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Abstract: We give a sufficient condition for admissibility of generalized Bayes estimators of the location vector of spherically symmetric distribution under squared error loss. Compared to the known results for the multivariate normal case, our sufficient condition is very tight and is close to being a necessary condition. In particular we establish the admissibility of generalized Bayes estimators with respect to the harmonic prior and priors with slightly heavier tail than the harmonic prior. We use the theory of regularly varying functions to construct a sequence of smooth proper priors approaching an improper prior fast enough for establishing the admissibility. We also discuss conditions of minimaxity of the generalized Bayes estimator with respect to the harmonic prior.

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1. Introduction

We consider estimation of the p-dimensional location parameter of a spherically symmetric distribution. Specifically, let $X = (X_1, \dots, X_p)'$ have a density function $f(\|x - \theta\|)$ and consider estimation of θ with a general quadratic loss function $L(\theta, \delta) = (\delta - \theta)' Q(\delta - \theta) = \|\delta - \theta\|_Q^2$ for a positive definite matrix Q. The usual minimax estimator X, which is generalized Bayes, is inadmissible for $p \geq 3$ as shown in Stein (1956) in the normal case, and in Brown (1966) under more general situation, respectively. In the decision-theoretic point of view, we are interested in proposing admissible estimators dominating X, that is, minimax admissible estimators. Note that the dominance over X means minimaxity in our setting because X is minimax with a constant risk. Note also that our results hold for the location vector of elliptically-contoured distributions, because we are considering a general quadratic loss function with arbitrary positive definite Q.

In the normal case, there already exists a broad class of admissible minimax estimators. Baranchik (1970) gave a sufficient condition for minimaxity of a shrinkage estimator of the form

$$\delta_\phi(X) = (1 - \phi(\|X\|)\|X\|^{-2})X. \tag{1.1}$$

Strawderman (1971) found a subclass of proper Bayes estimators of the form (1.1) satisfying the sufficient conditions for minimaxity. Brown (1971) gave a very powerful sufficient condition for admissibility of generalized Bayes estimators. Using Brown's (1971) condition, Berger (1976), Fourdrinier et al. (1998) and Maruyama (1998, 2004) enlarged a class of admissible minimax estimators which are of the Strawderman type and generalized Bayes. For a subclass of scale mixtures of multivariate normal distributions which includes multivariate-t distribution, some proper Bayes minimax estimators were proposed by Maruyama (2003) by using Strawderman's (1971) techniques.

However, for general spherically symmetric distributions, no minimax admissible estimators of the location vectors have been derived, although for the minimaxity, various Baranchik-type sufficient conditions of the estimator (1.1) were given by Berger (1975), Brandwein and Strawderman (1978, 1991) and Bock (1985). The main reason is the lack of a standard class of generalized or proper Bayes estimators of the form (1.1) like the Strawderman type in the normal case, which allows an easy check of the minimaxity condition. Furthermore no sufficient condition for admissibility of generalized Bayes estimators has been derived. In this paper, we will provide satisfactory solutions to these problems.

In Section 2, we give preliminary results including the properties of regularly varying functions and asymptotic behaviors of expected values when $\|\theta\|$ is sufficiently large. The former is useful for constructing a very convenient sequence of proper densities $g(\theta)h_i^2(\theta)$, $i = 1, 2, \ldots$, approaching an improper density $g(\theta)$, which is required in applying the method of Blyth (1951).

In Section 3, we will present a powerful sufficient condition for admissibility of generalized Bayes estimator and in particular show that the generalized Bayes estimators with respect to the harmonic prior $g(\theta) = \|\theta\|^{2-p}$ and with respect to a prior with a slightly heavier tail

$$g(\theta) = \|\theta\|^{2-p} \log(\|\theta\| + c), \quad c > 1, \tag{1.2}$$

are admissible under mild regularity conditions on f.

In Section 4, we show that the generalized Bayes estimator with respect to the harmonic prior is written as

$$\left(1 - \frac{\int_0^1 t^{p-1} F(t\|X\|)dt}{\int_0^1 t^{p-3} F(t\|X\|)dt}\right) X, \tag{1.3}$$

where $F(u) = \int_u^\infty sf(s)ds$. This form is simple enough to check various sufficient conditions for minimaxity and we demonstrate that (1.3) is minimax for some f. We believe that (1.3) is minimax for a broad subclass of spherically symmetric distributions. Notice that the generalized Bayes estimators with respect to priors except $\|\theta\|^{2-p}$ do not have such simple forms as far as we know.

Our proof of admissibility is mainly based on the techniques of Brown and Hwang (1982). Brown and Hwang (1982) considered the problem of estimating the natural mean vector of an exponential family under a quadratic loss function. Note that the intersection of their setting and our setting is the multivariate normal case. Their sufficient condition for admissibility in the normal case does not however permit $g(\theta)$ to diverge to infinity around the origin like $\|\theta\|^{2-p}$, while it permits $g(\theta) \leq \|\theta\|^a$ with $a \leq 2-p$ for sufficiently large $\|\theta\|$. Prior to Brown and Hwang (1982), Brown (1971) considered the estimation in the multivariate normal case and gave a powerful sufficient condition for minimaxity which are satisfied by the harmonic prior and (1.2), but his proof was based on many advanced mathematics. Our mathematical tool is much more familiar to the readers. Brown (1971) also gave sufficient condition for inadmissibility. By using it we see that the generalized Bayes estimator with respect to $\|\theta\|^{2-p}\log^2(\|\theta\|+2)$ is inadmissible. Hence our sufficient condition for admissibility should be very tight and close to being a necessary condition.

Brown (1979) considered a more general problem than ours: estimation of θ for a general density $p(x-\theta)$ and a general loss function $W(\delta-\theta)$. He conjectured that the prior $g(\theta) \sim \|\theta\|^a$ with $a \leq 2-p$ for sufficiently large $\|\theta\|$ leads to admissibility, regardless of the density p and the loss W. Hence our results support Brown (1979)'s conjecture for the case of elliptically-contoured family and a general quadratic loss function.

Finally we notice that the most important key for our proof for admissibility is the construction of a very convenient sequence $h_i(\theta)$ for approximating $g(\theta)$ by $g(\theta)h_i(\theta)^2$. Brown and Hwang (1982) used

$$h_i(\theta) = \begin{cases} 1 & \|\theta\| \leq 1 \\ 1 - \log\|\theta\|/\log i & 1 \leq \|\theta\| \leq i \\ 0 & \|\theta\| > i. \end{cases}$$

This $h_i(\theta)$ is not differentiable at $\|\theta\| = 1$ and truncated at $\|\theta\| = i$, which makes handling and extension difficult for our purposes. Our $h_i(\theta)$ given in Section 2 is smoother and not truncated. We believe that our $h_i(\theta)$ is very useful for showing admissibility of generalized Bayes estimators in various problems.

2. Preliminaries

In this section we prepare a sequence of proper densities using the theory of regularly varying functions and give some results on asymptotic behaviors of expected values when the location parameter diverges to infinity. For the theory of regularly varying and slowly varying functions the readers are referred to Geluk and de Haan (1987) and Bingham et al. (1987).

2.1. Regularly varying functions

A Lebesgue measurable function $f : R^+ \to R$ which is eventually positive is called regularly varying if for some $\alpha \in R$

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^\alpha, \quad \forall t > 0. \tag{2.1}$$

We sometimes use the notation $f \in \mathrm{RV}_\alpha$. The number α in the above is called the index of regular variation. A function satisfying (2.1) with $\alpha = 0$ is called slowly varying.

Let $\beta : R^+ \to R^+$ be positive, twice continuously differentiable, monotone decreasing, integrable (i.e. $\int_0^\infty \beta(r)dr < \infty$) and regular varying with index -1. A typical $\beta(\eta)$ is

$$\frac{1}{\eta + c}\frac{1}{\{\mathrm{Log}_n(\eta + c)\}^2}\prod_{i=0}^{n-1}\frac{1}{\mathrm{Log}_i(\eta + c)}, \tag{2.2}$$

where n is a positive integer, $\mathrm{Log}_0(\eta + c) \equiv 1$,

$$\mathrm{Log}_i(\eta + c) = \underbrace{\log\log\cdots\log}_{i}(\eta + c), \quad i \geq 1,$$

and c is chosen such that $\mathrm{Log}_n(c) > 0$. Note that for (2.2)

$$\int_\eta^\infty \beta(r)dr = \frac{1}{\mathrm{Log}_n(\eta + c)}.$$

The following results for $\beta(\eta)$ satisfying the above assumptions are known from the theory of regularly varying functions.

Lemma 2.1. *1.* $\int_\eta^\infty \beta(r)dr \in \mathrm{RV}_0$ *and* $\beta'(r) \in \mathrm{RV}_{-2}$.
2. $\lim_{\eta\to\infty} \eta\beta(\eta)/\int_\eta^\infty \beta(r)dr = 0$, $\lim_{\eta\to\infty} \eta\beta'(\eta)/\beta(\eta) = -1$, *and* $\lim_{\eta\to\infty} \eta\beta''(\eta)/\beta'(\eta) = -2$.

We now define functions $H_i(\eta)$, $i = 1, 2, \ldots$, based on $\beta(r)$ by

$$H_i(\eta) = \frac{\int_\eta^\infty e^{(\eta - r)/i}\beta(r)dr}{\int_\eta^\infty \beta(r)dr}. \tag{2.3}$$

These functions are very useful for constructing a sequence of proper prior densities approaching the target improper density in the next section. The properties of H_i are given in the following theorem.

Theorem 2.1. *1.* $0 \leq H_1(\eta) \leq H_2(\eta) \leq \cdots \leq 1$. *For any fixed* η, $\lim_{i\to\infty} H_i(\eta) = 1$.
2. For any fixed i, $\lim_{\eta\to\infty} \int_\eta^\infty \beta(r)dr\beta(\eta)^{-1}H_i(\eta) = i$ *and hence* $H_i(\eta) \in \mathrm{RV}_{-1}$.
3. For any fixed η, $\lim_{i\to\infty} H_i'(\eta) = 0$.
4. $|H_i'(\eta)| < 2\beta(\eta)/\int_\eta^\infty \beta(r)dr$ *for all* $\eta > 0$.
5. For any $\epsilon > 0$, *there exists* η_0 *such that* $-1 - \epsilon < \eta H_i'(\eta)/H_i(\eta) \leq 0$ *for all* $\eta \geq \eta_0$ *and for all* i.

Proof. It is obvious that $0 \leq H_1(\eta) \leq 1$ and $H_i(\eta)$ is increasing in i. For fixed η, $H_i(\eta) \uparrow 1$ by the monotone convergence theorem.

By integration by parts, the numerator of $H_i(\eta)$ is written as

$$\int_\eta^\infty e^{(\eta - r)/i}\beta(r)dr = i\beta(\eta) + i\int_\eta^\infty e^{(\eta - r)/i}\beta'(r)dr. \tag{2.4}$$

Therefore

$$H_i(\eta) = i\frac{\beta(\eta)}{\int_\eta^\infty \beta(r)dr} + i\frac{\int_\eta^\infty e^{(\eta-r)/i}\beta'(r)dr}{\int_\eta^\infty \beta(r)dr}. \tag{2.5}$$

(2.5) divided by (2.3) is

$$1 = i\frac{\beta(\eta)}{H_i(\eta)\int_\eta^\infty \beta(r)dr} + i\frac{\int_\eta^\infty e^{-r/i}\beta'(r)dr}{\int_\eta^\infty e^{-r/i}\beta(r)dr}.$$

For fixed i, the second term of the above equation converges to 0 as $\eta \to \infty$ by the L'Hospital theorem. $H_i(\eta) \in \mathrm{RV}_{-1}$ because $\int_\eta^\infty \beta(r)dr \in \mathrm{RV}_0$ and $\beta(\eta) \in \mathrm{RV}_{-1}$.

Using (2.4) again, differentiation of the numerator of $H_i(\eta)$ gives

$$\left(\int_\eta^\infty e^{(\eta-r)/i}\beta(r)dr\right)' = \frac{1}{i}\int_\eta^\infty e^{(\eta-r)/i}\beta(r)dr - \beta(\eta) = \int_\eta^\infty e^{(\eta-r)/i}\beta'(r)dr.$$

Therefore

$$H_i'(\eta) = \frac{\beta(\eta)\int_\eta^\infty e^{(\eta-r)/i}\beta(r)dr}{(\int_\eta^\infty \beta(r)dr)^2} - \frac{\int_\eta^\infty e^{(\eta-r)/i}\{-\beta'(r)\}dr}{\int_\eta^\infty \beta(r)dr}. \tag{2.6}$$

Note that $-\beta'(r) \geq 0$ by our assumption. Each term of the right hand side of (2.6) is nondecreasing in i and hence by the monotone convergence theorem

$$\lim_{i\to\infty} H_i'(\eta) = \frac{\beta(\eta)\int_\eta^\infty \beta(r)dr}{(\int_\eta^\infty \beta(r)dr)^2} - \frac{\int_\eta^\infty\{-\beta'(r)\}dr}{\int_\eta^\infty \beta(r)dr} = 0.$$

Furthermore we have

$$\begin{aligned} |H_i'(\eta)| &< \left|\frac{\beta(\eta)\int_\eta^\infty e^{(\eta-r)/i}\beta(r)dr}{(\int_\eta^\infty \beta(r)dr)^2}\right| + \left|\frac{\int_\eta^\infty e^{(\eta-r)/i}\{-\beta'(r)\}dr}{\int_\eta^\infty \beta(r)dr}\right| \\ &< 2\beta(\eta)/\int_\eta^\infty \beta(r)dr. \end{aligned}$$

Dividing (2.6) by (2.3), we have

$$\begin{aligned} \eta\frac{H_i'(\eta)}{H_i(\eta)} &= \eta\left(\frac{\int_\eta^\infty -\beta'(r)dr}{\int_\eta^\infty \beta(r)dr} - \frac{\int_\eta^\infty e^{-r/i}\{-\beta'(r)\}dr}{\int_\eta^\infty e^{-r/i}\beta(r)dr}\right) \\ &> \frac{\eta\beta(\eta)}{\int_\eta^\infty \beta(r)dr} - \frac{\int_\eta^\infty e^{-r/i}\{-r\beta'(r)/\beta(r)\}\beta(r)dr}{\int_\eta^\infty e^{-r/i}\beta(r)dr} \\ &> \frac{\eta\beta(\eta)}{\int_\eta^\infty \beta(r)dr} - \sup_{r>\eta}\frac{-r\beta'(r)}{\beta(r)}. \end{aligned} \tag{2.7}$$

By 2 of Lemma 2.1 the right hand side converges to -1. This implies that for any $\epsilon > 0$ there exists η_0 such that $\eta H_i'(\eta)/H_i(\eta) > -1-\epsilon$ for all $\eta \geq \eta_0$ and for all i. Finally by the covariance inequality (Lemma 6.6, Chapter 5 of Lehmann and Casella (1998)) we will prove that $H_i'(\eta)/H_i(\eta) \le 0$ for sufficiently large η independent of i. By 2 of Lemma 2.1

$$\eta^2 \left(\frac{\beta'(\eta)}{\beta(\eta)} \right)' = \eta \frac{\beta''(\eta)}{\beta'(\eta)} \, \eta \frac{\beta'(\eta)}{\beta(\eta)} - \left(\eta \frac{\beta'(\eta)}{\beta(\eta)} \right)^2 \to (-2)(-1) - (-1)^2 = 1.$$

This shows that, $\beta'(\eta)/\beta(\eta)$ is eventually monotone nondecreasing. Hence by redefining η_0 if necessary, we can assume that $\beta'(\eta)/\beta(\eta)$ is monotone nondecreasing for $\eta \ge \eta_0$. Then

$$\begin{aligned} &\int_\eta^\infty e^{(\eta-r)/i} \left(-\frac{\beta'(r)}{\beta(r)} \right) \frac{\beta(r)}{\int_\eta^\infty \beta(r')dr'} dr \\ &\qquad \ge \int_\eta^\infty e^{(\eta-r)/i} \frac{\beta(r)}{\int_\eta^\infty \beta(r')dr'} dr \times \int_\eta^\infty \left(-\frac{\beta'(r)}{\beta(r)} \right) \frac{\beta(r)}{\int_\eta^\infty \beta(r')dr'} dr. \end{aligned}$$

Comparing this to the first equality in (2.7), we see that $H_i'(\eta)/H_i(\eta) \le 0$ for $\eta \ge \eta_0$ and for all i. □

2.2. Asymptotic behavior of expectations

In the next section, we need evaluation of an asymptotic behavior of expectation

$$E_x[\rho(\theta)] = \int_{R^p} \rho(\theta) f(\|\theta - x\|) d\theta$$

for sufficiently large $\|x\|$, where a random vector θ has the density function $f(\|\theta - x\|)$. This is the expected value with respect to the posterior distribution. Interchanging the roles of x and θ, in this subsection, we consider the asymptotic behavior of expectation

$$E_\theta[\rho(X)] = \int_{R^p} \rho(x) f(\|x - \theta\|) dx$$

for sufficiently large $\|\theta\|$, where a random vector X has the density function $f(\|x - \theta\|)$. We believe that this does not confuse the readers.

We discuss some notations used in the following. In addition to the Euclidean norm $\|x\|^2 = x_1^2 + \cdots + x_p^2$, we consider the norm $\|x\|_d^2 = d_1^2 x_1^2 + \cdots + d_p^2 x_p^2$. For convenience we assume, without loss of generality, that $d_1 \ge \cdots \ge d_p \ge 1$. Under this assumption

$$\|x\| \le \|x\|_d \le d_1 \|x\|. \tag{2.8}$$

By introducing this norm our results hold for elliptically-contoured distributions. The gradient of $\rho(x)$ is denoted by

$$\nabla \rho(x) = (\frac{\partial}{\partial x_1} \rho(x), \ldots, \frac{\partial}{\partial x_p} \rho(x))'.$$

We also write $\nabla_j \rho(x) = (\partial / \partial x_j) \rho(x)$. Finally we write $c_p = 2\pi^{p/2}/\Gamma(p/2)$.

Now we make the following regularity conditions on the density f and the function ρ.

F1 There exist $r_0 > 0$, $L > 0$, and $s > 1$, such that $r^{p+s} f(r) \le L$ for all $r \ge r_0$.
B1 $\rho(x)$ is written as $\rho(x) = \varrho(\|x\|_d)$, where $\varrho(r)$ is continuously differentiable in $r > 0$.
B2 There exists $r_1 \ge 1$ and $0 \le t_2 \le t_1$ such that $\varrho(r) > 0$ and $-t_1 \le w\varrho'(r)/\varrho(r) \le -t_2$ for all $r \ge r_1$.

Note that

$$\nabla\rho(x) = \frac{\varrho'(\|x\|_d)}{\|x\|_d}(d_1^2 x_1, \dots, d_p^2 x_p)'$$

and

$$\|\nabla\rho(x)\| = \frac{|\varrho'(\|x\|_d)|}{\|x\|_d}(d_1^4 x_1^2 + \cdots + d_p^4 x_p^2)^{1/2} \le d_1 |\varrho'(\|x\|_d)|.$$

The following lemma is useful. The proof based on the integration of $(\log\varrho(r))' = \varrho'(r)/\varrho(r)$ is easy and omitted.

Lemma 2.2. *Under the assumption* ***B2***

$$(z/y)^{-t_1} \le \varrho(z)/\varrho(y) \le (z/y)^{-t_2}$$

for any $z > y \ge r_1$. *Moreover*

$$\limsup_{y\to\infty} \sup_{\alpha y \le z \le \beta y} \varrho(z)/\varrho(y) \le \alpha^{-t_1}$$

for any $0 < \alpha < 1 < \beta$.

We now state the following theorem concerning the asymptotic behavior of $E_\theta[\rho(X)]$ for large $\|\theta\|$.

Theorem 2.2. *Assume* **F1**, **B1** *and* **B2**. *Then, for* $a = 0$ *or* 1, *and* $j = 1, \dots, p$, *there exists* $\epsilon > 0$ *such that*

$$\|\theta\|_d^{\epsilon-a} \left| E_\theta[X_j^a \rho(X)] - \theta_j^a \rho(\theta) \right| < C\rho(\theta) \tag{2.9}$$

for sufficiently large $\|\theta\|_d$ *if* $s > \max(1, t_1 - a - p, t_1 - t_2)$ *and* $\int_0^1 r^{p+a-1}|\varrho(r)|dr < \infty$. *Moreover* C *depends on* ρ *(or* ϱ*) only through* $\varrho(r_1)$, t_1, t_2 *and* $\int_0^{r_1} r^{p+a-1}|\varrho(r)|dr$.

Proof. Note that

$$E_\theta[X_j^a \rho(X)] - \theta_j^a \rho(\theta) = E_\theta[X_j^a(\rho(X) - \rho(\theta))].$$

Fix $0 < \nu < 1$. Define

$$\begin{aligned} V_\nu &= \{x : \|x - \theta\| \le \nu\|\theta\|\} \\ V'_\nu &= \{x : (1-\nu)\|\theta\| \le \|x\| \le (1+\nu)\|\theta\|\} \\ V''_\nu &= \{x : d_1^{-1}(1-\nu)\|\theta\|_d \le \|x\|_d \le d_1(1+\nu)\|\theta\|_d\}. \end{aligned}$$

By (2.8) $V_\nu \subset V'_\nu \subset V''_\nu$. Then

$$
\begin{aligned}
&\|\theta\|_d^{\epsilon-a} \left| E[X_j^a(\rho(X)-\rho(\theta))] \right| \\
&\le \|\theta\|_d^{\epsilon-a} \int_{V_\nu} \|x\|_d^a \left|\rho(x)-\rho(\theta)\right| f(\|x-\theta\|)dx \\
&\quad + \|\theta\|_d^{\epsilon-a} \int_{V_\nu^C} \|x\|^a \left|\rho(x)-\rho(\theta)\right| f(\|x-\theta\|)dx \\
&\le \|\theta\|_d^{\epsilon} \sup_{x\in V''_\nu} \left(\frac{\|x\|_d}{\|\theta\|_d}\right)^a \int_{V_\nu} \left|\rho(x)-\rho(\theta)\right| f(\|x-\theta\|)dx \\
&\quad + \|\theta\|_d^{\epsilon-a}\rho(\theta) \int_{V_\nu^C} \|x\|^a f(\|x-\theta\|)dx \\
&\quad + \|\theta\|_d^{\epsilon-a} \int_{V_\nu^C} \|x\|^a |\rho(x)| f(\|x-\theta\|)dx \\
&= I_1 + I_2 + I_3. \quad \text{(say)}
\end{aligned}
\tag{2.10}
$$

Consider the first integral I_1. $\sup_{x\in V''_\nu}(\|x\|_d/\|\theta\|_d)^a \le d_1^a(1+\nu)^a$. If $s>1$, then $m_1 = \int_{R^p}\|x-\theta\| f(\|x-\theta\|)dx$ is finite. Therefore for $\|\theta\|_d \ge d_1(1-\nu)^{-1}r_1$ we have

$$
\begin{aligned}
&\|\theta\|_d^{\epsilon} \int_{V_\nu} \left|\rho(x)-\rho(\theta)\right| f(\|x-\theta\|)dx \\
&= \|\theta\|_d^{\epsilon} \int_{V_\nu} |(x-\theta)'\nabla\rho(x^*)| f(\|x-\theta\|)dx, \ x^* \in V_\nu \\
&\le m_1 \|\theta\|_d^{\epsilon} \sup_{x\in V_\nu} \|\nabla\rho(x)\| \\
&\le m_1 d_1 \|\theta\|_d^{\epsilon} \sup_{x\in V_\nu} |\varrho'(\|x\|_d)| \\
&\le m_1 d_1 \|\theta\|_d^{\epsilon-1} \sup_{x\in V''_\nu} \frac{\|\theta\|_d}{\|x\|_d} \sup_{x\in V''_\nu} \frac{\varrho(\|x\|_d)}{\varrho(\|\theta\|_d)} \sup_{x\in V''_\nu} \frac{\|x\|_d|\varrho'(\|x\|_d)|}{\varrho(\|x\|_d)} \times \rho(\theta) \\
&= t_1 m_1 d_1^{2+t_1}(1-\nu)^{-t_1-1}\|\theta\|_d^{\epsilon-1} \times \rho(\theta) \\
&\le t_1 m_1 d_1^{2+t_1}(1-\nu)^{-t_1-1} \times \rho(\theta) \qquad (0<\epsilon<1).
\end{aligned}
$$

Therefore we have $I_1 \le C_1\rho(\theta)$, where $C_1 = (1+\nu)^a t_1 m_1 d_1^{2+t_1+a}(1-\nu)^{-t_1-1}$.

Now we consider the integral outside of V_ν. We only consider $\|\theta\| \ge \max(d_1\nu^{-1}r_0, r_1)$. Then for $x \in V_\nu^C$

$$\|x-\theta\| \ge \nu\|\theta\| \ge \nu d_1^{-1}\|\theta\|_d \ge r_0.$$

Therefore for the second term I_2, omitting $\rho(\theta)$, we have

$$\begin{aligned}
\|\theta\|_d^{\epsilon-a}\int_{V_\nu^C}\|x\|^a f(\|x-\theta\|)dx &\le \|\theta\|_d^{\epsilon-a}\int_{V_\nu^C}(\|x-\theta\|^a+\|\theta\|^a)f(\|x-\theta\|)dx \\
&\le c_pL\|\theta\|_d^{\epsilon-a}\int_{\nu\|\theta\|}^{\infty}\{r^{-s+a-1}+\|\theta\|_d^a r^{-s-1}\}dr \\
&= c_pL\|\theta\|_d^{\epsilon-a}\left(\frac{(\nu\|\theta\|)^{-s+a}}{s-a}+\|\theta\|_d^a\frac{(\nu\|\theta\|)^{-s}}{s}\right) \\
&\le c_pL\frac{d_1^s}{\nu^s}\left(\frac{\nu^a d_1^{-a}}{s-a}+\frac{1}{s}\right)\|\theta\|_d^{\epsilon-s}.
\end{aligned}$$

Hence if $s>1$, then $I_2\le C_2\rho(\theta)$, where $C_2=c_pL(d_1/\nu)^s\left\{(\nu/d_1)^a(s-a)^{-1}+1/s\right\}$.

We have seen that I_1 and I_2 are bounded from above assuming only $s>1$.

The third term I_3 of (2.10) is more problematic. Write

$$\begin{aligned}
I_3 &= \|\theta\|_d^{\epsilon-a}\int_{V_\nu^C}\|x\|^a|\rho(x)|f(\|x-\theta\|)dx \\
&\le \|\theta\|_d^{\epsilon-a}\left(\int_{V_\nu^C\cap\{\|x\|_d<r_1\}}+\int_{V_\nu^C\cap\{r_1\le\|x\|_d\le\|\theta\|_d\}}\right. \\
&\qquad\left.+\int_{V_\nu^C\cap\{\|x\|_d>\|\theta\|_d\}}\right)\|x\|^a|\rho(x)|f(\|x-\theta\|)dx \\
&= I_{31}+I_{32}+I_{33}. \qquad \text{(say)}
\end{aligned}$$

We take care of I_{33} first. Since $\varrho(r)$ is monotone nonincreasing for $r\ge r_1$, $\rho(x)\le\rho(\theta)$ for $\|x\|_d>\|\theta\|_d$. Therefore as above we have

$$I_{33}\le\|\theta\|_d^{\epsilon-a}\rho(\theta)\int_{V_\nu^C}\|x\|^a f(\|x-\theta\|)dx\le C_2\rho(\theta).$$

Next we consider I_{31}. For $\|\theta\|\ge\max(d_1\nu^{-1}r_0,r_1)$ and $x\in V_\nu^C$

$$f(\|x-\theta\|)\le L\|x-\theta\|^{-p-s}\le L\nu^{-p-s}\|\theta\|^{-p-s}\le Ld_1^{p+s}\nu^{-p-s}\|\theta\|_d^{-p-s}.$$

Therefore

$$I_{31}\le\|\theta\|_d^{\epsilon-a}Ld_1^{p+s}\nu^{-p-s}\|\theta\|_d^{-p-s}\int_{\|x\|_d\le r_1}\|x\|_d^a\,\rho(x)dx.$$

Note that by simple change of variables we have

$$\frac{\partial}{\partial r}\int_{\|x\|_d\le r}dx=c_{p,d}r^{p-1},\qquad c_{p,d}=c_p\prod_{i=1}^p d_i^{-1}.$$

Then

$$\int_{\|x\|_d\le r_1}\|x\|_d^a\,\rho(x)dx=c_{p,d}\int_0^{r_1}r^{p+a-1}|\varrho(r)|dr.$$

Therefore

$$I_{31} \le \|\theta\|_d^{\epsilon-a-p-s} \times Ld_1^{p+s}\nu^{-p-s}c_{p,d}\int_0^{r_1} r^{p+a-1}|\varrho(r)|dr.$$

On the other hand for $\|\theta\|_d \ge \|\theta\| \ge r_1$, $\rho(\theta) = \varrho(\|\theta\|_d)$ is bounded from below as

$$\varrho(r_1)r_1^{t_1}\|\theta\|_d^{-t_1} \le \varrho(\|\theta\|_d).$$

Therefore $I_{31} \le C_{31}\|\theta\|_d^{\epsilon-a-p-s+t_1}\rho(\theta)$, where

$$C_{31} = \varrho(r_1)^{-1}r_1^{-t_1}Ld_1^{p+s}\nu^{-p-s}c_{p,d}\int_0^{r_1} r^{p+a-1}|\varrho(r)|dr.$$

Hence if $s > t_1 - a - p$, then we can choose $\epsilon > 0$ such that

$$I_{31} \le C_{31}\rho(\theta).$$

Finally we consider I_{32}. $\varrho(r) \le \varrho(r_1)r_1^{t_2}r^{-t_2}$ for $r \ge r_1$. Therefore

$$\begin{aligned} I_{32} &\le \|\theta\|_d^{\epsilon-a}\int_{r_1\le\|x\|_d\le\|\theta\|_d} \|x\|_d^a\,\rho(x)\,f(\|x-\theta\|)dx \\ &\le \|\theta\|_d^{\epsilon-a}Ld_1^{p+s}\nu^{-p-s}\|\theta\|_d^{-p-s}\varrho(r_1)r_1^{t_2}\int_{r_1\le\|x\|_d\le\|\theta\|_d} \|x\|_d^{-t_2+a}dx \\ &= \|\theta\|_d^{\epsilon-a-p-s} \times Ld_1^{p+s}\nu^{-p-s}\varrho(r_1)r_1^{t_2}c_{p,d} \times \int_{r_1}^{\|\theta\|_d} r^{p+a-t_2-1}dr. \end{aligned}$$

Consider the integral $Q = \int_{r_1}^{\|\theta\|_d} r^{p+a-t_2-1}dr$. If $p + a - t_2 < 0$, then

$$Q \le \frac{1}{t_2-a-p}r_1^{-(t_2-a-p)} \le \frac{1}{t_2-a-p}.$$

Therefore as in the case of I_{31}, for some C_{32} we have $I_{32} \le C_{32}\rho(\theta)$ if $s > t_1 - a - p$. If $p + a - t_2 = 0$, then $Q \le \log(\|\theta\|_d)$. Even in this case, if $s > t_1 - a - p$, we can choose $\epsilon > 0$ such that $\|\theta\|_d^{\epsilon-a-p-s+t_1}Q$ is bounded. On the other hand, if $p + a - t_2 > 0$, then $Q = O(\|\theta\|_d^{p+a-t_2})$ and we need $s > t_1 - t_2$ to choose ϵ such that $I_{32} \le C_{32}\rho(\theta)$.

We have now confirmed that if $s > \max(1, t_1 - a - p, t_1 - t_2)$, there exist $\epsilon > 0$ and C, such that (2.9) folds for $\|\theta\| \ge \max(d_1\nu^{-1}r_0, r_1)$. □

In the next section, we also consider the expectation $E[\rho(Y)]$ where $Y \sim F(\|y-\theta\|)/C_f$ and $C_f = \{\pi^{p/2}/\Gamma(p/2+1)\}\int_0^\infty z^{p+1}f(z)dz$. $F(\cdot)/C_f$ is a probability density function because

$$\begin{aligned}\int_{R^p} \|y-\theta\|^\alpha F(\|y-\theta\|)dy &= \int_{R^p} \|y\|^\alpha \{\int_{\|y\|} sf(s)ds\}dy \\ &= c_p \int_0^\infty r^{p-1+\alpha} \int_r^\infty sf(s)dsdr \\ &= c_p \int_0^\infty r^{p+1+\alpha} \int_1^\infty tf(rt)dtdr \\ &= c_p \int_1^\infty t\{\int_0^\infty r^{p+1+\alpha} f(rt)dr\}dt \\ &= c_p \int_1^\infty t^{-p-1-\alpha}dt \cdot \int_0^\infty z^{p+1+\alpha} f(z)dz \\ &= \frac{c_p}{p+\alpha} \int_0^\infty z^{p+1+\alpha} f(z)dz.\end{aligned}$$

Therefore we require more restricted $s > 2 + \max(1, t_1 - a - p, t_1 - t_2)$ to obtain corresponding results of Theorem 2.2 in the case where $Y \sim F(\|y-\theta\|)/C_f$.

3. Admissibility

In this section, we give a sufficient condition for admissibility of the generalized Bayes estimator with respect a the prior density $g(\theta)$. The assumptions on g are the following.

G1 $g(\theta) = G(\|\theta\|_d)$, where $G(\eta)$ is twice continuously differentiable in $\eta > 0$.

G2 $\int_0^1 \eta^{p-1} G(\eta)d\eta < \infty$ and $\int_1^\infty \eta^{p-1} G(\eta)d\eta = \infty$, that is, improperness of g occurs only at infinity.

G3 $G(\eta) \le \eta^{1-p}(\int_\eta^\infty \beta(r)dr)^2/\beta(\eta)$ for $\eta \ge 1$, where $\beta(r)$ is given in Section 2.1.

G4 $\int_0^1 \eta^{p-1}|G'(\eta)|d\eta < \infty$.

G5 There exist $\eta_1 > 0$, $0 \le t_2 \le t_1$, $0 \le t_4 \le t_3$, such that $-t_1 \le \eta G'(\eta)/G(\eta) \le -t_2$ and $-t_3 \le \eta G''(\eta)/G'(\eta) \le -t_4$ for all $\eta \ge \eta_1$.

FG1 $\int_0^1 r^{p-1} f(r)G(r)dr < \infty$ and $\int_0^1 r^{p-1} F(r)|G'(r)|dr < \infty$.

We discuss some implications of these assumptions. By 2 of Theorem 2.1 and the assumption **G3**, $g(\theta)H_i^2(\|\theta\|_d)$ for any fixed i is integrable and hence becomes a proper probability density by standardization. Since $H_i(\cdot)$ approaches 1 as $i \to \infty$, $g(\theta)H_i^2(\|\theta\|_d)$ is a sequence of proper densities approaching $g(\theta)$, which is essential for using Blyth' method.

By Lemma 2.2, $G(\eta) = O(\eta^{-t_2})$. Therefore if $t_2 > p$, then $g(\theta) = G(\|\theta\|_d)$ is a proper prior. Since we are considering an improper $g(\theta)$, we assume $t_2 \le p$ from now on. Then in Theorem 2.2, $t_1 - a - p \le t_1 - p \le t_1 - t_2$ and

$$\max(1, t_1 - a - p, t_1 - t_2) = \max(1, t_1 - t_2).$$

If $G(\eta)$ is regularly varying, then for any $\epsilon > 0$, we can choose $\eta_0, t_1, t_2, t_3, t_4$, such that $t_1 - t_2 < \epsilon$ and $t_3 - t_4 < \epsilon$. However in **G5** we are allowing the case that $\liminf_{\eta\to\infty} \eta G'(\eta)/G(\eta)$ is strictly less than $\limsup_{\eta\to\infty} \eta G'(\eta)/G(\eta)$. Hence we are dealing with a broader class of

$G(\eta)$ than the class of regularly varying functions. It should also be noted that t_4 and t_3 are not always greater than t_2 and t_1, respectively. See Geluk and de Haan (1987) for the detail.

The generalized Bayes estimator δ_g with respect to the improper density $g(\theta)$ is written as

$$\begin{aligned}\delta_g(x) &= \frac{\int_{R^p} \theta f(\|x-\theta\|)g(\theta)d\theta}{\int_{R^p} f(\|x-\theta\|)g(\theta)d\theta} \\ &= x + \frac{\int_{R^p} (\theta - x) f(\|x-\theta\|)g(\theta)d\theta}{\int_{R^p} f(\|x-\theta\|)g(\theta)d\theta} \\ &= x + \frac{\int_{R^p} F(\|x-\theta\|)\nabla g(\theta)d\theta}{\int_{R^p} f(\|x-\theta\|)g(\theta)d\theta}, \end{aligned} \tag{3.1}$$

which is well-defined if both $\int_{R^p} F(\|x-\theta\|)\nabla g(\theta)d\theta$ and $\int_{R^p} f(\|x-\theta\|)g(\theta)d\theta$ are finite for all x. These are guaranteed by the assumptions. Write

$$\begin{aligned} m(\psi|x) &= \int_{R^p} \psi(\theta) f(\|\theta - x\|)d\theta \\ M(\psi|x) &= \frac{1}{C_f}\int_{R^p} \psi(\theta) F(\|\theta - x\|)d\theta. \end{aligned}$$

Then δ_g is written as

$$\delta_g(x) = x + C_f \frac{M(\nabla g|x)}{m(g|x)}.$$

Note that by **G1** the j-th element of ∇g is given by

$$\nabla_j g(\theta) = d_j^2 \theta_j \frac{G'(\|\theta\|_d)}{\|\theta\|_d}.$$

We also write

$$h_i(x) = H_i(\|x\|_d).$$

Now we state the following lemma in preparation of our main theorem.

Lemma 3.1. *Assume* **G1**–**G5**, **F1**, *and* **FG1**. *If* $s > 4 + t_1 - t_2$, *then there exists* $\epsilon > 0$ *such that*

$$\|x\|_d^{\epsilon} |m(g|x) - g(x)| < C_1 \times g(x) \tag{3.2}$$

$$\|x\|_d^{\epsilon} \left| m(gh_i^2|x) - g(x)h_i^2(x) \right| < C_2 \times g(x)h_i^2(x) \tag{3.3}$$

$$\|x\|_d^{\epsilon} \left| M(gh_i^2|x) - g(x)h_i^2(x) \right| < C_3 \times g(x)h_i^2(x) \tag{3.4}$$

for all sufficiently large $\|x\|_d$. *Moreover if* $s > 4 + \max(t_3 - t_4, t_3 - p)$, *then there exists* $\epsilon > 0$ *such that*

$$\|x\|_d^{\epsilon-1} |M(\nabla_j g|x) - \nabla_j g(x)| < C_4 \times G'(\|x\|_d)/\|x\|_d \tag{3.5}$$

$$\|x\|_d^{\epsilon-1} \left| M(\nabla_j g h_i^2|x) - \nabla_j g(x)h_i^2(x) \right| < C_5 \times G'(\|x\|_d)h_i^2(x)/\|x\|_d \tag{3.6}$$

for all sufficiently large $\|x\|_d$. *Furthermore* C_2, C_3, C_5 *do not depend on* i.

Proof. In Theorem 2.2 we take ρ as g, gh_i^2 or $\rho(x) = G'(\|x\|_d)h_i(x)^2/\|x\|_d$. For a differentiable $l(\eta)$, we have

$$\eta\{l(\eta)H_i^2(\eta)\}'/\{l(\eta)H_i^2(\eta)\} = \eta l'(\eta)/l(\eta) + 2\eta H_i'(\eta)/H_i(\eta).$$

Hence under Assumption **G5** and by 5 of Theorem 2.1, we have

$$\begin{aligned} &-t_1 - 2 - \epsilon \le \eta\{G(\eta)H_i^2(\eta)\}'/\{G(\eta)H_i^2(\eta)\} \le -t_2 \\ &-t_3 - 3 - \epsilon \le \eta\{G'(\eta)H_i^2(\eta)/\eta\}'/\{G'(\eta)H_i^2(\eta)/\eta\} \le -t_4 - 1 \end{aligned}$$

for any $\epsilon > 0$ and $\eta \ge \max(\eta_0, \eta_1)$. We can therefore think of t_1, t_3, t_4 replaced by $t_1 + 2$, $t_3 + 3$, $t_4 + 1$, respectively. By Theorem 2.2 for the case $a = 0$, (3.2), (3.3) and (3.4) are simultaneously satisfied if

$$s > 2 + \max(1, 2 + t_1 - t_2) = 4 + t_1 - t_2.$$

Similarly by Theorem 2.2 for $a = 1$, (3.5) and (3.6) are simultaneously satisfied if

$$s > 2 + \max(1, 2 + t_3 - t_4, t_3 + 2 - p) = 4 + \max(t_3 - t_4, t_3 - p).$$

□

By Lemma 3.1, $M(\nabla g|x)/m(g|x)$ is finite for all x and hence the risk function of δ_g is finite under our assumptions because

$$\begin{aligned} R(\theta, \delta_g) &= E[\|X - \theta + C_f M(\nabla g|X)/m(g|X)\|_Q^2] \\ &\le Q_{\max} E[\|X - \theta + C_f M(\nabla g|X)/m(g|X)\|^2] \\ &\le 2Q_{\max}\{E[\|X - \theta\|^2] + C_f^2 E[\|M(\nabla g|X)/m(g|X)\|^2]\}, \end{aligned}$$

where $Q_{\max}$ is the largest eigenvalue of Q.

Now we state the main theorem of this paper.

Theorem 3.1. *Assume* **G1**–**G5**, **F1**, **FG1**. *Then the generalized Bayes estimator with respect to g is admissible either if $s > 4 + t_1 - t_2$ and $t_2 > p - 2$ or if $s > 4 + \max(t_1 - t_2, t_3 - t_4, t_3 - p)$ and $t_2 \le p - 2$.*

Before giving a proof of this theorem we present the following corollary, which was the motivating case for this paper. It follows from the main theorem by taking $\beta(\eta)$ in (2.2).

Corollary 3.1. *Assume $s > 4$. Then the generalized Bayes estimator with respect to $\|\theta\|_d^{2-p}\prod_{i=0}^n \mathrm{Log}_i(\|\theta\|_d + c)$, where n is a nonnegative integer and $\mathrm{Log}_n(c) > 0$, is admissible.*

In the normal case, Brown (1971)'s sufficient conditions for admissibility and inadmissibility are known. He showed that the generalized Bayes estimator with respect to g is admissible if

$$\int_{\|x\|_d > 1} \|x\|_d^{2-2p}\{m(g|x)\}^{-1} dx \tag{3.7}$$

diverges and inadmissible if (3.7) converges. By Lemma 3.1, we see that

$$(1/2)G(\|x\|_d) < m(g|x) < 2G(\|x\|_d)$$

for sufficiently large $\|x\|_d$ and hence that $G(\eta) \leq \eta^{2-p}\prod_{i=0}^{n}\mathrm{Log}_i(\eta+c)$ leads to admissibility and $G(\eta) \geq \eta^{2-p}\prod_{i=0}^{n-1}\mathrm{Log}_i(\eta+c)\mathrm{Log}_n^2(\eta+c)$ leads to inadmissibility. Therefore our sufficient condition in Theorem 3.1 is very close to being necessary. Now we give a proof of Theorem 3.1.

Proof of Theorem 3.1. Let δ_{gi} denote the Bayes estimator with respect to the proper prior density $g(\theta)h_i^2(\theta)$. Then the Bayes risk difference of δ_g and δ_{gi} with respect to the density $g(\theta)h_i^2(\theta)$ is written as

$$\begin{aligned}
\Delta &= \int_{R^p}\left[R(\theta,\delta_g) - R(\theta,\delta_{gi})\right]g(\theta)h_i^2(\theta)d\theta \\
&= \int_{R^p}\int_{R^p}[\|\delta_g-\theta\|_Q^2 - \|\delta_{gi}-\theta\|_Q^2]f(\|x-\theta\|)g(\theta)h_i^2(\theta)d\theta dx \\
&= \int_{R^p}\Big\{[\|\delta_g\|_Q^2 - \|\delta_{gi}\|_Q^2]\int_{R^p}f(\|x-\theta\|)g(\theta)h_i^2(\theta)d\theta \\
&\quad - 2(\delta_g-\delta_{gi})Q'\int_{R^p}\theta f(\|x-\theta\|)g(\theta)h_i^2(\theta)d\theta\Big\}dx \\
&= \int_{R^p}\|\delta_g-\delta_{gi}\|_Q^2\int_{R^p}f(\|x-\theta\|)g(\theta)h_i^2(\theta)d\theta dx \\
&= C_f^2\int_{R^p}\left\|\frac{M(\nabla g|x)}{m(g|x)} - \frac{M(\nabla\{gh_i^2\}|x)}{m(gh_i^2|x)}\right\|_Q^2 m(gh_i^2|x)dx \\
&= C_f^2\int_{R^p}\left\|\frac{M(\nabla g|x)}{m(g|x)} - \frac{M(\nabla gh_i^2|x)}{m(gh_i^2|x)} - \frac{M(g\nabla h_i^2|x)}{m(gh_i^2|x)}\right\|_Q^2 m(gh_i^2|x)dx.
\end{aligned}$$

In the same way as in Brown and Hwang (1982), we have

$$\begin{aligned}
\Delta &\leq 2C_f^2Q_{\max}\int_{R^p}\left\|\frac{M(\nabla g|x)}{m(g|x)} - \frac{M(\nabla gh_i^2|x)}{m(gh_i^2|x)}\right\|^2 m(gh_i^2|x)dx \\
&\quad + 2C_f^2Q_{\max}\int_{R^p}\left\|\frac{M(g\nabla h_i^2|x)}{m(gh_i^2|x)}\right\|^2 m(gh_i^2|x)dx \\
&= 2C_f^2Q_{\max}(B_i + A_i). \qquad \text{(say)}
\end{aligned}$$

Using the Cauchy-Schwartz inequality for A_i, we have

$$\begin{aligned}
A_i &= 4\int_{R^p}\|M(gh_i\nabla h_i|x)\|^2\{m(gh_i^2|x)\}^{-1}dx \\
&\leq 4\int_{R^p}\frac{M(gh_i^2|x)}{m(gh_i^2|x)}M(g\|\nabla h_i\|^2|x)dx.
\end{aligned}$$

Both $M(gh_i^2|x)$ and $m(gh_i^2|x)$ at $x=0$ are clearly finite. By (3.3) and (3.4), we have

$$\lim_{\|x\|\to\infty}\frac{M(gh_i^2|x)}{m(gh_i^2|x)}=1$$

uniformly in i. This implies that there exists c_1 such that $M(gh_i^2|x)/m(gh_i^2|x)<c_1$ for all x and for all i. Then

$$\begin{aligned}A_i &\le 4c_1\int_{R^p}M(g\|\nabla h_i\|^2|x)dx\\ &=4c_1\int_{R^p}\{1/C_f\}F(\|x-\theta\|)dx\int_{R^p}g(\theta)\|\nabla h_i(\theta)\|^2d\theta\\ &=4c_1\int_{R^p}g(\theta)\|\nabla h_i(\theta)\|^2d\theta.\end{aligned}$$

By 4 of Theorem 2.1 we have $\|\nabla h_i(\theta)\|<2d_1\beta(\|\theta\|_d)/\int_{\|\theta\|_d}^\infty\beta(r)dr$ and by **G3**

$$\int_{R^p}g(\theta)\left\{\frac{\beta(\|\theta\|_d)}{\int_{\|\theta\|_d}^\infty\beta(r)dr}\right\}^2d\theta\le\int_{R^p}\|\theta\|_d^{1-p}\beta(\|\theta\|_d)d\theta<\infty.$$

Furthermore $\|\nabla h_i(\theta)\|\to 0$ as $i\to\infty$ by 3 of Theorem 2.1. Therefore by the dominated convergence theorem A_i converges to 0 as $i\to\infty$.

Next we consider B_i. $M(\nabla g|x)$ and $M(\nabla gh_i^2|x)$ at $x=0$ are zero vectors because g and h_i^2 are function of $\|\theta\|_d$. So the integrand of B_i is finite around $x=0$. For the asymptotic property of the integrand of B_i, we need to distinguish two cases: $t_2<p-2$ and $t_2\ge p-2$. When $t_2<p-2$, we can bound the norm in the integrand of B_i from above somewhat roughly. For $s>4+t_1-t_2$, using Lemma 3.1 we have

$$\begin{aligned}&\frac{1}{d_j^2}\left|\frac{M(\nabla_j g|x)}{m(g|x)}-\frac{M(\nabla_j gh_i^2|x)}{m(gh_i^2|x)}\right|\\ &\quad=\frac{1}{d_j^2}\left|\frac{M(\nabla_j g|x)}{M(g|x)}\frac{M(g|x)}{m(g|x)}-\frac{M(\nabla_j gh_i^2|x)}{M(gh_i^2|x)}\frac{M(gh_i^2|x)}{m(gh_i^2|x)}\right|\\ &\quad\le 2\left|\frac{G'(\|x\|_d)}{G(\|x\|_d)}\right|+2\left|\frac{G'(\|x\|_d)}{G(\|x\|_d)}\right|\\ &\quad<c\|x\|_d^{-1}\end{aligned}$$

for all sufficiently large $\|x\|_d$ and for all i. When $t_2\ge p-2$, we have to bound it from above more strictly. For $s>4+\max(t_1-t_2,t_3-t_4,t_3-p)$, by Lemma 3.1, we have

$$\begin{aligned}&\frac{1}{d_j^2}\left|\frac{M(\nabla_j g|x)}{m(g|x)}-\frac{M(\nabla_j gh_i^2|x)}{m(gh_i^2|x)}\right|\\ &\quad=\left|\frac{G'(\|x\|_d)}{G(\|x\|_d)\|x\|_d}\frac{x_j+O(\|x\|_d^{1-\epsilon})}{1+O(\|x\|_d^{-\epsilon})}-\frac{G'(\|x\|_d)}{G(\|x\|_d)\|x\|_d}\frac{x_j+O(\|x\|_d^{1-\epsilon})}{1+O(\|x\|_d^{-\epsilon})}\right|\\ &\quad<c\|x\|_d^{-1-\epsilon},\end{aligned}$$

for all sufficiently large $\|x\|_d$ and for all i. Moreover $m(gh_i^2|x) \le m(g|x)$ and $m(g|x) < 2G(\|x\|_d)$ all sufficiently large $\|x\|_d$. Therefore, in both cases, there exist C_1, C_2 and $\epsilon > 0$ such that the integrand of B_i is less than

$$\min(C_1, C_2\|x\|_d^{-1-p-\epsilon})$$

for all sufficiently large $\|x\|_d$. Therefore B_i converges to 0 as $i \to \infty$ by the dominated convergence theorem, in view of

$$\int_0^\infty r^{p-1}\min(C_1, C_2 r^{-1-p-\epsilon})dr < \infty.$$

□

4. The generalized Bayes estimator with respect to the harmonic prior and its minimaxity

In this section, we show that the generalized Bayes estimator with respect to the harmonic prior has a form simple enough to check some sufficient conditions for minimaxity given in early studies. We demonstrate that it is minimax for some f.

In (3.1), the generalized Bayes estimator can be also written as

$$\delta_g(x) = x + C_f \frac{\nabla_x M(g|x)}{m(g|x)}.$$

For $p \ge 3$ and $g(\theta) = \|\theta\|^{2-p}$, we have

$$\begin{aligned} m(g|x) &= \int_{R^p} f(\|x-\theta\|)\|\theta\|^{2-p}d\theta = \int_{R^p} f(\|\eta\|)\|x-\eta\|^{2-p}d\eta \\ &= c_{p-1}\int_0^\infty\int_0^\pi f(\lambda)(\lambda^2+2\lambda r\cos\varphi + r^2)^{1-p/2}\lambda^{p-1}\sin^{p-2}\varphi d\lambda d\varphi \\ &= c_{p-1}r^2\int_0^\infty\int_0^\pi f(rt)(1+2t\cos\varphi+t^2)^{1-p/2}t^{p-1}\sin^{p-2}\varphi dt d\varphi \\ &= c_p\left(r^2\int_0^1 t^{p-1}f(rt)dt + r^2\int_1^\infty tf(rt)dt\right) \\ &= c_p\left(\left[-t^{p-2}F(rt)\right]_0^1 + (p-2)\int_0^1 t^{p-3}F(rt)dt + F(r)\right) \\ &= c_p(p-2)\int_0^1 t^{p-3}F(rt)dt, \end{aligned} \tag{4.1}$$

where $r = \|x\|$. The fifth equality in the above equation follows from the relation

$$\int_0^\pi (1+2t\cos\varphi+t^2)^{1-p/2}\sin^{p-2}\varphi d\varphi = B(p/2-1/2, 1/2)\min(t^{2-p}, 1),$$

which is proved in Lemma 4.1 in the end of this section. In the same way, we have

$$C_f\nabla_x M(g|x) = -xc_p(p-2)\int_0^1 t^{p-1}F(rt)dt.$$

Hence the generalized Bayes estimator is written as $\delta_*(X) = (1-\phi_*(\|X\|)/\|X\|^2)X$, where

$$\phi_*(r) = r^2 \frac{\int_0^1 t^{p-1}F(rt)dt}{\int_0^1 t^{p-3}F(rt)dt}.$$

Some properties of the behavior of $\phi_*(r)$ are easily derived as follows.

Theorem 4.1. *1.* $\lim_{r\to\infty}\phi_*(r) = (p-2)E_0(\|X\|^2)/p$.
2. $\phi_*(r)$ *is nondecreasing in* r *for any* f.
3. $\phi_*(r)/r^2$ *is nonincreasing in* r *if* $F(t)\{t^2 f(t)\}^{-1}$ *is nonincreasing.*

Proof. $\phi_*(r)$ can be written as $\int_0^r t^{p-1}F(t)dt/\int_0^r t^{p-3}F(t)dt$ and we have

$$\lim_{r\to\infty}\phi_*(r) = \frac{\int_0^\infty t^{p-1}F(t)dt}{\int_0^\infty t^{p-3}F(t)dt} = \frac{p-2}{p}\frac{\int_0^\infty t^{p+1}f(t)dt}{\int_0^\infty t^{p-1}f(t)dt} = \frac{p-2}{p}E_0[\|X\|^2].$$

The derivative of $\phi_*(r)$ is calculated as

$$\phi_*'(r) = \frac{r^{p-3}F(r)}{(\int_0^r t^{p-3}F(t)dt)^2}\int_0^r (r^2-t^2)t^{p-3}F(t)dt,$$

which is nonnegative for any f. The derivative of $\phi_*(r)/r^2$ is calculated as

$$\frac{d}{dr}\left(\phi_*(r)/r^2\right) = r\left(\int_0^1 t^{p-3}F(rt)dt\right)^{-2}\left(\int_0^1 t^{p-1}F(rt)dt\int_0^1 t^{p-1}f(rt)dt \right.$$
$$\left. - \int_0^1 t^{p+1}f(rt)dt\int_0^1 t^{p-3}F(rt)dt\right).$$

If $F(t)\{t^2 f(t)\}^{-1}$ is nonincreasing, the right-hand side of the equality above is nonpositive by the covariance inequality. □

Now we consider the minimaxity of δ_*. We present a brief list of known sufficient conditions for minimaxity given in previous papers, for the estimator of the form (1.1) with nonnegative and nondecreasing $\phi(r)$.

Author	p	$\phi(r)/r^2$	upper bound of ϕ
general			
Berger (1975)	$p\geq 3$		$2(p-2)\inf_{s\in U}F(s)/f(s)$
Brandwein (1979)	$p\geq 4$	↘	$2(p-2)(pE_0(\|X\|^{-2}))^{-1}$
unimodal or f is nonincreasing			
Brandwein and Strawderman (1978)	$p\geq 4$	↘	$2p((p+2)E_0(\|X\|^{-2}))^{-1}$
Ralescu et al. (1992)	$p=3$	↘	$0.93(E_0(\|X\|^{-2}))^{-1}$
$F(t)/f(t)$ is nondecreasing			
Bock (1985)	$p\geq 4$	↘	$2(E_0(\|X\|^{-2}))^{-1}$
scale mixtures of multivariate normal			
Strawderman (1974)	$p\geq 3$	↘	$2(E_0(\|X\|^{-2}))^{-1}$

In the table, $U = \{t \geq 0 | f(t) > 0\}$ and an arrow $\searrow$ means *nonincreasing*. It is noted that $f(t)$ is nonincreasing in t if $F(t)/f(t)$ is nondecreasing in t and that $F(t)/f(t)$ is nondecreasing in t if f is a scale mixtures of multivariate normal.

Combining Theorem 4.1 and the table above, we can derive a sufficient condition for minimaxity of $\delta_*(X)$ and we state it in the following theorem for $p \geq 4$.

Theorem 4.2. *1. Assume $t^{-2}F(t)/f(t)$ is nonincreasing.*

- *(a) δ_* is minimax if $E_0[\|X\|^2]E_0[\|X\|^{-2}] \leq 2$.*
- *(b) Assume also $f(t)$ is nonincreasing. Then δ_* is minimax if $(p^2-4)E_0[\|X\|^2]E_0[\|X\|^{-2}] \leq 2p^2$.*
- *(c) Assume also $F(t)/f(t)$ is nondecreasing. Then δ_* is minimax if $(p-2)E_0[\|X\|^2]E_0[\|X\|^{-2}] \leq 2p$.*

2. Assume $0 < \inf_{s\in U} F(s)/f(s) < \infty$. Then δ_* is minimax if $E_0[\|X\|^2] \leq 2p \inf_{s\in U} F(s)/f(s)$.

Berger (1975) and Bock (1985) gave several examples of f, checked the monotonicity of $f(t)$, $F(t)/f(t)$, and $t^{-2}F(t)/f(t)$ and calculated an upper bound of $\phi(r)$. In this paper we give just two examples but we believe that the estimator $\delta_*(X)$ is minimax for a broad class of spherically symmetric distributions.

Example 4.1. *We consider $f(s) = s^\alpha \exp(-\beta s^2)$ for $\alpha,\ \beta > 0$. We have*

$$\frac{F(t)}{t^2 f(t)} = \int_1^\infty u^{\alpha+1} \exp(\beta t^2(1-u))du,$$

which is decreasing in t. By an integration by parts, we have

$$\begin{aligned}\frac{F(t)}{f(t)} &= \frac{1}{2\beta} + \frac{\alpha}{2\beta}\frac{\int_t^\infty s^{\alpha-1}\exp(-\beta s^2)ds}{t^\alpha \exp(-\beta t^2)} \\ &= \frac{1}{2\beta} + \frac{\alpha}{2\beta}\int_1^\infty u^{\alpha-1}\exp(\beta t^2\{1-u^2\})du\end{aligned}$$

and hence $\inf_{t\geq 0} F(t)/f(t) = (2\beta)^{-1}$. We also have $E_0(\|X\|^2) = (p/2+\alpha/2)/\beta$ and $E_0(\|X\|^{-2})^{-1} = (p/2+\alpha/2-1)/\beta$. Therefore the generalized Bayes estimator is minimax if $\alpha \leq p$ for $p \geq 3$ by Berger (1975)'s conditions and if $\alpha \geq 4-p$ for $p \geq 4$ by Brandwein (1979)'s conditions regardless of β. Hence the estimator for $p \geq 4$ is minimax regardless of α and β.

Example 4.2. *We consider $f(t) = \exp(-t^2/2) - a\exp(-t^2/\{2b\})$ for $0 < a \leq 1$, $0 < b < 1$. Note that if $a \leq b$ then f is unimodal and if $a > b$ then f is not. We easily see that $\inf_{t\geq 0} F(t)/f(t) = 1$ and that $E_0[\|X\|^2] = p(1-ab^{p/2+1})/(1-ab^{p/2})$. Because $(1-ab^{p/2+1})/(1-ab^{p/2}) \leq 2$ for $0 < a \leq 1$, $0 < b < 1$, the generalized Bayes estimator is minimax by Berger (1975).*

The following lemma is stated in a more general form in 3.036 of Gradshteyn and Ryzhik (2000), but it is incorrectly stated with an errata posted on the book's web page. Maruyama pointed out this error and he is acknowledged in the errata for 3.036. Since a derivation of the formula is not easily accessible, we provide our own proof.

Lemma 4.1. *For $\alpha > -1/2$ and $|a| < 1$,*

$$\int_0^\pi (1 + 2a\cos\varphi + a^2)^{-\alpha} \sin^{2\alpha}\varphi d\varphi = B(\alpha + 1/2, 1/2).$$

Proof. Let $g(\varphi) = (1 + 2a\cos\varphi + a^2)^{-1}\sin^2\varphi$. Then we have the derivative

$$g'(\varphi) = 2\sin\varphi \frac{(a\cos\varphi + 1)(\cos\varphi + a)}{(1 + 2a\cos\varphi + a^2)^2}.$$

We see that $g(\varphi)$ is monotone increasing from $g(0) = 0$ to $g(\arccos(-a)) = 1$ and decreasing from $g(\arccos(-a)) = 1$ to $g(\pi) = 0$. Therefore we have

$$\begin{aligned}
&\int_0^\pi (1 + 2a\cos\varphi + a^2)^{-\alpha} \sin^{2\alpha}\varphi d\varphi \\
&= \left(\int_0^{\arccos(-a)} + \int_{\arccos(-a)}^\pi\right)(1 + 2a\cos\varphi + a^2)^{-\alpha}\sin^{2\alpha}\varphi d\varphi \\
&= \int_0^{\arccos(-a)} (1 + 2a\cos\varphi + a^2)^{-\alpha}\sin^{2\alpha}\varphi d\varphi \\
&\quad + \int_0^{\arccos(a)} (1 - 2a\cos\rho + a^2)^{-\alpha}\sin^{2\alpha}\rho d\rho \\
&= \int_0^1 t^\alpha (d\varphi/dt)dt + \int_0^1 s^\alpha (d\rho/ds)ds, \qquad (4.2)
\end{aligned}$$

where $t = (1 + 2a\cos\varphi + a^2)^{-1}\sin^2\varphi$ and $s = (1 - 2a\cos\rho + a^2)^{-1}\sin^2\rho$. Here $(d\varphi/dt)$ and $(d\rho/ds)$ are calculated as

$$\begin{aligned}
d\varphi/dt &= \frac{1}{2tA(t)}\left(1 - \{at - A(t)\}^2\right)^{1/2} \\
d\rho/ds &= \frac{1}{2sA(s)}\left(1 - \{as + A(s)\}^2\right)^{1/2},
\end{aligned}$$

where $A(t) = (1 - t)^{1/2}(1 - a^2t)^{1/2}$. Let

$$h(t) = \frac{1}{2tA(t)}\left\{\left(1 - \{at - A(t)\}^2\right)^{1/2} + \left(1 - \{at + A(t)\}^2\right)^{1/2}\right\}.$$

Then we have $h^2(t) = \{2tA(t)\}^{-2}\{2 - 2a^2t^2 - 2A(t)^2 + 2B(t)\}$, where

$$\begin{aligned}
B(t) &= \left(1 - \{at - A(t)\}^2 - \{at + A(t)\}^2 + \{a^2t^2 - A^2(t)\}^2\right)^{1/2} \\
&= t(1 - a^2),
\end{aligned}$$

which implies $h(t) = t^{-1/2}(1 - t)^{-1/2}$. Therefore we get

$$\text{the right hand side of (4.2)} = \int_0^1 t^\alpha h(t)dt = B(\alpha + 1/2, 1/2).$$

□

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