MATHEMATICAL ENGINEERING TECHNICAL REPORTS

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(Communicated by Kazuo Murota)

METR 2005–18   July 2005

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Robust PID Control via Generalized KYP Synthesis

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July 20, 2005

Abstract

This paper presents a systematic method for designing PID controllers through optimization of the PID parameters to directly shape the Nyquist plot of the open-loop transfer function without introducing frequency weightings. The method is based on a recently developed theoretical tool, the generalized Kalman-Yakubovich-Popov (GKYP) lemma, that provides a unified characterization of frequency domain inequalities (FDIs) in (semi)finite frequency ranges in terms of linear matrix inequalities (LMIs). A comprehensive summary of the GKYP results is given first without proofs, including a new FDI/LMI conversion formula in the δ-domain for digital controller synthesis as well as standard ones in the continuous-time and discrete-time settings. The GKYP synthesis method is then extended to include, in the control specifications, robustness with respect to parameter perturbations to encompass practical situations. The proposed design methods are illustrated and their effectiveness is demonstrated through several numerical examples of PID control designs.

1 INTRODUCTION

The PID control is a representative of the classical control schemes and has been adopted in many engineering applications due to its essential functionality and structural simplicity that allow for relatively easy manual tuning to achieve tracking and regulation. Systematic methods for designing PID controllers have been extensively studied in the literature, including self-tuning of the PID parameters [1]–[3], loop-shaping and H_∞ control [4]–[7], digital control [8], [9], and robust design [10], [11]. Recent advances and state of the art are discussed in a recent special issue in Control Engineering Practice [12], indicating everlasting importance of and interests in the PID control.

In the general paradigm of loop-shaping design for a unity feedback control system with plant $P(s)$ and controller $K(s)$, the open-loop transfer function $L(s) := K(s)P(s)$ is "shaped" to meet the requirements given in terms of frequency domain inequalities (FDIs) in various frequency ranges. For example, suppose $P(s)$ is marginally stable, and a set of design specifications on the controller $K(s)$ is provided in terms of the Nyquist plot $L(j\omega)$ as illustrated in Fig. 1. The three colored regions in the figure indicate where the Nyquist plot should lie to meet fundamental design requirements in three different frequency ranges. The blue half plane constraint relates to the high-gain requirement in the low frequency range for sensitivity reduction, while the small green disk centered at the origin corresponds to the roll-off or small-gain requirement in the high frequency range for robust stability. In addition, a certain stability margin specification should be satisfied in the middle frequency range, which can be set by the yellow region below the straight line lying to the right of the critical point $-1+j0$.

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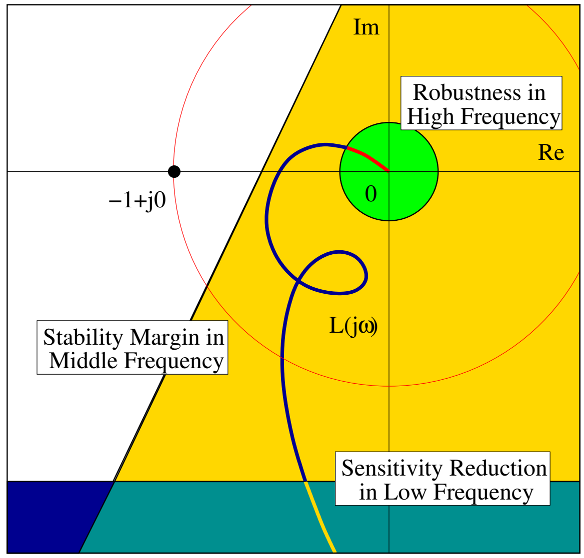

Figure 1: Open-loop shaping specifications. There are three fundamental design specifications on the Nyquist plot $L(j\omega)$, namely, high-gain requirement in the low frequency range (blue) for sensitivity reduction, stability margin requirement in the middle frequency range (yellow), and roll-off or small-gain requirement in the high frequency range (green) for robust stability.

This approach called "open-loop shaping" often leads to a convex optimization problem when applied to the PID control design as the open-loop transfer function is linearly dependent upon the PID gains [6]. The resulting problem, however, is infinite dimensional and may be difficult to solve exactly. In particular, the PID gains are constrained by infinitely many inequalities, parametrized by the frequency variable in certain ranges. For numerical tractability, these FDIs are typically approximated either by a finite number of FDIs on selected frequency points or by an H_∞ norm condition with frequency dependent weights.

The objective of this paper is to present a new approach to the open-loop PID shaping by directly dealing with the FDIs in (semi)finite frequency ranges. Our approach allows for an exact treatment of multiple FDI specifications, completely avoiding the approximations associated with the frequency gridding or the frequency weights. The resulting synthesis conditions are given as a finite dimensional convex optimization problem described by linear matrix inequalities (LMIs) for which commercial software packages are available. The key technical result underlying the proposed design method is the generalized Kalman-Yakubovich-Popov (GKYP) lemma [13]–[16] that equivalently transforms an FDI in a (semi)finite frequency range into a set of LMIs. This is an extension of the standard KYP lemma [17], [18] in which the entire frequency range is considered.

We will first provide a general and comprehensive summary of the GKYP results [15] without proofs. This summary includes some new results that have not been reported. First, in addition to the FDI/LMI conversion formulas in the continuous-time (imaginary axis) and discrete-time (unit circle) settings, a formula for the δ-domain (circle with center at $-1/T + j0$ and radius $1/T$ where T is the sampling period) is also given. Designing digital controllers in the δ-domain has several advantages such as consistency with the continuous-time setting and numerical conditioning [8], [19] when compared with the usual z-domain approach. Second, the GKYP lemma is extended to treat robust FDI condition subject to parametric uncertainties. The idea of Lyapunov functions that depend on the parameters in a linear fractional manner [20] is employed to obtain a sufficient LMI condition for a family of FDIs to hold. Several numerical examples will be given to illustrate the proposed design procedure and to demonstrate the effectiveness of the δ-domain and robust loop-shaping designs of PID controllers.

We use the following notation. For a Hermitian matrix, $M > 0$ and $M < 0$ denote positive definiteness and negative definiteness, respectively. The real and imaginary parts of matrix M are denoted by $\Re(M)$ and $\Im(M)$. The symbol $\mathbf{H}_n$ stands for the set of $n \times n$ Hermitian matrices. For matrices Φ and P, $\Phi \otimes P$ means their Kronecker product. The function $\sigma: \mathbb{C} \times \mathbf{H}_2 \to \mathbb{R}$ is defined by

$$\sigma(\lambda, \Omega) := \begin{bmatrix} \lambda \\ 1 \end{bmatrix}^* \Omega \begin{bmatrix} \lambda \\ 1 \end{bmatrix},$$

where $\lambda \in \mathbb{C}$ and $\Omega \in \mathbf{H}_2$.

2 Generalized KYP Lemma

2.1 Unified representations

In this section, we shall present the GKYP lemma and its dual version. The results are in slightly different forms than, but follow as special cases from, the ones reported earlier [15]. To state the results, let $G(\lambda)$ be a $p \times m$ matrix-valued rational function, Π be a Hermitian matrix, and define a subset of complex numbers

$$\mathbf{\Lambda}(\Phi, \Psi) := \{ \lambda \in \mathbb{C} \mid \sigma(\lambda, \Phi) = 0,\ \sigma(\lambda, \Psi) \geq 0 \}, \tag{1}$$

where $\Phi, \Psi \in \mathbf{H}_2$. Let $\bar{\mathbf{\Lambda}} := \mathbf{\Lambda}$ if $\mathbf{\Lambda}$ is bounded and $\bar{\mathbf{\Lambda}} := \mathbf{\Lambda} \cup \{\infty\}$ if unbounded. The set $\mathbf{\Lambda}(\Phi, \Psi)$ is seen as curve(s) on the complex plane that represents a finite frequency interval. We will later discuss what type of curve(s) can be represented by what choices of Φ and Ψ.

For now, let us present a unified form of the GKYP lemma [15], which establishes the equivalence between an FDI and LMIs.

Theorem 1 *Let $\Pi \in \mathbf{H}_p$, $\Phi, \Psi \in \mathbf{H}_2$, and a rational function*

$$G(\lambda) := C(\lambda E - A)^{-1}(B - \lambda O) + D \tag{2}$$

be given, where $A, E \in \mathbb{C}^{n \times n}$, $B, O \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{p \times n}$, and $D \in \mathbb{C}^{p \times m}$. Suppose (a) $\det(\lambda E - A) \neq 0$ for all $\lambda \in \mathbf{\Lambda}(\Phi, \Psi)$, and (b) either E is nonsingular or $\mathbf{\Lambda}$ is bounded. Then, the parametrized inequality condition

$$G(\lambda)^* \Pi G(\lambda) < 0, \quad \forall\, \lambda \in \bar{\mathbf{\Lambda}}(\Phi, \Psi) \tag{3}$$

holds if and only if there exist matrices $P, Q \in \mathbf{H}_n$ satisfying

$$Q > 0, \quad F^* Z F < 0, \quad Z := \operatorname{diag}(\Phi \otimes P + \Psi \otimes Q, \Pi),$$

where F is defined by

$$F := \begin{bmatrix} A & B \\ E & O \\ C & D \end{bmatrix}.$$

The class of $G(\lambda)$ in Theorem 1 contains possibly nonproper rational functions (e.g. polynomials) which can be realized as in (2) with singular matrix E. A version of the GKYP lemma that explicitly treats polynomial matrices are given in [15]. The following is the dual version of the generalized KYP lemma, which is more suitable than Theorem 1 for the controller synthesis and will be used later in this paper.

Theorem 2 *If G is a rational function given by*

$$G(\lambda) := (C - \lambda O)(\lambda E - A)^{-1}B + D, \tag{4}$$

where $A, E \in \mathbb{C}^{n\times n}$, $C, O \in \mathbb{C}^{p\times n}$, $B \in \mathbb{C}^{n\times m}$, and $D \in \mathbb{C}^{p\times m}$, then the parametrized inequality condition

$$G(\lambda)\Pi G(\lambda)^* < 0, \quad \forall\, \lambda \in \bar{\mathbf{\Lambda}}(\Phi, \Psi) \tag{5}$$

holds for given $\Pi \in \mathbf{H}_m$ and $\Phi, \Psi \in \mathbf{H}_2$, if and only if there exist $P, Q \in \mathbf{H}_n$ satisfying

$$Q > 0, \quad FZF^* < 0, \quad Z := \mathrm{diag}(\Phi^{\mathsf{T}} \otimes P + \Psi^{\mathsf{T}} \otimes Q, \Pi),$$

where F is defined by

$$F := \begin{bmatrix} A & E & B \\ C & O & D \end{bmatrix},$$

provided (a) $\det(\lambda E - A) \neq 0$ for all $\lambda \in \mathbf{\Lambda}(\Phi, \Psi)$, and (b) either E is nonsingular or $\mathbf{\Lambda}$ is bounded.

2.2 Various curves on the complex plane

The set $\mathbf{\Lambda}(\Phi, \Psi)$ is visualized as a curve (or curves) on the complex plane, depending on the choice of (Φ, Ψ). We shall discuss three classes that are important in the context of dynamical systems analysis. Specifically, $\mathbf{\Lambda}(\Phi, \Psi)$ is (part of) the imaginary axis, unit circle, or a circle of radius r with center at $-r + j0$. The first class defines the frequency range (s-transform) in the continuous-time setting, while the second and third classes define those (z- and δ-transforms) in the discrete-time setting. The matrix pair (Φ, Ψ) can also be chosen so that the set $\mathbf{\Lambda}(\Phi, \Psi)$ is a segment of the real axis. This version of the GKYP lemma will be useful for robust control with a single real parametric uncertainty [21], but will not be pursued in this paper. In what follows, it is assumed that the matrices Φ and Ψ are chosen so that the set $\mathbf{\Lambda}(\Phi, \Psi)$ is neither empty, a single point, nor the entire complex plane.

- *Frequency ranges in the continuous-time setting*

  The set $\mathbf{\Lambda}(\Phi, \Psi)$ becomes a subset of the imaginary axis for a specific choice of Φ as follows:

$$\mathbf{\Lambda}(\Phi_s, \Psi) = \{ j\omega :\ \omega \in \mathbf{\Omega}\ \}, \quad \Phi_s := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $\mathbf{\Omega} \subseteq \mathbb{R}$ is the frequency range specified by Ψ. Table 1 summarizes the choices of Ψ that lead to certain frequency ranges where $\varpi, \varpi_1, \varpi_2 \in \mathbb{R}$ are given numbers, and $\varpi_o := (\varpi_1 + \varpi_2)/2$.

Table 1: Frequency ranges in the continuous-time setting

$\mathbf{\Omega}$	$\pm(\lvert\varpi\rvert - \lvert\omega\rvert) \geq 0$	$\pm(\varpi - \omega) \geq 0$	$\pm(\omega - \varpi_1)(\omega - \varpi_2) \geq 0$
Ψ	$\pm \begin{bmatrix} -1 & 0 \\ 0 & \varpi^2 \end{bmatrix}$	$\pm \begin{bmatrix} 0 & -j \\ j & 2\varpi \end{bmatrix}$	$\pm \begin{bmatrix} 1 & -j\varpi_o \\ j\varpi_o & \varpi_1\varpi_2 \end{bmatrix}$

- *Frequency ranges in the discrete-time setting (z-transform)*

  Part of the unit circle can be represented by $\mathbf{\Lambda}(\Phi, \Psi)$ for a particular choice of Φ as follows:

$$\mathbf{\Lambda}(\Phi_z, \Psi) = \{\ e^{j\theta} :\ \theta \in \mathbf{\Theta}\ \}, \quad \Phi_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

Table 2: Frequency ranges in the discrete-time setting (z-transform)

$\mathbf{\Theta}$	$\pm(\lvert\vartheta\rvert - \lvert\theta\rvert) \geq 0$	$\pm(\vartheta - \theta) \geq 0$	$\pm(\theta - \vartheta_1)(\theta - \vartheta_2) \geq 0$
Ψ	$\pm \begin{bmatrix} 0 & 1 \\ 1 & -2\cos\vartheta \end{bmatrix}$	$\pm \begin{bmatrix} 0 & -je^{j\vartheta/2} \\ je^{-j\vartheta/2} & 2\sin(\vartheta/2) \end{bmatrix}$	$\pm \begin{bmatrix} 0 & -e^{j\vartheta_o} \\ -e^{-j\vartheta_o} & 2\cos\vartheta_d \end{bmatrix}$

where $\mathbf{\Theta} \subseteq \mathrm{I\!R}$ is specified by Ψ. Table 2 summarizes the choices of Ψ that lead to certain frequency ranges $\mathbf{\Theta}$ where $\vartheta, \vartheta_1, \vartheta_2 \in \mathrm{I\!R}$ are given numbers, $\vartheta_o := (\vartheta_1 + \vartheta_2)/2$, and $\vartheta_d := (\vartheta_2 - \vartheta_1)/2$. The set $\mathbf{\Theta}$ is characterized by each entry in the first row of the table, together with the additional condition $|\theta| \leq \pi$.

- *Frequency ranges in the discrete-time setting (δ-transform)*

The δ-transform is a modified version of the z-transform to describe discrete-time systems in the frequency domain, which is useful for digital control synthesis [19]. The main difference is that $\delta := (e^{j\omega T} - 1)/T$ does coincide with the Laplace variable $s := j\omega$ in the limit where the sampling time T approaches zero, while $z := e^{j\omega T}$ does not. This convergence property in the δ-transform helps us to link the continuous-time frequency domain design specifications to the discrete-time counter parts while preserving the physical meanings, and it may have an advantage in computations for small sampling period.

Table 3: Frequency ranges in the discrete-time setting (δ-transform)

$\mathbf{W}$	$\pm(\lvert\varpi\rvert - \lvert\omega\rvert) \geq 0$	$\pm(\omega - \varpi_1)(\omega - \varpi_2) \geq 0$
Ψ	$\frac{\pm 1}{T}\left\{\frac{1}{\varpi T^2}\begin{bmatrix} 0 & Ts \\ Ts & 2s(1-c) \end{bmatrix} - \Phi_\delta\right\}$	$\frac{\pm 1}{T}\left\{\frac{j}{\varpi_2 - \varpi_1}\begin{bmatrix} 0 & \delta_2 - \delta_1 \\ \delta_1^* - \delta_2^* & \delta_2^*\delta_1 - \delta_2\delta_1^* \end{bmatrix} - \Phi_\delta\right\}$

$\mathbf{W}$	$\pm(\varpi - \omega) \geq 0$
Ψ	$\frac{\pm 1}{T}\left\{\frac{1}{(\varpi T + \pi)T}\begin{bmatrix} 0 & (s - j(c+1))T \\ (s + j(c+1))T & 4s \end{bmatrix} - \Phi_\delta\right\}$

The set of the δ variables in the frequency range $\mathbf{W}$ is given by

$$\mathbf{\Lambda}(\Phi_\delta, \Psi) = \left\{ \frac{e^{j\omega T} - 1}{T} : \omega \in \mathbf{W} \right\}, \quad \Phi_\delta := \begin{bmatrix} T & 1 \\ 1 & 0 \end{bmatrix},$$

where $\mathbf{W} \subseteq \mathrm{I\!R}$ is specified by Ψ. Note that $\mathbf{\Lambda}(\Phi_\delta, 0)$ is the circle of radius $1/T$ with center at $-1/T + j0$, and $\mathbf{\Lambda}(\Phi_\delta, \Psi)$ is its arc. Table 3 summarizes the choices of Ψ leading to various frequency ranges $\mathbf{W}$ where $\varpi, \varpi_1, \varpi_2 \in \mathrm{I\!R}$ are given numbers such that $|\varpi T| \leq \pi$ and $-\pi \leq \varpi_1 T < \varpi_2 T \leq \pi$, and

$$s := \sin(\varpi T), \quad c := \cos(\varpi T), \quad \delta_i := \frac{e^{j\varpi_i T} - 1}{T} \quad (i = 1, 2).$$

Some details of the derivation can be found in the Appendix A. The set $\mathbf{W}$ is characterized by each entry in the first row of the table, together with the additional condition $|\omega T| \leq \pi$. Note that the case for $\pm(\omega - \varpi_1)(\omega - \varpi_2) \geq 0$ is the general result that basically covers the other two cases. It can be verified that these choices of Φ and Ψ recover those for the continuous-time setting in the limit when T approaches zero. Note that there are many other choices of Φ and Ψ which represent the set $\mathbf{W}$ but do not satisfy the convergence property.

3 Controller Synthesis via GKYP Lemma

3.1 LMI formulation for synthesis

This section is focused on how to apply the GKYP lemma to control system design. The design problem is to find a set of design parameters in the controller so that prescribed design specifications expressed in terms of multiple FDIs

$$G_k(\lambda)\Pi_k G_k(\lambda)^* < 0, \quad \forall\, \lambda \in \bar{\mathbf{\Lambda}}(\Phi_k, \Psi_k) \tag{6}$$

hold for all $k = 1, \ldots, \ell$, where all G_k relate to a single controller $K(\lambda)$ to be designed. We show how the problem can be reduced exactly (i.e., without conservatism) to LMIs under certain assumptions. For brevity of exposition, let us consider the case $\ell = 1$, i.e. we have just one FDI specification and drop the subscript "k" in equation (6). The case with multiple specifications can be handled by simultaneously solving the set of LMIs that result from all FDIs.

When designing a controller, the FDI specification is often given by (6) with the following form of $G(\lambda)$:

$$G(\lambda) = \begin{bmatrix} L(\lambda) & I_p \end{bmatrix}\ ;\ L(\lambda) := C_L(\lambda I - A_L)^{-1}B_L + D_L, \tag{7}$$

where $L(\lambda)$ is a $p \times q$ transfer function that depends on the controller parameters. Let us partition the weighting matrix $\Pi \in \mathbf{H}_{q+p}$ accordingly:

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}, \quad \Pi_{11} \in \mathbf{H}_q. \tag{8}$$

In the context of open-loop shaping design, $L(\lambda)$ corresponds to the open-loop transfer function. Various choices of Π define desired properties of the frequency response $L(\lambda)$, including positive real and small gain specifications. For SISO systems, this framework captures a general specification where a segment of the Nyquist plot in a certain frequency range is required to lie in an arbitrarily specified conic section (half plane, ellipse, parabola, etc.) on the complex plane. See [15] for details.

For tractability of the problem, let us impose the following:

Assumption 1

(a) *$L(\lambda)$ depends affinely on the design parameter vector ρ.*

(b) $\Pi_{11} \geq 0$.

When Assumption 1(a) holds true, a state space realization of $L(\lambda)$ can be chosen such that $B_L(\rho)$ and $D_L(\rho)$ are affine functions of ρ while A_L and C_L are independent of ρ. The condition in Assumption 1(b) ensures that the set of L satisfying the specification (5) is convex. Note in particular that, when $q = p = 1$, the design specification (5) defines a convex region on the complex plane in which $L(\lambda)$ is desired to lie. Under these assumptions, the feasible domain of the design parameter ρ, specified by (5) and (7), is clearly convex. Furthermore, Theorem 2 can be used to reduce the problem to LMIs, as summarized below.

Proposition 1 *Suppose Assumption 1 hold. Then the condition (5) holds if and only if there exist Hermitian matrices P and Q such that*

$$Q > 0, \quad \begin{bmatrix} W(P,Q,\rho) & T(\rho) \\ T(\rho)^* & -R \end{bmatrix} < 0 \tag{9}$$

hold, where R and S are full-rank factors of Π_{11} such that $\Pi_{11} = SR^{-1}S^*$ and $R > 0$, and

$$W(P,Q,\rho) := \begin{bmatrix} A_L & I \\ C_L & 0 \end{bmatrix} (\Phi^{\mathsf{T}} \otimes P + \Psi^{\mathsf{T}} \otimes Q) \begin{bmatrix} A_L & I \\ C_L & 0 \end{bmatrix}^* + V(\rho),$$

$$V(\rho) := \begin{bmatrix} 0 & B_L(\rho)\Pi_{12} \\ \Pi_{12}^* B_L(\rho)^* & D_L(\rho)\Pi_{12} + \Pi_{12}^* D_L(\rho)^* + \Pi_{22} \end{bmatrix}, \quad T(\rho) := \begin{bmatrix} B_L(\rho) \\ D_L(\rho) \end{bmatrix} S.$$

Clearly, condition (9) defines LMIs in terms of the variables P, Q, and ρ, and hence can be solved through numerical computations. It was shown in [15] that the following design problems can be treated within this framework: (i) open-loop shaping by a controller with fixed poles such as the PID controller, (ii) closed-loop control design with the Q-parametrization of all stabilizing or H_∞ controllers, and (iii) structure/control design integration.

We now explain how our method applies to the open-loop shaping problem. Suppose that state space realizations of the plant $P(\lambda)$ and the controller $K(\lambda)$ are given by

$$P(\lambda): \begin{bmatrix} \lambda x_p \\ y \end{bmatrix} = \begin{bmatrix} A_p & B_p \\ C_p & D_p \end{bmatrix} \begin{bmatrix} x_p \\ u \end{bmatrix}, \quad K(\lambda): \begin{bmatrix} \lambda x_k \\ u \end{bmatrix} = \begin{bmatrix} A_k & B_k(\rho) \\ C_k & D_k(\rho) \end{bmatrix} \begin{bmatrix} x_k \\ e \end{bmatrix}. \tag{10}$$

We assume that the controller transfer function $K(\lambda)$ is affinely dependent upon the design parameter ρ, and its state space realization has been chosen such that $B_k(\rho)$ and $D_k(\rho)$ are affine functions of ρ. This means that the poles of $K(\lambda)$ are fixed and the zeros are designed through the choice of ρ. A state space realization of the loop transfer function $L(\lambda) := P(\lambda)K(\lambda)$ is given by

$$L(\lambda): \begin{bmatrix} \lambda x_k \\ \lambda x_p \\ y \end{bmatrix} = \begin{bmatrix} A_k & 0 & B_k(\rho) \\ B_p C_k & A_p & B_p D_k(\rho) \\ D_p C_k & C_p & D_p D_k(\rho) \end{bmatrix} \begin{bmatrix} x_k \\ x_p \\ e \end{bmatrix}. \tag{11}$$

Let us consider L in (7) and define its state space matrices (A_L, B_L, C_L, D_L) by the above equation. Clearly, B_L and D_L, and hence $L(\lambda)$, depend on ρ in an affine manner, satisfying Assumption 1(a). Therefore, the synthesis problem is convex and can be reduced to LMIs whenever $\Pi_{11} \geq 0$.

3.2 PID controller synthesis

The transfer function of the PID controller is described by

$$K(s) = k_p + \frac{k_i}{s} + \frac{k_d s}{1 + T_d s}, \tag{12}$$

where $T_d > 0$ is a small parameter introduced to approximate the differentiator by a proper transfer function. If T_d is fixed, all the design parameters (k_p,k_i,k_d) appear affinely in the numerator of the open-loop transfer function $L(s)$, and hence its state space realization can be chosen such that $B_L(\rho)$ and $D_L(\rho)$ are affine functions of ρ by using the observable canonical form of $K(s)$ for instance.

Let us now demonstrate the effectiveness of the proposed method through numerical examples of PID controller design for a three-disk torsional system. The plant transfer function is given by

$$P(s) = \frac{440.5(s^2 + 0.748s + 540)(s^2 + 0.7493s + 3668)}{s(s + 2.002)(s^2 + 2.366s + 1277)(s^2 + 1.188s + 4099)}. \tag{13}$$

The system has two lightly damped flexible modes at $\omega = 36$ and 64 rad/s, and two anti-resonant modes at $\omega = 23$ and 61 rad/s.

- *Continuous-time PID controller synthesis*

DESIGN 1: Our objective here is to design a continuous-time PID controller (12) with $T_d = 0.001$ by minimizing γ_1 subject to the following multiple design specifications on the loop transfer function $L(s) := P(s)K(s)$

$$\begin{aligned} \Re[L(j\omega)] + \Im[L(j\omega)] \leq \gamma_1 &\quad : \quad 1 \leq \omega \leq 5. \\ -2\Re[L(j\omega)] + \Im[L(j\omega)] \leq 1 &\quad : \quad \omega \geq 5, \\ |L(j\omega)| \leq 0.2 &\quad : \quad |\omega| \geq 150, \end{aligned} \tag{14}$$

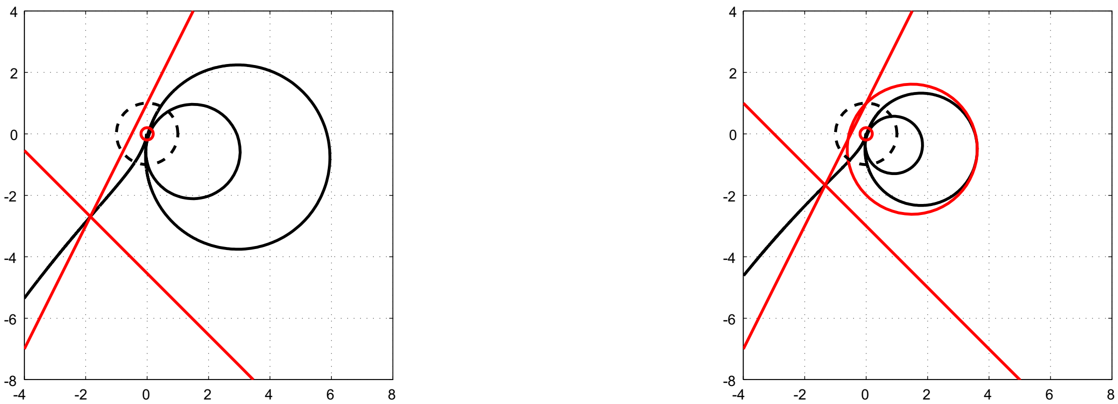

Figure 2: Nyquist plots of the loop transfer functions achieved by continuous-time synthesis. DESIGN 1 (left) and DESIGN 2 (right). The GKYP synthesis allows for direct shaping of the Nyquist plots through design specifications in various frequency ranges indicated by the red lines

The minimization of γ_1 implies maximizing the sensitivity reduction in the low frequency range, and the second and third constraints relate to the stability margin in the middle frequency range and the robustness requirement in the high frequency range, respectively. It turns out that the optimal controller has the PD structure (i.e., $k_i = 0$) due to the fact that the plant itself has an integrator. To ensure sufficient disturbance attenuation, we choose to add another constraint $k_i \geq 0.1$ in the design. As a result, we obtained the PID controller

$$K(s) = 0.4489 + \frac{0.1}{s} + \frac{0.0645s}{1 + 0.001s},$$

with the optimal value $\gamma_1 = -4.540$.

DESIGN 2: Although the above controller meets all the specifications, the Nyquist plot traces two large circles in the frequency range containing the two flexible modes of the plant as seen in Fig. 2 (left). In order to reduce the oscillation due to the flexible modes, we consider an extra specification that makes the circles smaller. In particular, we add the following constraint:

$$|L(j\omega) - c| \leq \gamma_2 \quad : \quad 30 \leq \omega \leq 150,$$

where $c = 1.5 - 0.5j$ and minimize γ_2 while fixing $\gamma_1 = -3$. Note that γ_2 is the radius of disk on which the Nyquist plot is required to lie in the specified frequency range. The optimal PID controller with these specifications has been found to be

$$K(s) = 0.3072 + \frac{0.1}{s} + \frac{0.0391s}{1 + 0.001s}.$$

We see from the Nyquist plot shown in Fig. 2 (right) that the circles are reasonably shrunk as expected. This example illustrates the effectiveness of multiple finite frequency constraints to describe the design specification.

- *Digital PID controller synthesis*

DESIGN 3: We design digital PID controllers through discretization of a continuous-time PID controller. Let us first design a continuous-time PID controller with the same specifications as (14) except that the third constraint is relaxed to be

$$|L(j\omega)| \leq 0.25 \quad : \quad |\omega| \geq 150.$$

The optimal PID controller is found to be

$$K_c(s) = 0.4207 + \frac{0.2771}{s} + \frac{0.0760s}{1 + 0.001s}.$$

Three digital PID controllers $K_d(z)$ are then obtained by discretizing $K_c(s)$ using the Tustin transform, i.e., $K_d(z) := K_c\left(\frac{2}{T}\cdot\frac{z-1}{z+1}\right)$, with the sampling period $T = 0.01$, 0.02, and 0.03. Figure 3 (left) shows the step responses of the digital feedback systems where $P(s)$ is controlled by each of the controllers $K_d(z)$ with ideal sampler and zero-order hold. The response for the small sampling period case ($T = 0.01$) is quite similar to that for the continuous-time case. However, the step response becomes worse as T becomes larger. Specifically, the response for $T = 0.03$ is very oscillatory and the design is considered unsatisfactory.

DESIGN 4: Next we design digital PID controllers by directly shaping the Nyquist plot of the δ-domain loop transfer function using the δ-domain GKYP lemma presented in Section 2. The loop transfer function is given by $L_d(\delta) := P_d(\delta)K_d(\delta)$, where $P_d(\delta)$ denotes the zero-order hold equivalent discretized plant in the δ domain and $K_d(\delta)$ is the the PID controller to be designed. Note that $K_d(z)$ and $K_d(\delta)$ are different functions but we use the same symbol K_d with a slight abuse of notation. The PID controller in the δ-domain is given by

$$K_d(\delta) := k_p + \frac{k_i}{\delta} + \frac{k_d\delta}{(1 - e^{-T/T_d})/(T/T_d) + T_d\delta}.$$

The synthesis process is completely the same as that for the continuous-time case. We minimize γ subject to the following specifications

$$\begin{aligned} \Re[L_d(\delta)] + \Im[L_d(\delta)] \le \gamma &\quad : \quad 1 \le \omega \le 5, \\ -2\Re[L_d(\delta)] + \Im[L_d(\delta)] \le 1 &\quad : \quad \omega \ge 5, \\ |L_d(\delta)| \le 0.25 &\quad : \quad |\omega| \ge 100, \end{aligned}$$

where $\delta := (e^{j\omega T} - 1)/T$. As seen in Fig. 3 (right), large amplitude oscillations are no longer observed in the step responses even when $T = 0.02$ and $T = 0.03$. It has also been confirmed that the method is still effective even when $T = 0.04$ for which the discretized controller makes the closed-loop system unstable. These less oscillatory responses are achieved by optimizing the sensitivity reduction in the low frequency range with the explicit knowledge of the sampling period. We see that the step response becomes slower when T becomes larger, indicating that the controllers are carefully designed by taking the limitation of the long sampling period into account. The comparison of DESIGNS 3 and 4 clearly indicates the advantage of the δ-domain design over the continuous-time design followed by discretization. It also demonstrates effectiveness of our proposed method for designing digital PID controllers with relatively long sampling periods.

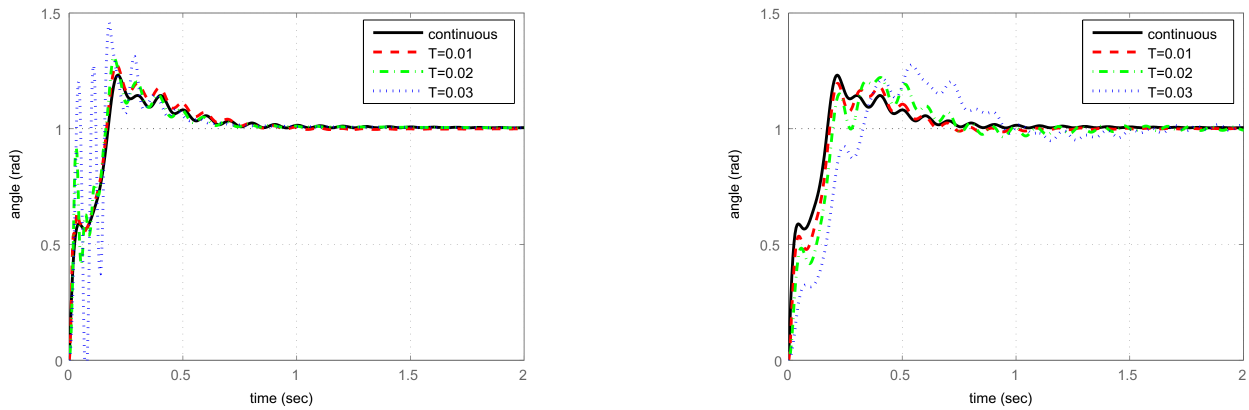

Figure 3: Step Responses of the closed-loop systems with digital PID controllers. DESIGN 3 (left) and DESIGN 4 (right). If a controller is designed in the continuous-time setting and then discretized by the Tustin transform (DESIGN 3), the closed-loop system becomes highly oscillatory as the sampling frequency reduces and comes close to the flexible modes of the plant. The GKYP synthesis enables the direct design of digital controllers in the δ-domain (DESIGN 4) so that the undesirable oscillations can be alleviated by explicitly taking the sampling-time limitation into account during the design process

Before closing this section, we show how to apply the GKYP synthesis technique to MIMO PID designs. The idea is to split the design into two steps; pseudo-diagonalization followed by multiple of single-loop PID designs. First we design an MIMO pre-compensator $\tilde{K}(\lambda)$ or a gain matrix $\tilde{K}$ which satisfies

$$\begin{aligned} \tilde{K}_{ii}(\lambda) = 1, \qquad & i = 1, \dots, n, \\ \left|\tilde{P}_{ij}(\lambda)\right| \leq \tilde{\varepsilon}_{ij}, \qquad & i \neq j \ \ ; \ \ \lambda \in \mathbf{\Lambda}_m, \end{aligned}$$

where $\tilde{P}(\lambda) := P(\lambda)\tilde{K}(\lambda)$ and $\mathbf{\Lambda}_m$ denotes a frequency range around the cross over frequency. The first constraint avoids the trivial solution of the second inequality. By minimizing the weighted sum or maximum value of $\tilde{\varepsilon}_{ij}$ with these constraints, a pseudo-diagonalized plant $\tilde{P}(\lambda)$ is obtained. The second step is to design a diagonal PID controller $K(\lambda)$, where the diagonal elements of open loop function $L(\lambda) = \tilde{P}(\lambda)K(\lambda)$ can be shaped by the same method as the SISO open-loop shaping with the following constraint

$$|L_{ij}(\lambda)| \leq \varepsilon_{ij} \quad i \neq j \ \ ; \ \ \lambda \in \mathbf{\Lambda}_m,$$

with small ε_{ij}. The final controller is implemented as $\tilde{K}(\lambda)K(\lambda)$. The basic idea presented above can be found in [22] where a method is proposed to achieve pseudo-diagonalization at a selected frequency. The main technological advance made here is the ability of pseudo-diagonalization in a frequency range.

4 Robustness Consideration

4.1 Robust GKYP analysis and synthesis

In many applications, the rational function $G(\lambda)$ in Theorem 2 is not completely known but contains some uncertain parameter vector δ that belongs to a given set $\boldsymbol{\delta} \subset \mathbb{C}^{\ell}$. One would then like to verify if condition (5) holds for all possible parameter variations $\delta \in \boldsymbol{\delta}$. This problem can be addressed (with some potential conservatism) as follows.

In view of Theorem 2, the problem is to check if, for each $\delta \in \boldsymbol{\delta}$, there exist P_δ and Q_δ such that

$$Q_\delta > 0, \quad F_\delta Z_\delta F_\delta^* < 0, \quad Z_\delta := \mathrm{diag}(\Phi^\mathsf{T} \otimes P_\delta + \Psi^\mathsf{T} \otimes Q_\delta, \Pi) \tag{15}$$

hold, where F_δ is a matrix defined as in Theorem 2 but now is dependent upon the parameter δ (the subscript $_\delta$ indicates this dependence). Let us consider the case where the dependence is linear fractional, i.e., each entry of F_δ is a rational function of δ.

We assume the following structures:

$$P_\delta = \mathcal{N}_\delta \mathcal{P} \mathcal{N}_\delta^*, \quad Q_\delta = \mathcal{N}_\delta \mathcal{Q} \mathcal{N}_\delta^*, \tag{16}$$

where $\mathcal{N}_\delta$ is a given rational matrix of δ while $\mathcal{P}$ and $\mathcal{Q}$ are variables independent of δ. In general, P_δ and Q_δ in (15) are not necessarily rational and hence these restrictions on P_δ and Q_δ may introduce potential conservatism. However, it may be possible to show, using the result in [23], that the rational dependence of P_δ and Q_δ on δ can be assumed without loss of generality (i.e., without introducing conservatism) if the order is chosen sufficiently high.

Theorem 3 *Consider the inequalities in (15) where F_δ is a matrix-valued rational function of $\delta \in \mathbb{C}^m$. Let a rational matrix $\mathcal{N}_\delta$ of δ be given and define $\mathcal{M}_\delta := F_\delta \mathrm{diag}(\mathcal{N}_\delta, \mathcal{N}_\delta, I)$. Let M_{ij} and N_{ij} be, respectively, the coefficient matrices of some realizations of $\mathcal{M}_\delta$ and $\mathcal{N}_\delta$:*

$$\mathcal{M}_\delta = M_{11} + (M_{12} - M_{13}\nabla)(M_{23}\nabla - M_{22})^{-1} M_{21}, \quad \mathcal{N}_\delta = N_{11} + (N_{12} - N_{13}\Delta)(N_{23}\Delta - N_{22})^{-1} N_{21}, \tag{17}$$

where Δ and ∇ are (arbitrarily chosen) matrices containing δ. Then, there exist Hermitian-valued, rational matrices P_δ and Q_δ of the form (16) satisfying (15) for all $\delta \in \boldsymbol{\delta}$ if and only if there exist Hermitian matrices $\mathcal{P}$, $\mathcal{Q}$, X, and Y such that

$$MXM^* < 0, \quad X := \mathrm{diag}(\Phi^\mathsf{T} \otimes \mathcal{P} + \Psi^\mathsf{T} \otimes \mathcal{Q}, \Pi, \mathcal{X}),$$

$$NYN^* > 0, \quad Y := \mathrm{diag}(\mathcal{Q}, \mathcal{Y}),$$

$$\begin{bmatrix} \nabla & I \end{bmatrix} \mathcal{X} \begin{bmatrix} \nabla & I \end{bmatrix}^* \geq 0, \quad \forall\, \nabla \in \boldsymbol{\nabla}, \tag{18}$$

$$\begin{bmatrix} \Delta & I \end{bmatrix} \mathcal{Y} \begin{bmatrix} \Delta & I \end{bmatrix}^* \leq 0, \quad \forall\, \Delta \in \boldsymbol{\Delta}, \tag{19}$$

where M and N are constant matrices whose (i,j) blocks are given by M_{ij} and N_{ij}, respectively, and $\boldsymbol{\Delta}$ and $\boldsymbol{\nabla}$ are the sets of $\Delta(\delta)$ and $\nabla(\delta)$ spanned by $\delta \in \boldsymbol{\delta}$, respectively.

Proof. Note that the condition in (15) can be written as

$$\mathcal{N}_\delta \mathcal{Q} \mathcal{N}_\delta^* > 0, \quad \mathcal{M}_\delta \mathcal{S} \mathcal{M}_\delta^* < 0, \quad \mathcal{S} := \mathrm{diag}(\Phi^\mathsf{T} \otimes \mathcal{P} + \Psi^\mathsf{T} \otimes \mathcal{Q}, \Pi). \tag{20}$$

We show that the second inequality holds for all $\delta \in \boldsymbol{\delta}$ if and only if there exists $\mathcal{X}$ satisfying $MXM^* < 0$ and (18). The first inequality can be treated similarly. Clearly, $\mathcal{M}_\delta \mathcal{S} \mathcal{M}_\delta^* < 0$ holds if and only if

$$v^* \mathcal{M}_\delta \mathcal{S} \mathcal{M}_\delta^* v < 0$$

holds for all nonzero vector v. Defining

$$w^* := v^*(M_{12} - M_{13}\nabla)(M_{23}\nabla - M_{22})^{-1},$$

we see that the original inequality holds if and only if

$$\begin{bmatrix} v \\ w \end{bmatrix}^* \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \mathcal{S} \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}^* \begin{bmatrix} v \\ w \end{bmatrix} < 0$$

holds for all nonzero v and w such that

$$\begin{bmatrix} v \\ w \end{bmatrix}^* \begin{bmatrix} M_{12} & M_{13} \\ M_{22} & M_{23} \end{bmatrix} \begin{bmatrix} I \\ -\nabla \end{bmatrix} = 0.$$

Then, using the quadratic separator in [20] (Lemma 10), we have the result. ■

When the uncertain parameter set $\boldsymbol{\delta}$ is real, it can often be normalized to be the set of vectors with each entry of magnitude less than or equal to one. The uncertain matrix Δ is usually taken as a diagonal matrix with repeated entries of δ_i, i.e., $\Delta = \mathrm{diag}(\delta_1 I_{k_1}, \ldots, \delta_m I_{k_m})$. In this case, the constraint in (19) can be satisfied by enforcing a certain structure on X or considering vertex conditions; the techniques include the (D, G)-scaling [24], LFT-scaling [25], quadratic separator [20], [26], and full block multiplier [27]. A similar comment applies to ∇ as well. The resulting problem is then a standard linear matrix inequality problem that can be solved numerically. Finally, the primal condition in Theorem 1 can also be "robustified" in a similar manner.

So far we have shown how a robust FDI condition can be reduced to LMIs at a somewhat conceptual level. Let us now provide some details of computations. In particular, we consider the case where F_δ is linearly dependent upon Δ, and give a formula for a realization of $\mathcal{M}_\delta$. In this case, it may be reasonable to look for P_δ and Q_δ with linear basis function $\mathcal{N}_\delta$. Hence we consider

$$F_\delta = F_o + F_1 \Delta F_2, \quad \mathcal{N}_\delta = \mathcal{N}_o + \mathcal{N}_1 \Delta \mathcal{N}_2. \tag{21}$$

It can be verified that realizations of $\mathcal{M}_\delta$ and $\mathcal{N}_\delta$ in (17) are given by

$$M = \left[\begin{array}{c|cc|cc} F_o \mathcal{H}_o & 0 & 0 & F_1 & F_o \mathcal{H}_1 \\ \hline F_2 \mathcal{H}_o & I & 0 & 0 & F_2 \mathcal{H}_1 \\ \mathcal{H}_2 & 0 & I & 0 & 0 \end{array}\right], \quad N = \begin{bmatrix} \mathcal{N}_o & 0 & \mathcal{N}_1 \\ \mathcal{N}_2 & I & 0 \end{bmatrix}, \tag{22}$$

$$\begin{array}{l} \mathcal{H}_o := \mathrm{diag}(\mathcal{N}_o, \mathcal{N}_o, I), \\ \nabla := \mathrm{diag}(\Delta, \Delta, \Delta), \end{array} \quad \mathcal{H}_1 := \begin{bmatrix} \mathcal{N}_1 & 0 \\ 0 & \mathcal{N}_1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{H}_2 := \begin{bmatrix} \mathcal{N}_2 & 0 & 0 \\ 0 & \mathcal{N}_2 & 0 \end{bmatrix}. \tag{23}$$

We now turn our attention to the synthesis problem. A technique similar to the one presented in Section 3 is valid for the robust control synthesis. To illustrate the idea, let us consider the specification given by (6)–(8) where the design parameter vector ρ linearly enters $B_L(\rho)$ and $D_L(\rho)$, and uncertain matrix $\Delta(\delta)$ linearly enters $A_L(\delta) := A_o + U\Delta(\delta)V$. In this case, F_δ and $\mathcal{N}_\delta$ can be given by (21) with

$$F_o := \begin{bmatrix} A_o & I & B_L & 0 \\ C_L & 0 & D_L & I \end{bmatrix}, \quad F_1 := \begin{bmatrix} U \\ 0 \end{bmatrix}, \quad F_2 := \begin{bmatrix} V & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{N}_o := \begin{bmatrix} 0 & I \end{bmatrix}, \quad \mathcal{N}_1 := U, \quad \mathcal{N}_2 := \begin{bmatrix} I & 0 \end{bmatrix}.$$

The choice of the basis function N_δ is not unique. The one suggested above shares the same basis matrix U as the system [20]. Another choice would be $\mathcal{N}_1 := I$.

The specifications are robustly satisfied against parameter perturbation $\delta \in \boldsymbol{\delta}$ if conditions in Theorem 3 are satisfied for M, N and ∇ defined by (22) and (23). The conditions are quadratic in terms of B_L and D_L with the coefficient matrix of the quadratic term being Π_{11}. Hence they can be made linear in the design parameter ρ through the Schur complement as has been done in Proposition 1. The design problem can thus be reduced to an LMI problem.

4.2 Robust PID controller synthesis

The following numerical examples confirm the effectiveness of the method outlined in the previous subsection. Let us consider two sets of plants given by

$$P_1(s) = \frac{10}{(s+1)(s^2+(1+\delta)s+10)},$$
$$P_2(s) = \frac{10}{(s+1+\delta)(s^2+s+10)},$$

where δ is the uncertain parameter. We shall design a PID controller $K(s)$ in (12) with $T_d = 0.05$ for each uncertain plant by minimizing β subject to the following design specifications on the loop transfer function $L(s) = P_i(s)K(s)$ with $i = 1$ or 2

$$\begin{aligned} |L(j\omega)| < 0.1 &\quad : \quad |\omega| \geq 7, \\ 3\Re(L(j\omega)) + \beta > \Im(L(j\omega)) &\quad : \quad \omega \geq 0.05, \\ \Im(L(j\omega)) < -2 &\quad : \quad 0.05 \leq \omega \leq 0.45, \end{aligned}$$

where these FDI requirements are enforced robustly against all perturbations satisfying $|\delta| \leq 0.3$. Note that the parameter δ linearly enters A_p in (10). Hence A_L defined by (11) can be expressed as $A_L(\delta) = A_o + U\Delta V$ with $\Delta := \delta$. The design equations explained in the previous section can then be applied to find an approximate optimal PID controller for each uncertain plant.

The PID controllers $K_i(s)$ with $i = 1, 2$, and the associated objective function values are found to be

$$k_p = 0.2133, \quad k_i = 1.0842, \quad k_d = 0.1579, \quad \beta = 0.9373: \quad \text{for } P_1(s),$$
$$k_p = 0.6271, \quad k_i = 1.3021, \quad k_d = 0.2331, \quad \beta = 1.8048: \quad \text{for } P_2(s).$$

The Nyquist plots of the loop transfer functions for the plants with various values of δ are shown in Fig. 4. It can be seen from these figures that the proposed method is effective for designing robust PID controllers. The design for $P_1(s)$, in particular, is quite tight in the sense that the straight line specifying the stability margin (almost) touches the Nyquist plots twice; once in the low frequency range for $\delta = 0.3$ and another time in the middle frequency range for $\delta = -0.3$. Noting that the minimization of β pushes down the straight line subject to the constraint that the Nyquist plots lie below the line, the degree of conservatism associated with the robust PID design appears small for this design.

5 Conclusion

We have presented a method for designing PID controllers to meet multiple FDI specifications on the open-loop transfer function in (semi)finite frequency ranges. The method is based on the GKYP lemma and enables direct loop shaping through LMI optimizations without frequency gridding or weights. The resulting synthesis condition is nonconservative; its infeasibility implies that no PID controller exists to meet the specifications. The design method is extended to deal with systems with parametric uncertainties using the Lyapunov function that depends on the parameter in a linear fractional manner. The robust PID control design is also reduced to an LMI optimization problem, albeit with some potential conservatism. The effectiveness of the proposed design methods has been demonstrated through several numerical examples.

The robust GKYP synthesis described in this paper applies not only to the PID control design but also to any open-loop shaping and filtering problems where the poles of the controller or filter are fixed (see [28], [29] for further design examples and software tools). The method does not apply directly to feedback control synthesis problems with closed-loop specifications and hence requires some modifications. Such extensions are currently under investigation [30], [31].

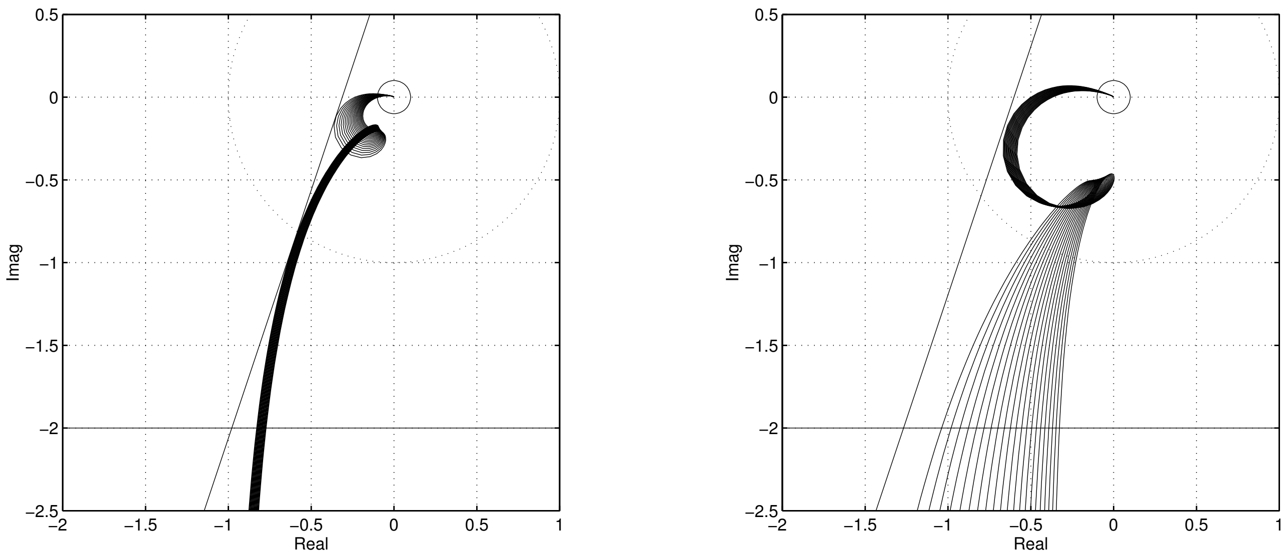

Figure 4: Nyquist plots of the loop transfer functions achieved by robust PID controllers. P_1K_1 (left) and P_2K_2 (right). For each design, several curves indicate the Nyquist plots for different values of the uncertain parameter within the assumed range of perturbation. The design specifications (indicated by the straight lines and circles) are robustly satisfied, with seemingly small conservatism

5.1 Acknowledgments

This work has been supported in part by CREST of JST(Japan Science and Technology Agency), The Ministry of Education, Science, Sport and Culture, Japan, under Grant No. 17360195, and the National Science Foundation under Grant No. 0237708.

A Derivation of the formulas for the δ-transform

Consider the set of frequencies in a (semi)finite range in the δ-domain:

$$\mathbf{\Delta} := \{ \frac{e^{j\omega T} - 1}{T} : \ \tau(\omega - \varpi_1)(\omega - \varpi_2) \geq 0, \ |\omega T| \leq \pi \ \}$$

where τ is $+1$ or -1, and $T, \varpi_1, \varpi_2 \in \mathbb{R}$ are given scalars such that $T > 0$ and $-\pi \leq \varpi_1 T < \varpi_2 T \leq \pi$. We would like to choose $\Phi, \Psi \in \mathbf{H}_2$ so that $\mathbf{\Delta} = \mathbf{\Lambda}(\Phi, \Psi)$ where

$$\mathbf{\Lambda}(\Phi, \Psi) := \{ \delta \in \mathbb{C} : \ \sigma(\delta, \Phi) = 0, \ \sigma(\delta, \Psi) \geq 0 \ \}.$$

Such choices are not unique and we are interested in choosing one that recovers the continuous-time result in the limit $T \to 0$. In particular, we require

$$\lim_{T \to 0} \Phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \lim_{T \to 0} \Psi = \tau \begin{bmatrix} 1 & -j\varpi_c \\ j\varpi_c & \varpi_1 \varpi_2 \end{bmatrix} \tag{24}$$

where $\varpi_c := (\varpi_1 + \varpi_2)/2$.

The set $\mathbf{\Delta}$ is an arc of the circle of radius $1/T$ with center at $-1/T$ described by the set of $\delta \in \mathbb{C}$ satisfying

$$|T\delta + 1| = 1 \quad \text{or} \quad T\delta^*\delta + \delta + \delta^* = 0.$$

Hence an obvious choice of Φ is

$$\Phi = \begin{bmatrix} T & 1 \\ 1 & 0 \end{bmatrix}. \tag{25}$$

Any multiple of Φ provides the same circle, but this is the one that has the desired property in (24).

To fix Ψ, note that $\mathbf{\Delta}$ can be seen as the intersection of the circle $\sigma(\delta, \Phi) = 0$ and a half plane with the boundary passing through the points $\delta_i := (e^{j\varpi_i T} - 1)/T$, $(i = 1, 2)$. The boundary is described by the set of points $\delta = \alpha + j\beta$ such that

$$(\alpha_2 - \alpha_1)(\beta - \beta_1) - (\beta_2 - \beta_1)(\alpha - \alpha_1) = 0$$

where $\alpha_i + j\beta_i := \delta_i$. This boundary equation can be written as $\sigma(\delta, \Psi_o) = 0$ with

$$\Psi_o := \begin{bmatrix} 0 & \beta_1 - \beta_2 - j(\alpha_1 - \alpha_2) \\ \beta_1 - \beta_2 + j(\alpha_1 - \alpha_2) & 2(\beta_2\alpha_1 - \alpha_2\beta_1) \end{bmatrix} = j \begin{bmatrix} 0 & \delta_2 - \delta_1 \\ \delta_1^* - \delta_2^* & \delta_2^*\delta_1 - \delta_2\delta_1^* \end{bmatrix}.$$

Noting that $\sigma(\delta, \Psi_o) \geq 0$ is the half plane below/above the line $\sigma(\delta, \Psi_o) = 0$ if $(\alpha_2 < \alpha_1)$/$(\alpha_2 > \alpha_1)$, we see that $\mathbf{\Delta} = \mathbf{\Lambda}(\Phi, \tau\Psi_o)$ holds. However, this choice of Ψ does not satisfy the limiting property in (24).

We now modify Ψ_o so that (24) holds. First note that

$$\mathbf{\Delta} = \mathbf{\Lambda}(\Phi, \tau\Psi_o) = \mathbf{\Lambda}(\Phi, \Psi), \quad \Psi := \tau(a\Phi + b\Psi_o)$$

holds for any $a, b \in \mathbb{R}$ such that $b > 0$. It turns out that the choices

$$a = \frac{1}{T}, \quad b = \frac{1}{(\varpi_2 - \varpi_1)T}$$

will make Ψ satisfy (24). This can be verified as follows. First note that

$$\tau\Psi = \begin{bmatrix} 1 & 1/T \\ 1/T & 0 \end{bmatrix} + \frac{1}{(\varpi_1 - \varpi_2)T^3} \begin{bmatrix} 0 & jT(e_1 - e_2) \\ -jT(e_1^* - e_2^*) & 2(s_1c_2 - c_1s_2 - s_1 + s_2) \end{bmatrix} \tag{26}$$

where, for $i = 1, 2$,

$$e_i := e^{j\varpi_i T}, \quad s_i := \sin(\varpi_i T), \quad c_i := \cos(\varpi_i T).$$

Then by direct calculations,

$$\lim_{T\to 0} \tau\Psi_{11} = 1$$

$$\begin{aligned} \lim_{T\to 0} \tau\Psi_{12} &= \lim_{T\to 0} \left(\frac{1}{T} + \frac{j(e_1 - e_2)}{(\varpi_1 - \varpi_2)T^2} \right) \\ &= \lim_{T\to 0} \frac{(\varpi_1 - \varpi_2)T + j(e_1 - e_2)}{(\varpi_1 - \varpi_2)T^2} \\ &= \lim_{T\to 0} \frac{(\varpi_1 - \varpi_2) - (\varpi_1 e_1 - \varpi_2 e_2)}{2(\varpi_1 - \varpi_2)T} \\ &= \lim_{T\to 0} \frac{-j(\varpi_1^2 e_1 - \varpi_2^2 e_2)}{2(\varpi_1 - \varpi_2)} \\ &= \frac{-j(\varpi_1^2 - \varpi_2^2)}{2(\varpi_1 - \varpi_2)} \\ &= -j\varpi_c. \end{aligned}$$

$$
\begin{aligned}
\lim_{T\to 0}\tau\Psi_{22} &= \lim_{T\to 0}\frac{\sin((\varpi_1-\varpi_2)T)-s_1+s_2}{(\varpi_1-\varpi_2)T^3/2} \\
&= \lim_{T\to 0}\frac{(\varpi_1-\varpi_2)\cos((\varpi_1-\varpi_2)T)-\varpi_1 c_1+\varpi_2 c_2}{3(\varpi_1-\varpi_2)T^2/2} \\
&= \lim_{T\to 0}\frac{-(\varpi_1-\varpi_2)^2\sin((\varpi_1-\varpi_2)T)+\varpi_1^2 s_1-\varpi_2^2 s_2}{3(\varpi_1-\varpi_2)T} \\
&= \lim_{T\to 0}\frac{-(\varpi_1-\varpi_2)^3\cos((\varpi_1-\varpi_2)T)+\varpi_1^3 c_1-\varpi_2^3 c_2}{3(\varpi_1-\varpi_2)} \\
&= \frac{-(\varpi_1-\varpi_2)^3+\varpi_1^3-\varpi_2^3}{3(\varpi_1-\varpi_2)} \\
&= \frac{(\varpi_1-\varpi_2)(\varpi_1^2+\varpi_1\varpi_2+\varpi_2^2)-(\varpi_1-\varpi_2)^3}{3(\varpi_1-\varpi_2)} \\
&= \frac{(\varpi_1^2+\varpi_1\varpi_2+\varpi_2^2)-(\varpi_1-\varpi_2)^2}{3} \\
&= \varpi_1\varpi_2.
\end{aligned}
$$

In summary, the solution to the problem is given by (25) and (26).

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