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Abstract

We deal with quadratic semi-assignment problems with symmetric distances. This symmetry reduces the number of variables in its mixed integer programming formulation. We investigate a polytope arising from the problem, and obtain some basic polyhedral properties, the dimension, the affine hull and certain facets through an isomorphic projection. We also present a nontrivial class of facets. Computational results show that LP relaxations of the symmetric formulation can be solved much faster than those of that does not incorporate symmetry at the reasonable expense of tightness of lower bounds.

Keywords: quadratic semi-assignment, polytope, facets

1 Introduction

The *quadratic semi-assignment problem* (QSAP) is a model often used in formulating scheduling or facility location problems. The QSAP is known to be NP-hard. An application of the problem can be found in the area of scheduling [6, 12, 13, 7]. Skutella dealt with the problem of scheduling unrelated parallel machines which is formulated as a special case of the QSAP and proposed an approximation algorithm [11]. The *hub location problem* is also a special case [10].

QSAP is formulated as a 0-1 quadratic programming as follows. Let M and N be mutually disjoint sets with $|M| = m$ and $|N| = n$, respectively. Given a_{ik} $(i \in M, k \in N)$, f_{ij} $(i, j \in M)$, and d_{kl} $(k, l \in N)$, the problem is

$$\text{min.} \sum_{i\in M}\sum_{k\in N} a_{ik}x_{ik} + \sum_{i,j\in M}\sum_{k,l\in N} f_{ij}d_{kl}x_{ik}x_{jl}$$

$$\text{s. t.} \sum_{k\in N} x_{ik} = 1 \quad (i \in M), \tag{1}$$

$$x_{ik} \in \{0,1\} \quad (i \in M, k \in N). \tag{2}$$

We may assume that $f_{ii} = 0$ $(i \in M)$ without loss of generality due to the constraints (1) and (2).

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A mixed integer programming (MIP) formulation of QSAP is given by Billionnet and Elloumi [1]. Saito, Matuura, and Matsui [10] independently studied the same MIP formulation in the context of a hub location problem. This MIP formulation is obtained through linearization of the objective function with new variables

$$y_{ikjl} := x_{ik}x_{jl} \qquad (i,j \in M, k,l \in N, i < j). \tag{3}$$

This linearization yields another equivalent formulation of QSAP:

$$\begin{aligned} \text{min.} \quad & \sum_{i \in M} \sum_{k \in N} a_{ik}x_{ik} + \sum_{\substack{i,j \in M \\ i<j}} \sum_{k,l \in N} (f_{ij} + f_{ji})d_{kl}y_{ikjl} \\ \text{s. t.} \quad & (1),\ (2),\ \text{and } (3), \end{aligned}$$

which will be referred to as QSAP in this paper. The MIP formulation of [1] and [10] is obtained from this by replacing the constraint (3) with certain linear constraints. The convex hull of the feasible solutions of QSAP, in the space of (x, y)-variables, is named the *quadratic semi-assignment polytope* (QSAP-polytope) by Saito, Fujie, Matsui, and Matuura [9]. Saito et al. showed that the linear constraints in the MIP formulation define the affine hull and facets of the polytope [9]. The computational experiences in [1], [10], and [9] show that the MIP formulation is very tight, i.e., several instances can be solved to optimality just by solving their LP relaxations. However it is very time-consuming to solve relatively large instances because the number of the new variables, which is $nm(m-1)/2 + m(m-1)n(n-1)/2$, becomes large. Hence the motivation of this paper is to propose another fast and tight lower bound for the sake of branch-and-bound methods.

Suppose that d is symmetric in the sense that $d_{kl} = d_{lk}$ for any $k, l \in N$. Since d often corresponds to distances in actual problems, this assumption will be acceptable in practical applications. Then it is possible to reduce the number of variables by putting

$$z_{ikjl} := x_{ik}x_{jl} + x_{il}x_{jk} \qquad (i,j \in M, k,l \in N, i < j, k < l), \tag{4}$$

$$y_{ikjk} := x_{ik}x_{jk} \qquad (i,j \in M, k \in N, i < j). \tag{5}$$

Note that the number of the variables is $nm(m-1)/2 + m(m-1)n(n-1)/4$, which almost halves that of the original formulation, even though the order of magnitude $\mathrm{O}(m^2n^2)$ is unchanged. Thus we are led to the following problem:

$$\begin{aligned} \text{min.} \quad & \sum_{i \in M} \sum_{k \in N} a_{ik}x_{ik} + \sum_{\substack{i,j \in M \\ i<j}} \sum_{k \in N} b_{ikjk}y_{ikjk} + \sum_{\substack{i,j \in M \\ i<j}} \sum_{\substack{k,l \in N \\ k<l}} c_{ikjl}z_{ikjl} \\ \text{s. t.} \quad & (1),\ (2),\ (4),\ \text{and } (5), \end{aligned}$$

where $b_{ikjk} = (f_{ij} + f_{ji})d_{kk}$ and $c_{ikjl} = (f_{ij} + f_{ji})d_{kl}$. We refer to this problem as SQSAP. The convex hull of the feasible solutions of SQSAP, in the space of (x, y, z)-variables, is called the *symmetric quadratic semi-assignment polytope* (SQSAP-polytope), which is the main object that we study in this paper.

The rest of this paper is organized as follows. In Section 2, we define the QSAP- and SQSAP-polytopes and present MIP formulations, respectively. In Section 3, we introduce the *star-transformation* and transform the SQSAP-polytope to another equivalent and tractable polytope. In Section 4, we determine the dimension and the affine hull of the SQSAP-polytope. In Section 5, we show a class of nontrivial facets. Finally, we present a computational results in Section 6.

2 QSAP-polytope and SQSAP-polytope

In this section, we formulate QSAP and SQSAP as optimization problems on certain graphs. We also define polytopes $\mathrm{QSAP}_{m,n}$ and $\mathrm{SQSAP}_{m,n}$ for respective problems and show their MIP reformulations. See [9] for details on $\mathrm{QSAP}_{m,n}$.

Let $\hat{G} = (V, E \cup \hat{E})$ denote a graph with node set $V := M \times N$ and edge set $E \cup \hat{E}$, where

$$E := \{\{(i,k),(j,k)\} \mid i,j \in M, k \in N, i < j\},$$
$$\hat{E} := \left\{ \{(i,k),(j,l)\} \;\middle|\; \begin{array}{l} i,j \in M, k,l \in N, \\ i < j, k \neq l \end{array} \right\}.$$

We can see that feasible solutions of QSAP correspond to m-cliques in $\hat{G}$. Thus, by assigning a_{ik} and b_{ikjl} for each node and edge, QSAP turns into a problem to find a minimum node- and edge- weighted m-clique in $\hat{G}$.

We remark on some notations. For any node subset $T \subseteq V$, $\chi_T \in \mathbb{R}^V$ denotes the characteristic vector of T. For $T \subseteq V$, we define $x(T) := \sum_{v \in T} x_v$ and $y(T) := y(E(T))$. For mutually disjoint subsets $S, T \subseteq V$, we denote the cut set by $E(S : T) := \{\{u,v\} \in E \mid u \in S,\ v \in T\}$ and we abbreviate $y(E(S : T))$ to $y(S : T)$. We omit brackets for singletons, e.g., we use χ_u instead of $\chi_{\{u\}}$. We also use χ_{uv} instead of $\chi_{\{\{u,v\}\}}$. For all $i \in M$, we define a subset of V by $\mathrm{row}_i := \{(i,k) \mid k \in N\}$. For any $v = (i,k) \in V$, we define subsets of V by $\mathrm{row}(v) := \{(i,k) \mid k \in N\}$ and $\mathrm{col}(v) := \{(i,k) \mid i \in M\}$. Recall that $m = |M|$ and $n = |N|$.

We define the polytope $\mathrm{QSAP}_{m,n}$ as the convex hull of feasible solutions:

$$\mathrm{QSAP}_{m,n} := \mathrm{conv}\{(\chi_C, \chi_{E(C)}, \chi_{\hat{E}(C)}) \mid C \text{ is an } m\text{-clique of } \hat{G}\}.$$

The following observation gives a MIP reformulation of QSAP.

Proposition 1. *A vector $(x, y, \hat{y})$ is a vertex of $\mathrm{QSAP}_{m,n}$ if and only if it satisfies*

$$x(\mathrm{row}_i) = 1 \quad (i \in M), \tag{6}$$
$$-x_{ik} + y_{ikjk} + \hat{y}((i,k) : \mathrm{row}_j) = 0 \quad (i,j \in M,\ k \in N,\ i \neq j), \tag{7}$$
$$y_e \geq 0 \quad (e \in E), \tag{8}$$
$$\hat{y}_{\hat{e}} \geq 0 \quad (\hat{e} \in \hat{E}), \tag{9}$$
$$x_v \in \{0,1\} \quad (v \in V). \tag{10}$$

□

Next we deal with the symmetric case. Let G be a hypergraph with node set V, edge set E, and hyperedge set

$$F := \left\{ \{(i,k),(j,l),(i,l),(j,k)\} \;\middle|\; \begin{array}{l} i,j \in M, \\ k,l \in N, \\ i<j, k<l \end{array} \right\}.$$

For simplicity, we introduce the symbols $[i,k,j,k]$ and $\langle i,k,j,l\rangle$ to each edge $\{(i,k),(j,k)\}$ and hyperedge $\{(i,k),(j,l),(i,l),(j,k)\}$, respectively. Note that a hyperedge $\langle i,k,j,l\rangle$ is obtained by identifying two edges $\{(i,k),(j,l)\}$ and $\{(i,l),(j,k)\}$ in the graph $\hat{G}$. We define the symmetric quadratic semi-assignment polytope.

Definition 2.

$$\mathrm{SQSAP}_{m,n} := \mathrm{conv} \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x(\mathrm{row}_i) = 1 \\ \quad (i \in M), \\ y_{ikjk} = x_{ik}x_{jk} \\ \quad ([i,k,j,k] \in E), \\ z_{ikjl} = x_{ik}x_{jl} + x_{il}x_{jk} \\ \quad (\langle i,k,j,l\rangle \in F), \\ x_{ik} \in \{0,1\} \\ \quad ((i,k) \in V) \end{array} \right\}.$$

The polytope $\mathrm{SQSAP}_{m,n}$ is a projection of $\mathrm{QSAP}_{m,n}$, i.e., the image of a linear map. Let sym be a linear map, $\mathrm{sym} : \mathbb{R}^V \times \mathbb{R}^E \times \mathbb{R}^{\hat{E}} \longrightarrow \mathbb{R}^V \times \mathbb{R}^E \times \mathbb{R}^F$, defined by $\mathrm{sym}(x,y,\hat{y}) = (x,y,z)$, where $z_{\langle i,k,j,l\rangle} = y_{[ikjl]} + y_{[iljk]}$ for any $\langle i,k,j,l\rangle \in F$. Then we have $\mathrm{SQSAP}_{m,n} = \mathrm{sym}(\mathrm{QSAP}_{m,n})$. We define a clique in the hypergraph G by a node subset induced by the image of an incidence vector of a clique in the graph $\hat{G}$. In the following we assume that $n \geq 3$ (Otherwise, $z \geq 0$ would be redundant). The next proposition gives a MIP reformulation of SQSAP, where Δ_{vw} denotes the set of all hyperedges which contain both of v and w for any $v, w \in V$.

Proposition 3. *A vector (x,y,z) is a vertex of $\mathrm{SQSAP}_{m,n}$ if and only if it satisfies*

$$x(\mathrm{row}_i) = 1 \quad (i \in M), \tag{11}$$
$$-x_{ik} - x_{jk} + 2y_{ikjk} + z(\Delta_{ikjk}) = 0 \quad (i,j \in M, k \in N, i<j), \tag{12}$$
$$z_f \geq 0 \quad (f \in F), \tag{13}$$
$$y_e \geq 0 \quad (e \in E), \tag{14}$$
$$-x_u + y_{uv} \leq 0 \quad (\{u,v\} \in E), \tag{15}$$
$$-x_v + y_{uv} \leq 0 \quad (\{u,v\} \in E), \tag{16}$$
$$x_v \in \{0,1\} \quad (v \in V). \tag{17}$$

Proof. We can see that $y_{ikjk} = x_{ik}x_{jk}$ and $z_{ikjl} = x_{ik}x_{jl} + x_{il}x_{jk}$ hold for any (x,y,z) which satisfies the above constraints. The converse inclusion is trivial. □

Next proposition ensures the existence of an optimal solution of the LP relaxation.

Proposition 4. *The inequalities* $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ *hold for any* (x, y, z) *which satisfies* (11)–(16). □

Finally, we remark on some polytopes related to $\mathrm{SQSAP}_{m,n}$. Padberg dealt with a quadratic programming problem with 0-1 constraints and introduced the Boolean quadric polytope defined by

$$\mathrm{BQP}_n = \mathrm{conv}\{(\chi_C, \chi_{\mathcal{E}_n(C)}) \in \mathbb{R}^{\mathcal{V}_n} \times \mathbb{R}^{\mathcal{E}_n} \mid C \text{ is a clique of } K_n\},$$

where $K_n = (\mathcal{V}_n, \mathcal{E}_n)$ is a complete graph with n nodes [8]. A generalization of BQP_n is studied by Fujie et al. [2]. Jünger and Kaibel studied the quadratic assignment polytope QAP_n which arises from the quadratic assignment problem [4]. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph with $\mathcal{V} = N \times N$ and $\mathcal{E} = \{\{(i,j),(k,l)\} \in \binom{\mathcal{V}}{2} \mid i \neq k, j \neq l\}$, where $\binom{\mathcal{V}}{2}$ is the set of all the 2-subsets of $\mathcal{V}$. Then the quadratic assignment polytope is defined by

$$\mathrm{QAP}_n = \mathrm{conv}\{(\chi_C, \chi_{\mathcal{E}(C)}) \in \mathbb{R}^{\mathcal{V}} \times \mathbb{R}^{\mathcal{E}} \mid C \text{ is an } n\text{-clique of } \mathcal{G}\}.$$

These polytopes are related as follows [9] (See also [4]).

Proposition 5. ([9]) $\mathrm{QSAP}_{m,n}$ *is isomorphic to a face of* BQP_{mn}, *and* QAP_n *is isomorphic to a face of* $\mathrm{QSAP}_{n,n}$. □

Jünger and Kaibel [3] studied the *symmetric quadratic assignment polytope* (SQAP-polytope) which is a projection of the QAP-polytope. We can show that SQAP-polytope is isomorphic to a face of SQSAP-polytope.

3 Star-transformation and SQSAP*-polytope

We introduce an isomorphic transformation (*star-transformation*) to derive a full-dimensional polytope which we call $\mathrm{SQSAP}^\star_{m,n^\star}$. Since $\mathrm{SQSAP}^\star_{m,n^\star}$ is essentially equivalent to $\mathrm{SQSAP}_{m,n}$ and more tractable, we first study $\mathrm{SQSAP}^\star_{m,n^\star}$ and determine the dimension and some trivial facets. The star-transformation was originally introduced for QAP- and SQAP-polytopes in [4, 3]. Ours is an adaptation to SQSAP.

By the equality constraints (11) and (12), certain variables, say those corresponding to $\mathrm{col}_n := \{(i,n) \mid i \in M\}$ in the graph G, are redundant in the sense that they are uniquely determined by others. Hence we eliminate these variables. In terms of polyhedron, we map $\mathrm{SQSAP}_{m,n}$ contained in an affine subspace

$$\mathcal{A} := \{(x, y, z) \mid (11) \text{ and } (12)\}$$

to another equivalent polytope contained in

$$\mathcal{U} := \left\{ (x,y,z) \;\middle|\; \begin{array}{ll} x_v = 0 & (v \in V \setminus V^\star), \\ y_e = 0 & (e \in E \setminus E^\star), \\ z_f = 0 & (f \in F \setminus F^\star) \end{array} \right\},$$

where every redundant variables are set to zero. We use the notations $N^\star := N \setminus \{n\}$, $V^\star := M \times N^\star$, $E^\star := \{[i,k,j,k] \in E \mid k \in N^\star\}$, and $F^\star := \{\langle i,k,j,l\rangle \in F \mid k,l \in N^\star\}$. The definitions of $\mathrm{row}_i^\star$, etc., are similar to those of $\mathrm{QSAP}_{m,n}$, respectively. For example,

$$F^\star(S:T) := \left\{ \langle i,k,j,l\rangle \;\middle|\; \begin{array}{c} (i,k)\in S, (j,l)\in T \\ \text{or} \\ (i,l)\in S, (j,k)\in T \end{array} \right\}$$

for any disjoint subsets $S, T \subseteq V^\star$.

We describe the above map formally. We define an affine map $\phi : \mathbb{R}^V \times \mathbb{R}^E \times \mathbb{R}^F \to \mathbb{R}^V \times \mathbb{R}^E \times \mathbb{R}^F$ by

$$\begin{aligned}
(\phi(x,y,z))_v &:= \begin{cases} x_v & (v \in V^\star) \\ x_v - \psi_v(x,y,z) & (v \in V \setminus V^\star), \end{cases} \\
(\phi(x,y,z))_e &:= \begin{cases} y_e & (e \in E^\star) \\ y_e - \psi_e(x,y,z) & (e \in E \setminus E^\star), \end{cases} \\
(\phi(x,y,z))_f &:= \begin{cases} z_f & (f \in F^\star) \\ z_f - \psi_f(x,y,z) & (f \in F \setminus F^\star), \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
\psi_v(x,y,z) &:= 1 - x(\mathrm{row}_i^\star), \\
\psi_e(x,y,z) &:= 1 - x(\mathrm{row}_i^\star \cup \mathrm{row}_j^\star) + y(\mathrm{row}_i^\star \cup \mathrm{row}_j^\star) + z(\mathrm{row}_i^\star \cup \mathrm{row}_j^\star), \\
\psi_f(x,y,z) &:= x_{ik} + x_{jk} - 2y_{ikjk} - z(\Delta_{ikjk}^\star).
\end{aligned}$$

Let $\pi : \mathbb{R}^V \times \mathbb{R}^E \times \mathbb{R}^F \to \mathbb{R}^{V^\star} \times \mathbb{R}^{E^\star} \times \mathbb{R}^{F^\star}$ be a linear map defined by $\pi(x,\bar{x},y,\bar{y},z,\bar{z}) = (x,y,z)$, where $x \in \mathbb{R}^{V^\star}$, $\bar{x} \in \mathbb{R}^{V\setminus V^\star}$, $y \in \mathbb{R}^{E^\star}$, $\bar{y} \in \mathbb{R}^{E\setminus E^\star}$, $z \in \mathbb{R}^{F^\star}$, and $\bar{z} \in \mathbb{R}^{F\setminus F^\star}$. Then we call the affine transformation $\pi\circ\phi$ a star-transformation, and define a polytope $\mathrm{SQSAP}^\star_{m,n^\star} := \pi\circ\phi(\mathrm{SQSAP}_{m,n})$. It is easy to see that the star-transformation is a bijection between the vertices of $\mathrm{SQSAP}_{m,n}$ and those of $\mathrm{SQSAP}^\star_{m,n^\star}$. Thus $\mathrm{SQSAP}^\star_{m,n^\star}$ is affinely isomorphic to $\mathrm{SQSAP}_{m,n}$.

Proposition 6. *The polytope* $\mathrm{SQSAP}^\star_{m,n^\star}$ *is full-dimensional, i.e.,* $\dim \mathrm{SQSAP}^\star_{m,n^\star} = |V^\star| + |E^\star| + |F^\star|$.

Proof. The vectors $(0,0,0)$, $(\chi_u,0,0)$ $(\forall u \in V^\star)$, $(\chi_u+\chi_v,\chi_e,0)$ $(\forall e = \{u,v\} \in E^\star)$, $(\chi_u+\chi_v,0,\chi_f)$ $(\forall f = \{u,v,u',v'\} \in F^\star)$ are affinely independent in $\mathrm{SQSAP}^\star_{m,n^\star}$. □

The following are trivial facet defining inequalities of $\mathrm{SQSAP}^\star_{m,n^\star}$. We will see that they correspond to (13)–(16) of $\mathrm{SQSAP}_{m,n}$ in the next section. Since $\mathrm{QSAP}^\star_{m,n^\star}$ is full-dimensional, these inequalities are essentially unique up to positive multiples.

Proposition 7. *The inequalities*

$$z_f \geq 0 \quad (f \in F^\star), \tag{18}$$

$$-x_{ik} - x_{jk} + 2y_{ikjk} + z(\Delta^\star_{ikjk}) \leq 0 \quad (i,j \in M, k \in N, i<j) \tag{19}$$

define facets of $\mathrm{SQSAP}^\star_{m,n^\star}$.

Proof. We prove that (18) defines a facet for any $f \in F^\star$. This follows from the fact that the face $\mathcal{F} := \{(x,y,z) \in \mathrm{SQSAP}^\star_{m,n^\star} | z_f = 0\}$ contains $|V^\star| + |E^\star| + |F^\star|$ affinely independent vectors: $(0,0,0)$, $(\chi_u, 0, 0)$ $(\forall u \in V^\star)$, $(\chi_u + \chi_v, \chi_e, 0)$ $(\forall e = \{u,v\} \in E^\star)$, $(\chi_u + \chi_v, 0, \chi_{f'})$ $(\forall f' = \{u,v,u',v'\} \in F^\star \setminus f)$. We can prove for (19) in a similar way. □

Next propositions can be proved similarly.

Proposition 8. *The inequalities*

$$y_e \geq 0 \quad (e \in E^\star), \tag{20}$$
$$x(\mathrm{row}^\star_i \cup \mathrm{row}^\star_j) - y(\mathrm{row}^\star_i \cup \mathrm{row}^\star_j) - z(\mathrm{row}^\star_i \cup \mathrm{row}^\star_j) \leq 1 \ (i,j \in M, i < j) \tag{21}$$

define facets of $\mathrm{SQSAP}^\star_{m,n^\star}$. □

Proposition 9. *The inequalities*

$$-x_{ik} + y_{ikjk} \leq 0 \quad ([i,k,j,k] \in E^\star), \tag{22}$$
$$-x_{jk} + y_{ikjk} \leq 0 \quad ([i,k,j,k] \in E^\star), \tag{23}$$
$$-x(\mathrm{row}^\star_i) + y(\mathrm{row}^\star_i \cup \mathrm{row}^\star_j) + z(\mathrm{row}^\star_i \cup \mathrm{row}^\star_j) \leq 0 \quad (i,j \in M, i < j), \tag{24}$$
$$-x(\mathrm{row}^\star_j) + y(\mathrm{row}^\star_i \cup \mathrm{row}^\star_j) + z(\mathrm{row}^\star_i \cup \mathrm{row}^\star_j) \leq 0 \quad (i,j \in M, i < j), \tag{25}$$

define facets of $\mathrm{SQSAP}^\star_{m,n^\star}$. □

4 Basic facial structure of $\mathrm{SQSAP}_{m,n}$

We present a basic facial structure of the polytope $\mathrm{SQSAP}_{m,n}$ by "pulling back" from the polytope $\mathrm{SQSAP}^\star_{m,n^\star}$.

Theorem 10.

$$\dim \mathrm{SQSAP}_{m,n} = \dim \mathbb{R}^V \times \mathbb{R}^E \times \mathbb{R}^F - \left(m + n\frac{m(m-1)}{2}\right).$$

Proof. Since the dimension is invariant under an affine isomorphism, we have

$$\begin{aligned} \dim \mathrm{SQSAP}_{m,n} &= \dim \mathrm{SQSAP}^\star_{m,n^\star} = |V^\star| + |E^\star| + |F^\star| \\ &= \dim \mathbb{R}^V \times \mathbb{R}^E \times \mathbb{R}^F - (m + nm(m-1)/2). \end{aligned}$$

□

Theorem 11. *The affine subspace $\mathcal{A}$ defines the affine hull of* $\mathrm{SQSAP}_{m,n}$.

Proof. From Theorem 10, it suffices to show that the rank of the coefficient matrix of $\mathcal{A}$ is equal to $m + n\binom{m}{2}$. We can see that the coefficient matrix of (11) and (12) is in the following form

$$\begin{array}{|ccc|ccc|c|}
\hline
1\cdots 1 & & & & & & \\
 & \ddots & & & O & & O \\
 & & 1\cdots 1 & & & & \\
\hline
 & & & 2 & & & \\
 & * & & & \ddots & & * \\
 & & & & & 2 & \\
\hline
\end{array},$$

which has full row rank. Thus, the rank is equal to $m + n\binom{m}{2}$. □

We obtain facets of $\mathrm{SQSAP}_{m,n}$ by a simple lifting, called *0-lifting*, from those of $\mathrm{SQSAP}^{\star}_{m,n^{\star}}$.

Lemma 12. *If* $a^{\top}x + b^{\top}y + c^{\top}z \leq d$ *defines a facet of* $\mathrm{SQSAP}^{\star}_{m,n^{\star}}$*, then* $(a,0)^{\top}(x,\bar{x}) + (b,0)^{\top}(y,\bar{y}) + (c,0)^{\top}(z,\bar{z}) \leq d$ *also defines a facet of* $\mathrm{SQSAP}_{m,n}$*, where the variables* $x, \bar{x}, y, \bar{y}, z$*, and* $\bar{z}$ *correspond to* $V^{\star}, V \setminus V^{\star}, E^{\star}, E \setminus E^{\star}, F^{\star}$*, and* $F \setminus F^{\star}$*, respectively.*

Proof. The validity is clear. To show that the inequality defines a facet of $\mathrm{QSAP}_{m,n}$, it is enough to take $\dim \mathrm{SQSAP}_{m,n} - 1$ affinely independent vectors on the face of $\mathrm{QSAP}_{m,n}$. We can take such vectors by pulling back $\dim \mathrm{SQSAP}^{\star}_{m,n^{\star}} - 1$ affinely independent vectors on the corresponding face of $\mathrm{SQSAP}^{\star}_{m,n^{\star}}$ by the inverse map of the star-transformation. □

From Lemma 12, we have facets of $\mathrm{SQSAP}_{m,n}$ by lifting (20)–(25). Note that the representation of the lifted inequalities is not unique because $\mathrm{SQSAP}_{m,n}$ is not full-dimensional. It follows that we can transform a lifted inequality, using the equations (11) and (12), into another form. For example, $z_f \geq 0$ for any $f \in F \setminus F^{\star}$ is derived from 0-lifting of (19) and together with (12).

Theorem 13. *For any* $f \in F$*, the inequality* $z_f \geq 0$ *defines a facet of* $\mathrm{SQSAP}_{m,n}$*.*

Proof. Obtained by 0-lifting of (18) and (19). □

Theorem 14. *For any* $e \in E$*, the inequality* $y_e \geq 0$ *defines a facet of* $\mathrm{SQSAP}_{m,n}$*.*

Proof. Obtained by 0-lifting of (20) and (21). □

Theorem 15. *For any* $e = \{u, v\} \in E$*, the inequalities* $-x_u + y_e \leq 0$ *and* $-x_v + y_e \leq 0$ *define facets of* $\mathrm{SQSAP}_{m,n}$*.*

Proof. Obtained by 0-lifting of (22), (23), (24), and (25). □

5 The curtain facets of $\mathrm{SQSAP}^{\star}_{m,n^{\star}}$

Since $\mathrm{SQSAP}^{\star}_{m,n^{\star}}$ is equivalent to $\mathrm{SQSAP}_{m,n}$, we treat $\mathrm{SQSAP}^{\star}_{m,n^{\star}}$.

For any $i, j \in M (i \neq j)$ and $S \subseteq N^{\star} (S \neq \emptyset)$ we define the *curtain inequality*

$$-x(\mathrm{row}^{\star}_i|_S) + y(\mathrm{row}^{\star}_i|_S : \mathrm{row}^{\star}_j|_S) + z(\mathrm{row}^{\star}_i|_S : \mathrm{row}^{\star}_j|_S) \leq 0, \tag{26}$$

where $\mathrm{row}^\star_i|_S := \{(i,k) \in \mathrm{row}^\star_i \mid k \in S\}$. It is easy to see the validity of (26) for the polytope $\mathrm{SQSAP}^\star_{m,n^\star}$. Curtain inequality was introduced for SQAP-polytope in [3, 5].

Theorem 16. *For any $i,j \in M (i \neq j)$ and $S \subseteq N^\star (S \neq \emptyset)$, the curtain inequality* (26) *defines a facet of* $\mathrm{SQSAP}^\star_{m,n^\star}$.

Proof. Suppose that there is an inequality $a^\top x + b^\top y + c^\top z \leq d$ valid for $\mathrm{QSAP}^\star_{m,n^\star}$ such that

$$\begin{aligned}\mathcal{F} &:= \left\{(x,y,z) \in \mathrm{SQSAP}^\star_{m,n^\star} \mid -x(I) + y(I:J) + z(I:J) = 0\right\} \\ &\subseteq \left\{(x,y,z) \in \mathrm{SQSAP}^\star_{m,n^\star} \;\middle|\; a^\top x + b^\top y + c^\top z = d\right\},\end{aligned}$$

where we put $I := \mathrm{row}^\star_i|_S$ and $J := \mathrm{row}^\star_j|_S$.

(a) Since $(0,0,0) \in \mathcal{F}$, we have $d = 0$.

(b) For any $v \notin I$, we have $(\chi_v, 0, 0) \in \mathcal{F}$. Thus, $a_v = 0$.

(c) For any edge $e = \{u,v\} \in E^\star(\overline{I} : \overline{I})$, where $\overline{I}$ denotes the complement of the set I, it holds that $(\chi_u + \chi_v, \chi_e, 0) \in F$. Hence, $b_e = 0$.

(d) For any edge $e = \{u,v\} \in E^\star(I : J)$ with $u \in I$ and $v \in J$, we have $a_u + b_e = 0$.

(e) For any edge $e = \{u,v\} \in E^\star(I : \overline{J})$ with $u \in I$ and $v \in \overline{J}$, there exists $w \in J$ such that u,v,w is a 3-clique of the hypergraph G. Since $(\chi_u + \chi_v + \chi_w, \chi_{uv} + \chi_{uw} + \chi_{vw}, 0) \in \mathcal{F}$, we have $b_e = 0$.

(f) For any hyperedge $f = \{u_1, v_1, u_2, v_2\} \in F^\star$ with $f \notin F^\star(I : \overline{I})$, we have $(\chi_{u_1} + \chi_{v_1}, 0, \chi_f) \in \mathcal{F}$. Thus, $c_f = 0$.

(g) For any hyperedge $f = \{u_1, v_1, u_2, v_2\} \in F^\star(I : \overline{I} \setminus J)$ with $u_1 \in I$ and $v_1, v_2 \notin J$, there exists $w \in J$ such that $\mathrm{col}^\star(w) = \mathrm{col}^\star(u_1)$. Thus, it holds that $(\chi_{u_1} + \chi_w + \chi_{v_1}, \chi_{u_1 w}, \chi_f + \chi_{w v_1 w' v_2}) \in \mathcal{F}$, where w' is the node uniquely determined by the conditions $\mathrm{row}^\star(w') = \mathrm{row}^\star(w)$ and $\mathrm{col}^\star(w') = \mathrm{col}^\star(u_2)$. Hence, we have $c_f = 0$.

(h) For any hyperedge $f = \{u_1, v_1, u_2, v_2\} \in F^\star(I : \overline{I} \setminus J)$ with $u_1 \in I$ and $v_2 \in J$, we have $(\chi_{u_2} + \chi_{v_2}, 0, \chi_f) \in \mathcal{F}$. Thus, $c_f = 0$.

(i) For any hyperedge $f = \langle i,k,j,l \rangle \in F^\star(I : J)$, we have $(\chi_{ik} + \chi_{jl}, 0, \chi_f) \in \mathcal{F}$ and $(\chi_{il} + \chi_{jk}, 0, \chi_f) \in \mathcal{F}$. Thus, $c_f = -a_{ik} = -a_{il}$.

(j) For any $k, l \in S$, we have $a_{ik} = a_{il}$ by (i). Together with (d), it holds that $a^\top x + b^\top y + c^\top z = \mu(-x(I) + y(I:J) + z(I:J))$ for some μ.

(k) Since $(\chi_v, 0, 0)$ for any $v \in I$ and validity of the inequality, it holds that $\mu > 0$.

From (a)–(k), $a^\top x + b^\top y + c^\top z \leq d$ is a nonnegative multiple of (26). Hence, (26) defines a facet of $\mathrm{SQSAP}^\star_{m,n^\star}$.

□

6 Computational results

We compare computational time and quality (tightness) between LP relaxations of symmetric and asymmetric MIP formulations.

For each $m = 10, \dots, 19$, we randomly generate instances for $n = 2, \dots, m$. The data f, d, c are generated within the following range: $f_{ij} \in \{0, \dots, 15\}$ $(i, j \in M)$, $d_{kl} \in \{0, \dots, 10\}$ $(k, l \in N)$, and $c_{ik} \in \{0, \dots, 100\}$ $(i \in M, l \in N)$, so that d is symmetric. We solved two kinds of LP relaxation problems obtained from QSAP and SQSAP MIP formulations to observe computational times and quality (gap of optimal values). Experiments were performed on PC with Pentium M 1.7 GHz CPU and 512 MB RAM using glpk 4.8 (http://www.gnu.org/software/glpk/glpk.html).

Table 1–Table 10 are the results where 'time', 'bound', and 'MIP' show computational time (in second), optimal value of an LP relaxation (lower bound for a MIP problem), and optimal value of a QSAP instance obtained by branch-and-bound for a MIP formulation, respectively. The symbol '$*$' is attached in the tables to each instance whose optimal solution was *not* integral. The results show that QSAP is very tight; almost every instance was solved just by solving its LP relaxation. We can see that SQSAP is relatively worse than QSAP in terms of a lower bound, but it is much faster than QSAP for n large.

Table 1: Comparison of computational times and optimal values ($m = 10$)

m	n	SQSAP		QSAP		ratios		MIP
		bound	time	bound	time	bound	time	
10	2	252.0	0.0	252.0	0.0	1.00	–	252
	3	366.0	0.0	366.0	0.0	1.00	–	366
	4	336.0	0.0	336.0	0.0	1.00	–	336
	5	263.0	0.0	263.0	0.0	1.00	–	263
	6	*277.7	0.0	*285.5	0.0	0.97	–	286
	7	*247.5	0.0	284.0	0.0	0.87	–	284
	8	247.0	0.0	247.0	0.0	1.00	–	247
	9	233.0	0.0	233.0	0.0	1.00	–	233
	10	299.0	1.0	299.0	0.0	1.00	–	299

7 Conclusion and future work

We investigate the SQSAP-polytope that arises from a compact formulation exploiting certain symmetry of the quadratic semi-assignment problem. We show some basic polyhedral properties such as the dimension, the affine hull, and certain trivial facets and presented a MIP formulation. We also give a class of nontrivial facets called curtain facets for the SQSAP-polytope. Computational results imply that the lower bound by an LP relaxation of the SQSAP MIP formulation can be a good alternative for that

Table 2: Comparison of computational times and optimal values ($m = 11$)

m	n	SQSAP		QSAP		ratios		MIP
		bound	time	bound	time	bound	time	
11	2	557.0	0.0	557.0	0.0	1.00	–	557
	3	461.0	0.0	461.0	0.0	1.00	–	461
	4	321.0	0.0	321.0	0.0	1.00	–	321
	5	352.0	0.0	352.0	0.0	1.00	–	352
	6	332.0	1.0	332.0	0.0	1.00	–	332
	7	253.0	0.0	253.0	1.0	1.00	0.00	253
	8	331.0	1.0	331.0	0.0	1.00	–	331
	9	296.0	0.0	296.0	1.0	1.00	0.00	296
	10	267.0	2.0	267.0	1.0	1.00	2.00	267
	11	256.0	1.0	256.0	1.0	1.00	1.00	256

Table 3: Comparison of computational times and optimal values ($m = 12$)

m	n	SQSAP		QSAP		ratios		MIP
		bound	time	bound	time	bound	time	
12	2	587.0	0.0	587.0	1.0	1.00	0.00	587
	3	556.0	0.0	556.0	0.0	1.00	–	556
	4	383.0	0.0	383.0	0.0	1.00	–	383
	5	453.0	0.0	453.0	0.0	1.00	–	453
	6	345.0	0.0	345.0	0.0	1.00	–	345
	7	369.0	1.0	369.0	1.0	1.00	1.00	369
	8	318.0	1.0	318.0	0.0	1.00	–	318
	9	356.0	1.0	356.0	1.0	1.00	1.00	356
	10	336.0	2.0	336.0	2.0	1.00	1.00	336
	11	373.0	3.0	373.0	3.0	1.00	1.00	373
	12	308.0	3.0	308.0	3.0	1.00	1.00	308

Table 4: Comparison of computational times and optimal values ($m = 13$)

m	n	SQSAP		QSAP		ratios		MIP
		bound	time	bound	time	bound	time	
13	2	457.0	0.0	457.0	0.0	1.00	–	457
	3	447.0	1.0	447.0	0.0	1.00	–	447
	4	418.0	0.0	418.0	0.0	1.00	–	418
	5	344.0	0.0	344.0	1.0	1.00	0.00	344
	6	358.0	1.0	358.0	0.0	1.00	–	358
	7	385.0	1.0	385.0	1.0	1.00	1.00	385
	8	283.0	1.0	283.0	1.0	1.00	1.00	283
	9	373.0	1.0	373.0	1.0	1.00	1.00	373
	10	296.0	2.0	296.0	2.0	1.00	1.00	296
	11	*320.0	2.0	359.0	3.0	0.89	0.67	359
	12	*343.2	4.0	366.0	4.0	0.94	1.00	366
	13	*308.5	5.0	350.0	9.0	0.88	0.56	350

Table 5: Comparison of computational times and optimal values ($m = 14$)

m	n	SQSAP		QSAP		ratios		MIP
		bound	time	bound	time	bound	time	
14	2	567.0	0.0	567.0	0.0	1.00	–	567
	3	426.0	0.0	426.0	0.0	1.00	–	426
	4	413.0	1.0	413.0	0.0	1.00	–	413
	5	573.0	0.0	573.0	0.0	1.00	–	573
	6	538.0	1.0	538.0	0.0	1.00	–	538
	7	494.0	2.0	494.0	1.0	1.00	2.00	494
	8	304.0	1.0	304.0	1.0	1.00	1.00	304
	9	302.0	2.0	302.0	3.0	1.00	0.67	302
	10	*371.0	3.0	381.0	3.0	0.97	1.00	381
	11	*304.0	3.0	*340.5	5.0	0.89	0.60	351
	12	*338.0	5.0	339.0	8.0	1.00	0.62	339
	13	*302.5	7.0	342.0	15.0	0.88	0.47	342
	14	379.0	14.0	379.0	14.0	1.00	1.00	379

Table 6: Comparison of computational times and optimal values ($m = 15$)

m	n	SQSAP		QSAP		ratios		MIP
		bound	time	bound	time	bound	time	
15	2	604.0	0.0	604.0	1.0	1.00	0.00	604
	3	502.0	0.0	502.0	0.0	1.00	–	502
	4	333.0	0.0	333.0	0.0	1.00	–	333
	5	460.0	0.0	460.0	0.0	1.00	–	460
	6	451.0	1.0	451.0	0.0	1.00	–	451
	7	350.0	2.0	350.0	1.0	1.00	2.00	350
	8	317.0	2.0	317.0	2.0	1.00	1.00	317
	9	*391.5	3.0	431.0	3.0	0.91	1.00	431
	10	346.0	4.0	346.0	5.0	1.00	0.80	346
	11	*356.5	4.0	398.0	7.0	0.90	0.57	398
	12	*339.0	9.0	406.0	13.0	0.83	0.69	406
	13	*310.5	13.0	352.0	14.0	0.88	0.93	352
	14	*303.5	14.0	378.0	19.0	0.80	0.74	378
	15	*244.5	14.0	363.0	30.0	0.67	0.47	363

Table 7: Comparison of computational times and optimal values ($m = 16$)

m	n	SQSAP		QSAP		ratios		MIP
		bound	time	bound	time	bound	time	
16	2	516.0	0.0	516.0	0.0	1.00	–	516
	3	641.0	0.0	641.0	0.0	1.00	–	641
	4	531.0	0.0	531.0	0.0	1.00	–	531
	5	446.0	1.0	446.0	0.0	1.00	–	446
	6	376.0	1.0	376.0	0.0	1.00	–	376
	7	398.0	2.0	398.0	2.0	1.00	1.00	398
	8	374.0	4.0	374.0	3.0	1.00	1.33	374
	9	363.0	5.0	363.0	4.0	1.00	1.25	363
	10	*365.0	4.0	492.0	6.0	0.74	0.67	492
	11	*367.0	7.0	410.0	8.0	0.90	0.88	410
	12	*368.0	11.0	392.0	11.0	0.94	1.00	392
	13	*352.0	14.0	387.0	12.0	0.91	1.17	387
	14	*324.0	15.0	418.0	31.0	0.78	0.48	418
	15	*277.0	20.0	*351.5	46.0	0.79	0.43	357
	16	*232.5	25.0	289.0	97.0	0.80	0.26	289

Table 8: Comparison of computational times and optimal values ($m = 17$)

m	n	SQSAP		QSAP		ratios		MIP
		bound	time	bound	time	bound	time	
17	2	463.0	0.0	463.0	0.0	1.00	–	463
	3	601.0	0.0	601.0	0.0	1.00	–	601
	4	527.0	1.0	527.0	0.0	1.00	–	527
	5	*539.5	0.0	*539.5	1.0	1.00	0.00	546
	6	400.0	1.0	400.0	1.0	1.00	1.00	400
	7	383.0	1.0	383.0	2.0	1.00	0.50	383
	8	477.0	3.0	477.0	1.0	1.00	3.00	477
	9	367.0	3.0	367.0	3.0	1.00	1.00	367
	10	465.0	5.0	465.0	4.0	1.00	1.25	465
	11	477.0	7.0	477.0	7.0	1.00	1.00	477
	12	397.0	13.0	397.0	9.0	1.00	1.44	397
	13	432.0	16.0	432.0	24.0	1.00	0.67	432
	14	*391.5	28.0	*433.0	88.0	0.90	0.32	449
	15	*313.0	36.0	411.0	36.0	0.76	1.00	411
	16	*301.5	44.0	406.0	179.0	0.74	0.25	406
	17	*250.0	35.0	*416.0	145.0	0.60	0.24	419

of the QSAP MIP one in branch-and-bound methods in terms of computational time. Our future work is to develop a branch-and-cut procedure by facets of the SQSAP-polytope.

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Table 9: Comparison of computational times and optimal values ($m = 18$)

m	n	SQSAP		QSAP		ratios		MIP
		bound	time	bound	time	bound	time	
18	2	868.0	0.0	868.0	0.0	1.00	–	868
	3	823.0	0.0	823.0	0.0	1.00	–	823
	4	639.0	1.0	639.0	0.0	1.00	–	639
	5	733.0	1.0	733.0	1.0	1.00	1.00	733
	6	616.0	3.0	616.0	2.0	1.00	1.50	616
	7	634.0	3.0	634.0	2.0	1.00	1.50	634
	8	661.0	6.0	661.0	4.0	1.00	1.50	661
	9	652.0	7.0	652.0	7.0	1.00	1.00	652
	10	535.0	15.0	535.0	13.0	1.00	1.15	535
	11	476.0	24.0	476.0	22.0	1.00	1.09	476
	12	*409.5	34.0	418.0	43.0	0.98	0.79	418
	13	*429.5	31.0	450.0	71.0	0.95	0.44	450
	14	*410.0	32.0	424.0	81.0	0.97	0.40	424
	15	*345.5	59.0	418.0	122.0	0.83	0.48	418
	16	*339.0	104.0	384.0	135.0	0.88	0.77	384
	17	*398.5	149.0	435.0	198.0	0.92	0.75	435
	18	*315.0	148.0	431.0	695.0	0.73	0.21	431

Table 10: Comparison of computational times and optimal values ($m = 19$)

m	n	SQSAP		QSAP		ratios		MIP
		bound	time	bound	time	bound	time	
19	2	921.0	0.0	921.0	0.0	1.00	–	921
	3	771.0	0.0	771.0	0.0	1.00	–	771
	4	821.0	1.0	821.0	1.0	1.00	1.00	821
	5	848.0	2.0	848.0	1.0	1.00	2.00	848
	6	761.0	2.0	761.0	1.0	1.00	2.00	761
	7	682.0	5.0	682.0	3.0	1.00	1.67	682
	8	832.0	8.0	832.0	5.0	1.00	1.60	832
	9	581.0	10.0	581.0	18.0	1.00	0.56	581
	10	475.0	14.0	475.0	26.0	1.00	0.54	475
	11	437.0	19.0	437.0	61.0	1.00	0.31	437
	12	*490.5	28.0	512.0	64.0	0.96	0.44	512
	13	*507.0	44.0	597.0	87.0	0.85	0.51	597
	14	*401.0	103.0	436.0	155.0	0.92	0.66	436
	15	*403.5	118.0	*503.5	287.0	0.80	0.41	505
	16	*414.5	112.0	475.0	547.0	0.87	0.20	475
	17	*276.0	335.0	418.0	1109.0	0.66	0.30	418
	18	*323.0	260.0	445.0	895.0	0.73	0.29	445
	19	*325.5	791.0	454.0	8770.0	0.72	0.09	454

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