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Abstract

For the design of networked control systems, we employ a subband coding technique to efficiently use the available data rate. Such coding schemes have widely been used in the area of signal processing for data compression in communication and storage. Compared to conventional subband coding methods, the proposed approach is more suitable from the control perspective in that frequency characteristics of a controller is directly used. In particular, the proposed scheme has three features. First, on the coder side, it uses a controller consisting of a filter bank whose outputs can be viewed as subband signals of the control input. Second, on the decoder side is another filter bank for reconstruction of those subband signals. This decoder is capable of taking account of random message losses that occur during the communication over a channel and is designed via an H^∞ type method. Third, for the quantizers in the coder, an optimal bit allocation scheme is developed.

Keywords: H^∞ control, Networked control, Quantization, Random message losses, Stochastic systems, Subband coding.

1 Introduction

In control systems employing communication networks, the consideration of data rate is one of the important issues. The main question here is how to utilize the available bit rate efficiently for control purposes. Various quantization schemes have recently been developed that take account of the dynamics in the system to be controlled (e.g., [2, 3, 6, 10, 16]).

In this paper, we propose an alternative approach to the problem and consider the so-called subband coding from the area of signal processing

[4, 8, 15]. This coding scheme has widely been used in signal compression in transmission and storage such as in MPEG standards for audio and video. The basic idea in subband coding can be described as follows: The signal to be sent over a channel is first divided into subsignals based on the frequency characteristics, and then each of them is quantized at a precision according to its relative size; on the receiver side, the coded signals are processed to give an approximate of the original signal. We note that the efficiency of coding or quantization is determined by the level of knowledge on the frequency characteristics of the original signal.

Motivated by the subband coding technique, we develop a modified scheme suitable for control. A preliminary study on this approach can be found in [7]. It is emphasized that, compared to signal processing applications, the requirement for real-time communication is much more critical. We consider a networked system where a network connects the sensor and actuator sides and assume that sufficient computation is available on both sides.

The main idea is to focus on the controller in the frequency domain and to view it as a filter generating subband signals. For example, in a PID controller, the proportional and derivative (PD) parts generate fast modes whereas the integrator (I) part gives only slow modes; these two modes play different roles since the PD signal has more influence on the transient response while the I signal affects the steady state response. The advantage is that the controller characteristics are directly exploited and used as part of a filter bank. Hence, no need for extra filters to obtain subband signals arises, which in turn enables us to keep the system order low and the time delay in coding small.

We propose a design procedure for the overall controller having three features. First, on the transmitter side, it uses a controller which consists of a filter bank and is obtained by a controller decomposition method. Specifically, we employ a method of [17], which originally is used for reduction of computation load in a controller. Second, on the receiver side, the subsignals are processed for reconstruction of the control inputs. The system here is capable of taking account of random message losses that occur during the communication over a channel and is designed via an H^∞ type method [5, 14]. Third, we develop an optimal bit allocation scheme for the quantizers in the coder.

The paper is organized as follows: In the next section, we provide some material on the general class of systems to which the networked system belongs. In Section 3, we describe the system setup and formulate the problem of the paper. The design procedure is presented in Section 4. A numerical example is given in Section 5 to show the effectiveness of the approach. This is followed by some concluding remarks in Section 6.

We use the following notation: The set of real numbers is denoted by $\mathbb{R}$, the set of nonnegative integers by $\mathbb{Z}_+$, and the set of natural numbers by $\mathbb{N}$.

2 Preliminaries

The class of systems that we consider in this paper is in general a periodic system with random switchings in the system matrices. Here, we introduce several notions for this class of systems and related results [5, 13, 14].

Consider the following periodically time-varying system G_0 with random switchings:

$$\begin{aligned} x_{k+1} &= A_{k,\Theta(k)}x_k + B_{k,\Theta(k)}w_k, \\ z_k &= C_{k,\Theta(k)}x_k + D_{k,\Theta(k)}w_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^m$ is the input, $z_k \in \mathbb{R}^p$ is the output, $\Theta_k \in \mathcal{I}_M$ is the mode of the system with the index set $\mathcal{I}_M := \{0, 1, \ldots, M-1\}$. The mode Θ_k is assumed to be a stochastic process determined by the probability $\alpha_i = \mathrm{Prob}\{\Theta_k = i\} \geq 0$, $i \in \mathcal{I}_M$, where $\sum_{i=0}^{M-1} \alpha_i = 1$. The system matrices are periodic with period $N \in \mathbb{N}$, i.e., $A_{k+N,i} = A_{k,i}$ for all $k \in \mathbb{Z}_+$ and $i \in \mathcal{I}_M$, and so on.

Let $\mathcal{F}_k$ be the sigma-field generated by $\{\Theta_0, \ldots, \Theta_k\}$. We assume that the input w_k is $\mathcal{F}_k$-measurable for each k. Further, w_k is assumed to be in l^2, that is, $E[\sum_{k=0}^{\infty} |w_k|^2] < \infty$, where the expectation $E[\,\cdot\,]$ is taken over the statistics of $\{\Theta_k\}_{k\in\mathbb{Z}_+}$. The space of such signals is denoted by $\mathcal{W}$. Let the norm of signals in $\mathcal{W}$ be $\|w\| := E[\sum_{k=0}^{\infty} |w_k|^2]^{1/2}$.

We employ the following notion of stability. For the system (1) with $w_k \equiv 0$, the origin or the system is said to be *stochastically stable* if for every initial condition x_0,

$$E\Big[\sum_{k=0}^{\infty} |x_k|^2 \Big|\ x_0\Big] < \infty.$$

The following result gives a necessary and sufficient condition for stochastic stability of this system [5].

Proposition 2.1 The system G_0 in (1) is stochastically stable if and only if there exists an N-periodic matrix P_k such that $0 < P_k = P_k^T \in \mathbb{R}^{n\times n}$ and

$$\sum_{i\in\mathcal{I}_M} \alpha_i A_{k,i}^T P_{k+1} A_{k,i} - P_k < 0, \quad k \in \mathcal{I}_N.$$

Suppose that the system (1) is stochastically stable and the initial condition of the state is $x_0 = 0$. Then, we define the l^2-induced norm of the system (1) as follows:

$$\|G_0\|_\infty := \sup_{w\in\mathcal{W}, w\neq 0} \frac{\|z\|}{\|w\|}.$$

Next, we provide with a bounded real lemma for N-periodic systems with random switchings [5]. For this result, we should introduce the next definition.

Definition 2.2 The system G_0 in (1) is said to be *weakly controllable* if for each $\xi \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}_+$, there exist $T \in \mathbb{Z}_+$ and an input $w_{[0,T-1]}$ such that $\mathrm{mod}(T, N) = k_0$ and under the input $w_{[0,T-1]}$, the state x_T satisfies $\mathrm{Prob}\{x_T = \xi\} > 0$.

Theorem 2.3 [5] Assume that the system G_0 in (1) is weakly controllable. Then, for a given scalar $\gamma > 0$, the system G_0 is stochastically stable and $\|G_0\|_\infty < \gamma$ holds if and only if there exists an N-periodic matrix P_k such that $0 < P_k = P_k^T \in \mathbb{R}^{n\times n}$ and

$$\sum_i \alpha_i \begin{bmatrix} A_{k,i} & B_{k,i} \\ C_{k,i} & D_{k,i} \end{bmatrix}^T \begin{bmatrix} P_{k+1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{k,i} & B_{k,i} \\ C_{k,i} & D_{k,i} \end{bmatrix} - \begin{bmatrix} P_k & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0 \quad \text{for } k \in \mathcal{I}_N.$$

3 Problem formulation

Consider the networked control system depicted in Fig. 1. The generalized plant G is a discrete-time system and has the following state-space equation:

$$\begin{aligned} x_{k+1} &= Ax_k + B_1 w_k + B_2 u_k, \\ z_k &= C_1 x_k + D_{11} w_k + D_{12} u_k, \\ y_k &= C_2 x_k + D_{21} w_k, \end{aligned} \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state, $w_k \in \mathbb{R}^{m_1}$ is the exogenous input, $u_k \in \mathbb{R}$ is the scalar control input, $z_k \in \mathbb{R}^{p_1}$ is the controlled output, and $y_k \in \mathbb{R}$ is the scalar measurement output. The underlying sampling period is denoted by $h > 0$. We also make standard assumptions that (A, B_2) is controllable and (A, C_2) is observable. Note that the controller to be designed is SISO.

The overall controller is realized over a network channel, where signals are subject to quantization and random message losses. Before giving a detailed description, we introduce the controller components in Fig. 1. The measurement output y_k first goes to the controller K. The output $\hat{u}_k$ of K is to be sent over a channel to the remote actuator side: First, it is preprocessed in E_k, which is called the analysis filter bank (AFB) and outputs the p-dimensional subband signal $v_k \in \mathbb{R}^p$. The subband signal is then quantized in the quantizer Q and is transmitted over the channel. However, it may get lost due to congestion or delay; the variable $\Theta_k \in \{0, 1\}$ represents whether the message is received or lost at time k. Finally, on the actuator side, $R_{k,\Theta(k)}$ denotes the synthesis filter bank (SFB), which gives the control input u. We call the AFB-SFB pair $(E_k, R_{k,\Theta(k)})$ the filter bank system.

We now describe each controller component in the following:

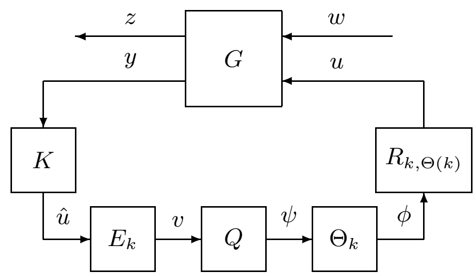

Figure 1: Networked control system

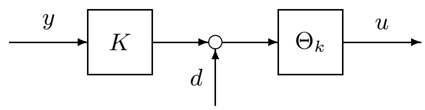

Figure 2: Simple controller with signal+noise quantization model

(1) Controller: The controller K is linear time invariant (LTI). This system is the controller in the usual sense, whereas other components function as part of the communication system.

(2) Quantizer: We employ a component-wise uniform quantizer given by $Q = \mathrm{diag}(Q_1, \dots, Q_p)$. The scalar quantizer Q_i, $i = 1, \dots, p$, is determined by two parameters, the quantization width $\delta_i > 0$ and the number $b_i \in \mathbb{N}$ of bits. (We assume that it has a bounded saturation level.) To simplify the analysis, we assume the so-called *signal+noise* model for quantization; such a model has often been used in both signal processing and control systems (e.g., [1,8,11]). Under this model, the quantization noise is considered as an exogenous noise. In this paper, this noise is denoted by d. Hence the output ψ of Q is expressed as $\psi_k = v_k + d_k$ in Fig. 1. It is assumed that the frequency domain properties of d are unknown, but its energy is bounded.

(3) Message loss: Θ_k is a random variable that determines the message loss at time k. It takes values in $\{0, 1\} \subset \mathbb{R}^{p \times p}$. If $\Theta_k = 0$, then the message at time k is lost, and otherwise the message arrives. We assume that $\{\Theta_k\}_{k \in \mathbb{Z}_+}$ is an i.i.d. Bernoulli process specified by $\alpha = \mathrm{Prob}\{\Theta_k = 0\}$ and $1 - \alpha = \mathrm{Prob}\{\Theta_k = 1\}$ for all k.

(4) Filter banks: The AFB E_k and the SFB $R_{k,\Theta(k)}$ are both periodically time varying with period N, and further $R_{k,\Theta(k)}$ may depend on the message loss process Θ_k at each k; this information is clearly available at the SFB side.

For these components, we propose three types as follows:

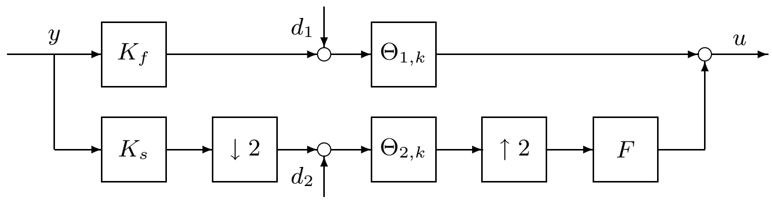

Figure 3: Decomposed controller in a subband coder form

a) Simple type: The most simple case is when E_k and $R_{k,\Theta(k)}$ are both identity. Fig. 2 shows this with the signal+noise model for quantization.

b) Signal processing type: We may use a conventional filter bank in the signal processing literature (e.g., [8, 15]). For an application of such coding to control systems, we refer to [7].

c) Controller decomposition type: This scheme is motivated by the subband coding systems and also by [17] and is shown in Fig. 3. The idea is to use K directly as part of the AFB E_k. Let $K(z)$ be its transfer function form. We first decompose the controller as

$$K(z) = K_f(z) + K_s(z).$$

Here, K_f contains the fast modes of K, and K_s has the slow modes. More specifically, K_s is chosen so that its output is sufficiently bandlimited and hence downsampling does not cause serious aliasing. Then K_f can be defined as $K - K_s$. For example, if K is a PID controller, $K(z) = K_P + K_I/(1-z^{-1}) + K_D(1-z^{-1})$ with $K_P, K_I, K_D \in \mathbb{R}$, then we may take $K_f(z) = K_P + K_D(1-z^{-1})$ and $K_s(z) = K_I/(1-z^{-1})$.

The output of K_s is assumed to be sufficiently bandlimited to $\pi/2h$. The sampling period of this signal is then reduced to $2h$ by the downsampler defined as

$$\downarrow 2 : \tilde{u} \mapsto v : v(k) = \tilde{u}(2k).$$

After being sent over a channel, on the SFB side, this signal is put through the upsampler given by

$$\uparrow 2 : \phi \mapsto \eta : \eta(k) = \begin{cases} \phi(k/2) & \text{if } k \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

As a result, the sampling period is doubled and becomes h. Finally, being through the synthesis filter F and added to the output of K_f,

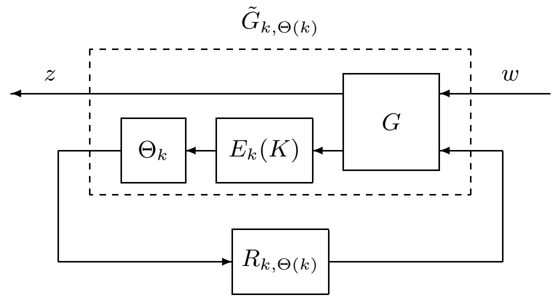

Figure 4: Design of SFB $R_{k,\Theta(k)}$

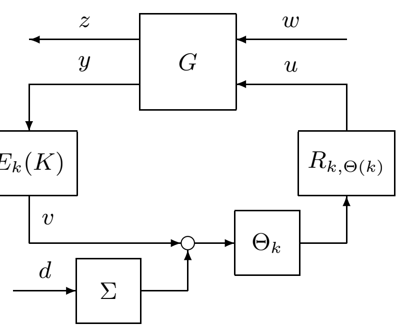

Figure 5: Networked control system with quantization noise

the control input u is obtained. By an appropriate choice of F, u would be sufficiently close to the original control input. We note that this system is periodic with period 2 due to the multirate operations. This system in Fig. 3 can be generalized in terms of the number of channels and the downsampling ratio as we will see later.

In this paper, we mainly consider the design of the third filter bank scheme. We demonstrate the differences in the performance of the schemes through examples in Section 5. Regarding the exogenous input w_k and the quantization noise d_k, we assume that they are $\mathcal{F}_k$-measurable for each k, where $\mathcal{F}_k$ is the sigma-field generated by $\{\Theta_0, \ldots, \Theta_k\}$, and that they are in $\mathcal{W}$.

We are now ready to formulate the design of the controller using subband coding for the plant G in (2). The design is divided in three steps.

Step 1: Design the controller K and the AFB E_k.

Step 2: Design the SFB $R_{k,\Theta(k)}$ as follows: Let $\tilde{G}_{k,\Theta(k)}$ be the system including G, K, E_k, and Θ_k as shown in Fig. 4. Assume that there is no quantization noise ($d_k \equiv 0$). Then, given a scalar $\gamma > 0$, design the

SFB so that

$$\|F_l(\tilde{G}_{k,\Theta(k)}, R_{k,\Theta(k)})\|_\infty < \gamma,$$

where $F_l(\tilde{G}_{k,\Theta(k)}, R_{k,\Theta(k)})$ denotes the closed-loop system in Fig. 4.

Step 3: Consider the system in Fig. 5. Allocate the available bits for the quantizers as follows. Given the number b of average bits per sample, find the parameters b_i and δ_i for each quantizer Q_i, $i = 1, \ldots, p$, such that the H^∞ norm of the system from d to z is minimized, where Σ serves as a weight.

In the next section, we describe each step of the design.

4 Controller design based on subband coding

Step 1: Controller and analysis filter bank (AFB)

For the design of the controller K, we may consider the system in Fig. 6 and use standard H^∞ design methods: Given $\gamma > 0$, find an LTI controller K such that the closed-loop system $F_l(G, K)$ satisfies the norm condition $\|F_l(G, K)\|_\infty < \gamma$, where $\|\cdot\|_\infty$ is the (deterministic) l^2-induced norm. In this case, the γ here sets the best performance level achievable in the overall system since adding the communication system to the original system in Fig. 6 would in general degrade the performance.

Next, denoting the transfer function of K by $K(z)$, we decompose $K(z)$ into p subsystems $K_f(z), K_{s,1}(z), \ldots, K_{s,p-1}(z)$ such that $K(z) = K_f(z) + K_{s,1}(z) + \cdots + K_{s,p-1}(z)$. This is done in such a way that, the first subsystem $K_f(z)$ has the full bandwidth π/h, while for $i = 1, \ldots, p-1$, the subsystem $K_{s,i}(z)$ has a limited bandwidth of $\pi/N_i h$. For later use, let $N_0 = 1$. We then construct the AFB shown in Fig. 7, where $\downarrow N_i$ is the downsampler with factor N_i, $i = 1, \ldots, p-1$. This system is also denoted by $E_k(K)$ to explicitly show its dependence on K. Here, let N be the least common multiple of $N_1, \ldots, N_{p-1}$.

To illustrate the controller decomposition, we consider the simple case of $p = 2$. We follow the procedure outlined in [17]: Assume that $K(z)$ is stable. The downsample ratio here is $N = N_1$, and the corresponding Nyquist frequency is $\omega_{Nh} := \pi/Nh$. A mode in $K(z)$ is considered slow if its natural frequency is below this Nyquist frequency ω_{Nh} by a certain factor η; a suitable choice for η is between 5 and 10. Hence, we define the *critical frequency* by $\omega_c := \omega_{Nh}/\eta$. Write $K(z)$ in its partial fractional expression as

$$K(z) = \sum_{i=1}^{N_p} \frac{a_i}{z - \lambda_i} + D_K.$$

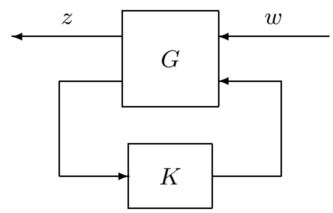

Figure 6: Design of K

For a nonnegative real pole λ_i, its natural frequency ω_i can be obtained by

$$\omega_i = -1/h \ln \lambda_i,$$

and for a purely complex pole λ_i, its natural frequency ω_i is determined as

$$\omega_i = -1/h[(\ln|\lambda_i|)^2 - (\ln \lambda_i/|\lambda_i|)^2]^{1/2}.$$

Now, if $\omega_i \leq w_c$, then λ_i is classified as a slow mode, and otherwise λ_i is a fast mode (including all negative real poles). Hence, the two controllers $K_f(z)$ and $K_s(z)$ corresponding to fast and slow modes, respectively, are given by

$$K_f(z) = \sum_{\lambda_i\text{: fast}} \frac{a_i}{z-\lambda_i} + D_K, \quad K_s(z) = \sum_{\lambda_i\text{: slow}} \frac{a_i}{z-\lambda_i}.$$

In general, the task to decompose the controller $K(z)$ in an effective way can be difficult. Especially, for high order cases, some combinations of the subsystems in $K(z)$ may result in bandpass filters.

Finally, we note that the system E_k as described so far is a multirate one. At this point, we convert it in a straightforward way to an equivalent single rate system with the sampling period h. This is done by simply assuming zero values for signals of lower rates at each instant when no value is given. With a slight abuse of notation, this single rate system is denoted also by E_k; this is an N-periodic system.

Step 2: Synthesis filter bank (SFB)

The system $\tilde{G}_{k,\Theta(k)}$, including the AFB E_k and the message loss process Θ_k, is shown in Fig. 4. This is N-periodic with random switchings, and the state-space equation can be expressed as

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}_1 w_k + \tilde{B}_2 u_k, \\ z_k &= \tilde{C}_1 \tilde{x}_k + \tilde{D}_{11} w_k + \tilde{D}_{12} u_k, \\ \phi_k &= \Theta_k(\tilde{C}_{2,k}\tilde{x}_k + \tilde{D}_{21,k} w_k), \end{aligned} \tag{3}$$

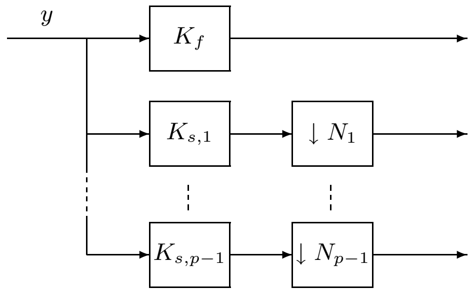

Figure 7: AFB $E_k(K)$ with decomposed controllers

where $\tilde{x} \in \mathbb{R}^{\tilde{n}}$. The system matrices are obtained from G in (2), which is LTI, and the N-periodic system E_k. Note that only $\tilde{C}_{2,k}$ and $\tilde{D}_{21,k}$ are N-periodic.

We allow the SFB $R_{k,\Theta(k)}$ to also be N-periodic. Further, as mentioned earlier, for each k, the control u_k is $\mathcal{F}_k$-measurable. That is, the SFB takes account of the message loss at each step; clearly, this information is available there. It specifically takes a state-space form as follows:

$$\begin{aligned} \hat{x}_{k+1} &= \hat{A}_{k,\Theta(k)}\hat{x}_k + \hat{B}_k\phi_k, \\ \hat{u}_k &= \hat{C}_{k,\Theta(k)}\hat{x}_k + \hat{D}_k\phi_k, \end{aligned} \tag{4}$$

where $\hat{x}_k$ is the state and has the dimension $\tilde{n}$, the same as that of (3). All system matrices are N-periodic in k: E.g., $\hat{A}_{k+N,i} = \hat{A}_{k,i}$ for $k \in \mathbb{Z}_+$ and $i \in \{0,1\}$. We note that the B- and D-matrices are independent of Θ_k because when the message is lost, $\phi_k = 0$ and hence the terms for these matrices are equal to zero. Denote the closed-loop system in Fig. 4 by $F_l(\tilde{G}_{k,\Theta(k)}, R_{k,\Theta(k)})$. This system is N-periodic and has random switchings with 2 modes, $\Theta_k \in \{0,1\}$. For later use, let $\alpha_0 := \alpha$ and $\alpha_1 := 1-\alpha$.

We next present the solution to the SFB design using the H^∞ characterization result in Section 2. We first give the state-space equation for the closed-loop system $F_l(\tilde{G}_{k,\Theta(k)}, R_{k,\Theta(k)})$:

$$\begin{aligned} \xi_{k+1} &= \bar{A}_{k,\Theta(k)}\xi_k + \bar{B}_{k,\Theta(k)}w_k, \\ z_k &= \bar{C}_{k,\Theta(k)}\xi_k + \bar{D}_{k,\Theta(k)}w_k, \end{aligned}$$

where $\xi_k \in \mathbb{R}^{2\tilde{n}}$ is the state given by $\xi_k := [\tilde{x}_k^T \; \hat{x}_k^T]^T$ and

$$\begin{aligned}
\bar{A}_{k,\Theta(k)} &:= \begin{bmatrix} \tilde{A} + \Theta_k \tilde{B}_2 \hat{D}_k \tilde{C}_{2,k} & \tilde{B}_2 \hat{C}_{k,\Theta(k)} \\ \Theta_k \hat{B}_k \tilde{C}_{2,k} & \hat{A}_{k,\Theta(k)} \end{bmatrix}, \\
\bar{B}_{k,\Theta(k)} &:= \begin{bmatrix} \tilde{B}_1 + \Theta_k \tilde{B}_2 \hat{D}_k \tilde{D}_{21,k} \\ \Theta_k \hat{B}_k \tilde{D}_{21,k} \end{bmatrix}, \\
\bar{C}_{k,\Theta(k)} &:= \begin{bmatrix} \tilde{C}_1 + \Theta_k \tilde{D}_{12} \hat{D}_k \tilde{C}_{2,k} & \tilde{D}_{12} \hat{C}_{k,\Theta(k)} \end{bmatrix}, \\
\bar{D}_{k,\Theta(k)} &:= \tilde{D}_{11} + \Theta_k \tilde{D}_{12} \hat{D}_k \tilde{D}_{21,k}.
\end{aligned}$$

Notice that all system matrices depend on Θ_k.

We can apply Theorem 2.3 to this system to obtain a synthesis result. To make the final condition convex, we invoke an approach that has been developed in the context of H^∞ control for linear time-invariant systems. In particular, we use a linearizing change of controller variables and a congruence transformation [9, 12]; see also [5, 14].

Theorem 4.1 Given a scalar $\gamma > 0$, there exists an SFB $R_{k,\Theta(k)}$ of the form (4) such that the closed-loop system $F_l(\tilde{G}_{k,\Theta(k)}, R_{k,\Theta(k)})$ is stochastically stable and $\|F_l(\tilde{G}_{k,\Theta(k)}, R_{k,\Theta(k)})\|_\infty < \gamma$ if and only if there exist N-periodic matrices $X_k, Y_k, \check{A}_{k,i}, \check{B}_k, \check{C}_{k,i}, \check{D}_k$, $i \in 0, 1$, such that the following matrix inequalities hold:

$$\begin{bmatrix} R_{1,k} & Q_{0,k}^T & Q_{1,k}^T \\ Q_{0,k} & R_{2,k+1} & 0 \\ Q_{1,k} & 0 & R_{2,k+1} \end{bmatrix} > 0, \qquad k \in \mathcal{I}_N, \tag{5}$$

where for k and $i \in \{0, 1\}$, the submatrices are given by

$$\begin{aligned}
R_{1,k} &= \begin{bmatrix} X_k & I & 0 \\ I & Y_k & 0 \\ 0 & 0 & \gamma^2 I \end{bmatrix}, \quad R_{2,k+1} = \begin{bmatrix} X_{k+1} & I & 0 \\ I & Y_{k+1} & 0 \\ 0 & 0 & I \end{bmatrix}, \\
Q_{i,k} &= \sqrt{\alpha_i} \begin{bmatrix} X_{k+1}\tilde{A} + i\check{B}_k \tilde{C}_{2,k} & \check{A}_{k,i} & X_{k+1}\tilde{B}_1 + i\check{B}_k \tilde{D}_{21,k} \\ \tilde{A} + i\tilde{B}_2 \check{D}_k \tilde{C}_{2,k} & \tilde{A} Y_k + \tilde{B}_2 \check{C}_{k,i} & \tilde{B}_1 + i\tilde{B}_2 \check{D}_k \tilde{D}_{21,k} \\ \tilde{C}_1 + i\tilde{D}_{12} \check{D}_k \tilde{C}_{2,k} & \tilde{C}_1 Y_k + \tilde{D}_{12} \check{C}_{k,i} & \tilde{D}_{11} + i\tilde{D}_{12} \check{D}_k \tilde{D}_{21,k} \end{bmatrix}.
\end{aligned}$$

We note that the inequalities in (5) are affine in the variable matrices and hence are LMIs. After obtaining the N-periodic matrices such that the inequalities in the theorem hold, we can construct the controller (4). The following are the formulae for its system matrices:

$$
\begin{aligned}
\hat{D}_k &= \check{D}_k, \\
\hat{C}_{k,i} &= \left(\check{C}_{k,i} - i\hat{D}_k\tilde{C}_{2,k}Y_k\right)\left(X_k^{-1} - Y_k\right)^{-1}, \\
\hat{B}_k &= X_{k+1}^{-1}\check{B}_k - \tilde{B}_2\check{D}_k, \\
\hat{A}_{k,i} &= \left[X_{k+1}^{-1}\check{A}_{k,i} - \left(\tilde{A} + i\tilde{B}_2\hat{D}_k\tilde{C}_{2,k}\right)Y_k\right. \\
&\qquad \left. - i\hat{B}_k\tilde{C}_{2,k}Y_k\right]\left(X_k^{-1} - Y_k\right)^{-1} - \tilde{B}_2\hat{C}_{k,i}
\end{aligned}
$$

for $k \in \mathcal{I}_N$ and $i \in \{0, 1\}$.

Step 3: Quantizer

So far, the controller K and the filter bank $(E_k, R_{k,\Theta(k)})$ are designed. The final step is to determine the bit allocation for the quantizers.

Consider the system depicted in Fig. 5. Here d is the exogenous noise which, after being weighted by Σ, represents the quantization noise. Denote the system from d_i to z by T_{zd_i}, $i = 1, 2, \ldots, p$.

Recall that b is the number of available bits at every step k on average and that b_i is the bits for quantizer Q_i. The aim at this stage is to reduce the effect of quantization in the controlled output. For this purpose, we introduce the following cost function:

$$
J(b_1, \ldots, b_p) := \frac{m_1}{2^{b_1}}\|T_{zd_1}\|_\infty + \cdots + \frac{m_p}{2^{b_p}}\|T_{zd_p}\|_\infty, \tag{6}
$$

where m_i, $i = 1, 2, \ldots, p$, is a bound on v_i when $d_k \equiv 0$: $\sup_{k\in\mathbb{Z}_+}|v_i(k)| \le m_i$. The constraint on b_i is

$$
b = \sum_{i=1}^{p} \frac{b_i}{N_{i-1}}. \tag{7}
$$

The problem at this step is to find the optimal bit allocation b_i^*, $i = 1, 2, \ldots, p$, that minimizes the cost function (6) subject to (7). We obtain the following result.

Proposition 4.2

$$
b_i^* = \log_2 m_i\|T_{zd_i}\|_\infty + \left(\sum_{j=1}^{p}\frac{1}{N_{j-1}}\right)^{-1}\left(b - \sum_{j=1}^{p}\frac{1}{N_{j-1}}\log_2 m_j\|T_{zd_j}\|_\infty\right).
$$

Proof: Let $m_i' := m_i\|T_{zd_i}\|_\infty$. By the inequality between the arithmetic and geometric means, the cost J is lower bounded as

$$
J(b_1, \ldots, b_p) \ge p\left(\prod_{i=1}^{p}\frac{m_i'}{2^{b_i}}\right)^{1/p}.
$$

Here, the equality holds if

$$\frac{m_1'}{2^{b_1}} = \cdots = \frac{m_p'}{2^{b_p}}.$$

Using the optimal bit b_1^*, define $\nu := m_1'/2^{b_1^*}$. Then from the equalities above, we can write

$$b_i^* = \log_2 m_i' - \log_2 \nu, \quad i = 1, 2, \ldots, p. \tag{8}$$

By substituting this into the constraint on b_i in (7), we obtain

$$-\log_2 \nu = \left(\sum_{i=1}^{p} \frac{1}{N_{i-1}} \right)^{-1} \left(b - \sum_{i=1}^{p} \frac{1}{N_{i-1}} \log_2 m_i' \right).$$

Now, by (8), we have the optimal bits b_i^* for $i = 1, 2, \ldots, p$. $\square$

We have several remarks regarding the bit allocation problem. We first note that the bit allocation scheme in this proposition can result in fractional bit values. This issue can be resolved either by rounding the values or by implementing a periodically time-varying allocation with the average being equal to the designed values.

In our development, the parameters m_i are specified as the maximum magnitude of the input to the quantizer Q_i. This is a useful and practical choice in the context of control. In contrast, in signal processing, these parameters often represent the energy of the input signal to Q when an exogenous input is applied [8,15]. However, if we take this approach in our setup, the energy may not be finite when, for example, the input is a step and the controller has an integrator.

5 Numerical example

In this section, we present a numerical example to illustrate the proposed design method.

Consider a continuous-time, third-order plant $P_c(s)$ and a weight function $W_c(s)$ given by

$$P_c(s) = \frac{s+1}{s(s^2+s+1)}, \quad W_c(s) = \frac{(0.5s+1)^2}{10}.$$

These systems are discretized to $P(z)$ and $W(z)$ with the sampling period $h = 0.1$.

Step 1: We performed a modified mixed sensitivity design for the controller K: Let S and T be the sensitivity and the complimentary sensitivity functions, respectively. The design criterion is

$$\left\| W \begin{bmatrix} T & \beta PS \end{bmatrix} \right\|_\infty < \gamma,$$

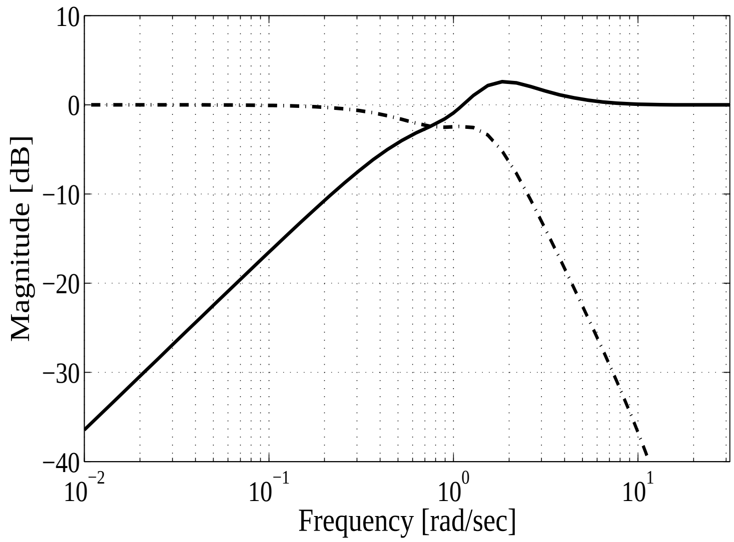

Figure 8: Frequency responses of S and T

where the parameter β is chosen as $\beta = 1$. The minimization of γ resulted in the controller K as follows:

$$K(z) = \frac{-0.732(z-1.06)(z-0.981)}{(z-0.883)(z-0.826)(z-0.218)} - 0.712.$$

With this, we obtained $\gamma = 0.201$. The frequency responses of S and T are shown in Fig. 8.

Next, this controller was decomposed following the procedure in Section 4. We took the downsampling factor as $N = 4$. The critical frequency for determining the fast/slow modes was $\omega_c = \pi/20h$ (that is, with factor $\eta = 5$). The two controllers $K_f(z)$ and $K_s(z)$ are

$$K_f(z) = \frac{-0.712(z-1.24)(z+0.755)}{(z-0.826)(z-0.218)},$$
$$K_s(z) = \frac{-0.331}{z-0.883}.$$

Fig. 9 shows the gain plots for K, K_f, and K_s. We notice the high gains in the low frequencies for K_f and K_s. Indeed, this will result in signals with larger magnitudes and, in turn, may demand finer quantization. However, we shortly see that the optimal bit allocation of the proposed scheme can still improve the performance.

Step 2: For the design of $R_{k,\Theta(k)}$, we considered two cases: One is a simply N-periodic one, which does not account for message losses; we denote this one by $R_{1,k}$. The minimum performance level was $\gamma = 0.239$.

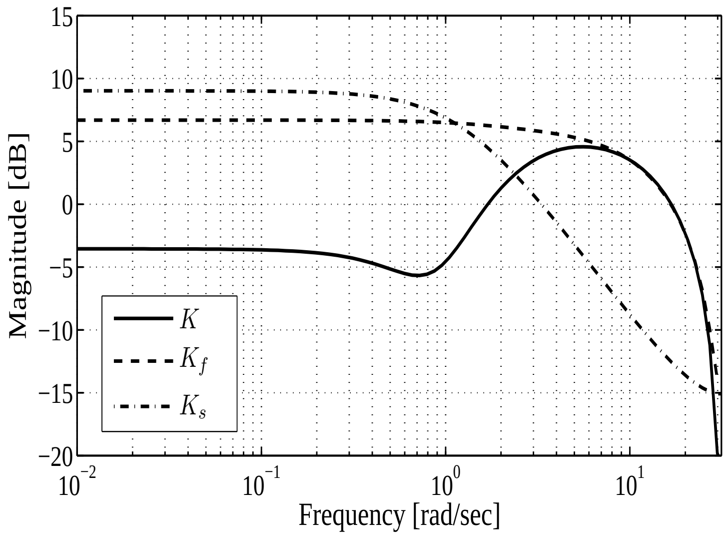

Figure 9: Frequency responses of the original controller K, the fast controller K_f, and the slow controller K_s

We note that the two systems with K and with $R_{1,k}$ showed very similar characteristics in the time domain in simulations.

The other one, denoted $R_{2,k,\Theta(k)}$, is N-periodic, but also exploits the information Θ_k of the message losses at each time step; in particular, we used the message loss probability of $\alpha = 0.05$. The obtained performance level was $\gamma = 0.240$. In the design, the γ-level was slightly increased to obtain similar performance with $R_{1,k}$ under no message loss.

Step 3: The number b of bits per time step on average was taken as $b = 5$. We followed the procedure in Section 4 and in particular used a unit step input in obtaining the weights m_i, $i = 1, 2$, in the cost function $J(b_1, b_2)$ in (6). For both cases of $R_{1,k}$ and $R_{2,k,\Theta(k)}$, the optimal bit allocation scheme followed by appropriate rounding (to avoid any fractional bit values) yielded the number of bits for the fast controller K_f as $b_1 = 3$ and the slow controller K_s as $b_2 = 8$. Note that, in particular, 256 quantization levels were allocated for K_s compared to 32 levels in the non-subband controller.

Fig. 10 shows the step responses for 4 cases: (i) The original system, i.e., with K only and no subband coding, (ii) the original system with quantization, (iii) the proposed scheme using $R_{1,k}$, and (iv) the proposed scheme using $R_{1,k}$ with quantization. We observe that for the quantized cases, there is a steady state error due to the coarse quantization for the non-subband case, while the proposed scheme achieves almost perfect tracking.

In Fig. 11, we show similar step responses as those in Fig. 10 but with

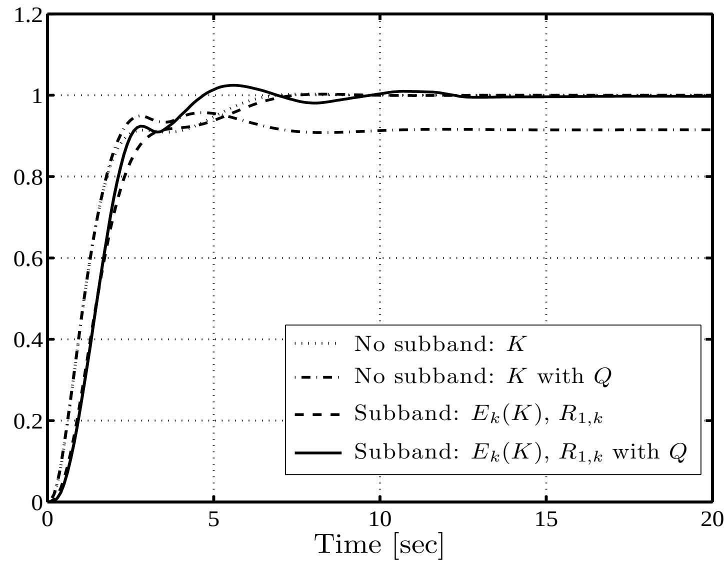

Figure 10: Step responses under perfect communication ($\Theta_k \equiv 1$)

message losses. A loss sample path Θ_k with $\alpha = 0.05$ was used for the following cases: (i) The original system, (ii) the original system with quantization, (iii) the proposed scheme using $R_{1,k}$ with quantization, and (iv) the proposed scheme using $R_{2,k,\Theta(k)}$ with quantization. Recall that the last scheme with the SFB $R_{2,k,\Theta(k)}$ takes account of the random losses. As in Fig. 10, for the non-subband case with Q, steady state error remains. On the other hand, $R_{1,k}$ is capable to obtain asymptotic tracking though the transient response may not be acceptable. In the last case for $R_{2,k,\Theta(k)}$, desirable transient and tracking performance can be observed, and the losses seem to have little effect. In summary, we confirmed that the proposed subband coding schemes perform well under the communication constraints of coarse quantization and random message losses.

6 Conclusion

In this paper, we have proposed a networked control system with subband coding to efficiently use the number of bits available for communication and take account of message losses. The coding scheme is based on the decomposition of a pre-designed controller and exploits its frequency domain characteristics. A three-step design procedure has been developed for the controller including the receiver and the optimal bit allocation. Future research includes further performance analysis of the proposed control scheme.

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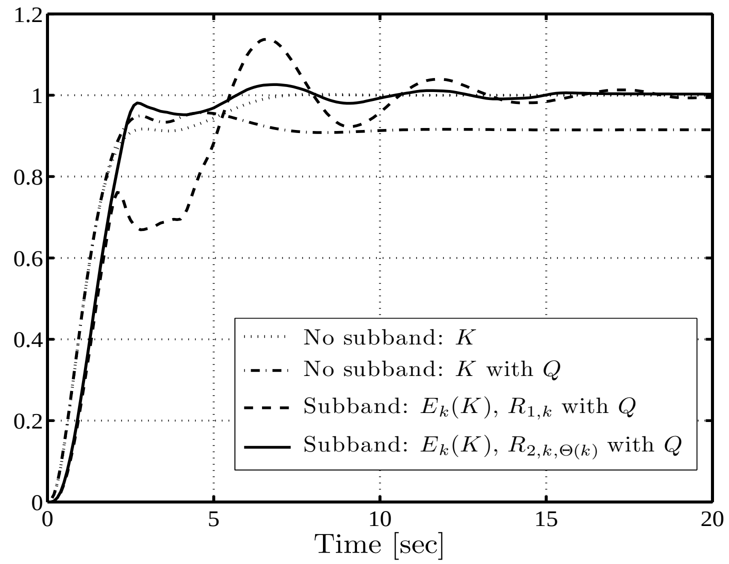

Figure 11: Step responses with random message losses ($\alpha = 0.05$)

in [7], we initiated the line of research reported in this paper. This work was supported in part by the Ministry of Education, Culture, Sports, Science and Technology, Japan, under Grant No. 17760344 and Japan Science and Technology Agency under the CREST program.

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