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Abstract

In this expository paper we illustrate the generality of game theoretic probability protocols of Shafer and Vovk (2001) in finite-horizon discrete games. By restricting ourselves to finite-horizon discrete games, we can explicitly describe how discrete distributions with finite support and the discrete pricing formulas, such as the Cox-Ross-Rubinstein formula, are naturally derived from game-theoretic probability protocols. Corresponding to any discrete distribution with finite support, we construct a finite-horizon discrete game, a replicating strategy of Skeptic, and a neutral forecasting strategy of Forecaster, such that the discrete distribution is derived from the game. Construction of a replicating strategy is the same as in the standard arbitrage arguments of pricing European options in the binomial tree models. However the game theoretic framework is advantageous because no a priori probabilistic assumption is needed.

Keywords and phrases: binomial distribution, Cox-Ross-Rubinstein formula, hypergeometric distribution, lower price, Polya's distribution, probability protocol, replicating strategy, upper price

1 Introduction

In the game theoretic probability of Shafer and Vovk (2001), probability distributions and probability models are not assumed a priori but derived as logical consequences of certain protocol of a game between two players "Skeptic" and "Reality". In this game Skeptic tries to become rich by exploiting patterns in the moves of Reality. In order to prevent Skeptic from becoming rich, Reality is in a sense forced to behave probabilistically. Therefore probability distributions are determined by the protocol of the game. This feature of the game theoretic probability is well illustrated by Shafer and Vovk (2001) in their derivation of Skeptic's strategy forcing the strong law of large numbers (Chapter 3) and the derivation of Black-Scholes formula (Chapter 9). Also in Takeuchi's exposition of the game theoretic probability and finance (Takeuchi (2004)) this point is discussed with many interesting examples. Recently Kumon and Takemura (2005) gave a very simple strategy forcing the strong law of large numbers.

In the standard stochastic derivation of option pricing formulas, empirical probability is assumed first, but then by arbitrage arguments, the empirical probability is replaced by the risk neutral probability and the price of an option is given as the expected value with respect to the risk neutral probability. The risk neutral probability is often explained as a purely operational device useful in expressing the option price in a convenient form. On the other hand in the game theoretic probability the risk neutral probability is more substantial, in the sense that Reality is forced to behave according to the risk neutral probability to avoid arbitrage by Skeptic. We should mention here that in Shafer and Vovk (2001) "forcing" is used only for infinite-horizon games. In this paper we somewhat informally use the word to mean that Reality should avoid arbitrage by Skeptic in the setting of finite-horizon games.

Additional flexibility of game theoretic probability is gained by introducing the third player "Forecaster" into the game. At the beginning of each round Forecaster sets the price for Reality's move. By appropriately specifying the strategy of Forecaster, Reality's moves can be forced to follow any prespecified distribution.

In this paper we demonstrate the above features of the game theoretic probability in the setting of finite-horizon discrete games. For expository purposes we start with the simplest setting of the coin-tossing game and derive binomial distribution in Section 2 and give an analogous derivation of the Cox-Ross-Rubinstein formula in Section 3. We discuss derivation of hypergeometric distribution and Polya's distribution in Section 4 in order to illustrate the role of Forecaster. Then in Section 5 we discuss derivation of an arbitrary discrete distribution with finite support. Multivariate extension is given in Section 6. Some preliminary material on game theoretic probability is given in Appendix.

2 Derivation of binomial distribution

Consider the finite-horizon fair-coin game in Section 6.1 of Shafer and Vovk (2001). Its protocol is given as follows.

FAIR-COIN GAME
Protocol:

$\mathcal{K}_0 = \alpha$: given
FOR $n = 1, \ldots, N$
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-1, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
END FOR

In this protocol a game theoretic version of Chebyshev inequality is proved in (6.9) of

S&V in the following form:

$$\underline{\mathbb{P}}\left\{\left|\frac{S_N}{N}\right| \le \epsilon\right\} \ge 1 - \frac{1}{N\epsilon^2},$$

where $S_N = x_1 + \cdots + x_N$ and $\underline{\mathbb{P}}$ denotes the lower probability. Actually the equality of the upper probability and the lower probability

$$\underline{\mathbb{P}}\left\{\left|\frac{S_N}{N}\right| \le \epsilon\right\} = \bar{\mathbb{P}}\left\{\left|\frac{S_N}{N}\right| \le \epsilon\right\} \tag{1}$$

holds here and this probability is given by binomial distribution. Although this fact is contained in a more general statement of Proposition 8.5 of S&V, we give a full proof of this fact employing standard arbitrage arguments.

In order to treat success probability $p \neq 1/2$, let us consider the following biased-coin game.

Biased-Coin Game
Protocol:

$\mathcal{K}_0 = \alpha \in \mathbb{R},\ a, b > 0$: given
FOR $n = 1, \dots, N$
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{a, -b\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
END FOR

As above we write $S_n = x_1 + \cdots + x_n$. Let $S_0 = 0$. Consider a random variable $x(\xi) = \eta(S_N)$ which depends only on S_N (European option). Then we have the following basic result.

Theorem 2.1 *The upper and the lower expected values of $\eta(S_N)$ coincide and given by*

$$\bar{\mathbb{E}}(\eta(S_N)) = \underline{\mathbb{E}}(\eta(S_N)) = \sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m} \eta(ma - (N-m)b), \tag{2}$$

where $p = b/(a+b)$ is the risk neutral probability.

Proof: The first step of our proof consists of defining a "candidate" price of the European option. In the second step we verify that the candidate price is actually the precise price, by constructing a replicating strategy.

Let $\bar{\eta}(n, S_n)$, $S_n = -nb, -(n-1)b + a, \dots na$, denote the price of $\eta(S_N)$ at time n. We require $\eta(n, S_n)$ to satisfy the following "partial difference equation"

$$\bar{\eta}(n, S_n) = p\bar{\eta}(n+1, S_n + a) + q\bar{\eta}(n+1, S_n - b), \quad 0 \le n < N, \tag{3}$$

where $q = 1 - p$. Note that (3) with $p = 1/2$ is a discrete version of the heat equation. The terminal condition for $\bar{\eta}(n, S_n)$ is given by

$$\bar{\eta}(N, S_N) = \eta(S_N), \quad S_N = ma - (N - m)b, \; m = 0, \dots, N. \tag{4}$$

Starting with the terminal condition (4) we can solve for $\bar{\eta}(n, S_n)$ in (3) by backward induction $n = N - 1, N - 2, \dots, 0$. Then the initial value $\bar{\eta}(0, 0)$ is easily calculated as

$$\begin{aligned}
\bar{\eta}(0,0) &= p\bar{\eta}(1,a) + q\bar{\eta}(1,-b) \\
&= p(p\bar{\eta}(2,2a) + q\bar{\eta}(2,a-b)) + q(p\bar{\eta}(2,a-b) + q\bar{\eta}(2,-2b)) \\
&= p^2\bar{\eta}(2,2a) + 2pq\bar{\eta}(2,a-b) + q^2\bar{\eta}(2,-2b) \\
&= \dots \\
&= \sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m} \eta(ma - (N-m)b).
\end{aligned}$$

Now we describe a replicating strategy for $\eta(S_N)$ with the the replicating initial capital $\bar{\eta}(0, 0)$. For $n = 1, \dots, N$, let

$$M_n = \frac{\bar{\eta}(n, S_{n-1} + a) - \bar{\eta}(n, S_{n-1} - b)}{a + b}. \tag{5}$$

Note that $a + b$ can be written as

$$a + b = (S_{n-1} + a) - (S_{n-1} - b).$$

Therefore M_n is the ratio of the increments of $\bar{\eta}(n, \mathcal{S}_n)$ and S_n and is called the "delta hedge". We now check that this M_n gives a replicating strategy $\mathcal{P}$. This can be confirmed by forward induction. At the end of the first round $n = 1$,

$$\begin{aligned}
\bar{\eta}(0,0) + \mathcal{K}_1^{\mathcal{P}} &= \bar{\eta}(0,0) + M_1 x_1 \\
&= \bar{\eta}(0,0) + \frac{\bar{\eta}(1,a) - \bar{\eta}(1,-b)}{a+b} x_1 \\
&= \begin{cases} \bar{\eta}(0,0) + q(\bar{\eta}(1,a) - \bar{\eta}(1,-b)), & \text{if } x_1 = a, \\ \bar{\eta}(0,0) - p(\bar{\eta}(1,a) - \bar{\eta}(1,-b)), & \text{if } x_1 = -b \end{cases} \\
&= \begin{cases} \bar{\eta}(1,a), & \text{if } x_1 = a, \\ \bar{\eta}(1,-b), & \text{if } x_1 = -b, \end{cases} \\
&= \bar{\eta}(1, S_1).
\end{aligned}$$

Similarly at the end of round $n = 2$, we have

$$\begin{aligned}
\bar{\eta}(0,0) + \mathcal{K}_2^{\mathcal{P}} &= \begin{cases} \bar{\eta}(2, S_1 + a), & \text{if } x_2 = a, \\ \bar{\eta}(2, S_1 - b), & \text{if } x_2 = -b, \end{cases} \\
&= \bar{\eta}(2, S_2).
\end{aligned}$$

Now by induction we arrive at

$$\bar{\eta}(0,0) + \mathcal{K}_N^{\mathcal{P}} = \bar{\eta}(N, S_N) = \eta(S_N).$$

We have confirmed that M_n in (5) with the replicating initial capital (2) gives a replicating strategy for $\eta(S_N)$. Hence the theorem holds by Proposition A.1 in Appendix. □

In particular if we take

$$\eta(S_N) = \begin{cases} 1, & \text{if } |S_N|/N \le \epsilon, \\ 0, & \text{otherwise}, \end{cases}$$

we see that the equality holds in (1) and the probability is given by binomial distribution.

In this section we took Reality's move space as $\{a, -b\}$. This is convenient in comparing Theorem 2.1 with the Cox-Ross-Rubinstein formula in the next section. However for generalization of binomial distribution to hypergeometric distribution in Section 4, it is more convenient to rescale Reality's move space to $\{0, 1\}$. Then we need to introduce the price p for the "ticket" x_n. The rescaled protocol is written as follows.

Rescaled Biased-Coin Game
Protocol:

$\mathcal{K}_0 = \alpha, 0 < p < 1$: given
FOR $n = 1, \ldots, N$
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{0, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - p)$.
END FOR

It is clear that the biased-coin game and the rescaled biased-coin game is equivalent by the affine correspondence $x_n \leftrightarrow (a+b)(x_n - p), p = b/(a+b)$. In the rescaled version the expected value in (2) is simply written as

$$\bar{\mathbb{E}}(\eta(S_N)) = \underline{\mathbb{E}}(\eta(S_N)) = \sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m} \eta(m). \tag{6}$$

Furthermore, since the increment of S_n is normalized to be 1, the replicating strategy in (5) is simply written as

$$M_n = \bar{\eta}(n, S_{n-1} + 1) - \bar{\eta}(n, S_{n-1}). \tag{7}$$

It is also conceptually very important to consider the single step game i.e. the game with $N = 1$. Note that each round n of the N step biased-coin game can be viewed as a single step game. In the single step game binomial distribution reduces to a Bernoulli trial. This implies that given the price p, Reality's move x_n for each round n is exactly the same as a single Bernoulli trial with success probability p. Furthermore this behavior of Reality is dictated solely by the value of p, independently from the past moves $x_1, \ldots, x_{n-1}$ of Reality. Therefore in the Rescaled Biased-Coin Game, Reality's moves $x_1, \ldots, x_N$ are independent Bernoulli trials.

3 Derivation of the Cox-Ross-Rubinstein formula

Here we present a game theoretic formulation and derivation of the Cox-Ross-Rubinstein formula (Cox, Ross and Rubinstein (1979)), which is fully discussed in many introductory textbooks on option pricing (e.g., Shreve (2004), Chapter 2 of Baxter and Rennie (1996), Chapter 8 of Capiński and Zastawniak (2003)). Once an appropriate game is formulated, the rest of the argument is the same as in the previous section.

Our protocol for Cox-Ross-Rubinstein game is as follows.

Cox-Ross-Rubinstein Game
Protocol:
$S_0 > 0, u > r > d > 0$: given
FOR $n = 1, \ldots, N$
Skeptic announces $M_n \in \mathbb{R}$.
Reality announces $x_n \in \{u, d\}$.
$S_n := S_{n-1} \times x_n$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(S_n - S_{n-1}) + (r-1)(\mathcal{K}_{n-1} - M_n S_{n-1})$.
END FOR

Here $\mathcal{K}_{n-1} - M_n S_{n-1}$ is the amount of riskless bond held by Skeptic for the round n and $r-1$ is the fixed riskless interest rate. Although by appropriate discounting we may put $r = 1$ without essential loss of generality (Section 12.1 of S&V), here we leave the interest rate r as in standard derivation of the Cox-Ross-Rubinstein formula. $p = (r-d)/(u-d)$ is called the risk neutral probability.

Let $\eta(S_N)$ denote a payoff function of a European option depending on S_N. Corresponding to Theorem 2.1 we have the following result.

Theorem 3.1 *(the Cox-Ross-Rubinstein formula) The upper and the lower expected values of $\eta(S_N)$ coincide and given by*

$$\bar{\mathbb{E}}(\eta(S_N)) = \underline{\mathbb{E}}(\eta(S_N)) = \frac{1}{r^N} \sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m} \eta(u^m d^{N-m} S_0), \tag{8}$$

where $p = (r-d)/(u-d)$ is the risk neutral probability.

Proof: As in the previous section we define $\bar{\eta}(n, S_n)$ by backward induction. Put $\bar{\eta}(N, S_N) = \eta(S_N)$ and for $n = N-1, \ldots, 0$, define

$$\bar{\eta}(n, S_n) = \frac{1}{r}\Big(p\bar{\eta}(n+1, uS_n) + (1-p)\bar{\eta}(n+1, dS_n)\Big)$$

Then the initial value $\bar{\eta}(0, S_0)$ is easily calculated as

$$\bar{\eta}(0, S_0) = \frac{1}{r^N} \sum_{m=0}^{N} \binom{N}{m} p^m (1-p)^{N-m} \eta(u^m d^{N-m} S_0).$$

This becomes the replicating initial capital of the following replicating strategy:

$$M_n = \frac{\bar{\eta}(n, uS_{n-1}) - \bar{\eta}(n, dS_{n-1})}{(u-d)S_{n-1}}.$$

Since the game is coherent by the requirement $u > r > d$, the theorem follows from Proposition A.1. □

4 Hypergeometric distribution and Polya's distribution

In the rescaled biased-coin game of Section 2, the price p of the ticket x_n was a constant. Therefore the third player "Forecaster" did not enter the protocol. Now we introduce Forecaster, who sets the price of the ticket at the beginning of each round in the rescaled biased-coin game. We illustrate the role of Forecaster below by deriving the hypergeometric distribution. We also derive Polya's distribution.

Rescaled Biased-Coin Game With Forecaster
Protocol:

$\mathcal{K}_0 = \alpha$: given
FOR $n = 1, \dots, N$
  Forecaster announces $p_n \in \mathbb{R}$.
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{0, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - p_n)$.
END FOR

Note that if Forecaster announces $p_n > 1$ or $p_n < 0$, then Skeptic can become infinitely rich immediately by taking $|M_n|$ arbitrarily large. Therefore we can restrict the move space of Forecaster to be $[0, 1]$. Furthermore if $p_n = 0$, Skeptic can still take M_n arbitrarily large, which forces Reality to choose $x_n = 0$. Similarly if $p_n = 1$, then Reality is forced to choose $x_n = 1$.

Now consider a strategy of Forecaster. A strategy of Forecaster is called *neutral* (Section 8.2 of S&V) if p_n is determined by the past moves of Reality $x_1 \dots x_{n-1}$. From now on we only consider neutral strategies for Forecaster. Furthermore for simplicity we consider neutral strategy depending on $S_{n-1} = x_1 + \cdots + x_{n-1}$ and write $p_n = p_n(S_{n-1})$, which we may call "Markovian neutral strategy". In Markovian neutral strategy Forecaster only needs to keep S_{n-1} in memory to choose his move.

Consider an urn with ν_1 red balls and ν_2 black balls, where $\nu_1 + \nu_2 \geq N$. Let $x_n = 1$ correspond to drawing a red ball and let $x_n = 0$ correspond to drawing a black ball from the urn by Reality. Let p_n be the ratio of red balls in the urn at the n-th round. Then

$$p_n = p_n(S_{n-1}) = \frac{\max(0, \nu_1 - S_{n-1})}{\nu_1 + \nu_2 - (n-1)}.$$

Actually here we do need to take the positive part of $\nu_1 - S_{n-1}$, because as remarked above once the boundary $S_{n-1} = \nu_1$ is attained, then $0 = p_n = p_{n+1} = \dots$ and Reality is forced to choose $0 = x_n = x_{n+1} = \dots$, which results in $\nu_1 = S_n = S_{n+1} = \dots$. Now we write out a game of sampling without replacement from an urn.

Game of Sampling Without Replacement From An Urn

Protocol:

$\mathcal{K}_0 = \alpha, \nu_1 > 0, \nu_2 > 0, \nu_1 + \nu_2 \geq N$, $S_0 = 0$: given

FOR $n = 1, \dots, N$

Forecaster announces $p_n = (\nu_1 - S_{n-1})/(\nu_1 + \nu_2 - n + 1)$

Skeptic announces $M_n \in \mathbb{R}$.

Reality announces $x_n \in \{0, 1\}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - p_n)$.

$S_n := S_{n-1} + x_n$.

END FOR

For this game the upper and the lower values of the payoff $\eta(\mathcal{S}_N)$ coincide and are given by the expected value with respect to the hypergeometric distribution:

$$\bar{\mathbb{E}}(\eta(S_N)) = \underline{\mathbb{E}}(\eta(S_N)) = \sum_{m=\max(0,N-\nu_2)}^{\min(\nu_1,N)} \eta(m) \frac{\binom{\nu_1}{m}\binom{\nu_2}{N-m}}{\binom{\nu_1+\nu_2}{N}}. \tag{9}$$

This result is actually almost obvious from the discussion at the end of Section 2, namely, at each round n Reality's move x_n is like drawing a ball from an urn with $\nu_1 - S_{n-1}$ red balls and $\nu_2 - (n - 1 - S_{n-1})$ black balls. However it is instructive to look at a formal proof of (9).

Define a candidate price of $\eta(S_N)$ at time n by backward induction:

$$\bar{\eta}(n, S_n) = p_{n+1}(S_n)\bar{\eta}(n+1, S_n+1) + (1 - p_{n+1}(S_n))\bar{\eta}(n+1, S_n), \ n = N-1, \dots, 0,$$

with the terminal condition $\bar{\eta}(N, S_N) = \eta(S_N)$. Then by fully expanding the recurrence relation we have

$$\bar{\eta}(0,0) = \sum_{\substack{(x_1,\dots,x_N)\in\{0,1\}^N \\ \max(0,N-\nu_2)\leq S_N\leq \min(\nu_1,N)}} \prod_{n=1}^{N} p_n(S_{n-1})^{x_n}(1 - p_n(S_{n-1}))^{1-x_n}\eta(S_N). \tag{10}$$

Actually we do not need the restriction $\max(0, N - \nu_2) \leq S_N \leq \min(\nu_1, N)$ in the summation, because $\prod_{n=1}^{N} p_n(S_{n-1})^{x_n}(1 - p_n(S_{n-1}))^{1-x_n} = 0$ for S_N outside of this range. Now it is easily seen that

$$\prod_{n=1}^{N} p_n(S_{n-1})^{x_n}(1-p_n(S_{n-1}))^{1-x_n}$$

$$= \frac{\nu_1(\nu_1-1)\cdots(\nu_1-S_N+1)\ \cdot\ \nu_2(\nu_2-1)\cdots(\nu_2-N+S_N+1)}{(\nu_1+\nu_2)(\nu_1+\nu_2-1)\cdots(\nu_1+\nu_2-N+1)}$$

$$= \frac{\frac{\nu_1!}{(\nu_1-S_N)!}\frac{\nu_2!}{(\nu_2-(N-S_N))!}}{\frac{(\nu_1+\nu_2)!}{(\nu_1+\nu_2-N)!}}.$$

Therefore

$$\bar{\eta}(0,0) = \sum_{\substack{(x_1,\ldots,x_N)\in\{0,1\}^N \\ \max(0,N-\nu_2)\le S_N\le\min(\nu_1,N)}} \frac{\frac{\nu_1!}{(\nu_1-S_N)!}\frac{\nu_2!}{(\nu_2-(N-S_N))!}}{\frac{(\nu_1+\nu_2)!}{(\nu_1+\nu_2-N)!}}\eta(S_N).$$

Now for a given value of S_N, the summation just counts the number of ways of choosing S_N 1's among $x_1,\ldots,x_N$. It follows that

$$\bar{\eta}(0,0) = \sum_{\max(0,N-\nu_2)\le S_N\le\min(\nu_1,N)} \frac{N!}{S_N!(N-S_N)!}\frac{\frac{\nu_1!}{(\nu_1-S_N)!}\frac{\nu_2!}{(\nu_2-(N-S_N))!}}{\frac{(\nu_1+\nu_2)!}{(\nu_1+\nu_2-N)!}}\eta(S_N),$$

which proves (9).

The above argument can be immediately applied to Polya's urn model (Section V.2 of Feller (1968)). In this scheme, when a ball is drawn from an urn, it is replaced and, moreover, c balls of the same color are added. Then the game corresponding to Polya's urn model differs from the game of sampling without replacement only in the specification of Forecaster's neutral strategy. In Polya's urn model

$$p_n = p_n(S_{n-1}) = \frac{\nu_1 + cS_{n-1}}{\nu_1+\nu_2+(n-1)c}.$$

Then as in (2.4) of Section V.2 of Feller (1968), the expected value of $\eta(S_N)$ in this game is written as

$$\bar{\mathbb{E}}(\eta(S_N)) = \underline{\mathbb{E}}(\eta(S_N)) = \sum_{m=0}^{N}\eta(m)\frac{\binom{-\nu_1/c}{m}\binom{-\nu_2/c}{N-m}}{\binom{-(\nu_1+\nu_2)/c}{N}}, \tag{11}$$

where the binomial coefficient $\binom{r}{n}$ for a real r and nonnegative integer k denotes

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}. \tag{12}$$

Note that (6) and (9) are special cases of (11) with $c = 0$ and $c = -1$, respectively. In (9) the range of summation can be taken as $m = 0,\ldots,N$, with the convention (12).

5 Arbitrary discrete distribution with finite support

So far we have discussed how to derive some classical distributions. We now show that given any distribution on $\{0,\dots,N\}$, we can specify a neutral strategy of Forecaster in a game with N rounds such that Reality follows the distribution.

Let $q_m \ge 0, m = 0,\dots,N$, $\sum_{m=0}^N q_m = 1$, denote an arbitrary probability distribution on $\{0,\dots,N\}$. By decreasing N if necessary, we assume $q_N > 0$. Define

$$\bar{q}_m = \frac{q_m + \cdots + q_N}{q_{m-1} + q_m + \cdots + q_N}, \quad m = 1,\dots,N.$$

Then

$$q_m = \bar{q}_1 \cdots \bar{q}_m (1 - \bar{q}_{m+1}), \quad m = 1,\dots,N, \ \bar{q}_{N+1} = 0. \tag{13}$$

Let

$$p_n = \begin{cases} \bar{q}_n, & \text{if } S_{n-1} = n-1, \\ 0, & \text{otherwise.} \end{cases}$$

The idea here is to let Reality increase S_{n-1} by 1 with probability $\bar{q}_{n-1}$ if $S_{n-1} = n-1$ or otherwise let him stop at the current level for the rest of the rounds. Note that $p_n = p_n(S_{n-1})$ is indeed a function of S_{n-1}, because it is written as

$$p_n = \bar{q}_n \times I(S_{n-1} = n-1),$$

where $I(\cdot)$ is the indicator function.

Biased-Coin Game With Forecaster For Arbitrary Distribution
Protocol:

$\mathcal{K}_0 = \alpha$, $S_0 = 0$, $q_m \ge 0, m = 0,\dots,N$, $\sum_{m=0}^N q_m = 1$: given
FOR $n = 1,\dots,N$
  Forecaster announces $p_n = \bar{q}_n \times I(S_{n-1} = n-1)$.
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{0,1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n(x_n - p_n)$.
  $S_n := S_{n-1} + x_n$.
END FOR

The tree of this game is illustrated in Figure 1. For this game we have the following result.

Theorem 5.1 *The upper and the lower expected values of $\eta(S_N)$ coincide and given by*

$$\bar{\mathbb{E}}(\eta(S_N)) = \underline{\mathbb{E}}(\eta(S_N)) = \sum_{m=0}^N \eta(m) q_m. \tag{14}$$

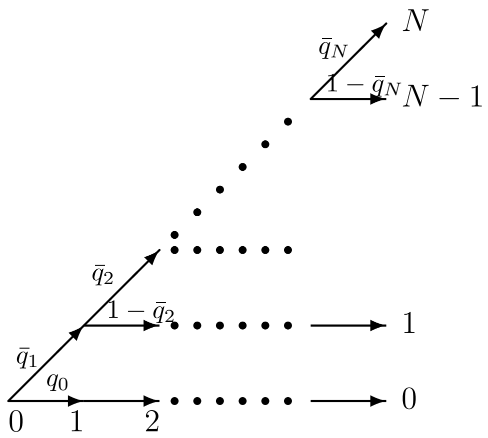

Figure 1: Tree of the game for arbitrary distribution

Proof: As in the case of hypergeometric distribution

$$\bar{\eta}(0,0) = \sum_{(x_1,\ldots,x_N)\in\{0,1\}^N} \prod_{n=1}^{N} p_n(S_{n-1})^{x_n}(1-p_n(S_{n-1}))^{1-x_n}\eta(S_N).$$

In this game, the path leading to S_m is uniquely determined as

$$x_1 \ldots x_N = 1 \ldots 1\, 0 \ldots 0$$

with m initial 1's. By (13), for this path $\prod_{n=1}^{N} p_n(S_{n-1})^{x_n}(1-p_n(S_{n-1}))^{1-x_n} = q_m$. Therefore $\bar{\eta}(0,0) = \sum_{m=0}^{N}\eta(m)q_m$. The replicating strategy confirming this candidate price is given as

$$M_n = \begin{cases} \frac{\sum_{m=n}^{N}\eta(m)q_m}{\sum_{m=n}^{N} q_m} - \eta(S_{n-1}), & \text{if } S_{n-1} = n-1, \\ \infty, & \text{otherwise.} \end{cases}$$

□

In Theorem 5.1 we have considered a discrete distribution with the support $\{0,\ldots,N\}$. We can deal with the support of the form $\{a, a+1, \ldots, b\}$, by letting $N = b-a$ and setting the initial value $S_0 = a$.

In Section 8.3 of their book, Shafer and Vovk discuss "adding tickets" to make the upper expected value and the lower expected value to coincide. Note that if Reality's move space has more than two elements in a single step game, then the upper expected value is generally larger than the lower expected value. Theorem 5.1 shows that if we add sufficient number of steps to a single step game, the equality of the upper and the lower prices is achieved.

6 Multivariate extension

In the previous sections we have considered univariate random variable S_N. In this section we give a straightforward multivariate extension of the results of the previous sections. We employ the multi-label classification protocol discussed in Vovk, Nouretdinov, Takemura and Shafer (2005).

Our extension corresponds to generalizing Binomial distribution to multinomial distribution. Let $S_N = (S_N^1, \dots, S_N^d)$ be a d-dimensional vector. As in multinomial distribution, for the sake of symmetry, we leave one-dimensional redundancy $N = S_N^1 + \cdots + S_N^d$ in the components of S_N. Therefore x_n in Rescaled Biased-Coin Game now corresponds to a 2-dimensional vector $(x_n, 1-x_n)$. For the general d-dimensional case the move space of Reality

$$\mathbf{X} = \{e_1, \dots, e_d\} = \{(1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,\dots,0,1)\}$$

consists of d standard coordinate vectors.

Multilabel Classification Game With Neutral Forecasting Strategy

Protocol:

$\mathcal{K}_0 = \alpha$, $S_0 = 0$: given

FOR $n = 1, \dots, N$

Forecaster announces $p_n = p_n(S_{n-1}) \in \mathbb{R}^d$.

Skeptic announces $M_n \in \mathbb{R}^d$.

Reality announces $x_n \in \mathbf{X}$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n \cdot (x_n - p_n)$.

$S_n := S_{n-1} + x_n$.

END FOR

Here "$\cdot$" denotes the standard inner product of $\mathbb{R}^d$.

In the above protocol we took the whole $\mathbb{R}^d$ as the move space of Forecaster. Let

$$\Delta(\mathbf{X}) = \{(p^1, \dots, p^d) \mid p^i \geq 0, \sum_{i=1}^d p^i = 1\}$$

denote the probability simplex spanned by the standard coordinate vectors. If Forecaster announces $p_n \notin \Delta(\mathbf{X})$, then by the hyperplane separation theorem Skeptic can choose $M_n \in \mathbb{R}^d$ such that he becomes infinitely rich immediately, no matter what move Reality chooses. See Vovk, Nouretdinov, Takemura and Shafer (2005) for a discussion of this point. Therefore we can restrict Forecaster's move space to the probability simplex $\Delta(\mathbf{X})$. Also if $p_n^i = 0$ for some i, Skeptic can choose M_n^i arbitrarily large and Reality is forced to choose $x_n^i = 0$.

We also note that there is a redundancy in the move space of Skeptic, once p_n is restricted to lie in $\Delta(\mathbf{X})$. $M_n + c(1, \dots, 1)$ for any $c \in \mathbb{R}$ leads to the same increment of the capital process $\mathcal{K}_n$. However it is often convenient to ignore this redundancy in specifying Reality's move M_n.

For notational simplicity write

$$p_n^{x_n} = (p_n^1)^{x_n^1} \cdots (p_n^d)^{x_n^d} = p_n^i \qquad \text{for } x_n = e_i.$$

As a straightforward generalization of results in the previous sections we have the following theorem.

Theorem 6.1 *The upper and the lower expected values of $\eta(S_N)$ coincide and given by*

$$\bar{\mathbb{E}}(\eta(S_N)) = \underline{\mathbb{E}}(\eta(S_N)) = \sum_{(x_1,\ldots,x_N)\in\mathbf{X}^N} \prod_{n=1}^{N} p_n(S_{n-1})^{x_n}\eta(S_N). \tag{15}$$

The line of the proof is the same as in the previous theorems and we omit the details. The price $\bar{\eta}(n, S_n)$ at time n is defined recursively by

$$\bar{\eta}(n, S_n) = \sum_{i=1}^{d} p^i_{n+1}(S_n)\bar{\eta}(n+1, S_n + e_i)$$

and the replicating strategy M^i_n is simply given by

$$M^i_n = \bar{\eta}(n, S_{n-1} + e_i).$$

From Theorem 6.1 we can easily derive multinomial distribution, multivariate hypergeometric distribution as well as multivariate Polya's distribution.

A generalization of Theorem 5.1 to an arbitrary $(d-1)$-dimensional discrete distribution of $(S^1_N, \ldots .S^{d-1}_N)$ with finite support can be explained as follows. We first use Theorem 5.1 on the first component S^1_n to derive the one-dimensional marginal distribution of S^1_N. Then, given the realization of the first component, we derive the conditional distribution of S^2_N given S^1_N by another application of Theorem 5.1 to the second component. We can continue this process up to the $(d-1)th$ component. The last component S^d_n is used as a slack variable.

Finally as an illustration of Theorem 6.1 we show how the Cox-Ross-Rubinstein formula of Section 3 is reduced to our multivariate framework. Define

$$x_n \equiv \frac{S_n - rS_{n-1}}{r^n}.$$

Furthermore by discounting define

$$K^*_n \equiv \frac{K_n}{r^n}.$$

Then the recurrence relation of the Cox-Ross-Rubinstein game

$$\begin{aligned} \mathcal{K}_n &= \mathcal{K}_{n-1} + M_n(S_n - S_{n-1}) + (r-1)(\mathcal{K}_{n-1} - M_n S_{n-1}) \\ &= r\mathcal{K}_{n-1} + (S_n - rS_{n-1})M_n \end{aligned}$$

is written as

$$\mathcal{K}^*_n = \mathcal{K}^*_{n-1} + M_n x_n.$$

Here x_n can take two values

$$(x^1_n, x^2_n) = \left(\frac{S_{n-1}(u-r)}{r^n}, \frac{S_{n-1}(d-r)}{r^n}\right).$$

Rescaling the values we define $d = 2$, $x_n^* \in \{e_1, e_2\} = \{(1,0),(0,1)\}$ and

$$M_n^* = (x_n^1, x_n^2),$$
$$p_n^* = \left(\frac{-x_n^2}{x_n^1 - x_n^2}, \frac{x_n^1}{x_n^1 - x_n^2} \right).$$

Then $M_n x_n$ is written as

$$M_n x_n = M_n^* \cdot (x_n^* - p_n^*).$$

Therefore the iteration part of the the Cox-Ross-Rubinstein game is written as

FOR $n = 1, \ldots, N$
  Skeptic announces $M_n^* \in \mathbb{R}^2$.
  Reality announces $x_n^* \in \{e_1, e_2\}$.
  $\mathcal{K}_n^* := \mathcal{K}_{n-1}^* + M_n^* \cdot (x_n^* - p_n^*)$.
END FOR

This shows that the Cox-Ross-Rubinstein formula is also a special case of our multivariate extension.

A Preliminaries on game theoretic probability

Here we summarize preliminary material (Chapter 1 of S&V) of the game theoretic probability. We also state a basic proposition on the existence of a replicating strategy and the existence of the game theoretic expectation in a coherent game.

In this paper all the games are finite-horizon games with N rounds. Therefore a path of the game is a finite sequence $\xi = x_1 \ldots x_N$ of Reality's moves. A random variable $x(\xi)$ denotes a payoff to Skeptic, when Reality chooses the path ξ. Given a strategy $\mathcal{P}$ of Skeptic, $\mathcal{K}^{\mathcal{P}}$ denotes the capital process of $\mathcal{P}$ with zero initial capital. Furthermore in this paper we only consider symmetric games, in the sense that if $\mathcal{P}$ is a strategy of Skeptic, then $-\mathcal{P}$ is also a strategy of Skeptic and

$$\mathcal{K}^{-\mathcal{P}} = -\mathcal{K}^{\mathcal{P}}.$$

The upper expected value $\bar{\mathbb{E}}x$ and the lower expected value $\underline{\mathbb{E}}x$ of x is defined as

$$\begin{aligned} \bar{\mathbb{E}}x &= \inf\{\alpha \mid \exists \mathcal{P}, \forall \xi,\ \mathcal{K}_N^{\mathcal{P}}(\xi) \geq x(\xi) - \alpha\}, \\ \underline{\mathbb{E}}x &= \sup\{\alpha \mid \exists \mathcal{P}, \forall \xi,\ \mathcal{K}_N^{\mathcal{P}}(\xi) \geq \alpha - x(\xi)\}. \end{aligned}$$

In a symmetric game $\underline{\mathbb{E}}x$ can also be written as

$$\underline{\mathbb{E}}x = \sup\{\alpha \mid \exists \mathcal{P}, \forall \xi,\ \alpha + \mathcal{K}_N^{\mathcal{P}}(\xi) \leq x(\xi)\}. \tag{16}$$

A game is coherent if Skeptic is not allowed to make money for certain, i.e.,

$$\forall \mathcal{P},\ \exists \xi,\ \mathcal{K}_N^{\mathcal{P}}(\xi) < 0.$$

If a game is coherent, then $\bar{\mathbb{E}}x \geq \underline{\mathbb{E}}x$ for every random variable x (Proposition 7.2 of S&V).

We call $\mathcal{P}$ a *replicating strategy* for x with the replicating initial capital $\alpha \in \mathbb{R}$ if

$$\alpha + \mathcal{K}_N^{\mathcal{P}}(\xi) = x(\xi), \ \forall \xi.$$

We now state the following basic fact.

Proposition A.1 *In a coherent symmetric game, suppose that $\mathcal{P}^*$ is a replicating strategy for x with the replicating initial capital α^*. Then*

$$\bar{\mathbb{E}}x = \underline{\mathbb{E}}x = \alpha^*.$$

Proof: By definition of $\bar{\mathbb{E}}x$ we have $\bar{\mathbb{E}}x \leq \alpha^*$. Furthermore in a symmetric game $\underline{\mathbb{E}}x \geq \alpha^*$ follows from (16). Therefore $\bar{\mathbb{E}}x \leq \alpha^* \leq \underline{\mathbb{E}}x$. Combining this with the inequality $\bar{\mathbb{E}}x \geq \underline{\mathbb{E}}x$ we obtain the proposition. □

We should note that the proof of the inequality $\bar{\mathbb{E}}x \geq \underline{\mathbb{E}}x$ in S&V and the above proof are standard arbitrage arguments.

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References

[1] Martin Baxter and Andrew Rennie. *Financial Calculus, An Introduction to Derivative Pricing.* Cambridge University Press, Cambridge, 1996.

[2] Marek Capiński and Tomasz Zastawniak. *Mathematics for Finance, An Introduction to Financial Engineering.* Springer, London, 2003.

[3] John C. Cox, Stephen A. Ross, and Mark Rubinstein. Option pricing: A simplified approach. *Journal of Financial Economics*, **3**, 229–263, 1979.

[4] William Feller. *An Introduction to Probability Theory and Its Applications.* third edition, volume 1, Wiley, New York, 1968.

[5] Masayuki Kumon and Akimichi Takemura. On a simple strategy weakly forcing the strong law of large numbers in the bounded forecasting game. Technical Report METR 05-20, Department of Mathematical Informatics, University of Tokyo. 2005.

[6] Glenn Shafer and Vladimir Vovk. *Probability and Finance: It's Only a Game!* Wiley, New York, 2001.

[7] Steven E. Shreve. *Stochastic Calculus for Finance I. The Binomial Asset Pricing Model.* Springer, New York. 2004.

[8] Kei Takeuchi. *Kake no suuri to kinyu kogaku* (Mathematics of betting and financial engineering). Saiensusha, Tokyo, 2004. (in Japanese)

[9] Vladimir Vovk, Ilia Nouretdinov, Akimichi Takemura, and Glenn Shafer. Defensive forecasting for linear protocols. The Game-Theoretic Probability and Finance project, http://probabilityandfinance.com, Working paper #10, 2005. To appear in Proceedings of the Sixteenth International Conference on Algorithmic Learning Theory (ed. by Sanjay Jain, Hans Ulrich Simon, and Etsuji Tomita), *Lecture Notes in Artificial Intelligence*, Springer, Berlin.