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Abstract

The concept of M-convex functions has recently been generalized for functions defined on constant-parity jump systems. The b-matching problem and its generalization provide canonical examples of M-convex functions on jump systems. In this paper, we propose a steepest descent algorithm for minimizing M-convex functions on constant-parity jump systems.

1 Introduction

The concept of M-convex functions introduced by Murota [10] gives a framework for well-solved discrete optimization problems with nonlinear objective functions. In particular, many problems of network flow type are included in the framework provided by M-convex functions. Moreover, a common generalization of the submodular flow problem of Edmonds and Giles and the valuated matroid intersection problem of Murota is provided by the M-convex submodular problem, which covers a reasonably wide class of well-solved discrete optimization problems.

Let V be a nonempty finite set and B be the set of integer points of a base polyhedron of a submodular function $\rho : 2^V \to \mathbf{Z} \cup \{+\infty\}$. A function $f : B \to \mathbf{R}$ is said to be M-convex if f satisfies

(M-EXC[B]) $\forall x, y \in B$, $\forall u$ with $x(u) > y(u)$, $\exists v$ with $x(v) < y(v)$ such that $x - \chi_u + \chi_v \in B,\ y + \chi_u - \chi_v \in B$, and

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v),$$

where $\chi_v \in \{0,1\}^V$ is the characteristic vector of $v \in V$. This definition of M-convex functions is motivated by valuated matroids introduced by Dress and Wenzel [4][5]. M-convex functions have various desirable properties of "discrete convexity" such as extensibility to ordinary convex functions, conjugacy, and duality, etc.

Recently, the concept of M-convex functions is generalized by Murota [12] for functions defined on constant-parity jump systems with a view to providing a general framework for the minsquare factor problem considered by Apollonio and Sebő [2]. A canonical example of such functions arises from minimum weight perfect b-matchings and from a separable convex function (sum of univariate convex functions) on the degree sequences of an undirected graph. Fundamental properties of M-convex functions on constant-parity jump systems are investigated in [12], such as equivalence between different exchange axioms, a local optimality criterion guaranteeing global optimality, and some ideas for minimization algorithms. Minimization of a separable convex function over a jump system has been studied in [1], where a local criterion for optimality as well as a greedy algorithm is given.

In this paper, we propose a steepest descent algorithm for minimizing M-convex functions on constant-parity jump systems, which is an extension of the algorithm of [10][11] for M-convex functions on base polyhedra.

The organization of this paper is as follows. In Section 2, we prepare the definitions and fundamental properties of M-convex functions on constant-parity jump systems. Canonical examples of M-convex functions are shown in Section 3. In Section 4, our steepest descent algorithm is described. Generalizations of the minimizer-cut property and the tie-breaking rule are keys to the validity of the algorithm.

2 Definitions and Fundamental Properties

Let V be a finite set. For $x = (x(v)),\ y = (y(v)) \in \mathbf{Z}^V$ define

$$x(V) = \sum_{v \in V} x(v), \quad \|x\|_1 = \sum_{v \in V} |x(v)|,$$
$$\mathrm{supp}(x) = \{v \in V \mid x(v) \neq 0\},$$
$$[x, y] = \{z \in \mathbf{Z}^V \mid \min(x(v), y(v)) \leq z(v) \leq \max(x(v), y(v)) \ (\forall v \in V)\}.$$

The characteristic vector of $u \in V$ is denoted by χ_u, i.e., $\chi_u(v) = 1$ or 0 according as $v = u$ or $v \in V \setminus \{u\}$. For $s = \pm\chi_u$ we define $\mathrm{sign}(s) \in \{+1, -1\}$ by $s = \mathrm{sign}(s) \cdot \chi_u$.

A vector $s \in \mathbf{Z}^V$ is called an (x, y)-*increment* if $s = \chi_u$ or $s = -\chi_u$ for some $u \in V$ and $x + s \in [x, y]$. We denote the set of all (x, y)-increments by $\mathrm{Inc}(x, y)$. Jump systems are defined as follows.

Definition 2.1. A nonempty set $J \subseteq \mathbf{Z}^V$ is said to be a *jump system* if it satisfies the *2-step exchange axiom*:

$\forall x, y \in J$, $\forall s \in \mathrm{Inc}(x, y)$ with $x + s \notin J$, $\exists t \in \mathrm{Inc}(x + s, y)$ such that $x + s + t \in J$.

A set $J \subseteq \mathbf{Z}^V$ is a *constant-sum system* if $x(V) = y(V)$ for any $x, y \in J$, and a *constant-parity system* if $x(V) - y(V)$ is even for any $x, y \in J$.

A stronger exchange axiom:

(J-EXC) $\forall x, y \in J$, $\forall s \in \mathrm{Inc}(x, y)$, $\exists t \in \mathrm{Inc}(x + s, y)$ such that $x + s + t \in J$ and $y - s - t \in J$

characterizes a constant-parity jump system, a fact communicated to the first author by J. Geelen (see Section 6.1 of [12] for a proof).

Lemma 2.2 (Geelen [7]). *A nonempty set J is a constant-parity jump system if and only if it satisfies* (J-EXC).

M-convex functions on constant-parity jump systems are defined in [12] as follows:

Definition 2.3. A function $f : J \to \mathbf{R}$ on a constant-parity jump system J is said to be *M-convex* if it satisfies the following exchange axiom:

(M-EXC[J]) $\forall x, y \in J$, $\forall s \in \mathrm{Inc}(x, y)$, $\exists t \in \mathrm{Inc}(x + s, y)$ such that $x + s + t \in J$, $y - s - t \in J$, and

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t).$$

We adopt the convention that $f(x) = +\infty$ for $x \notin J$.

For examples of M-convex functions, see [12] or Section 3. Note that addition of a linear function preserves M-convexity. That is, for an M-convex function f and a vector $p = (p(v)) \in \mathbf{R}^V$, the function $f[p]$ defined by $f[p](x) = f(x) + \langle p, x\rangle$ with $\langle p, x\rangle = \sum_{v \in V} p(v)x(v)$ is M-convex.

Remark 2.4. A jump system is a common generalization of a delta-matroid and a base polyhedron of an integral polymatroid, which are generalizations of a matroid in two different directions. According to Definition 2.3, this hierarchy of discrete structures is extended to discrete functions; an M-convex function on a constant-parity jump system is a common generalization of a valuated delta-matroid [6] and an M-convex function on a base polyhedron.

The following theorem gives an optimality criterion of an M-convex function on a constant-parity jump system. More specifically, global optimality is guaranteed by local optimality in the neighborhood of l_1-distance two.

Theorem 2.5 (Murota [12]). *Let $f : J \to \mathbf{R}$ be an M-convex function on a constant-parity jump system J, and let $x \in J$. Then $f(x) \leq f(y)$ for all $y \in J$ if and only if $f(x) \leq f(y)$ for all $y \in J$ with $\|x - y\|_1 \leq 2$.*

Remark 2.6. Let $f : J \to \mathbf{R}$ be an M-convex function on a constant-parity jump system J. Define $J_k = \{x \in J \mid x(V) = k\}$ with $k \in \mathbf{Z}$ and suppose that $J_k \neq \emptyset$. It is also pointed out by Murota [12] that global optimality of f on J_k is guaranteed by local optimality in the neighborhood of l_1-distance four. Note that J_k is not necessarily a jump system. This is a generalization of the result of [2] for the minsquare factor problem.

3 Examples

Let $G = (V, E)$ be an undirected graph that may have loops, but no parallel edges. Let $w : E \to \mathbf{R}$ be an edge weight function, and $c : E \to \mathbf{Z}_+$ be an edge capacity function. Put $n = |V|$ and $m = |E|$. We define $J \subseteq \mathbf{Z}^V$ as the set of vectors $x \in \mathbf{Z}^V$ such that a c-capacitated perfect x-matching exists in G, i.e., such that there exists $\lambda \in \mathbf{Z}^E$ satisfying

$$\sum_{e \in \delta(v)} \lambda(e) = x(v) \ (\forall v \in V), \ 0 \leq \lambda(e) \leq c(e) \ (\forall e \in E). \tag{3.1}$$

Then J is a constant-parity jump system. Moreover, we define a function $f_{\mathrm{M}} : J \to \mathbf{R}$ as the minimum weight of a c-capacitated perfect x-matching, i.e.,

$$f_{\mathrm{M}}(x) = \min \left\{ \sum_{e \in E} \lambda(e) w(e) \;\middle|\; \sum_{e \in \delta(v)} \lambda(e) = x(v) \ (\forall v \in V), \right.$$
$$\left. 0 \leq \lambda(e) \leq c(e) \ (\forall e \in E), \ \lambda(e) \in \mathbf{Z}_+ \ (\forall e \in E) \right\}, \tag{3.2}$$

and $f : J \to \mathbf{R}$ as

$$f(x) = f_{\mathrm{M}}(x) + \sum_{v \in V} \varphi_v(x(v)), \tag{3.3}$$

where $\{\varphi_v \mid \varphi_v : \mathbf{R} \to \mathbf{R} \cup \{+\infty\} \ (v \in V)\}$ is a family of univariate convex functions. The function f in (3.3) is an M-convex function on J, as is observed by Murota [12] in the special case of $c \in \{0, 1\}^E$ (the case of factor problem).

Theorem 3.1. *The function f in (3.3) is M-convex.*

Proof. The proof is a natural generalization of an observation by Murota [12] for the case of b-factor problem. The detail is given in Appendix A, where a generalization of (3.3) given in Remark 3.2 is treated. ■

The problem of minimizing f in (3.3) contains, as a special case, the general matching problem with upper and lower degree bounds discussed, e.g., in p. 191 of [3] and in [8]. To see this, choose φ_v to be the indicator function of the admissible interval of degrees at $v \in V$.

Remark 3.2. The function f in (3.3) remains M-convex even when the edge cost is convex rather than linear. That is, the function

$$f(x) = f_{\mathrm{M}}(x) + \sum_{v \in V} \varphi_v(x(v)) \tag{3.4}$$

is M-convex, where

$$f_{\mathrm{M}}(x) = \min \left\{ \sum_{e \in E} \psi_e(\lambda(e)) \;\middle|\; \sum_{e \in \delta(v)} \lambda(e) = x(v) \ (\forall v \in V), \right.$$
$$\left. 0 \le \lambda(e) \le c(e) \ (\forall e \in E),\ \lambda(e) \in \mathbf{Z}_+ \ (\forall e \in E) \right\}, \tag{3.5}$$

with $\{\psi_e \mid \psi_e : \mathbf{R} \to \mathbf{R} \cup \{+\infty\} \ (e \in E)\}$ being a family of univariate convex functions. The proof of this claim is given in Appendix A.

Remark 3.3. Minimization of f in (3.4) can be done independently of any minimization algorithms for M-convex functions on constant-parity jump systems. In fact, it can be reduced to a minimum weight factor problem by modifying G, c, $\{\varphi_v \mid v \in V\}$, and $\{\psi_e \mid e \in E\}$.

4 Steepest Descent Algorithm

The characterization given in Theorem 2.5 naturally suggests the following algorithm of steepest descent type for minimizing an M-convex function f on a constant-parity jump system J.

Steepest Descent Algorithm

S0 Find a vector $x \in J$.

S1 Find $s, t \in \{\pm\chi_u \mid u \in V\}$ $(s + t \neq 0)$ that minimize $f(x + s + t)$.

S2 If $f(x) \le f(x + s + t)$, then stop (x is a minimizer of f).

S3 Set $x := x + s + t$ and go to S1.

Our objective is to elaborate on this natural idea to obtain a pseudopolynomial complexity bound on the number of iterations in the steepest descent algorithm. The following theorem, a generalization of Theorem 6.28 of [10], is a key fact for this complexity analysis.

Theorem 4.1 (M-minimizer cut). *Let $f : J \to \mathbf{R}$ be an M-convex function on a constant-parity jump system J with* $\operatorname{argmin} f \neq \emptyset$.

(1) For $x \in J$ and $t \in \{\pm\chi_u \mid u \in V\}$, let $s_0 \in \{\pm\chi_u \mid u \in V\}$ be such that

$$f(x + s_0 + t) = \min_{s \in \{\pm\chi_u \mid u \in V\}} f(x + s + t).$$

Put $x' = x + s_0 + t$ and $\{u\} = \mathrm{supp}(s_0)$. Then there exists $x^* \in \mathrm{argmin}\, f$ with

$$x^*(u) \begin{cases} \leq x'(u) & (\textit{if } \mathrm{sign}(s_0) < 0), \\ \geq x'(u) & (\textit{if } \mathrm{sign}(s_0) > 0). \end{cases}$$

(2) For $x \in J \setminus \mathrm{argmin}\, f$, let $s_0, t_0 \in \{\pm\chi_u \mid u \in V\}$ be such that

$$f(x + s_0 + t_0) = \min_{s,t \in \{\pm\chi_u \mid u \in V\}} f(x + s + t).$$

Put $x' = x + s_0 + t_0$, $\{u\} = \mathrm{supp}(s_0)$, and $\{v\} = \mathrm{supp}(t_0)$. Then there exists $x^* \in \mathrm{argmin}\, f$ with

$$x^*(u) \begin{cases} \leq x'(u) & (\textit{if } \mathrm{sign}(s_0) < 0), \\ \geq x'(u) & (\textit{if } \mathrm{sign}(s_0) > 0), \end{cases} \qquad x^*(v) \begin{cases} \leq x'(v) & (\textit{if } \mathrm{sign}(t_0) < 0), \\ \geq x'(v) & (\textit{if } \mathrm{sign}(t_0) > 0). \end{cases}$$

Proof. (1) Suppose $s_0 = -\chi_u$, while the other case can be treated in a similar manner. To prove the assertion by contradiction, we assume that $x^*(u) > x'(u)$ for every $x^* \in \mathrm{argmin}\, f$. Let $x^* \in \mathrm{argmin}\, f$ be a minimizer of f such that $x^*(u)$ is minimum. By applying (M-EXC[J]) to x^*, x', and $s_0 \in \mathrm{Inc}(x^*, x')$, we have

$$f(x') - f(x' - s_0 - t') \geq f(x^* + s_0 + t') - f(x^*)$$

for some $t' \in \mathrm{Inc}(x^* + s_0, x')$. Noting that $x' - s_0 - t' = x - t' + t$ and $x^* + s_0 + t' \notin \mathrm{argmin}\, f$, we have

$$f(x') - f(x - t' + t) > 0,$$

which contradicts the choice of x'.

(2) Due to $x \in J \setminus \mathrm{argmin}\, f$ and Theorem 2.5, we have $s_0 + t_0 \neq 0$. Note that

$$f(x + s_0 + t_0) = \min_{s \in \{\pm\chi_u \mid u \in V\}} f(x + s + t_0). \tag{4.1}$$

If $u = v$, we must have $\mathrm{sign}(s_0) = \mathrm{sign}(t_0)$, and therefore the assertion follows from (1). Suppose $s_0 = -\chi_u$ and $t_0 = -\chi_v$ with $u \neq v$, while the other cases can be treated in a similar manner. By (4.1) and (1), we have $x^*(u) \leq x'(u)$ for some $x^* \in \mathrm{argmin}\, f$. We assume that x^* minimizes $x^*(v)$ among all such vectors. If $x^*(v) > x'(v)$, by applying (M-EXC[J]) to x^*, x', and $t_0 \in \mathrm{Inc}(x^*, x')$, we have

$$f(x') - f(x' - t_0 - t') \geq f(x^* + t_0 + t') - f(x^*)$$

for some $t' \in \mathrm{Inc}(x^* + t_0,\, x')$. Noting that $x' - t_0 - t' = x + s_0 - t'$ and $x^* \in \arg\min f$, we have $f(x^*+t_0+t') - f(x^*) = 0$ because $0 \geq f(x') - f(x'-t_0-t')$ and $f(x^*+t_0+t') - f(x^*) \geq 0$. Hence we have $x^* + t_0 + t' \in \arg\min f$, which contradicts the choice of x^*. ■

The following theorem gives an upper bound on the number of iterations in the steepest descent algorithm.

Theorem 4.2. *If f has a unique minimizer x^*, the number of iterations in the steepest descent algorithm is bounded by $\|x^\circ - x^*\|_1/2$, where x° denotes the initial vector found in step* S0.

Proof. Put $x' = x + s + t$, $\{u\} = \mathrm{supp}(s)$, and $\{v\} = \mathrm{supp}(t)$ in step S2. By Theorem 4.1, we have

$$\begin{cases} x^*(u) \geq x(u) + 2 = x'(u) & (u = v,\ s = t = +\chi_u), \\ x^*(u) \leq x(u) - 2 = x'(u) & (u = v,\ s = t = -\chi_u), \\ x^*(u) \geq x(u) + 1 = x'(u),\ x^*(v) \geq x(v) + 1 = x'(v) & (u \neq v,\ s = +\chi_u,\ t = +\chi_v), \\ x^*(u) \geq x(u) + 1 = x'(u),\ x^*(v) \leq x(v) - 1 = x'(v) & (u \neq v,\ s = +\chi_u,\ t = -\chi_v), \\ x^*(u) \leq x(u) - 1 = x'(u),\ x^*(v) \geq x(v) + 1 = x'(v) & (u \neq v,\ s = -\chi_u,\ t = +\chi_v), \\ x^*(u) \leq x(u) - 1 = x'(u),\ x^*(v) \leq x(v) - 1 = x'(v) & (u \neq v,\ s = -\chi_u,\ t = -\chi_v). \end{cases}$$

Hence we have $\|x' - x^*\|_1 = \|x - x^*\|_1 - 2$. Note that $\|x^\circ - x^*\|_1$ is an even integer because J is a constant-parity jump system. ■

When given an M-convex function f, which may have multiple minimizers, we consider a perturbation of f so that we can use Theorem 4.2. Assume now that the effective domain is bounded and denote its l_1-size by

$$K_1 = \max\{\|x - y\|_1 \mid x, y \in J\}. \tag{4.2}$$

We arbitrarily fix a bijection $\varphi : V \to \{1, 2, \ldots, n\}$ to represent an ordering of the elements of V, put $v_i = \varphi^{-1}(i)$ for $i = 1, 2, \ldots, n$, and define a vector $p \in \mathbf{R}^V$ by $p(v_i) = \varepsilon^i$ for $i = 1, 2, \ldots, n$, where $\varepsilon > 0$. The function $f_\varepsilon = f[p]$ is M-convex and, for a sufficiently small ε, it has a unique minimizer that is also a minimizer of f. More precisely, we have

$$f(x) > f(y) \Rightarrow f_\varepsilon(x) > f_\varepsilon(y) \quad (x, y \in J), \tag{4.3}$$

$$x \neq y \Rightarrow f_\varepsilon(x) \neq f_\varepsilon(y) \quad (x, y \in J). \tag{4.4}$$

The details of the above assertions are explained in Remark 4.4.

Suppose that the steepest descent algorithm is applied to the perturbed function f_ε. Since

$$f_\varepsilon(x + s + t) = f(x + s + t) + \sum_{i=1}^{n} \varepsilon^i x(v_i) + \mathrm{sign}(s)\varepsilon^{\varphi(u)} + \mathrm{sign}(t)\varepsilon^{\varphi(v)},$$

where $\{u\} = \mathrm{supp}(s)$ and $\{v\} = \mathrm{supp}(t)$, this amounts to employing a tie-breaking rule:

$$\text{take } (s,t) \text{ that lexicographically minimizes } \Psi(s,t), \tag{4.5}$$

where

$$\Psi(s,t) = \begin{cases} (\mathrm{sign}(s),\ -\mathrm{sign}(s)\varphi(u),\ \mathrm{sign}(t),\ -\mathrm{sign}(t)\varphi(v)) & (\varphi(u) \le \varphi(v)), \\ (\mathrm{sign}(t),\ -\mathrm{sign}(t)\varphi(v),\ \mathrm{sign}(s),\ -\mathrm{sign}(s)\varphi(u)) & (\varphi(u) > \varphi(v)), \end{cases} \tag{4.6}$$

in the case of multiple candidates in step S1 of the steepest descent algorithm applied to f.

With the tie-breaking rule (4.5) we have the following complexity bound.

Theorem 4.3. *For an M-convex function f on a constant-parity jump system J with finite K_1 in* (4.2)*, the number of iterations in the steepest descent algorithm with tie-breaking rule* (4.5) *is bounded by $K_1/2$. Hence, if a vector in J is given, the algorithm finds a minimizer of f with* $\mathrm{O}(n^2 K_1)$ *evaluations of f.*

Remark 4.4. We show (4.3) and (4.4) by taking a sufficiently small $\varepsilon > 0$. Let $0 < \varepsilon < 1/3$ be a real number such that

$$\sum_{i=1}^{n} (x(v_i) - y(v_i))\varepsilon^i \neq 0 \quad (\forall x, y \in J \text{ with } x \neq y), \tag{4.7}$$

$$\varepsilon < \frac{1}{2K_\infty} \min\{|f(x) - f(y)| \mid x, y \in J,\ f(x) - f(y) \neq 0\}, \tag{4.8}$$

where

$$K_\infty = \max\{\|x - y\|_\infty \mid x, y \in J\}. \tag{4.9}$$

For the proof of (4.3), assume that $f(x) > f(y)$ $(x, y \in J)$. Then we have

$$\sum_{i=1}^{n} \varepsilon^i (y(v_i) - x(v_i)) \le \sum_{i=1}^{n} \varepsilon^i K_\infty < \frac{\varepsilon}{1-\varepsilon} K_\infty < 2\varepsilon K_\infty < f(x) - f(y),$$

where the third inequality is due to $\varepsilon < 1/3$ and the last is due to (4.8). Hence

$$f_\varepsilon(y) < f_\varepsilon(x).$$

Next, for the proof of (4.4), we may assume $f(x) = f(y)$ for $x, y \in J$ with $x \neq y$. Then we have

$$f_\varepsilon(x) - f_\varepsilon(y) = \sum_{i=1}^{n} \varepsilon^i (x(v_i) - y(v_i)) \neq 0$$

due to (4.7).

Remark 4.5. It is known that no strongly polynomial time algorithm exists for minimizing M-convex functions on constant-parity jump systems. This follows from a result of Hochbaum [9], showing a weakly polynomial time lower bound for minimization of a separable convex function on a simplex. This means that even a special type of M-convex functions on constant-parity jump systems cannot be minimized in strongly polynomial time.

5 Concluding Remarks

The proposed algorithm involves iterations the number of which is polynomial in K_1. For a polynomial time algorithm, the number of iterations must be bounded by a polynomial in $\log K_1$. A scaling algorithm, which is available for M-convex functions on base polyhedra [14][15], is a promising candidate for a polynomial time algorithm. This is left for a future work.

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A Proof of M-convexity of f in (3.4)

It suffices to show that f_{M} in (3.5) is M-convex. Noting that ψ_e is convex for each $e \in E$, we can reduce the problem defining $f_{\mathrm{M}}(x)$ to the minimum weight $\hat{x}$-factor problem for some $\hat{x}$. Let $\mathrm{dom}\,\psi_e = [l(e), u(e)] \subseteq \mathbf{R}$ with $l(e), u(e) \in \mathbf{Z} \cup \{\pm\infty\}$ for each $e \in E$. Without loss of generality, we can assume $[0, c(e)] \cap [l(e), u(e)] \neq \emptyset$ for each $e \in E$ (otherwise $f_{\mathrm{M}}(x) = +\infty$ for any x). Let $\hat{l}(e) = \max\{0, l(e)\}$ and $\hat{u}(e) = \min\{c(e), u(e)\}$. Note that $\hat{l}(e)$ and $\hat{u}(e)$ are finite since $c(e) < +\infty$. Replacing each edge $e \in E$ with $\hat{c}(e) = \hat{u}(e) - \hat{l}(e)$ multiple edges $\hat{e}_1, \ldots, \hat{e}_{\hat{c}(e)}$, setting new linear weights $\hat{w}$ on the new edges by

$$\hat{w}(\hat{e}_i) = \psi_e(\hat{l}(e) + i) - \psi_e(\hat{l}(e) + i - 1) \quad (i = 1, \ldots, \hat{c}(e)),$$

and replacing x with $\hat{x} = x - \hat{l}_{\mathrm{d}}$, where $\hat{l}_{\mathrm{d}}(v) = \sum_{e \in \delta_G(v)} \hat{l}(e)$ for each $v \in V$, we have

$$f_{\mathrm{M}}(x) = \hat{f}_{\mathrm{M}}(x - \hat{l}_{\mathrm{d}}) + \sum_{e \in E} \psi_e(\hat{l}(e))$$

with

$$\hat{f}_{\rm M}(\hat{x}) = \min\left\{ \hat{w}(\hat{F}) \;\middle|\; \hat{H} = (V,\hat{F}) \text{ is a subgraph of } \hat{G} \text{ and } \deg_{\hat{H}} = \hat{x} \right\},$$

where $\hat{G} = (V, \hat{E})$ denotes the new graph made as above. Note that we have $f_{\rm M}(x) = +\infty$ unless $\hat{x} \geq 0$, and that $\hat{f}_{\rm M}(\hat{x})$ is defined in terms of a minimum weight $\hat{x}$-factor problem on $\hat{G}$.

The proof of M-convexity of $\hat{f}_{\rm M}$ [12][13] is as follows. Let $\hat{F}_{\hat{x}}$ (resp. $\hat{F}_{\hat{y}}$) be an optimal $\hat{x}$-factor (resp. $\hat{y}$-factor). Without loss of generality, we can choose $v_0 \in V$ such that $\hat{x}(v_0) < \hat{y}(v_0)$, and therefore $s = \chi_{v_0} \in {\rm Inc}(\hat{x}, \hat{y})$. Here we claim that there exists an alternating path $P = (v_0, \hat{e}_1, v_1, \ldots, \hat{e}_k, v_k)$ on $(V, \hat{F}_{\hat{x}} \Delta \hat{F}_{\hat{y}})$ with $\hat{e}_1 \in \hat{F}_{\hat{y}} \setminus \hat{F}_{\hat{x}}$ such that

$$t = \begin{cases} +\chi_{v_k} & (\text{if } k \text{ is odd}), \\ -\chi_{v_k} & (\text{if } k \text{ is even}) \end{cases}$$

satisfies $t \in {\rm Inc}(\hat{x} + \chi_{v_0}, \hat{y})$. Starting with an edge in $\hat{F}_{\hat{y}} \setminus \hat{F}_{\hat{x}}$ incident to v_0 we construct an alternating path P by adding an edge in $\hat{F}_{\hat{x}} \setminus \hat{F}_{\hat{y}}$ and an edge in $\hat{F}_{\hat{y}} \setminus \hat{F}_{\hat{x}}$ alternately. The path P is not necessarily simple so that it may contain the same vertex more than once, whereas it consists of distinct edges. We assume that P is maximal in the sense that it cannot be extended further beyond the end vertex, say, v_k. Then P has the desired property mentioned above. Noting that $\hat{x} + s + t = \deg_{\hat{H}'_{\hat{x}}}$ and $\hat{y} - s - t = \deg_{\hat{H}'_{\hat{y}}}$, where $\hat{H}'_{\hat{x}} = (V, \hat{F}_{\hat{x}} \Delta P)$ and $\hat{H}'_{\hat{y}} = (V, \hat{F}_{\hat{y}} \Delta P)$, we have

$$\begin{aligned} & \hat{f}_{\rm M}(\hat{x}) + \hat{f}_{\rm M}(\hat{y}) - \hat{f}_{\rm M}(\hat{x} + s + t) - \hat{f}_{\rm M}(\hat{y} - s - t) \\ & \geq \hat{w}(\hat{F}_{\hat{x}}) + \hat{w}(\hat{F}_{\hat{y}}) - \hat{w}(\hat{F}_{\hat{x}} \Delta P) - \hat{w}(\hat{F}_{\hat{y}} \Delta P) = 0. \end{aligned}$$

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