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Approximation Algorithms for Minimum Span Channel Assignment Problems

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Abstract

We propose polynomial time approximation algorithms for minimum span channel (frequency) assignment problems, which is known to be NP-hard. Let α be the approximation ratio of our algorithm and $W \geq 2$ be the maximum of numbers of channels required in vertices. If an instance is defined on a perfect graph G, then $\alpha \leq 1 + (1 + \frac{1}{W-1})\mathrm{H}_{\omega(G)}$, where H_i denotes the i-th harmonic number. For any instance defined on a unit disk graph G, α is less than or equal to $(1 + \frac{1}{W-1})(3\mathrm{H}_{\omega(G)} - 1)$. If a given graph is 4 or 3 colorable, α is bounded by $(2.5 + \frac{1.5}{W-1})$ and $(2 + \frac{1}{W-1})$, respectively. We also discuss a well-known practical instances called Philadelphia instances and propose an algorithm with $\alpha \leq 83/30$, which improves a trivial algorithm with $\alpha \leq 14/5$.

1 Introduction

In this paper, we denote the set of non-negative (positive) integers by $\mathbb{Z}_+$ ($\mathbb{N}$), respectively. Let $G = (V, E)$ be an undirected simple graph. We introduce a non-negative integer vertex weight function (vector) $w : V \rightarrow$

$\mathbb{Z}_+$, a non-negative integer edge length function (vector) $l : E \to \mathbb{Z}_+$ and a non-negative integer $k \in \mathbb{Z}_+$. A *channel assignment* of (G, w, l, k) is an assignment $\phi : V \to 2^{\mathbb{N}}$ such that

$$\begin{array}{ll} |\phi(v)| \geq w(v) & (\forall v \in V), \\ |c_1 - c_2| \geq k & (\forall c_1, \forall c_2 \in \phi(v),\ \forall v \in V), \\ |c_1 - c_2| \geq l(\{u, v\}) & (\forall c_1 \in \phi(u),\ \forall c_2 \in \phi(v),\ \forall \{u, v\} \in E). \end{array}$$

Given a channel assignment ϕ, a *span* of the channel assignment ϕ, denoted by span(ϕ), is defined by span$(\phi) \stackrel{\text{def.}}{=} \max\{c_1 - c_2 + 1 \mid c_1, c_2 \in \cup_{v \in V} \phi(v)\}$. A minimum span channel assignment problem (G, w, l, k) finds a channel assignment ϕ of (G, w, l, k) which minimizes span(ϕ).

The minimum span channel assignment problem is a kind of discrete versions of channel assignment problems which originated from wireless communication networks [1, 2]. Channel assignment problems are also called frequency assignment problems and/or radio channel assignment problem. For problems related to radio channel assignment problems, see McDiarmid's survey papers [3, 4]. The minimum span channel assignment problem includes the ordinary vertex coloring problem as a special case. It is known that if there exists a polynomial time algorithm for the vertex coloring problem which provides a performance guarantee of $\mathrm{O}(n^{\epsilon})$ for a constant $\epsilon > 0$, then P=NP [5]. When a given graph is perfect, we can solve the vertex coloring problem in polynomial time [6].

In this paper, we propose approximation algorithms for minimum span channel assignment problems. Let α be the approximation ratio of our algorithm and $W \geq 2$ be the maximum of numbers of vertex weights. If an instance is defined on a perfect graph G, then $\alpha \leq 1 + (1 + \frac{1}{W-1})\mathrm{H}_{\omega(G)}$, where H_i denotes the i-th harmonic number. For any instance defined on a unit disk graph G, α is less than or equal to $(1 + \frac{1}{W-1})(3\mathrm{H}_{\omega(G)} - 1)$. If a given graph is 4 or 3 colorable, α is bounded by $(2.5 + \frac{1.5}{W-1})$ and $(2 + \frac{1}{W-1})$, respectively. We also discuss a well-known practical instances called Philadelphia instances and propose an algorithm with $\alpha \leq 83/30$, which improves a trivial algorithm with $\alpha \leq 14/5$.

2 Algorithms

In this section, we describe our algorithm for the case that $0 < \exists l \leq k,\ \forall e \in E,\ l(e) = l$. In this case, the minimum span channel assignment problem remains NP-hard, since the problem still includes the vertex coloring problem. First, we introduce a naive algorithm. Given a graph G and a vertex weight w, $(G, w)_+$ denotes a subgraph of G induced by a set of vertices with positive weights. Throughout this paper, W denotes the maximum of vertex weights.

Algorithm 1

Step 1: Find a coloring of graph $(G, w)_{+}$. Let c be a number of used colors.

Step 2: For each vertex v whose color is $i \in \{1, \ldots, c\}$, we assign channels

$$\{1 + (i-1)l, 1 + (i-1)l + \max\{cl, k\}, \\ \ldots, 1 + (i-1)l + (w(v)-1)\max\{cl, k\}\}.$$

It is clear that Algorithm 1 gives a channel assignment to the problem (G, w, l, k) whose span is less than or equal to

$$1+(c-1)l+(W-1)\max\{cl,k\} \leq cl+(W-1)\max\{cl,k\} \leq W\max\{cl,k\}.$$

Since $l \leq k$, the span is bounded by $1 + (W-1)k$, if $c = 1$. The sequence

$$(1+(i-1)l, 1+(i-1)l+\max\{cl,k\}, \ldots, 1+(i-1)l+(w(v)-1)\max\{cl,k\})$$

is an arithmetic sequence with the first term $1+(i-1)l$, common difference of $\max\{cl, k\}$ and length of $w(v)$. Thus we can output the set of channels by a triplet $(1+(i-1)l, \max\{cl, k\}, w(v))$. If we use a polynomial time coloring algorithm in Step 1 and output a channel assignment by a set of triplets representing arithmetic sequences, then the time complexity of Algorithm 1 is bounded by a polynomial of the input size of problem (G, w, l, k). Here we note that the input size of the problem (G, w, l, k) is $\mathrm{O}(|V|\lceil \log W + 1\rceil + |E|\lceil \log L + 1\rceil + \lceil \log k\rceil)$, where L denotes the maximum of edge lengths.

Next, we propose our algorithm. Let $(G, w)_{\geq d}$ be a subgraph of G induced by $\{v \in V \mid w(v) \geq d\}$. For any graph G', $\omega(G')$ and $\chi(G')$ denote the clique number and the chromatic number of G', respectively. Clearly, $d \leq d'$ implies that $\omega((G, w)_{\geq d}) \geq \omega((G, w)_{\geq d'})$. Throughout this paper, n denotes the number of vertices in G. We define a sequence $(W(1), W(2), \ldots, W(n))$ by

$$W(q) \stackrel{\text{def.}}{=} \begin{cases} 0 & (\text{ if } q > \omega(G)), \\ \max\{d \mid \omega((G, w)_{\geq d}) \geq q\} & (\text{ if } q \leq \omega(G)\}). \end{cases}$$

The definition above implies that $W(1) = W$. Let $(w^1, w^2, \ldots, w^n)$ be a sequence of vertex weight vectors defined by

$$w^n(v) + w^{n-1}(v) + \cdots + w^q(v) = \min\{W(q), w(v)\} \;\; (\forall v \in V,\ 1 \leq \forall q \leq n).$$

Clearly, the equality $w^1 + w^2 + \cdots + w^n = w$ holds. Our algorithm finds a channel assignment ϕ^q for each problem (G, w^q, l, k) by applying Algorithm 1, independently. Lastly, we output a channel assignment ϕ defined by

$$\phi(v) = \cup_{q=1}^{n}\{c + p^0 + p^1 + \cdots + p^{q-1} \mid c \in \phi^q(v)\} \;\; (\forall v \in V)$$

where

$$p^q = \begin{cases} 0 & (\text{ if } q = 0 \text{ or } \mathrm{span}(\phi^q) = 0), \\ \mathrm{span}(\phi^q) + k - 1 & (\text{ if } q \geq 1 \text{ and } \mathrm{span}(\phi^q) > 0). \end{cases}$$

The span of an assignment ϕ is bounded by $\sum_{q=1}^{n} \mathrm{span}(\phi^q) + (k-1)(W-1)$, since the inequality $|\{q \in \{1, 2, \ldots, n\} \mid \mathrm{span}(\phi^q) > 0\}| \leq W$ holds. We briefly describe our algorithm below.

Algorithm 2

Step 1: Obtain the sequence $(W(1), \ldots, W(n))$.

Step 2: Construct the sequence of vertex weights $(w^1, w^2, \ldots, w^n)$ defined by $w^q(v) := \min\{W(q), w(v)\} - \min\{W(q+1), w(v)\}$ $(\forall v \in V,\ 1 \leq \forall q < n)$ and $w^n(v) := \min\{W(n), w(v)\}$ $(\forall v \in V)$.

Step 3: For each $q \in \{1, 2, \ldots, n\}$, we solve the problem (G, w^q, l, k) by Algorithm 1 and output an assignment obtained by merging n assignments.

3 Approximation Ratios

In this section, we discuss the approximation ratio of our algorithm. Throughout this section, we assume that $0 < \exists l \leq k, \forall e \in E, l(e) = l$.

3.1 Lower Bounds

First, we give some lower bounds which plays an important role to analyze the approximation ratio.

Lemma 1. *The optimal span Z^* of the problem (G, w, l, k) satisfies that*

$$k(W-1) \leq Z^* - 1, \text{ and } qlW(q) \leq Z^* + l - 1 \ \ (\forall q \in \{2, 3, \ldots, n\}).$$

Proof. The existence of a vertex v satisfying $w(v) = W$ implies that $1 + k(W-1) \leq Z^*$. From the definition of $W(q)$ $(2 \leq q \leq n)$, the graph $(G, w)_{\geq W(q)}$ contains a clique $Q \subseteq V$ satisfying $|Q| = q$ and $w(v) \geq W(q)$ $(\forall v \in Q)$. Thus $1 + l(qW(q) - 1) \leq 1 + l(\sum_{v \in Q} w(v) - 1) \leq Z^*$ holds. □

When $\omega(G) = 1$, we can use the trivial 1-coloring at Step 1 of Algorithm 1, and Algorithm 2 finds an optimal solution. Now we discuss the approximation ratio of Algorithm 2 in case $\omega(G) \geq 2$ and $W = 1$.

Lemma 2. *When $W = 1$ and $\omega(G) \geq 2$, the approximation ratio of Algorithm 2 is less than or equal to $(c-1)/(\omega(G)-1)$.*

Proof. In case $W = 1$, Algorithm 2 is essentially equivalent to Algorithm 1. When $W = 1$, Algorithm 1 finds a channel assignment whose span is bounded by $1 + (c-1)l$. Lemma 1 shows that $Z^* \geq 1 - l + lqW(q)$ $(2 \leq \forall q \leq n)$. By setting $q = \omega(G)$, the inequality $c \geq \omega(G) \geq 2$ implies that the approximation ratio is bounded by

$$\frac{1+(c-1)l}{Z^*} \leq \frac{1+(c-1)l}{1-l+l\omega(G)W(\omega(G))} = \frac{1+(c-1)l}{1+(\omega(G)-1)l} \leq \frac{c-1}{\omega(G)-1}. \qquad \square$$

Let $w^1, w^2, \ldots, w^n$ be the decomposition of vertex weight vector w defined in the previous section. The the following lemma gives a lower bound of a number of colors required for coloring $(G, w^q)_+$.

Lemma 3. *The clique number of the graph $(G, w^q)_+$ is less than or equal to q.*

Proof. Assume that $(G, w^q)_+$ has a clique Q whose size is $q+1$. For any vertex $v \in Q$, $0 < w^q(v) = \min\{W(q), w(v)\} - \min\{W(q+1), w(v)\}$ and thus $w(v) > W(q+1)$. It implies that clique Q is contained in the graph $(G, w)_{\geq W(q+1)+1}$. Contradiction. $\square$

3.2 Perfect Graphs

A graph G is perfect if and only if every induced subgraph of G satisfies $\chi(G) = \omega(G)$. In Algorithm 1, we can find a coloring of $(G, w^q)_+$ with $\chi((G, w^q)_+) = \omega((G, w^q)_+)$ colors at Step 1 in polynomial time [6]. We can obtain the sequence $(W(1), W(2), \ldots, W(n))$ by using ordinary binary search technique in polynomial time. Thus, the total time complexity of Algorithm 2 is bounded by a polynomial of the input size of the problem, where we output a channel assignment by a set of triplets for representing arithmetic sequences.

Theorem 1. *Let G be a perfect graph and $\mathcal{I}_G$ a class of instances of minimum span channel assignment problems defined on G.*

For any instance $I \in \mathcal{I}_G$ with $W = 1$, Algorithm 2 finds an optimal solution of I. For any instance $I \in \mathcal{I}_G$ with $W \geq 2$, the approximation ratio of Algorithm 2 is bounded by $1 + (1 + \frac{1}{W-1})H_{\omega(G)}$, where H_i denotes the i-th harmonic number.

Proof. We omit the trivial case that $\omega(G) = 1$. When $W = 1$ and $\omega(G) \geq 2$, Lemma 2 implies that the approximation ratio is bounded by $(\chi(G) - 1)/(\omega(G) - 1) = 1$ and thus Algorithm 2 finds an optimal solution.

Next, we consider the case that $W \geq 2$. Lemma 3 and perfectness of G implies that $\chi((G, w^q)_+) = \omega((G, w^q)_+ \leq q$. For each $q \in \{1, 2, \ldots, n\}$, Algorithm 1 finds an optimal coloring of graph $(G, w^q)_+$ and outputs a channel assignment whose span is bounded by $W^q \max\{ql, k\}$, where W^q

denotes the maximum of vertex weights $w^q(v)$ $(v \in V)$. In the following, we define $W(n+1)=0$. Let A be the span of a channel assignment obtained by Algorithm 2 and Z^* the optimal span. Then the following inequalities hold;

$$\begin{aligned}
A &\le (k-1)(W-1) + \textstyle\sum_{q=1}^{n} W^q \max\{ql, k\} \\
&\le k(W-1) + \textstyle\sum_{q=1}^{n} (W(q) - W(q+1)) \max\{ql, k\} \\
&= k(W-1) + \textstyle\sum_{q=1}^{\omega(G)} (W(q) - W(q+1)) \max\{ql, k\} \\
&\le Z^* + W(1)\max\{l, k\} + \textstyle\sum_{q=2}^{\omega(G)} W(q)(\max\{ql, k\} - \max\{(q-1)l, k\}) \\
&\le Z^* + Wk + \textstyle\sum_{q=2}^{\omega(G)} W(q)l \le Z^* + (Z^* + k - 1) + (Z^* + l - 1)\textstyle\sum_{q=2}^{\omega(G)} (1/q) \\
&\le Z^* + (Z^* + k) + (Z^* + k)\textstyle\sum_{q=2}^{\omega(G)} (1/q) \\
&\le Z^* + (Z^* + \tfrac{Z^*-1}{W-1})\textstyle\sum_{q=1}^{\omega(G)} (1/q) \le (1 + (1 + \tfrac{1}{W-1})\mathrm{H}_{\omega(G)})Z^* \qquad \square
\end{aligned}$$

3.3 Unit Disk Graphs

A graph $G=(V,E)$ is called a *unit disk graph* if the vertex set V is a point-set on 2-dimensional space and a pair of vertices is adjacent if and only if the Euclidean distance between them is less than or equal to 1. There exists a polynomial time algorithm for finding a vertex coloring of a given unit disk graph G satisfying that the required number of colors is bounded by $3\omega(G)-2$ (see [7, 8]). We can employ the algorithm in Step 1 of Algorithm 1. Since the maximum clique problem defined on a unit disk graph G is solvable in polynomial time (see [9]), we can find the sequence $(W(1), \dots, W(n))$ in polynomial time by employing ordinary binary search technique. If we adopt the procedures above, Algorithm 2 becomes a polynomial time algorithm.

Theorem 2. *Let G be a unit disk graph and $\mathcal{I}_G$ a class of instances of minimum span channel assignment problems defined on G.*

For any $I \in \mathcal{I}_G$ with $W=1$, the approximation ratio of Algorithm 2 is bounded by 3. For any instance $I \in \mathcal{I}_G$ with $W \ge 2$, the approximation ratio of Algorithm 2 is bounded by $(1+\frac{1}{W-1})(3\mathrm{H}_{\omega(\mathrm{G})}-1)$.

Proof. We omit the trivial case that $\omega(G)=1$. In case $W=1$ and $\omega(G)\ge 2$, Lemma 2 implies that the approximation ratio is bounded by $(c-1)/(\omega(G)-1) \le (3\omega(G)-2-1)/(\omega(G)-1) = 3$.

Next, we consider the case that $W \ge 2$. For each $q \in \{1,2,\dots,n\}$, Lemma 3 implies that Algorithm 1 finds a coloring of graph $(G, w^q)_+$ with at most $3q-2$ colors and outputs an assignment whose span is bounded by $W^q \max\{(3q-2)l, k\}$, where W^q denotes the maximum of vertex weights $w^q(v)$ $(v \in V)$. In the following, we define $W(n+1)=0$. The span A of a channel assignment obtained by Algorithm 2 and the optimal span Z^*

satisfy the following,

$$\begin{aligned}
A &\le (k-1)(W-1) + \textstyle\sum_{q=1}^{n} W^q \max\{(3q-2)l, k\} \\
&\le k(W-1) + \textstyle\sum_{q=1}^{n} (W(q) - W(q+1)) \max\{(3q-2)l, k\} \\
&= k(W-1) + \textstyle\sum_{q=1}^{\omega(G)} (W(q) - W(q+1)) \max\{(3q-2)l, k\} \\
&= Z^* + W(1)\max\{l,k\} + \textstyle\sum_{q=2}^{\omega(G)} W(q)(\max\{(3q-2)l,k\} - \max\{(3q-5)l,k\}) \\
&\le Z^* + Wk + \textstyle\sum_{q=2}^{\omega(G)} W(q)3l \le Z^* + (Z^*+k-1) + (Z^*+l-1)\sum_{q=2}^{\omega(G)}(3/q) \\
&\le 2(Z^*+k) + (Z^*+k)\textstyle\sum_{q=2}^{\omega(G)}(3/q) \le (Z^*+k)3\sum_{q=1}^{\omega(G)}(1/q) - (Z^*+k) \\
&\le (Z^* + \tfrac{Z^*-1}{W-1})(3\mathrm{H}_{\omega(G)} - 1) \le (1 + \tfrac{1}{W-1})(3\mathrm{H}_{\omega(G)} - 1)Z^* \qquad \square
\end{aligned}$$

3.4 General Case

In this section, we consider a case that we have a c-coloring of a given (general) graph G. In the following, we describe a modified version of Algorithm 2. The definition of $W(2)$ implies that $W(2)$ attains the minimum value of W' subject to the condition that the graph $G_{\ge W'+1}$ is a set of isolated vertices. Thus, we can calculate the value $W(2)$ in polynomial time. We introduce vertex weight vectors $\widetilde{w}^1, \widetilde{w}^2$ defined by $\widetilde{w}^2(v) = \min\{W(2), w(v)\}$ and $\widetilde{w}^1 = w - \widetilde{w}^2$. Our algorithm applies Algorithm 1 to problems $(G, \widetilde{w}^1, l, k)$, $(G, \widetilde{w}^2, l, k)$, and obtain two assignments ϕ^1 and ϕ^2, independently. When we solve $(G, \widetilde{w}^2, l, k)$, we use a given c-coloring of G in Step 1 of Algorithm 1. When we solve $(G, \widetilde{w}^1, l, k)$, we use the trivial 1-coloring of $(G, \widetilde{w}^1)_+$. Then we output a channel assignment defined by $\phi(v) = \phi^1(v) \cup \{c + (k-1) + \mathrm{span}(\phi^1) \mid c \in \phi^2(v)\}$ $(\forall v \in V)$.

Theorem 3. *Let G be a c-colorable graph with $c \ge 3$ and $\mathcal{I}_G$ a class of instances of minimum span channel assignment problems defined on G.*

For any $I \in \mathcal{I}_G$ with $W = 1$, the approximation ratio of Algorithm 2 is bounded by $c - 1$. For any $I \in \mathcal{I}_G$ with $W \ge 2$, the approximation ratio of Algorithm 2 is bounded by $(1/2)(1 + c + \frac{c-1}{W-1})$.

Proof. We omit the trivial case that $W(2) = 0$. If $W(2) > 0$, then $\omega(G) \ge 2$. In case that $W = 1$ and $W(2) > 0$, Lemma 2 implies that the approximation ratio is bounded by $(c-1)/(\omega(G)-1) \le c - 1$.

Next, we consider the case that $W \ge 2$ and $W(2) > 0$. Algorithm 1 finds a channel assignment of $(G, \widetilde{w}^1, l, k)$ whose span is bounded by $1 + (\widetilde{W}^1 - 1)k$ where $\widetilde{W}^1$ is the maximum of vertex weights $\widetilde{w}^1(v)$ $(v \in V)$. For the problem $(G, \widetilde{w}^2, l, k)$, Algorithm 1 outputs a channel assignment whose span is bounded by $1 + (c-1)l + (\widetilde{W}^2 - 1)\max\{cl, k\}$ where $\widetilde{W}^2$ denotes the maximum of vertex weights $\widetilde{w}^2(v)$ $(v \in V)$. The span A of a channel assignment obtained by Algorithm 2 and the optimal span Z^* satisfy that

$$\begin{aligned}A &\leq (1+(\widetilde{W}^1-1)k)+(k-1)+(1+(c-1)l+(\widetilde{W}^2-1)\max\{cl,k\}) \\ &\leq (1+(W-W(2)-1)k)+(k-1)+(1+(c-1)l+(W(2)-1)\max\{cl,k\}) \\ &= (1+Wk-k)+(c-1)l+(W(2)-1)(\max\{cl,k\}-k) \\ &= (1+Wk-k)+(c-1)l+(W(2)-1)(\max\{cl,k\}-\max\{l,k\}) \\ &\leq Z^*+(c-1)l+(W(2)-1)(cl-l)=Z^*+W(2)l(c-1) \\ &\leq Z^*+\tfrac{Z^*+l-1}{2}(c-1)=Z^*+(1/2)(c-1)Z^*+(1/2)(c-1)(l-1) \\ &\leq (1/2)(1+c)Z^*+(1/2)(c-1)k \leq (1/2)\left(1+c+\tfrac{c-1}{W-1}\right)Z^*. \end{aligned}$$

□

When a given graph G is planer, G is 4-colorable. Some instances of the minimum span channel assignment problem is defined on a triangular lattice, which is 3-colorable. The above theorem gives bounds of the approximation ratio of our algorithm for these cases.

Corollary 1. *When a given graph G is 4-colorable, the approximation ratio of Algorithm 2 is bounded by $2.5+1.5/(W-1)$. When we have a 3-coloring of a given graph, the approximation ration of Algorithm 2 is bounded by $2+1/(W-1)$.*

4 Philadelphia Instances

In this section, we discuss a class of practical instances in *Philadelphia Instances* [10]. A triangular lattice graph $T_{m,n}$ has a vertex set $\{(x\boldsymbol{e}_1+y\boldsymbol{e}_2) \mid x \in \{0,1,2,\ldots,m-1\},\ y \in \{0,1,2,\ldots,n-1\}\}$ where $\boldsymbol{e}_1 \stackrel{\text{def.}}{=} (1,0)$, $\boldsymbol{e}_2 \stackrel{\text{def.}}{=} (1/2,\sqrt{3}/2)$, and an edge set consists of pairs of vertices with unit distance. In this section, we denote $T_{m,n}$ by T for simplicity. We denote the 3rd power of graph T by T^3. Here, we consider a class of Philadelphia instances, which is defined on T^3. Let $l^*(e)$ be a weight of an edge e in T^3 defined by

$$l^*(e) \stackrel{\text{def.}}{=} \begin{cases} 2 & (e \text{ is an edge in } T), \\ 1 & (\text{otherwise}). \end{cases}$$

The minimum span channel assignment problem $(T^3,w,l^*,5)$ includes a class of the Philadelphia instances [10]. The minimum span channel assignment problem $(T^3,w,l^*,5)$ is NP-hard, since the problem includes the multicoloring problem defined on T^3 which is known to be NP-hard.

Figure 1 shows a coloring c^* of T^3 defined by

$$c^*(v) \stackrel{\text{def.}}{=} \begin{cases} 1, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (0,0) \mid x,y \in \mathbb{Z}\}, \\ 2, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (2,0) \mid x,y \in \mathbb{Z}\}, \\ 3, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (0,2) \mid x,y \in \mathbb{Z}\}, \\ 4, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (0,-1) \mid x,y \in \mathbb{Z}\}, \\ 5, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (1,0) \mid x,y \in \mathbb{Z}\}, \\ 6, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (-1,0) \mid x,y \in \mathbb{Z}\}, \\ 7, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (1,-1) \mid x,y \in \mathbb{Z}\}, \\ 8, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (-1,-1) \mid x,y \in \mathbb{Z}\}, \\ 9, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (-1,2) \mid x,y \in \mathbb{Z}\}, \\ 10, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (2,-1) \mid x,y \in \mathbb{Z}\}, \\ 11, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (0,1) \mid x,y \in \mathbb{Z}\}, \\ 12, & v \in \{(2x+4y)\boldsymbol{e}_1 + (2x-2y)\boldsymbol{e}_2 + (-1,1) \mid x,y \in \mathbb{Z}\}. \end{cases}$$

Since T^3 has a clique of size 12, the coloring c^* is an optimal coloring of T^3.

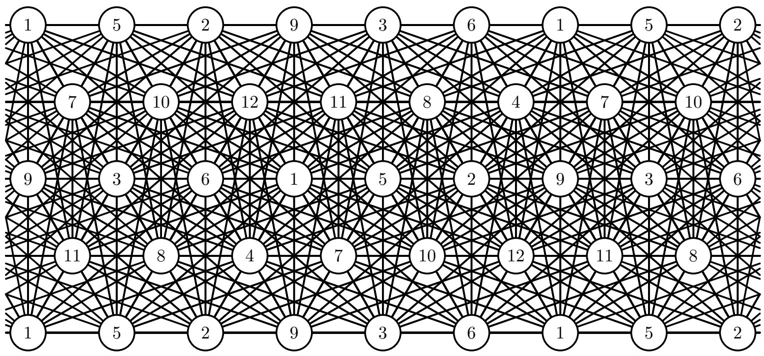

Figure 1: The coloring c^* of a graph $T^3_{m,5}$

In the following, we denote the vertex-set of T by $V(T)$ and edge-sets of graphs T and T^3 by $E(T)$ and $E(T^3)$, respectively. For any edge $\{u,v\} \in E(T)$, $\{11,12\} \neq \{c^*(u), c^*(v)\} \neq \{12,1\}$ implies that $|c^*(u) - c^*(v)| \geq 2$. Here we note that there exists a 3-clique in T whose colors are 1, 11 and 12.

Then we have a naive algorithm.

Algorithm P1

Step 1: Obtain the 12-coloring c^* of T^3.

Step 2: For each vertex v in T, we set the channels of v by

$$\phi(v) := \begin{cases} \{c^*(v), c^*(v)+14, \ldots, c^*(v)+14(w(v)-1)\} & (\text{ if } c^*(v) \leq 11), \\ \{13, 13+14, \ldots, 13+14(w(v)-1)\} & (\text{ if } c^*(v) = 12). \end{cases}$$

It is clear that the above algorithm gives a channel assignment of $(T^3, w, l^*, 5)$. Since there exists edges $\{u, u'\}, \{u', u''\} \in E(T)$ satisfying $(c^*(u), c^*(u'), c^*(u'')) = (11, 12, 1)$, we need to set the common difference of the arithmetic sequence to 14 (not 13). The above algorithm finds an assignment whose span is bounded by $13 + 14(W - 1) = 14W - 1$. When we output the channel assignment by a set of triplets for representing arithmetic sequences, the above algorithm becomes a polynomial time algorithm.

We can shows that the optimal span Z^* of a given Philadelphia instance satisfies $Z^* \geq 1 + 5(W - 1) = 5W - 4$ in a similar way with the proof of Lemma 1. Thus, we have the following.

Lemma 4. *Algorithm P1 finds a channel assignment of* $(T^3, w, l^*, 5)$ *whose span* A *and the optimal span* Z^* *satisfies*

$$A \leq 14W - 1 = (14/5)(5W - 4) + 51/5 \leq (14/5)Z^* + 51/5.$$

Next, we introduce a decomposition of vertex weight vector w. Let $\overline{W} = \max\{d \mid \omega((T, w)_{\geq d}) \geq 3\}$ and $(\overline{w}^1, \overline{w}^2)$ be a pair of vertex weights defined by $\overline{w}^2(v) = \min\{w(v), \overline{W}\}$ $(\forall v \in V(T))$ and $\overline{w}^1 = w - \overline{w}^2$. Clearly, graph $(T, \overline{w}^1)_+$ does not include any 3-clique. It implies that if graph $(T, \overline{w}^1)_+$ has a vertex v with $c^*(v) = 12$, then v has at most one adjacent vertex in $(T, \overline{w}^1)_+$ whose color is either 1 or 11. Thus we can construct a new channel assignment algorithm.

Algorithm P2

Step 1: Construct the vertex weight vectors $\overline{w}^1$ and $\overline{w}^2$.

Step 2: Solve $(T^3, \overline{w}^2, l^*, 5)$ by Algorithm 1 and obtain a channel assignment ϕ^2.

Step 3: Construct a coloring c^{**} of $(T, \overline{w}^1)_+$ defined by

$$c^{**}(v) := \begin{cases} c^*(v) & (\text{ if } c^*(v) \in \{1, 2, \ldots, 11\}), \\ 12 & (\text{ if } c^*(v) = 12 \text{ and } \exists \{u, v\} \text{ in } (T, \overline{w}^1)_+, c^*(u) = 1), \\ 13 & (\text{ if } c^*(v) = 12 \text{ and } \nexists \{u, v\} \text{ in } (T, \overline{w}^1)_+, c^*(u) = 1). \end{cases}$$

For each vertex v in T, we set the channels of v to

$$\phi^1(v) = \{c^{**}(v), c^{**}(v) + 13, \ldots, c^{**}(v) + 13(\overline{w}^1(v) - 1)\}.$$

Step 4: For each vertex v in T, we set $\phi(v) = \phi^1(v) \cup \{c + 4 + \text{span}(\phi^1) | c \in \phi^2(v)\}$.

It is clear that the above algorithm finds a channel assignment. Lastly, we discuss the approximation ratio.

Theorem 4. *Algorithm P2 finds a channel assignment of* $(T^3, w, l^*, 5)$ *whose span* A *and the optimal span* Z^* *satisfies* $A \leq (83/30)Z^* + (407/30)$.

Proof. Algorithm P2 finds a channel assignment whose span is bounded by $(14\overline{W}-1)+4+(13+13(W-\overline{W}-1)) = \overline{W}+13W+3$. Clearly, $5W-4 \leq Z^*$ holds. The graph $(T, \overline{w}^2)_+$ has a 3-clique $Q = \{u, u', u''\}$ satisfying $w(v) \geq \overline{W}$ $(\forall v \in Q)$, and thus the inequality $Z^* \geq 1+2(w(u)+w(u')+w(u'')-1) \geq 1+2(3\overline{W}-1) = 6\overline{W}-1$ holds. From the above, we have that

$$\begin{aligned} A &\leq \overline{W}+13W+3 = (1/6)(6\overline{W}-1)+(13/5)(5W-4)+(407/30) \\ &\leq (1/6)Z^*+(13/5)Z^*+(407/30) = (83/30)Z^*+(407/30). \end{aligned}$$ □

Here we note that $83/30 = 2.766\cdots < 2.8 = 14/5$.

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