily smooth. (However, the continuity of tangent planes is guaranteed. So, it is true that the surface is C^1-continuous.) For example, see Fig. 6.

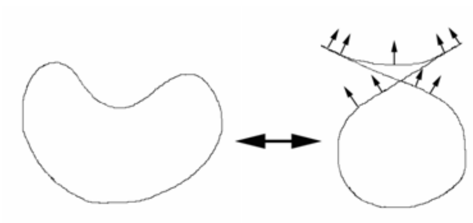

Figure 6: The left object in 2D has inflection points. The right object, which is the dual of the left object, has reversals of the normal at dual inflection points. In 3D, such thing happens, too.

Although the dual surface in the dual space is smooth, we see that the surface is not smooth in the primal space. This arises from reversals of the normal at dual inflection points. So, although the dual surface is C^1-continuous, the associated surface is not necessarily smooth.

Then we get a mesh in primal space as like Fig. 7. Applying smooth subdivision scheme to this mesh, the limit surface in primal space is smooth. So, the surface is not equal to the right shape in Fig. 6. Moreover, the surface is not a star-shape. Even if the original mesh in primal space approximate the right shape very well, similarly, the limit surface in primal space is not a star-shape. So, the dual limit surface in dual space is not bounded. This is a big issue.

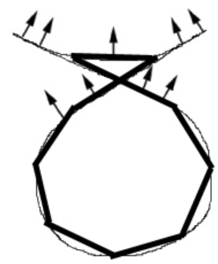

Figure 7: A dual mesh in primal space.

Here, there is a dual object which we want to represent in dual space. To represent that by dual subdivision, we represent the associated object in primal space by ordinary subdivision. So, we want to make such an object which has dual inflection points in primal space by ordinary subdivision.

Here, we define "inflection plane". We want to decide tangent planes at dual inflection points at limit. We call the envelope surface made by the tangent planes at dual inflection points "inflection surface". Now, we add two-ply faces as shown in Fig. 8. These added faces decide inflection surface at the limit of ordinary subdivision. So, we call the two-ply faces inflection planes.

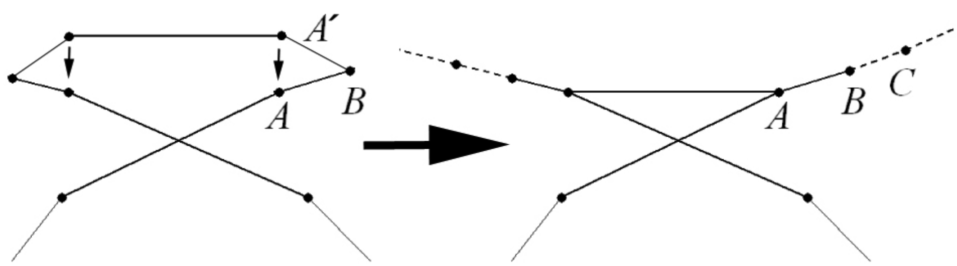

Figure 8: Adding an inflection plane AB. We conform the position of vertex A' to that of vertex A. So, both meshes have the same topology. We call the two-ply face AB "inflection plane".

Now, we define these.

Definition 5.1 (Inflection surface)

We denote the boundary of inflection plane D_{inf} *as* $D_{inf,b} \subset D_{inf} \subset |K|$. *We define the inflection surface as the envelope surface made by tangent planes at* $y \in D_{inf,b}$ *at limit.*

Note that the mesh added inflection planes may not be star-shape. So, we must consider the position of point B in Fig. 8.

Then, we subdivide the mesh to which the inflection planes are added. The mesh is C^0-continuous. So, we must check the continuity of tangent planes at the boundary of the inflection planes. If the tangent planes at the boundary are continuous, we can get the object in the dual space which has inflection points.

Now, if all supports of the basis functions are 1-disc or 2-disc, clearly, the tangent planes coincide at its boundary. If all supports of those are over 3-disc, we add points $C, D, \cdots$. Then, we get the continuity of tangent planes at the boundary. Here, we must consider the positions of points $C, D, \cdots$, similarly.

Indeed, the tangent planes coincide independent of the supports. However, we want to make reversals of normals. So, we must consider the supports.

Thus, using inflection planes, we can get smooth surfaces which have inflection points in dual space (see Fig. 9).

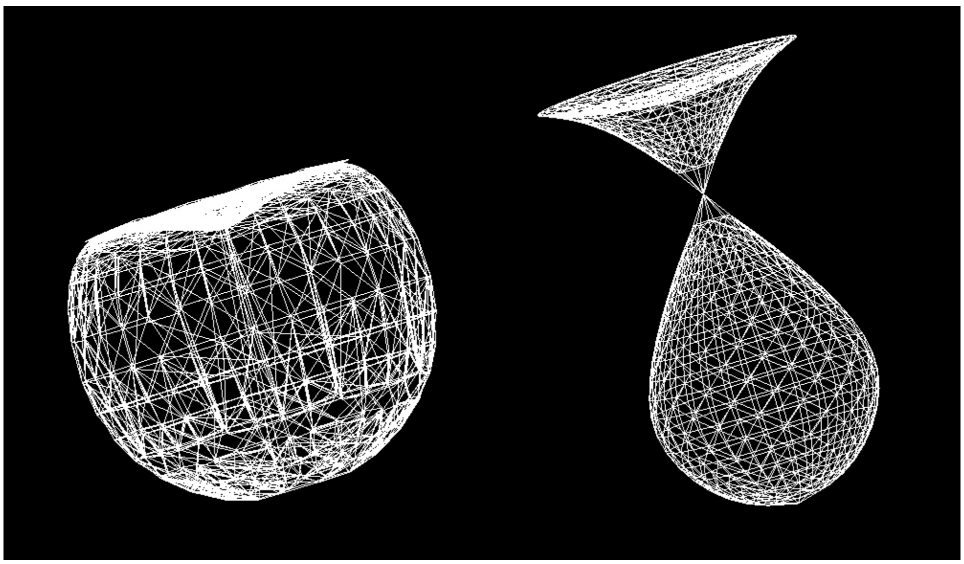

Figure 9: The left mesh which is made by dual subdivision is the dual of the right mesh and not convex and has inflection points. The right mesh which is made by ordinary subdivision using inflection plane has dual inflection points.

6 Analysis of ordinary subdivision

In this section, we analyze ordinary subdivisions.

Reif [16] already argued a sufficient condition of C^1-continuity of the subdivision surface on irregular meshes. Moreover, Prautzsch [15] derived some necessary conditions of that.

Zorin [22] derived a sufficient and necessary condition of C^k-continuity of the subdivision surface on irregular meshes with some assumptions. That is based on a fact that the subdivision surface is locally a projection of a higher-dimensional surface called “universal surface”.

Here, we explain that based on [22]. We assume triangular meshes in this chapter. However, we can argue following theory similarly on other meshes.

6.1 Universal surface

The limit surface $f : |K| \to \mathbf{R}^3$ of ordinary stationary subdivision is written by affine combination of vertices:

$$f[p](y) = \sum_i v_i \phi_i(y),$$

where $\phi_i(y),\ i = 0, 1, 2, \cdots$ are basis functions, K is a simplicial complex, $|K|$ is a topological space based on K, that is, a mesh. i is an index of a vertex, $v_i \in \mathbf{R}^3$ is a position of i-th vertex, $p = (v_0, v_1, v_2, \cdots)$. The subscript of $\phi_i(y)$ denotes that a basis function attaches to a vertex.

On smoothness of ordinary subdivision surface, we may only consider the neighborhoods of vertices of all connectivity. In short, locally any surface generated by a stationary subdivision scheme on an arbitrary simplicial complex can be thought of as a part of a subdivision surface defined on a k-regular complex with respect to which the scheme is invariant. k-regular complex is a simplicial complex which consists of a vertex whose degree is k and vertices whose degree is 6.

So, we define ψ as a row vector whose element is a basis function which affects the subdivision surface in neighborhood U_1 of the vertex whose degree is k. The neighborhood is an area which consists of triangles which have the vertex. We call the area a region of 1-disc. The region of 2-disc is an area which consists of triangles which have a vertex in 1-dusc. Like this, we can define a region of k-disc. Here, we assume that U_1 is the region of 1-disc. However, we can argue similarly on other neighborhood.

Here, p is a column vector whose element is a vertex corresponding to $\phi_i(y)$ in ψ. Then, $f[p](y) = (p, \psi(y)),\ y \in\ U_1$. We call $\psi : U_1 \to \mathbf{R}^\gamma$ “universal surface”. γ is the number of elements of ψ. For example, if all supports of basis functions are 1-disc, $\gamma = k + 1$. Here, we can see that any subdivision surface is locally a projection of the universal surface.

Here, we define a condition (Condition A in [22]). Condition A is that parameterization of the surface generated by subdivision on k-regular complex except the vertex whose degree is k is regular, that is C^1-continuous and with Jacobi matrix of maximal rank 2.

Then, following theorem (theorem 2.1 in [22]) is known.

Theorem 6.1

For a subdivision scheme satisfying Condition A to be tangent plane continuous on a k-regular complex, it is necessary and sufficient that the universal surface *be tangent plane continuous; for the subdivision scheme to be C^k-continuous, it is necessary and sufficient that the universal surface be C^k-continuous.*

In this theorem, we assume Condition A in [22]. Subdivision scheme without satisfying Condition A is very rare. Most practical subdivision schemes satisfy Condition A.

6.2 Normals of the subdivision surface

Here, we know a simple formula relating the Jacobian of a mapping $U_1 \to \mathbf{R}^2$ generated by subdivision to the wedge product $\partial_1\psi \wedge \partial_2\psi$. Then, the Jacobian of a mapping $f[p_x, p_y] = ((p_x, \psi), (p_y, \psi))$ is

$$J[f[p_x, p_y]] = (p_x \wedge p_y, \partial_1\psi \wedge \partial_2\psi).$$

So, we can write a normal at any point except the vertex whose degree is k for a limit surface $f[p_x, p_y, p_z] : U_1 \to \mathbf{R}^3$ as

$$\begin{aligned} & N(y) \\ = \ & [(p_y \wedge p_z, w(y)), (p_z \wedge p_x, w(y)), (p_x \wedge p_y, w(y))], \end{aligned}$$

where $w(y) = \partial_1\psi(y) \wedge \partial_2\psi(y)$.

7 Conditions of C^k-continuity for the limit surface of dual subdivision

Now, we derive a necessary condition of C^k-continuity for the limit surface of dual subdivision and a sufficient condition of that.

Now, we can write a tangent plane of the limit surface of ordinary subdivision on U_1:

$$N(y) \cdot (X - f(y)) = 0.$$

So,

$$\frac{N(y)}{N(y) \cdot f(y)} \cdot X - 1 = 0.$$

Therefore, we get the limit surface of dual subdivision as

$$\frac{N(y)}{N(y) \cdot f(y)}.$$

Then, we get following theorem.

Theorem 7.1

Assume that Condition A and any subset of surfaces generated by ordinary and dual subdivision is not flat and the limit surfaces of ordinary and dual subdivision have no inflection points. Then,

$$C^{k+1}_{\mathbf{ordinary}} \Rightarrow C^{k}_{\mathbf{dual}} \Rightarrow C^{k}_{\mathbf{ordinary}}.$$

Proof *If an ordinary subdivision scheme is C^{k+1}-continuous, by theorem 6.1, the universal surface ψ is C^{k+1}-continuous. So, $w(y)$ is C^k-continuous. Moreover, $N(y)$ is C^k-continuous, because $N(y)$ is a projection of $w(y)$. Moreover, any subset of the surface generated by ordinary subdivision is not flat. So, the parameterization of dual subdivision surface is non-degenerate. Therefore, we see that the limit surface $\frac{N(y)}{N(y)\cdot f(y)}$ of dual subdivision is C^k-continuous. Similarly, the right arrow is clear.* □

Note that, as is shown in theorem 4.6, we have $C^1_{\mathbf{ordinary}} \Leftrightarrow C^1_{\mathbf{dual}}$.

8 Boundedness of the limit surface of dual subdivision

In this section, we derive a sufficient condition of boundedness for the limit surface of dual subdivision using an idea called "subdivision kernel".

We want to make a monotone region in kernel of star-shape for ordinary subdivision. By corollary 4.1, the limit surface of dual subdivision is bounded if the limit surface of ordinary subdivision is a star-shape and if the origin is in interior of the kernel of the limit surface of ordinary subdivision.

So, we want to know a sufficient condition for the limit surface of ordinary subdivision to be star-shape. Moreover, we want to know the position of the origin, subject to the origin is in the kernel of the limit surface of ordinary subdivision.

Whether the limit surface of ordinary subdivision is star-shape can be checked using following equation:

$$N(y) \cdot (X - f(y)) = 0,$$

where $N(y)$ is the normal at $y \in |K|$, $f(y)$ is the limit surface of ordinary subdivision. Here, we define the direction of $N(y)$ such that the origin is in lower half-space. In short, we must check the existence of the point X subject to

$$\forall y \in |K|,\ N(y) \cdot (X - f(y)) < 0.$$

However, generally, it is not easy to write basis functions analytically. So, the above check is difficult. We can approximately check that using approximating basis functions by well-known functions, for example, Bezier surfaces or B-spline surfaces. However, using the approximation, it is difficult to get the intersection of infinite half-spaces (however, at specific surfaces, we may check only the intersection of finite half-spaces.).

So, we derive a sufficient condition for the mesh subdivided by ordinary subdivision to be a star-shape, where the initial mesh is a star-shape.

Our target is to construct a monotone region in the kernel of star-shape for ordinary subdivision. We call the region "subdivision kernel".

8.1 Normal subdivision matrix

We compute new vertices by subdivision matrix S_k:

$$\begin{pmatrix} v_0^{j+1} \\ v_1^{j+1} \\ \vdots \\ v_k^{j+1} \end{pmatrix} = S_k \begin{pmatrix} v_0^{j} \\ v_1^{j} \\ \vdots \\ v_k^{j} \end{pmatrix}.$$

Then, we define a matrix Δ as

$$\Delta = \begin{pmatrix} 1 & 0 & \cdots & & \\ -1 & 1 & 0 & \cdots & \\ -1 & 0 & 1 & 0 & \cdots \\ \vdots & & & \ddots & \end{pmatrix}$$

Using a matrix $D'_k = \Delta S_k \Delta^{-1}$, we get

$$\begin{pmatrix} v_0^{j+1} \\ v_1^{j+1} - v_0^{j+1} \\ \vdots \\ v_k^{j+1} - v_0^{j+1} \end{pmatrix} = D'_k \begin{pmatrix} v_0^{j} \\ v_1^{j} - v_0^{j} \\ \vdots \\ v_k^{j} - v_0^{j} \end{pmatrix}.$$

Here, the sum of each row of S_k is 1. So, the subdivision scheme is affine invariant. Therefore, the first element v_0^j does not affect elements $v_1^{j+1} - v_0^{j+1}, v_2^{j+1} - v_0^{j+1}, \cdots, v_k^{j+1} - v_0^{j+1}$. So, we denote elements $v_1^j - v_0^j, v_2^j - v_0^j, \cdots, v_k^j - v_0^j$ as d^j and the associated submatrix of D'_k as D_k. Then, $d^{j+1} = D_k d^j$. We call this different scheme.

Moreover, we denote the column vector in $\mathbf{R}^k$ which is a set of x elements of d^j as d_x^j. Similarly, we denote the column vectors corresponding to y elements, z elements as d_y^j, d_z^j. Here, using $u_1, u_2 \in \mathbf{R}^k$, we define a matrix ΛD_k as

$$\Lambda D_k(u_1 \wedge u_2) = D_k u_1 \wedge D_k u_2,$$

where $\wedge$ is wedge product. Then,

$$\Lambda D_k(d_y^j \wedge d_z^j) = D_k d_y^j \wedge D_k d_z^j.$$

Now, we define $N^j = (d_y^j \wedge d_z^j, d_z^j \wedge d_x^j, d_x^j \wedge d_y^j)$. Then,

$$N^{j+1} = \Lambda D_k \, N^j,$$

where the element of N^j is a cross product of between $v_i^j - v_0^j$ and $v_l^j - v_0^j$, that is, normal on the neighborhood of v_0^j. So, we see that the matrix ΛD_k subdivides normals of faces which connects a vertex whose degree is k. Note that N^j contains normals of unreal faces in the mesh.

Therefore, we call this matrix "normal subdivision matrix".

8.2 Subdivision kernel

Here, we consider $\frac{k(k-1)}{2}$ equations of planes:

$$N^j(X - v_0^j) = 0,$$

where $X = (x, y, z)^\top$, N^j is a $\frac{k(k-1)}{2} \times 3$ matrix, $v_0^j = (v_{0x}^j, v_{0y}^j, v_{0z}^j)^\top$. Now, we subdivide faces such as Fig. 2. Then, the equations of plane are

$$N^{j+1}(X - v_0^{j+1}) = 0.$$

So, we get

$$\Lambda D_k \ N^j(X - v_0^{j+1}) = 0.$$

Therefore, if

$$\begin{array}{rcl} N^j(X - v_0^j) & < & 0, \\ N^j(X - v_1^j) & < & 0, \\ & \vdots & \\ N^j(X - v_p^j) & < & 0 \end{array}$$

and matrices $S_k, \ \Lambda D_k$ are nonnegative matrix, we get

$$N^{j+1}(X - v_0^{j+1}) < 0.$$

Here, we can see that N^j is a cone with translation.

Next, we consider normals of faces which have a vertex which is generated at $j+1$st step. In Fig. 10, normals of faces which have the vertex v_1^j depend on vertices in 1-disc of vertices in $j-1$st mesh which decide the position of v_1^j.

Here, we denote normals which are cross products between $v_i^j - v_1^j$ and $v_l^j - v_1^j$ as N_m^j, where v_i^j, v_l^j are vertices in 2-disc of v_1^j. Then, using matrix ΛD_{m1}, normal N_{m1}^{j+1} which is made by vertices in 2-disc of $v_{1,m1}^{j+1}$ which is generated at $j+1$st is computed:

$$N_{m1}^{j+1} = \Lambda D_{m1} \ N_m^j.$$

Similarly, N_{mi}^{j+1} is computed using ΛD_{mi}:

$$N_{mi}^{j+1} = \Lambda D_{mi} \ N_m^j, 1 \le i \le 6.$$

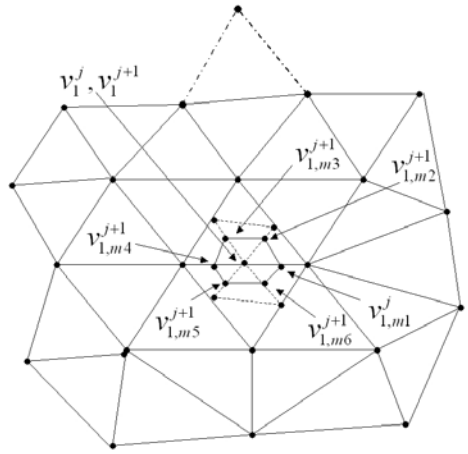

Figure 10: vertices which affect the positions of vertices $v_{1,m1}^{j+1}, \cdots, v_{1,m6}^{j+1}$. For example, the top vertex partitioned by chain lines does not affect the positions of vertices v_1^j and $v_1^{j+1}, v_{1,m1}^{j+1}, \cdots, v_{1,m6}^{j+1}$. Moreover, positions of vertices except the top vertex decide normals of faces which connect $v_{1,m1}^{j+1}, \cdots, v_{1,m6}^{j+1}$.

Here, $\Lambda D_{mi}, 1 \leq i \leq 6$ are essentially the same. On irregular mesh, $\Lambda D_{mi}, 1 \leq i \leq 6$ are different. When we consider all connectivities, it is sufficient that we consider only ΛD_{m1}. So, we denote this matrix as ΛD_m.

So, we can derive following lemma.

Lemma 8.1 (Monotone region SK^j)

At $j(\neq 0)$th mesh, on vertices which exist at $j-1$st mesh, a cone N^j made from 1-disc of a vertex attaches to vertices in 1-disc of the vertex. On the other hand, on vertices which do not exist at $j-1$st mesh, a cone N_m^j made from 2-disc of a vertex attaches to vertices in 2-disc of the vertex. If there exists a region SK^j which is the intersection of all the cones, then there exists a similar region SK^{j+1} at $j+1$st mesh ($SK^j \subseteq SK^{j+1}$). Here, matrices $S_k,\ \Lambda D_k,\ \Lambda D_m$ with all connectivities are nonnegative matrices.

Proof *Clearly, on vertices which exist at $j-1$st mesh, the intersection made by cones corresponding to N^{j+1} includes the intersection made by cones corresponding to N^j. On vertices which do not exist at $j-1$st mesh, the intersection made by cones corresponding to N^{j+1} includes the intersection made by cones corresponding to N_m^{j+1}. So, the cone at $j+1$st mesh, made by cones which are linked to vertices which exist at jth mesh, includes SK^j.*

Next, we must prove that on vertices which do not exist jth mesh the cone N_m^{j+1} includes SK^j. This is clear if matrices $S_k,\ \Lambda D_m$ are nonnegative matrices.

Therefore, all cones which compose SK^{j+1} include SK^j. So, $SK^j \subseteq SK^{j+1}$.
□

Here, we translate this lemma into following theorem.

Theorem 8.1 (Monotone region SK^j)
If the region SK^j $(j \neq 0)$ exists and if matrices $S_k,\ \Lambda D_k,\ \Lambda D_m$ with all connectivities are nonnegative matrices, then the region SK^{j+1} exists ($SK^j \subseteq SK^{j+1}$).

Moreover, we see following theorem.

Theorem 8.2 (Kernel and monotone region)
The kernel of j-th mesh includes the monotone region SK^j.

This is obvious. Of course, if the kernel does not exist, SK^j does not exist.

So, we call SK^j subdivision kernel of j-th mesh. Then, we easily get following theorem.

Theorem 8.3 (Subdivision kernel)
If subdivision kernel SK^j of $j(\neq 0)$th mesh exists and if matrices $S_k,\ \Lambda D_k,\ \Lambda D_m$ with all connectivities are nonnegative matrices, then the limit surface of ordinary subdivision has the subdivision kernel SK^∞ ($SK^j \subseteq SK^\infty$).

So, using this theorem and corollary 4.1 and theorem 8.2, we get following important corollary.

Corollary 8.1 (Boundedness of dual subdivision)
On the primal space, if subdivision kernel SK^j of $j(\neq 0)$th mesh exists and if matrices $S_k,\ \Lambda D_k,\ \Lambda D_m$ with all connectivities are nonnegative matrices and if the origin is in the interior of SK^j, then the limit surface of dual subdivision is bounded.

This is obvious.

We defined subdivision kernel $SK^j, j \neq 0$. However, whether the limit surface of ordinary subdivision is star-shape only depends on the initial mesh and the subdivision scheme. So, we can derive subdivision kernel SK^0. Here, we easily see that having SK^0 is a strongly condition than having SK^1. Moreover, we see that the effort of the check of having SK^0 is less than that of SK^1.

In this section, we assumed that subdivision matrices depends on vertices in 1-disc. However, on other subdivision matrices, we can define subdivision kernel similarly.

Here, we consider strength of this condition. The computation of the subdivision kernel requires $\mathrm{O}(n4^j \log(n4^j))$ time, where n is a number of faces of the original mesh. Here, we assume that the subdivision scheme is C^1-continuous. Then, at $j \to \infty$, the cone converges to half-space. The plane which divides this half-space is a tangent plane. So, we can see that, at $j \to \infty$, SK^∞ is equal to the kernel of the limit surface. Therefore, if the limit surface of ordinary subdivision is boundaryless and closed and tangent plane continuous, the sufficient condition converges to the necessary and sufficient condition.

However, at $j \to \infty$, the number $n4^j$ of faces of j-th mesh diverges. This means SK^∞ is the intersection of infinite number of half-space. To compute the subdivision kernel in finite time, we approximate the necessary and sufficient condition to the sufficient condition. So, the speed of convergence of the subdivision scheme determines the quality of the approximation. On most subdivision schemes, the speed depends on second eigenvalues of subdivision matrices.

9 Arbitrary topology

In previous section, we explain that the smooth limit surface of dual subdivision surface is bounded if and only if the smooth limit surface of ordinary subdivision is a star-shape. Similarly, the smooth limit surface of ordinary subdivision surface is bounded if and only if the smooth limit surface of dual subdivision is a star-shape.

Here, we consider ordinary subdivision. If ordinary subdivision surface is smooth, ordinary subdivision can represent only bounded surfaces with arbitrary topology. Therefore, we can see that if the dual subdivision surface is smooth, the dual subdivision can represent only star-shape.

Therefore, we must extend the dual subdivision scheme to represent smooth surfaces with arbitrary topology.

9.1 Projective duality

Now, we consider subdivision schemes on projective space. In the 3-dimensional real projective space P^3, points are represented by homogeneous coordinate vectors $p = (p_x, p_y, p_z, p_w)$. So, we subdivide the points: $p^{j+1} = Sp^j$, where

$$p^j = \begin{pmatrix} p^j_{0x} & p^j_{0y} & p^j_{0z} & p^j_{0w} \\ p^j_{1x} & p^j_{1y} & p^j_{1z} & p^j_{1w} \\ & \vdots & & \end{pmatrix}.$$

Therefore, we get subdivision schemes on P^3. Then, the subdivision scheme can represent smooth surfaces with arbitrary topology which are not necessarily bounded. For, example, if p^0_i is bounded, then $p^0_{iw} = 1$, if p^0_i is a point at infinity, then $p^0_{iw} = 0$. Then, if all supports of basis functions are 1-disc, we get a smooth subdivision surface which is not bounded. Here, using the projective duality $(p_x, p_y, p_z, p_w) \leftrightarrow p_x x + p_y y + p_z z - p_w w = 0$, we get dual subdivision schemes. Then, the subdivision schemes can represent smooth surfaces with arbitrary topology.

Now, we consider properties of the dual subdivision schemes using the projective duality. If p_w is in $\mathbf{R}_+ \cup \{0\}$, then the projective duality satisfies the properties of polar transformation. So, we get proposition 4.1, 4.2, similarly. Therefore, this dual subdivision can represent surfaces by non-triangular “flat” faces. Moreover, the subdivision scheme and the dual subdivision scheme on P^3 have the nested functional spaces, so we use multiresolution analysis. Moreover, on smoothness, we easily get theorem 4.6, 7.1, similarly.

10 Conclusion

In this paper, we proposed a new subdivision method. This is a dual framework of ordinary subdivision based on polar transformation which is a type of projective duality. Because of the duality, dual subdivision admits many useful properties that ordinary subdivision has.

First, we derived the duality of smoothness, nested functional spaces, and kernel expansion. The duality of smoothness enables us to represent smooth surface by "flat" non-trigonal polygons. It is useful for the mesh editing that the duality of nested functional spaces derives the multiresolution analysis of dual subdivision schemes.

Second, we derived important properties based on star-shape and polar transformation and 2-manifold. These properties hold independently of the dual subdivision.

Third, we defined inflection plane. Using this, we can represent smooth surfaces by dual subdivision even if they have inflection points.

Fourth, we derived conditions for the limit surface of dual subdivision surfaces to be C^k-continuous. Using "universal surface" [22], we derived relations of smoothness between ordinary subdivision and dual subdivision.

Fifth, we proposed an idea called "subdivision kernel". Subdivision kernel is an monotone open set for ordinary subdivision. Moreover, subdivision kernel is in the kernel of star-shape. Using subdivision kernel, we derived a sufficient condition for the limit surface of dual subdivision to be bounded.

Finally, we extended the dual subdivision to make smooth surfaces with arbitrary topology using projective duality. This dual subdivision scheme inherits similar properties. So, dual subdivision schemes on P^3 can represent smooth surfaces with arbitrary topology by non-triangular "flat" faces.

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