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Satoru IWATA and Naonori KAKIMURA

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.i.u-tokyo.ac.jp/mi/mi-e.htm

Solving Linear Programs from Sign Patterns

Satoru IWATA* Naonori KAKIMURA †

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Abstract

This paper is an attempt to provide a connection between qualitative matrix theory and linear programming. A linear program $\max\{cx \mid Ax = b, x \geq 0\}$ is said to be *sign-solvable* if the set of sign patterns of the optimal solutions is uniquely determined by the sign patterns of A, b, and c. It turns out to be NP-complete to decide whether a given linear program is not sign-solvable. We then introduce a class of sign-solvable linear programs in terms of totally sign-nonsingular matrices, which can be recognized in polynomial time. For a linear program in this class, we devise an efficient combinatorial algorithm to obtain the sign pattern of an optimal solution from the sign patterns of A, b, and c. The algorithm runs in $\mathrm{O}(m\gamma)$ time, where m is the number of rows of A and γ is the number of all nonzero entries in A, b, and c.

1 Introduction

This paper deals with a linear program in the standard form:

$$\begin{array}{ll} \text{maximize} & cx \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array}$$

denoted by $\mathrm{LP}(A, b, c)$. When we formulate a linear programming model, we need to specify entries in A, b, and c. However, there are many situations in which it is hard to estimate the exact quantities of these entries, while their signs are obvious. This paper aims at analyzing linear programs only from the combinatorial arrangements of the positive and negative entries in A, b, and c.

The *sign* of a real number a, denoted by $\operatorname{sgn} a$, is defined to be $+1$ for $a > 0$, -1 for $a < 0$, and 0 for $a = 0$. The *sign pattern* of a real matrix A is the $\{+1, 0, -1\}$-matrix obtained from A by replacing each entry by its sign. For a matrix A, we denote by $\mathcal{Q}(A)$ the set of all matrices having the same sign pattern as A, called the *qualitative class* of A. The qualitative class of a vector is defined similarly. A linear program $\mathrm{LP}(A, b, c)$ is said to be *sign-solvable* if there exists a set $\mathcal{S}$ of sign patterns of vectors such that the set of sign patterns of all the optimal solutions of $\mathrm{LP}(\tilde{A}, \tilde{b}, \tilde{c})$ coincides with $\mathcal{S}$ for any $\tilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$, and $\tilde{c} \in \mathcal{Q}(c)$.

Matrix analysis by sign patterns is often called qualitative matrix theory, which was originated in economics by Samuelson [17]. Since then, qualitative matrix theory and its applications have been studied. Various results are compiled in the book of Brualdi and Shader [3]. A matrix A is said to be an *L-matrix* if $\tilde{A}$ has row-full rank for any $\tilde{A} \in \mathcal{Q}(A)$. A square matrix A is called *sign-nonsingular* if all matrices in $\mathcal{Q}(A)$ are nonsingular. Klee, Radner, and Manber [10] showed that it is NP-complete to discern whether a given rectangular matrix is not an L-matrix. However, it can be decided in polynomial time whether a given square matrix is sign-nonsingular or not. The problem of recognizing sign-nonsingular matrices

*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan. iwata@kurims.kyoto-u.ac.jp

†Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan. naonori_kakimura@mist.i.u-tokyo.ac.jp

has many equivalent problems in combinatorics [12, 15, 19, 20]. In particular, it is equivalent to testing whether a given bipartite graph has a Pfaffian orientation, which problem was suggested by Pólya [15] in 1913. In 1999, Robertson, Seymour and Thomas [16] presented an algorithm for solving this problem in polynomial time. Their algorithm is based on a structural theorem for the special bipartite graphs called braces, which was also proved independently by McCuaig [13, 14]. See [14] for a survey on these equivalent problems.

In this paper, we investigate sign-solvable linear programs in terms of qualitative matrix theory. We first show that recognizing sign-solvability of a given linear program is co-NP-complete (Theorem 3.2). We then introduce a class of sign-solvable linear programs.

An $m \times n$ matrix with $m \leq n$ is said to be *totally sign-nonsingular* if the sign of the determinant of each submatrix of order m is determined uniquely by the sign pattern. Totally sign-nonsingular matrices were investigated in the context of sign-solvability of linear systems [3, 8, 9, 18]. Totally sign-nonsingular matrices can be recognized in polynomial time by testing sign-nonsingularity of related square matrices (Theorem 4.2).

A linear program $\mathrm{LP}(A, b, c)$ is *totally sign-nonsingular* if A has row-full rank and both $A_{\mathrm{p}} = (A \ \ -b)$ and $A_{\mathrm{d}} = \begin{pmatrix} -c \\ A \end{pmatrix}$ are totally sign-nonsingular. This condition can be tested in polynomial time. The total sign-nonsingularity of A_{p} implies that the set of sign patterns of the feasible solutions is uniquely determined by the sign patterns of A and b. Moreover, a totally sign-nonsingular linear program is sign-solvable (Theorem 5.1).

If a linear program is sign-solvable, we can obtain the sign pattern of an optimal solution in strongly polynomial time. Indeed, since the magnitudes of the nonzero entries of A, b, and c do not affect the sign patterns of optimal solutions, it suffices to solve a linear program $\mathrm{LP}(\tilde{A}, \tilde{b}, \tilde{c})$ for some $\tilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$, and $\tilde{c} \in \mathcal{Q}(c)$. By replacing all the nonzero entries with their signs, the sign pattern of an optimal solution can be found in polynomial time by means of the ellipsoid method [7] or the interior point method [6]. Since the sizes of the numbers in $\tilde{A}$, $\tilde{b}$, and $\tilde{c}$ are constant, this algorithm is in fact strongly polynomial.

In this paper, we present a more efficient combinatorial algorithm to solve a totally sign-nonsingular linear program from the sign patterns. In order to measure the complexity of our algorithm, let m denote the number of rows of A, and γ denote the number of all nonzero entries in A, b, and c. We first show that the feasibility can be tested in $\mathrm{O}(\gamma)$ time. The boundedness can be tested by applying the same procedure to the dual program. For a totally sign-nonsingular linear program that is feasible and bounded, we devise a recursive algorithm for obtaining the sign pattern of an optimal solution. The algorithm is based on an optimality criterion in terms of the bipartite graph associated with the matrix $L = \begin{pmatrix} 1 & -c & 0 \\ 0 & A & -b \end{pmatrix}$. By finding a certain path in this bipartite graph, the linear program can be reduced to a smaller one. The running time of this recursive algorithm is $\mathrm{O}(m\gamma)$.

This paper is organized as follows. Section 2 provides some notations and preliminaries about matrices and bipartite graphs. Section 3 is devoted to sign-solvability of linear programs. We show that recognizing sign-solvability is co-NP-complete. Section 4 introduces totally sign-nonsingular matrices. In Section 5, we show that totally sign-nonsingular linear programs are sign-solvable. In Sections 6 to 8, we discuss totally sign-nonsingular linear programs. In Section 6, we investigate the bipartite graph associated with totally sign-nonsingular matrices to give a combinatorial optimality condition. We show in Section 7 that the feasibility can be tested in polynomial time. Finally, in Section 8, we devise an efficient combinatorial algorithm to obtain the sign pattern of an optimal solution.

2 Matrices and Bipartite Graphs

Throughout this paper, we deal with a real matrix A with row set U and column set V. The (i, j)-entry of a matrix A is denoted by a_{ij}. For $I \subseteq U$ and $J \subseteq V$, we denote by $A[I, J]$ the submatrix in A with row set I and column set J. The submatrix $A[U, J]$ is abbreviated as $A[J]$. For a vector x, we mean by $x[J]$ the subvector with support J.

We say that a square matrix A is *term-nonsingular* if the determinant of A contains at least one nonvanishing term, that is, if $a_{i\pi(i)} \neq 0$ for each $i \in U$ for some bijection $\pi : U \to V$. A square matrix is said to be *term-singular* if it is not term-nonsingular. For a nonsingular matrix A, at least one of the expansion terms of $\det A$ must be distinct from zero. Thus the nonsingularity implies term-nonsingularity. For a matrix A, the *term-rank* of A, denoted by $\text{t-rank}A$, is defined to be the maximum size of a term-nonsingular submatrix in A. It is easily deduced that $\text{t-rank}A \geq \text{rank}A$ holds. A matrix A is said to have *row-full term-rank* if $\text{t-rank}A = |U|$.

We say that a square matrix A is *sign-nonsingular* if all matrices in $\mathcal{Q}(A)$ are nonsingular. A square matrix A is sign-nonsingular if and only if A is term-nonsingular and every nonvanishing term of the determinant of A has the same sign [3]. Thus, if A is sign-nonsingular, the determinant of every matrix in $\mathcal{Q}(A)$ has the same sign.

Let $G = (U, V; E)$ be a bipartite graph with vertex sets U, V and edge set $E \subseteq U \times V$. For an edge subset $F \subseteq E$, we denote the set of end vertices of F by $U(F) = \{i \in U \mid (i, j) \in F\}$ and $V(F) = \{j \in V \mid (i, j) \in F\}$. A *path* $P \subseteq E$ is a sequence of consecutive edges in a graph. In this paper, we deal with only elementary paths, in which edges and vertices are all distinct. An *i-j* path means a path from vertex i to vertex j. We call a path of even length an *even* path, and odd length an *odd* path. A *cycle* $C \subseteq E$ is a path which ends at the vertex it begins with.

An edge subset $M \subseteq E$ is said to be a *matching* if $|M| = |U(M)| = |V(M)|$. We call a matching M *perfect* if it satisfies $|M| = \min\{|U|, |V|\}$. For an edge subset M, we say a path P of G is M*-alternating* if the elements of P alternate between elements of M and $E \setminus M$ along P. For edge subsets F_1 and F_2, we denote by $F_1 \triangle F_2$ the symmetric difference between F_1 and F_2, that is, $F_1 \triangle F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. Notice that for a matching M and an M-alternating path P, the symmetric difference $M \triangle P$ is also a matching in G.

For $X \subseteq U$, $\Gamma_G(X)$ means the set of neighbors of X, that is, $\Gamma_G(X) = \{v \in V \mid \exists u \in X, (u, v) \in E\}$. If there is no ambiguity, we simply use $\Gamma(X)$. The following proposition is known as Hall's theorem for bipartite graphs.

Proposition 2.1. *For a bipartite graph $G = (U, V; E)$ with $|U| \leq |V|$, there exists a perfect matching in G if and only if $|\Gamma(X)| \geq |X|$ for all $X \subseteq U$.*

For a bipartite graph G, an *orientation* $\vec{G}$ of G is a directed graph obtained from G by orienting its edges. For an orientation $\vec{G}$ of G, an even path P of G is said to be *oddly oriented* (*evenly oriented*) in $\vec{G}$ if an odd (even) number of its edges are directed in the same direction along P. A path P is *central* if the subgraph obtained from G by deleting the vertices in P has a perfect matching. For a bipartite graph $G = (U, V; E)$ with $|U| = |V|$, we say that an orientation of G is *Pfaffian* if every central cycle of even length is oddly oriented. As mentioned in Section 1, it can be decided in polynomial time whether a given directed bipartite graph is Pfaffian or not.

With a matrix A, we associate a directed bipartite graph $G(A) = (U, V; E)$. The vertex sets U and V correspond to the row and column sets, respectively. The edge set E is defined by $E = \{(i, j) \mid a_{ij} \neq 0, i \in U, j \in V\}$. An edge in $G(A)$ represents a nonzero entry in A. An edge $(i, j) \in E$ is oriented from $i \in U$ to $j \in V$ if a_{ij} is positive, and from $j \in V$ to $i \in U$ if a_{ij} is negative. We simplify $\Gamma_{G(A)}(X)$ by $\Gamma(X)$, which means the column subset having nonzero entries in $A[X, V]$.

For a square matrix A, a perfect matching in $G(A)$ corresponds to a nonzero expansion term of $\det A$. Hence A is term-nonsingular if and only if $G(A)$ has a perfect matching. The term-rank of a matrix A is equal to the maximum size of a matching in $G(A)$. It is known that a square matrix A is sign-nonsingular if and only if $G(A)$ is Pfaffian.

3 Sign-Solvability of Linear Programs

Consider a linear program in the standard form $\mathrm{LP}(A, b, c)$, where b is a column vector, c is a row vector, and A is a matrix with row set U and column set V. In this section, we simply denote $A_J = A[J]$, $c_J = c[J]$ and $x_J = x[J]$ for $J \subseteq V$.

We say that x is a *feasible solution* if x satisfies all the constraints. A linear program is said to be *feasible* if it has feasible solutions, and *infeasible* otherwise. A feasible linear program is *unbounded* if, for any $\lambda \in \mathbb{R}$, there exists a feasible solution x such that $cx \geq \lambda$. A linear program which is not unbounded is called *bounded*. For a feasible and bounded linear program, a feasible solution that maximizes the objective function is said to be an *optimal solution*. The objective value of an optimal solution is called the *optimal value*.

Proposition 3.1. *A feasible linear program* $\mathrm{LP}(A, b, c)$ *is unbounded if and only if the linear system* $\begin{pmatrix} c \\ A \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *has nonnegative solutions.*

A *basis* B of a linear program $\mathrm{LP}(A, b, c)$ is the column subset such that $|B| = |U|$ and A_B is nonsingular. The remaining column subset $V \setminus B$ is denoted by N. The *basic solution* x for a basis B is defined by $x_B = A_B^{-1} b$ and $x_N = 0$. If a basic solution is nonnegative, the corresponding basis is called *feasible*. A basis is *optimal* if the basic solution is optimal. The *relative weight vector* z for a basis B is defined by $z_B = 0$ and $z_N = c_N - c_B A_B^{-1} A_N$. It is known that a basis B is optimal if and only if a basis B is feasible and the relative weight vector is nonpositive.

We say that a linear program $\mathrm{LP}(A, b, c)$ is *sign-solvable* if there exists a set $\mathcal{S}$ of sign patterns of vectors such that the set of sign patterns of all the optimal solutions of $\mathrm{LP}(\tilde{A}, \tilde{b}, \tilde{c})$ coincides with $\mathcal{S}$ for any $\tilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$, and $\tilde{c} \in \mathcal{Q}(c)$. If a linear program is infeasible or unbounded, the set of the optimal solutions is defined to be empty. Note that the definition of sign-solvability does not require the uniqueness of the sign of the optimal value.

We give examples of sign-solvable linear programs.

Example 1. Consider the following linear program of two linear equations in four unknowns:

$$\begin{array}{lll} \text{maximize} & \begin{pmatrix} 0 & 0 & -c_1 & -c_2 \end{pmatrix} & x \\ \text{subject to} & \begin{pmatrix} a_1 & 0 & -a_2 & -a_3 \\ 0 & a_4 & -a_5 & -a_6 \end{pmatrix} & x = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \\ & x \geq 0, & \end{array}$$

where $c_i\ (i = 1, 2)$, $a_i\ (i = 1, \ldots, 6)$, and $b_i\ (i = 1, 2)$ are all positive constants. The sign patterns imply that the first two columns form an optimal basis, and the others not. Thus this linear program is sign-solvable.

Example 2. Consider the following linear program of three linear equations in six unknowns:

$$\begin{array}{lll} \text{maximize} & \begin{pmatrix} 0 & 0 & c_1 & 0 & 0 & -c_2 \end{pmatrix} & x \\ \text{subject to} & \begin{pmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 \\ -a_3 & a_4 & 0 & a_5 & a_6 & 0 \\ 0 & a_7 & -a_8 & 0 & -a_9 & -a_{10} \end{pmatrix} & x = \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}, \\ & x \geq 0, & \end{array}$$

where $c_i\ (i = 1, 2)$, $a_i\ (i = 1, \ldots, 10)$, and b_1 are all positive constants. Then the first three columns form an optimal basis B. Indeed, the basic solution x for B is

$$x = \frac{b_1}{a_1} \begin{pmatrix} 1 & \frac{a_3}{a_4} & \frac{a_3 a_7}{a_4 a_8} & 0 & 0 & 0 \end{pmatrix}^\top \geq 0,$$

and the relative weight vector z is

$$z_N = -\frac{1}{a_1a_4a_8}\begin{pmatrix} c_1a_7(a_2a_3+a_1a_5) \\ a_1(a_6a_7+a_4a_9)c_1 \\ a_1a_4a_{10}c_1+a_1a_4a_8c_2 \end{pmatrix}^\top \leq 0.$$

Therefore, B is optimal independently of the magnitudes of the given constants. Similarly, the basic solutions and the relative weight vectors for the other bases imply that the other bases are not optimal independently of the magnitudes of the given constants. Thus this linear program is sign-solvable. Since the optimal value is

$$\frac{a_3a_7b_1c_1}{a_1a_4a_8} > 0,$$

the sign of the optimal value is positive independently of the magnitudes of the given constants.

We now discuss the complexity of recognizing sign-solvability of a linear program. A matrix A is *sign-central* if $\tilde{A}x = 0$ has a nonzero, nonnegative solution for any $\tilde{A} \in \mathcal{Q}(A)$. It is NP-complete to decide whether A is not sign-central [1]. On the other side, we call A *sign-extreme* if $\tilde{A}x = 0$ does not have such a solution for any $\tilde{A} \in \mathcal{Q}(A)$. It can be decided in polynomial time whether $\tilde{A}$ is sign-extreme [11].

Theorem 3.2. *It is NP-complete to decide whether a given linear program* $\mathrm{LP}(A, b, c)$ *is not sign-solvable.*

Proof. We first show that the problem is in NP. If the linear program is not sign-solvable, then there exist $\tilde{A}, \hat{A} \in \mathcal{Q}(A)$, $\tilde{b}, \hat{b} \in \mathcal{Q}(b)$, and $\tilde{c}, \hat{c} \in \mathcal{Q}(c)$ such that the linear program $\mathrm{LP}(\tilde{A}, \tilde{b}, \tilde{c})$ has an optimal solution x^*, but $\mathrm{LP}(\hat{A}, \hat{b}, \hat{c})$ has no optimal solutions with the same sign pattern as x^*. If $\mathrm{LP}(\hat{A}, \hat{b}, \hat{c})$ is infeasible or unbounded, then we know $\mathrm{LP}(A, b, c)$ is not sign-solvable. Suppose that $\mathrm{LP}(\hat{A}, \hat{b}, \hat{c})$ has an optimal solution $\hat{x}$. Let S be the set of the optimal solutions of $\mathrm{LP}(\hat{A}, \hat{b}, \hat{c})$, that is, $S = \{x \mid \hat{A}x = \hat{b}, \hat{c}x = \hat{c}\hat{x}, x \geq 0\}$. Then we have $\mathcal{Q}(x^*) \cap S = \emptyset$, which implies that there exists a separating hyperplane $wx = 0$ between the convex cone $\mathcal{Q}(x^*)$ and the polytope S. The hyperplane serves as a polynomial certificate that $\mathrm{LP}(A, b, c)$ is not sign-solvable. Indeed, we can verify in polynomial time that the hyperplane separates $\mathcal{Q}(x^*)$ and S by solving the linear programs $\max\{wx \mid x \in S\}$ and $\min\{wx \mid x_Z = 0, x \geq 0\}$, where $Z = \{j \mid x_j^* = 0\}$.

We next show that the problem of recognizing sign-central matrices can be polynomially reduced to that of recognizing sign-solvable linear programs. Let A be an $m \times n$ matrix, and consider the linear program $\mathrm{LP}(A, 0, \mathbf{1})$, where $\mathbf{1}$ is the row vector whose entries are all one. Note that zero is a feasible solution of $\mathrm{LP}(A, 0, \mathbf{1})$. Proposition 3.1 implies that $\mathrm{LP}(A, 0, \mathbf{1})$ is unbounded or zero is the unique feasible solution of $\mathrm{LP}(A, 0, \mathbf{1})$. Therefore, $\mathrm{LP}(A, 0, \mathbf{1})$ is sign-solvable if and only if A is either sign-central or sign-extreme. If there were a polynomial time algorithm for testing sign-solvability of $\mathrm{LP}(A, 0, \mathbf{1})$, then one could decide whether A is sign-central or not in polynomial time with the aid of the algorithm for recognizing sign-extreme matrices. Thus it is NP-complete to decide whether a given linear program is not sign-solvable. □

4 Totally Sign-Nonsingular Matrices

We say that an $m \times n$ matrix ($m \leq n$) is *totally sign-nonsingular* if all term-nonsingular submatrices of order m are sign-nonsingular, namely, if the sign of the determinant of each submatrix of order m is determined uniquely by the sign pattern of the matrix.

A matrix A is said to have *signed null space* if there exists a set $\mathcal{S}$ of sign patterns of vectors such that the set of sign patterns of all the solutions in $\tilde{A}x = 0$ coincides with $\mathcal{S}$ for any $\tilde{A} \in \mathcal{Q}(A)$. A matrix A has *signed row space* if there exists a set $\mathcal{S}$ of sign patterns such that the set of sign patterns of all the vectors in $\{y\tilde{A} \mid y \in \mathbb{R}^m\}$ coincides with $\mathcal{S}$ for any $\tilde{A} \in \mathcal{Q}(A)$. Equivalence among these three concepts has been established as follows.

Theorem 4.1 (Kim and Shader [8], Shao and Ren [18]). *Let A be a matrix with row-full term-rank. Then the following three conditions are equivalent.*

(i) *The matrix A is totally sign-nonsingular.*

(ii) *The matrix A has signed null space.*

(iii) *The matrix A has signed row space.*

This theorem can also be derived in terms of oriented matroid theory. See [2] for the terminology in the theory of oriented matroids. Let $\mathcal{M}(A)$ be the oriented matroid represented by a matrix A. The condition (i) of Theorem 4.1 means that, for any $\tilde{A} \in \mathcal{Q}(A)$, the oriented matroid $\mathcal{M}(\tilde{A})$ has the same chirotope as $\mathcal{M}(A)$. The condition (ii) means that $\mathcal{M}(\tilde{A})$ has the same vector family as $\mathcal{M}(A)$ for any $\tilde{A} \in \mathcal{Q}(A)$. And the condition (iii) means that $\mathcal{M}(\tilde{A})$ has the same covector family as $\mathcal{M}(A)$ for any $\tilde{A} \in \mathcal{Q}(A)$. Therefore, each of these three conditions is equivalent to that $\mathcal{M}(\tilde{A})$ is the same oriented matroid as $\mathcal{M}(A)$ for any $\tilde{A} \in \mathcal{Q}(A)$.

We now discuss the computational complexity of recognizing totally sign-nonsingular matrices. A matrix which does not have row-full term-rank is totally sign-nonsingular. The following theorem shows that testing total sign-nonsingularity of an $m \times n$ matrix with row-full term-rank can be reduced to recognizing a sign-nonsingular matrix of order $m + n$. By the result of Robertson, Seymour, and Thomas [16], this can be done in $\mathrm{O}(n^3)$ time.

Theorem 4.2. *Let A be an $m \times n$ matrix with row-full term-rank, and U and V the row and column sets of A. Then A is totally sign-nonsingular if and only if $T = \begin{pmatrix} O & A \\ A^\top & D \end{pmatrix}$ is sign-nonsingular, where D is a diagonal matrix with positive diagonal entries.*

Proof. Suppose that A is totally sign-nonsingular. Let $\tilde{T} \in \mathcal{Q}(T)$ be a matrix in the form of $\tilde{T} = \begin{pmatrix} O & \tilde{A} \\ \hat{A}^\top & \tilde{D} \end{pmatrix}$, where $\tilde{A}$ and $\hat{A}$ are matrices in $\mathcal{Q}(A)$, and $\tilde{D}$ is a matrix in $\mathcal{Q}(D)$. Since the sign of $\det \tilde{T}$ is invariant under multiplying any positive scalar to a column, we assume without loss of generality that $\tilde{D}$ is the identity matrix. Then it holds that

$$\det \tilde{T} = \det(-\tilde{A}\hat{A}^\top) = (-1)^m \sum_{\substack{J \subseteq V, \\ |J| = |U|}} (\det \tilde{A}[J])(\det \hat{A}[J]),$$

where the last equality follows from the Binet-Cauchy formula. Since $A[J]$ is sign-nonsingular for any $J \subseteq V$ such that $\det A[J] \neq 0$, the signs of $\det \tilde{A}[J]$ and $\det \hat{A}[J]$ are the same. Hence the sign of $\det \tilde{T}$ equals to $(-1)^m$. Thus T is sign-nonsingular.

Conversely, suppose that A is not totally sign-nonsingular. Then there exists $\tilde{A} \in \mathcal{Q}(A)$ such that $\tilde{A}[J]$ is term-nonsingular but not sign-nonsingular for some $J \subseteq V$. This implies that $\det \tilde{A}[J]$ has at least two nonzero expansion terms with different signs. We denote these two terms by $\operatorname{sgn} \pi \prod_{i \in U} a_{i\pi(i)}$ and $\operatorname{sgn} \pi' \prod_{i \in U} a_{i\pi'(i)}$, where π and π' are the bijections from U to J. Choose the matrix $\tilde{T} = \begin{pmatrix} O & \tilde{A} \\ \tilde{A}^\top & \tilde{D} \end{pmatrix}$ in $\mathcal{Q}(T)$, where $\tilde{D}$ is the identity matrix of order n. Suppose that the rows and columns of $\tilde{T}$ are indexed by $U \cup V$. Define a bijection σ over $U \cup V$ by $\sigma(i) = \pi(i)$ for $i \in U$, $\sigma(i) = \pi^{-1}(i)$ for $i \in J$, and $\sigma(i) = i$ for $i \in V \setminus J$. Another bijection σ' is defined by $\sigma'(i) = \pi'(i)$ for $i \in U$, and $\sigma'(i) = \sigma(i)$ for the other indices $i \in V$. Then $\operatorname{sgn} \sigma \prod_{i \in U \cup V} a_{i\sigma(i)}$ and $\operatorname{sgn} \sigma' \prod_{i \in U \cup V} a_{i\sigma'(i)}$ are both nonzero expansion terms in $\det \tilde{T}$, but they have the opposite signs. Thus T is not sign-nonsingular. $\square$

5 Totally Sign-Nonsingular Linear Programs

We have shown in Theorem 3.2 that recognizing sign-solvability of a given linear program is co-NP-complete. In contrast, Theorem 4.2 implies that there is a strongly polynomial-time algorithm for recognizing totally sign-nonsingular matrices. In this section, we introduce a class of sign-solvable linear programs using totally sign-nonsingular matrices.

For the linear program $\mathrm{LP}(A, b, c)$, we denote by A_{p} the matrix in the form of

$$A_{\mathrm{p}} = \begin{pmatrix} A & -b \end{pmatrix}, \tag{1}$$

where the column vector $-b$ is indexed by g. Then the linear program $\mathrm{LP}(A, b, c)$ can be represented as $\max\{cx \mid A_{\mathrm{p}}\binom{x}{x_g} = 0, x_g = 1, x \geq 0\}$.

The dual program of $\mathrm{LP}(A, b, c)$ is $\min\{yb \mid yA \geq c\}$. Let A_{d} be the matrix defined by

$$A_{\mathrm{d}} = \begin{pmatrix} -c \\ A \end{pmatrix}, \tag{2}$$

where the row vector $-c$ is indexed by r. Then the dual program can be represented as $\min\{yb \mid (y_r\ y)A_{\mathrm{d}} = z, y_r = 1, z \geq 0\}$.

A basic solution and a relative weight vector can be rewritten by using the determinants of submatrices of A_{p} and A_{d}. Let B be a basis. Cramer's rule implies that the basic solution x for B is obtained by

$$x_i = -\frac{\det A_{\mathrm{p}}[B - i + g]}{\det A_{\mathrm{p}}[B]}, \ \forall i \in B, \tag{3}$$

where $B - i + g$ means $B \setminus \{i\} \cup \{g\}$ with g being put at the position of i in B. Similarly, the relative weight vector z for B is obtained by

$$z_j = c_j - c_B A_B^{-1} A_j = -\frac{\det A_{\mathrm{d}}[B + j]}{\det A_B}, \ \forall j \in N, \tag{4}$$

where $B + j$ means $B \cup \{j\}$ with j being inserted into the first position and A_j is the column vector of A indexed by j.

A linear program $\mathrm{LP}(A, b, c)$ is said to be *totally sign-nonsingular* if A has row-full term-rank and both A_{p} and A_{d} are totally sign-nonsingular. If $\mathrm{LP}(A, b, c)$ is totally sign-nonsingular, then A_{p} has signed null space by Theorem 4.1, which means that the set of the sign patterns of the feasible solutions is determined independently of the magnitudes of nonzero entries in A_{p}. Moreover, the total sign-nonsingularity of $\mathrm{LP}(A, b, c)$ implies that A and $K = \begin{pmatrix} 1 & -c \\ 0 & A \end{pmatrix}$ are both totally sign-nonsingular matrices with row-full term-rank. The following theorem asserts that the total sign-nonsingularity implies sign-solvability.

Theorem 5.1. *A totally sign-nonsingular linear program* $\mathrm{LP}(A, b, c)$ *is sign-solvable.*

Proof. Suppose that the totally sign-nonsingular linear program $\mathrm{LP}(A, b, c)$ is infeasible, which implies that $\max\{cx \mid A_{\mathrm{p}}\binom{x}{x_g} = 0, x_g = 1, x \geq 0\}$ is infeasible. Since A_{p} has signed null space, a linear program $\mathrm{LP}(\tilde{A}, \tilde{b}, \tilde{c})$ is infeasible for any $\tilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$, and $\tilde{c} \in \mathcal{Q}(c)$. Thus the linear program $\mathrm{LP}(A, b, c)$ is sign-solvable.

Suppose that $\mathrm{LP}(A, b, c)$ is feasible and unbounded. Since K is a totally sign-nonsingular matrix with row-full term-rank, Theorem 4.1 implies that K has signed null space. Hence the existence of a nonnegative solution of the linear system $A_{\mathrm{d}}x = -\binom{1}{0}$ is independent of the magnitudes of nonzero entries in K. Therefore, by Proposition 3.1, a linear program $\mathrm{LP}(\tilde{A}, \tilde{b}, \tilde{c})$ is unbounded for any $\tilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$, and $\tilde{c} \in \mathcal{Q}(c)$, which shows that $\mathrm{LP}(A, b, c)$ is sign-solvable.

Suppose that $\mathrm{LP}(A, b, c)$ is feasible and bounded. Let $\tilde{A}$, $\tilde{b}$, and $\tilde{c}$ be a matrix and vectors in $\mathcal{Q}(A)$, $\mathcal{Q}(b)$, and $\mathcal{Q}(c)$, respectively. The total sign-nonsingularity of $\mathrm{LP}(A, b, c)$ implies that a column subset J

is a basis of $\mathrm{LP}(A,b,c)$ if and only if J is a basis of $\mathrm{LP}(\tilde{A},\tilde{b},\tilde{c})$. Let B be a basis of $\mathrm{LP}(A,b,c)$. Since the basic solution is represented by (3), the total sign-nonsingularity of A_{p} implies that the basic solution for B of $\mathrm{LP}(A,b,c)$ has the same sign pattern as that of $\mathrm{LP}(\tilde{A},\tilde{b},\tilde{c})$. Similarly, the relative weight vector is represented by (4). Since A_{d} and A are totally sign-nonsingular, the relative weight vector for B of $\mathrm{LP}(A,b,c)$ has the same sign pattern as that of $\mathrm{LP}(\tilde{A},\tilde{b},\tilde{c})$. Therefore, the basic solution for B of $\mathrm{LP}(A,b,c)$ is optimal if and only if so is that for B of $\mathrm{LP}(\tilde{A},\tilde{b},\tilde{c})$.

Let x_i for $i=1,\ldots,p$ be all the optimal basic solutions of $\mathrm{LP}(A,b,c)$. An optimal solution is represented as $\sum_{i=1}^{p}\lambda_i x_i$ for some $\lambda_i \geq 0\,(i=1,\ldots,p)$ with $\sum_{i=1}^{p}\lambda_i = 1$. Since $\mathrm{LP}(\tilde{A},\tilde{b},\tilde{c})$ has an optimal basic solution whose sign pattern is $\operatorname{sgn} x_i$ for $i=1,\ldots,p$, $\mathrm{LP}(\tilde{A},\tilde{b},\tilde{c})$ also has an optimal solution with the same sign pattern as $\operatorname{sgn}\sum_{i=1}^{p}\lambda_i x_i$. Similarly, an optimal solution of $\mathrm{LP}(\tilde{A},\tilde{b},\tilde{c})$ has the same sign pattern as some optimal solution of $\mathrm{LP}(A,b,c)$. Thus $\mathrm{LP}(A,b,c)$ is sign-solvable. □

Sign-solvable linear programs are not necessarily totally sign-nonsingular. Indeed, the sign-solvable linear program of Example 1 is not totally sign-nonsingular, whereas the linear program of Example 2 is totally sign-nonsingular.

The sign of the optimal value may not be determined uniquely by the sign patterns, even if a given linear program is totally sign-nonsingular. Let B^* be an optimal basis. The optimal value equals to $c_{B^*}A_{B^*}^{-1}b$, that is, the determinant of $\begin{pmatrix} 0 & c_{B^*} \\ -b & A_{B^*} \end{pmatrix}$. Hence the sign of the optimal value is uniquely determined by the sign pattern if and only if this matrix is sign-nonsingular.

6 Combinatorial Optimality Criterion

This section provides an optimality condition for a totally sign-nonsingular linear program $\mathrm{LP}(A,b,c)$. Let L be the matrix in the form of

$$L = \begin{array}{c} \\ r \\ U \end{array}\!\!\begin{array}{c} \begin{array}{ccc} f & V & g \end{array} \\ \begin{pmatrix} 1 & -c & 0 \\ 0 & A & -b \end{pmatrix} \end{array}, \tag{5}$$

where the row and column sets are indexed by $\{r\}\cup U$ and $\{f\}\cup V\cup\{g\}$ as above. Notice that, even if both A_{p} and A_{d} are totally sign-nonsingular, L may fail to be totally sign-nonsingular.

The optimality criterion is given by the following theorem.

Theorem 6.1. *For a totally sign-nonsingular linear program* $\mathrm{LP}(A,b,c)$, *let B be a basis and M_B be a perfect matching in $G(L[B\cup\{f\}])$. Then the following statements hold.*

- *The basis B is feasible if and only if all i-g M_B-alternating paths in $G(L)$ are evenly oriented for any $i\in B$.*
- *The basis B is optimal if and only if B is feasible and all f-j M_B-alternating paths in $G(L)$ are oddly oriented for any $j\in V\setminus B$.*

In order to prove Theorem 6.1, we investigate combinatorial aspects of totally sign-nonsingular matrices.

Let A be a totally sign-nonsingular matrix with row set U and column set V. Suppose that A has row-full term-rank. The total sign-nonsingularity of A implies that, if $A[J]$ is term-nonsingular, then $J\subseteq V$ is a basis. Let B be a basis in A, and N denote the remaining column subset $V\setminus B$. We denote by M a perfect matching in $G(A[B])$.

Lemma 6.2. *For any $u\in B$ and $v\in N$, the column subset $B\setminus\{u\}\cup\{v\}$ is a basis if and only if there exists an M-alternating path P from u to v in $G(A)$.*

Proof. Assume that $B\setminus\{u\}\cup\{v\}$ is a basis. Then there exists a perfect matching M' in $G(A[B\setminus\{u\}\cup\{v\}])$. The union $M\cup M'$ consists of M-alternating cycles and a u-v M-alternating path. Conversely, assume that there exists a u-v M-alternating path P. The symmetric difference $M\triangle P$ is a matching with $V(M\triangle P)=B\setminus\{u\}\cup\{v\}$. Thus $B\setminus\{u\}\cup\{v\}$ is a basis. □

The next lemma says that the difference between the signs of $\det A[B]$ and $\det A[B-u+v]$ is obtained by the sign of a u-v path, where $B-u+v$ means $B \setminus \{u\} \cup \{v\}$ with v being put at the position of u in B.

Lemma 6.3. *Let P be an M-alternating path from $u \in B$ to $v \in N$. Then P is oddly (evenly) oriented if and only if* $\det A[B]$ *and* $\det A[B-u+v]$ *have the same (opposite) signs.*

Proof. By Lemma 6.2, $B-u+v$ is also a basis in A, and $M' = M \triangle P$ is a perfect matching in $G(A[B-u+v])$. Let $\pi : U \to B$ be the bijection corresponding to M, and $\pi' : U \to B-u+v$ corresponding to M'. Since both $A[B]$ and $A[B-u+v]$ are sign-nonsingular, in order to compare the signs of $\det A[B]$ and $\det A[B-u+v]$, it suffices to compare the signs of $\operatorname{sgn}\pi \prod_{i \in U} a_{i\pi(i)}$ and $\operatorname{sgn}\pi' \prod_{i \in U} a_{i\pi'(i)}$, namely, $\operatorname{sgn}\pi \prod_{(i,j)\in M\cap P} a_{ij}$ and $\operatorname{sgn}\pi' \prod_{(i,j)\in M'\cap P} a_{ij}$.

Let $2p$ be the length of P with an integer $p \geq 1$, and let $u_1, u_2, \ldots, u_p$ be the vertices in $U(P)$ along P. Then the bijection π' is represented as $\pi'(u_s) = \pi(u_{s+1})$ for $s = 1, \ldots, p-1$, $\pi'(u_p) = v$, and $\pi'(t) = \pi(t)$ for the other vertices $t \in U \setminus U(P)$. This implies that π' is a product of π and the cyclic permutation of length $p+1$. Hence, if p is even, then the signs of π and π' are different, and if p is odd, then these signs are the same. Examples of M-alternating paths and the corresponding submatrices are depicted in Figures 1 and 2.

In traversing P from u to v, the number of edges in the direction of traversal is the sum of the numbers of negative edges in $M \cap P$ and positive edges in $M' \cap P$. Suppose that p is even. Then the numbers of positive and negative edges in $M' \cap P$ have the same parity. Hence, P is oddly oriented if and only if the number of negative edges in P is odd, that is, $\prod_{(i,j)\in P} a_{ij}$ is negative. Thus $\prod_{(i,j)\in M\cap P} a_{ij}$ and $\prod_{(i,j)\in M'\cap P} a_{ij}$ have the opposite signs. Next suppose that p is odd. The parity of the numbers of positive and negative edges in $M' \cap P$ is different. Hence, P is oddly oriented if and only if $\prod_{(i,j)\in P} a_{ij}$ is positive, namely, $\prod_{(i,j)\in M\cap P} a_{ij}$ and $\prod_{(i,j)\in M'\cap P} a_{ij}$ have the same signs. Therefore, P is oddly oriented if and only if $\operatorname{sgn}\pi \prod_{(i,j)\in M\cap P} a_{ij}$ and $\operatorname{sgn}\pi' \prod_{(i,j)\in M'\cap P} a_{ij}$ have the same signs. $\square$

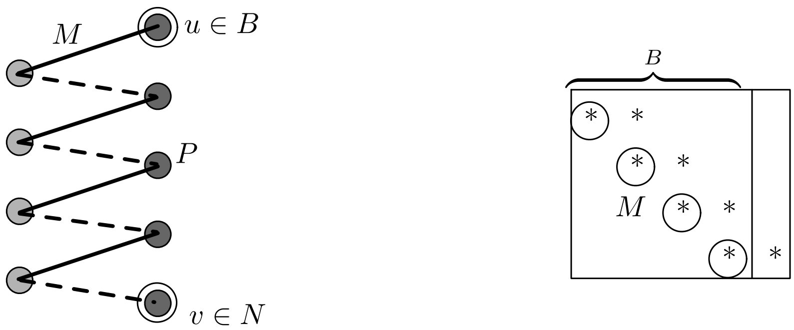

Figure 1: An M-alternating path of length $2p$ with an even integer $p \geq 2$.

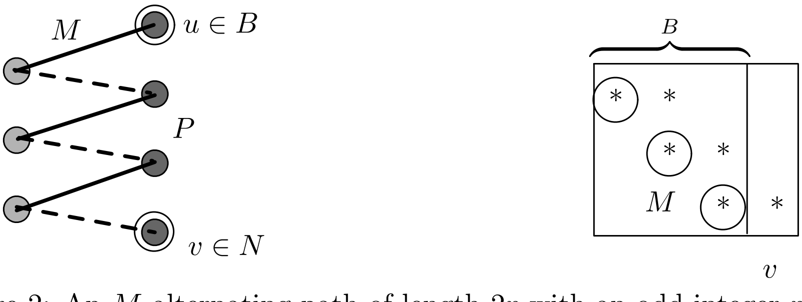

Figure 2: An M-alternating path of length $2p$ with an odd integer $p \geq 1$.

Corollary 6.4. *For any $u \in B$ and $v \in N$, all u-v M-alternating paths have the same sign.*

Proof. This directly follows from Lemma 6.3. $\square$

We are now ready to prove Theorem 6.1.

Proof of Theorem 6.1. For a basis B, the basic solution can be represented by (3). Since A_p is totally sign-nonsingular, it follows from Lemma 6.3 and Corollary 6.4 that the basic solution is nonnegative if and only if all i-g M-alternating paths in $G(A_\mathrm{p})$ are evenly oriented for any $i \in B$. Thus the first part of this theorem holds.

By (4), the sign pattern of the relative weight vector is obtained from the signs of $\det A_\mathrm{d}[B + j]$ and $\det A[B]$ for all $j \in N$. Since both A_p and A_d are totally sign-nonsingular, $K = L[V \cup \{f\}]$ is also totally sign-nonsingular. Moreover, it holds that $\det A[B] = \det K[B + f]$ and $\det A_\mathrm{d}[B + j] = \det K[B + j]$. Then Lemma 6.3 and Corollary 6.4 imply that the relative weight vector is nonpositive if and only if all f-j M_B-alternating paths in $G(K)$ are oddly oriented for any $j \in N$. Thus the second part holds. $\square$

7 Feasibility of Totally Sign-Nonsingular Linear Programs

This section is devoted to testing feasibility of a totally sign-nonsingular linear program $\mathrm{LP}(A, b, c)$, and to finding a feasible basis if $\mathrm{LP}(A, b, c)$ is feasible. Recall that A has row-full term-rank and A_p in the form of (1) is totally sign-nonsingular.

7.1 Testing Feasibility

We say that a row of a matrix is *unisigned* if it has a nonzero entry and all nonzero entries have the same sign. A row of a matrix is *mixed* if it has both positive and negative entries. A matrix is said to be *row-mixed* if every row is mixed.

We first show that we can reduce the feasibility problem of $\mathrm{LP}(A, b, c)$ to the one with row-mixed A_p. Assume that $A_\mathrm{p}\,(\neq O)$ is not row-mixed. Then there exists a row $u \in U$ such that $A[\{u\}, V]$ is nonnegative and $-b_u \geq 0$. If $b_u < 0$, then $A[\{u\}, V] \geq 0$ implies that $\mathrm{LP}(A, b, c)$ is infeasible. If $b_u = 0$, then any feasible solution satisfies that $x_v = 0$ for all $v \in \Gamma(\{u\})$. Thus $Ax = b$ has nonnegative solutions if and only if so does $A[U \setminus \{u\}, V \setminus \Gamma(\{u\})]x[V \setminus \Gamma(\{u\})] = b[U \setminus \{u\}]$.

More concretely, an algorithm for the reduction works as follows. Set $A_\mathrm{p}^0 = A_\mathrm{p}$. Assume that A_p^{l-1} is not row-mixed for a positive integer l. Let U_l be the set of rows that are not mixed in A_p^{l-1}. Define $V_l = \Gamma(U_l) \setminus \Gamma(U_{l-1})$, where $\Gamma(U_0) = \emptyset$. Let A_p^l be the submatrix obtained from A_p^{l-1} by deleting U_l and V_l. Repeat this for $l = 1, 2, \ldots$ until A_p^l is row-mixed. Let k be the number of iterations. We denote $U^h = \bigcup_{l=1}^h U_l$ and $V^h = \bigcup_{l=1}^h V_l$ for $h = 1, \ldots, k$, and $U_\infty = U \setminus U^k$ and $V_\infty = V \cup \{g\} \setminus V^k$.

The obtained partition of row and column sets into $U_1, \ldots, U_k, U_\infty$ and $V_1, \ldots, V_k, V_\infty$ is called the *unisigned partition* of A_p. The unisigned partition satisfies that $A_\mathrm{p}[U_l, V_h] = O$ for $1 \leq l < h \leq k$. Moreover, each row in $A_\mathrm{p}[U_h, V_h]$ is unisigned or zero, and $A_\mathrm{p}[U \setminus U^h, V \cup \{g\} \setminus V^{h-1}]$ is row-mixed for $h = 1, \ldots, k$, where $V^0 = \emptyset$. Therefore, by row and column permutations, the unisigned partition derives a staircase structure of A_p. An example of the unisigned partition is depicted in Figure 3.

If $g \in V^k$, then we know the linear program $\mathrm{LP}(A, b, c)$ is infeasible. If $g \in V_\infty$, then any feasible solution x satisfies that $x[V^k] = 0$, and the feasibility of A_p is reduced to that of $A_\mathrm{p}[U_\infty, V_\infty]$ that is row-mixed. The following lemma implies that $A_\mathrm{p}[U_\infty, V_\infty]$ is totally sign-nonsingular and $A[U_\infty, V_\infty \setminus \{g\}]$ has row-full term-rank.

Lemma 7.1. *If A_p is totally sign-nonsingular, then $A_\mathrm{p}[U_\infty, V_\infty]$ is also totally sign-nonsingular and $A_\mathrm{p}[U_\infty, V_\infty \setminus \{j\}]$ has row-full term-rank for any $j \in V_\infty$.*

Proof. By Theorem 4.1, A_p has signed null space. Since $A_\mathrm{p}[U^k, V_\infty] = O$, this implies that $A_\mathrm{p}[U_\infty, V_\infty]$ also has signed null space.

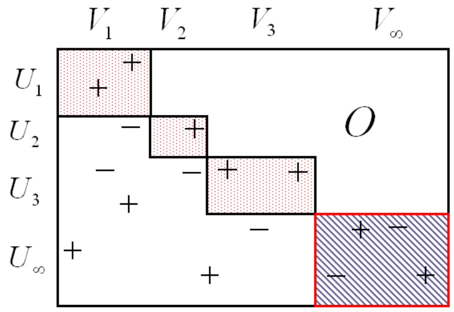

Figure 3: The unisigned partition of a matrix.

We then claim that the submatrix $A_{\rm p}[U_\infty, V_\infty \setminus \{j\}]$ has row-full term-rank for any $j \in V_\infty$. Assume that $\text{t-rank}A_{\rm p}[U_\infty, V_\infty \setminus \{j\}] < |U_\infty|$ for some $j \in V_\infty$. Then Proposition 2.1 implies that there exists $X \subseteq U_\infty$ such that $|Y| \leq |X|$ holds for $Y = \Gamma(X) \cap V_\infty$. We choose X such that $|Y| - |X|$ is minimum. Then we have $\text{t-rank}A_{\rm p}[X, Y] = |Y|$. Since $A_{\rm p}[U_\infty, V_\infty]$ is row-mixed, there exists $\tilde{A}_{\rm p} \in \mathcal{Q}(A_{\rm p})$ such that the sum of the columns of $\tilde{A}_{\rm p}[U_\infty, V_\infty]$ is zero, that is, $\tilde{A}_{\rm p}[U_\infty, V_\infty]x = 0$ has a solution $x = \mathbf{1}$, where $\mathbf{1}$ is the column vector whose entries are all one. Since $\text{t-rank}A_{\rm p}[X, Y] = |Y|$, there exists $\hat{A}_{\rm p} \in \mathcal{Q}(A_{\rm p})$ such that the columns indexed by Y are linearly independent, which means that $\hat{A}_{\rm p}[U_\infty, V_\infty]x = 0$ has no positive solution. This contradicts that $A_{\rm p}[U_\infty, V_\infty]$ has signed null space. Thus $A_{\rm p}[U_\infty, V_\infty \setminus \{j\}]$ has row-full term-rank for any $j \in V_\infty$.

Therefore, the submatrix $A_{\rm p}[U_\infty, V_\infty]$ also has row-full term-rank. Since $A_{\rm p}[U_\infty, V_\infty]$ has signed null space, it follows from Theorem 4.1 that $A_{\rm p}[U_\infty, V_\infty]$ is totally sign-nonsingular. □

For the feasibility of a linear program $\mathrm{LP}(A, b, c)$ with row-mixed $A_{\rm p}$, we have the following lemma. Shao and Ren [18] proved this to derive Theorem 4.1. Conversely, we describe a simple proof based on Theorem 4.1.

Lemma 7.2. *A totally sign-nonsingular linear program* $\mathrm{LP}(A, b, c)$ *has a positive solution if* $A_{\rm p}$ *is row-mixed.*

Proof. Theorem 4.1 implies that $A_{\rm p}$ has signed null space. Hence it suffices to find some matrix $\tilde{A}_{\rm p}$ in $\mathcal{Q}(A_{\rm p})$ such that the corresponding linear system $\tilde{A}x = \tilde{b}$ has positive solutions. Since $A_{\rm p}$ is row-mixed, there exists $\tilde{A}_{\rm p} \in \mathcal{Q}(A_{\rm p})$ such that the sum of the columns of $\tilde{A}_{\rm p}$ is zero, which implies that the linear system $\tilde{A}_{\rm p}x = 0$ has a solution $x = \mathbf{1}$, where $\mathbf{1}$ is the column vector whose entries are all one. This means that the corresponding linear system $\tilde{A}_{\rm p}x = \tilde{b}$ also has a solution $x = \mathbf{1}$. Thus $\mathrm{LP}(A, b, c)$ has a positive solution. □

These results naturally suggest an efficient algorithm for testing feasibility of a totally sign-nonsingular linear program $\mathrm{LP}(A, b, c)$. The algorithm starts with finding the unisigned partition $U_1, \ldots, U_k, U_\infty$ and $V_1, \ldots, V_k, V_\infty$ of $A_{\rm p}$. If $g \in V^k$, then $\mathrm{LP}(A, b, c)$ is infeasible. Otherwise, $\mathrm{LP}(A, b, c)$ is feasible by Lemmas 7.1 and 7.2.

The running time bound of this algorithm is now given as follows. To obtain the unisigned partition, we check in each iteration whether each row remains row-mixed or not. Hence the lth iteration for $l = 1, \ldots, k$ requires $\mathrm{O}(\gamma_l)$ time, where γ_l is the number of nonzero entries in $A_{\rm p}[V_l]$. Therefore, it takes $\mathrm{O}(\gamma)$ time to find the unisigned partition. Thus we have the following theorem.

Theorem 7.3. *For a totally sign-nonsingular linear program* $\mathrm{LP}(A, b, c)$*, the feasibility of* $\mathrm{LP}(A, b, c)$ *can be decided in* $\mathrm{O}(\gamma)$ *time, where* γ *is the number of nonzero entries in* $A_{\rm p}$.

7.2 Row-Mixed Totally Sign-Nonsingular Matrices

Kim and Shader gave a characterization on row-mixed totally sign-nonsingular matrices. We say that a path P in a directed bipartite graph $G = (U, V; E)$ is *mixed* if, for each inner vertex u in $U(P)$, the two edges incident to u have different signs. A *signing* D of an $m \times n$ matrix is a $\{1, -1\}$-diagonal matrix of order n. Notice that total sign-nonsingularity is preserved by multiplying -1 to some columns and rows.

Theorem 7.4 (Kim and Shader [8]). *Let A be a row-mixed matrix. Then A is a totally sign-nonsingular matrix with row-full term-rank if and only if $G(AD)$ has no mixed cycles for any signing D such that AD is row-mixed.*

It follows from Lemma 7.1 that, for a row-mixed totally sign-nonsingular matrix A, $A[V \setminus \{j\}]$ has row-full term-rank for any $j \in V$. Theorem 7.4 and the following lemma imply that, for a row-mixed submatrix $A[I, J]$ in A, the submatrix $A[I, J \setminus \{j\}]$ has row-full term-rank for any $j \in J$.

Lemma 7.5. *Let A be a row-mixed matrix such that $G(A) = (U, V; E)$ has no mixed cycles. Then $A[V \setminus \{j\}]$ has row-full term-rank for any $j \in V$.*

Proof. We first show that $|\Gamma(X)| \geq |X| + 1$ holds for any nonempty subset $X \subseteq U$. Since $A[X, \Gamma(X)]$ is row-mixed, each vertex in X of $G(A[X, \Gamma(X)])$ has at least two edges with positive and negative signs. The number of these edges is $2|X|$. Since $G(A[X, \Gamma(X)])$ has no mixed cycles, these $2|X|$ edges form a forest covering all vertices in X, and hence $G(A[X, \Gamma(X)])$ has at least $2|X| + 1$ vertices. This implies that $|\Gamma(X)|$ is at least $|X| + 1$. Therefore, it follows from Proposition 2.1 that $A[V \setminus \{j\}]$ has row-full term-rank for any $j \in V$. $\square$

7.3 Primal Contraction

We introduce an operation, called *primal contraction*, on a row-mixed totally sign-nonsingular matrix A_{p}. We say that a square submatrix $A[I, J]$ in A is *primal contractible* if $A_{\mathrm{p}}[I, J \cup \{g\}]$ is row-mixed.

Let $H = A[I, J]$ be a primal contractible submatrix in A. Since $A_{\mathrm{p}}[I, J \cup \{g\}]$ is row-mixed and $G(A_{\mathrm{p}}[I, J \cup \{g\}])$ has no mixed cycles by Theorem 7.4, it follows from Lemma 7.5 that $G(H)$ has a perfect matching M_H.

Lemma 7.6. *For a primal contractible submatrix $H = A[I, J]$ in A, the following* (a) *and* (b) *hold.*

(a) *For any $v \in J$, the bipartite graph $G(A_{\mathrm{p}})$ has a g-v M_H-alternating mixed path.*

(b) *For any two distinct vertices $u, v \in J$, the bipartite graph $G(A_{\mathrm{p}})$ has a u-v mixed path in $G(H)$.*

Proof. **(a)** Since $A_{\mathrm{p}}[I, J \cup \{g\}]$ is row-mixed and $G(A_{\mathrm{p}}[I, J \cup \{g\}])$ has no mixed cycles, there exists an M_H-alternating mixed path in $G(A_{\mathrm{p}}[I, J \cup \{g\}])$ from g to any vertex v in J.
(b) For any two distinct vertices $u, v \in J$, (a) implies that $G(A_{\mathrm{p}}[I, J \cup \{g\}])$ has two M_H-alternating mixed paths P_u and P_v from g to u and v, respectively. Then $P_u \triangle P_v$ includes one mixed path, which implies (b). $\square$

Suppose that A_{p} is in the form of

$$A_{\mathrm{p}} = \begin{pmatrix} H & A_{\mathrm{p}}[I, V'] & -b[I] \\ A_{\mathrm{p}}[U', J] & A' & -b[U'] \end{pmatrix},$$

where $U' = U \setminus I$, $V' = V \setminus (J \cup \{g\})$, and $A' = A_{\mathrm{p}}[U', V']$. We then construct a new matrix

$$A'_{\mathrm{p}} = \begin{pmatrix} A' & -b' \end{pmatrix} \tag{6}$$

with row set U' and column set $V' \cup \{g\}$, where the last column vector $-b'$ is indexed by g. The vector $-b'$ is obtained by adding all column vectors in $A_{\mathrm{p}}[U', J]$ to $-b[U']$. Since $G(A_{\mathrm{p}})$ has no mixed cycles, Lemma 7.6 (b) implies that each row in $A_{\mathrm{p}}[U', J \cup \{g\}]$ is unisigned or zero. Hence the sign pattern of $-b'$ is uniquely determined by the sign pattern of A_{p}. Such a new matrix A'_{p} is said to be the *primal contraction of* A_{p} *by* H.

Theorem 7.7. *Let* A_{p} *be a row-mixed totally sign-nonsingular matrix, and* $H = A_{\mathrm{p}}[I, J]$ *a primal contractible submatrix in* A. *Then the primal contraction* A'_{p} *by* H *is row-mixed and totally sign-nonsingular.*

Proof. Since $A_{\mathrm{p}}[U', V]$ is row-mixed and the vector $-b'$ has the same sign pattern as the sum of columns of $A_{\mathrm{p}}[U', J \cup \{g\}]$, A'_{p} is row-mixed.

We will show that A'_{p} is totally sign-nonsingular. Assume that A'_{p} is not totally sign-nonsingular. Then Theorem 7.4 implies that there exists a signing D' of A'_{p} such that $A'_{\mathrm{p}}D'$ is row-mixed and $G(A'_{\mathrm{p}}D')$ has a mixed cycle C'. Let d'_i be the diagonal entry of D' indexed by $i \in V' \cup \{g\}$. Define the signing D of A_{p} from D' as follows. Each diagonal entry d_i of D is defined by $d_j = d'_g$ for $j \in J \cup \{g\}$ and $d_j = d'_j$ for $j \in V'$. Then $A_{\mathrm{p}}D$ is row-mixed. If $V(C')$ does not contain the vertex g, then C' is also a mixed cycle in $G(A_{\mathrm{p}}D)$, which contradicts that A_{p} is row-mixed and totally sign-nonsingular by Theorem 7.4. Hence we assume that $V(C')$ contains the vertex g. Let $i, j \in U'$ be vertices with $(i, g),\ (j, g) \in C'$. Since (i, g) and (j, g) are edges in $G(A'_{\mathrm{p}})$, there exist $s, t \in J$ such that $\operatorname{sgn} a_{is} = -\operatorname{sgn} b'_i$ and $\operatorname{sgn} a_{jt} = -\operatorname{sgn} b'_j$. Moreover, Lemma 7.6 (b) implies that $G(H)$ has a mixed path P from s to t. Then $C' \setminus \{(i, g), (j, g)\} \cup P \cup \{(i, s), (j, t)\}$ is a mixed cycle in $G(A_{\mathrm{p}}D)$. This contradicts that A_{p} is row-mixed and totally sign-nonsingular. Thus A'_{p} is totally sign-nonsingular. $\square$

Kim and Shader [8] introduced a similar operation preserving total sign-nonsingularity on row-mixed matrices. If a primal contractible submatrix $A[I, J]$ satisfies $|I| = 1$ and $A_{\mathrm{p}}[I, V'] = O$, then the primal contraction corresponds to their *conformal contraction*. Theorem 7.7 implies that if A_{p} is totally sign-nonsingular, then the conformal contraction is totally sign-nonsingular. Kim and Shader showed that the converse also holds for conformal contractions.

7.4 Finding a Feasible Basis

We now describe an algorithm for finding a feasible basis of a totally sign-nonsingular feasible linear program $\mathrm{LP}(A, b, c)$.

We first assume that A_{p} is row-mixed. Then we find a feasible basis recursively as follows. If $b = 0$, then any basis is feasible. If b has a nonzero entry b_u, then there exists $v \in V$ such that $\operatorname{sgn} a_{uv} = \operatorname{sgn} b_u$. The submatrix $H = A[\{u\}, \{v\}]$ is primal contractible. The primal contraction A'_{p} by H is row-mixed and totally sign-nonsingular by Theorem 7.7. Moreover, Lemma 7.1 implies that A' has row-full term-rank. We apply this algorithm recursively to the smaller linear system $A'x = b'$, and obtain a feasible basis B' of $A'x = b'$. The following lemma implies that $B' \cup \{v\}$ is a feasible basis of $\mathrm{LP}(A, b, c)$.

Lemma 7.8. *For a totally sign-nonsingular linear program* $\mathrm{LP}(A, b, c)$ *with row-mixed* A_{p}, *let* $H = A[I, J]$ *be a primal contractible submatrix, and* B' *a feasible basis of the primal contraction by* H. *Then* $B' \cup J$ *is a feasible basis of* $\mathrm{LP}(A, b, c)$.

Proof. Let M' be a perfect matching in $G(A'_{\mathrm{p}}[B'])$, and M_H a perfect matching in $G(H)$. Since $M = M' \cup M_H$ is a perfect matching in $G(A)$, $B' \cup J$ is a basis of $\mathrm{LP}(A, b, c)$.

It follows from Corollary 6.4 and Lemma 7.6 (a) that, for any $i \in J$, all g-i M_H-alternating paths are evenly oriented. Let P be a g-i M-alternating path in $G(A_{\mathrm{p}})$ for $i \in B'$. Suppose that P has no vertices in J. Then P is also a g-i M'-alternating path in $G(A'_{\mathrm{p}})$. Since B' is a feasible basis of the primal contraction, P is evenly oriented by Theorem 6.1. Next suppose that P has a vertex in J, namely, that P has some edge (u, v) with $u \in U'$ and $v \in J$. Then P is partitioned into the g-v M_H-alternating path P_v in $G(H)$ and the v-i M'-alternating path P_i in $G(A_{\mathrm{p}})$. The path P_v is evenly oriented. Since P_i appears in $G(A'_{\mathrm{p}})$ as a g-i M'-alternating path, Theorem 6.1 implies that P_i is also evenly oriented. Hence P is evenly oriented. Thus all g-i M-alternating paths are evenly oriented for any $i \in B$, which implies that B is feasible by Theorem 6.1. $\square$

In each recursive step, we obtain the primal contraction in $\mathrm{O}(\gamma_v)$ time, where γ_v is the number of nonzero entries in column v. Hence it takes $\mathrm{O}(\gamma)$ time until the recursion terminates, where γ is the number of nonzero entries in A_{p}. If $b = 0$, then we obtain a feasible basis with the aid of an efficient maximum bipartite matching algorithm [5], which runs in $\mathrm{O}(\sqrt{m}\gamma)$ time, where m is the number of rows of A. Thus it requires $\mathrm{O}(\sqrt{m}\gamma)$ time to find a feasible basis of $\mathrm{LP}(A, b, c)$ with row-mixed A_{p}.

Even if A_{p} is not row-mixed, we find a feasible basis efficiently as follows. We first obtain the unisigned partition of A_{p} into $U_1, \ldots, U_k, U_\infty$ and $V_1, \ldots, V_k, V_\infty$, which requires $\mathrm{O}(\gamma)$ time. The feasibility of $\mathrm{LP}(A, b, c)$ implies $g \in V_\infty$. Let B_∞ be a feasible basis of the remaining linear system $A[U_\infty, V_\infty \setminus \{g\}]x = b[U_\infty]$. Since $A[U^k, V^k]$ has row-full term-rank, $A[U^k, V^k]$ has a basis, denoted by B_*. Then $B = B_\infty \cup B_*$ is a feasible basis of $\mathrm{LP}(A, b, c)$. Since $A_{\mathrm{p}}[U_\infty, V_\infty]$ is row-mixed and totally sign-nonsingular, we can find B_∞ as described above in $\mathrm{O}(\sqrt{m}\gamma)$ time. The basis B_* can be obtained by finding a perfect matching in $G(A[U^k, V^k])$, which takes $\mathrm{O}(\sqrt{m}\gamma)$ time. Thus we have the following theorem.

Theorem 7.9. *For a totally sign-nonsingular linear program $\mathrm{LP}(A, b, c)$ that is feasible, we can find a feasible basis in $\mathrm{O}(\sqrt{m}\gamma)$ time, where m is the number of rows of A and γ is the number of nonzero entries in A_{p}.*

8 Solving Totally Sign-Nonsingular Linear Programs

This section discusses how to obtain the sign pattern of an optimal solution of a totally sign-nonsingular linear program $\mathrm{LP}(A, b, c)$. Recall that A_{p}, A_{d}, and L are in the forms of (1), (2), and (5), respectively.

8.1 Algorithm Outline

We first describe the outline of an algorithm for finding the sign pattern of an optimal solution of the totally sign-nonsingular linear program $\mathrm{LP}(A, b, c)$.

The algorithm starts with testing the feasibility of $\mathrm{LP}(A, b, c)$. Let $U_1, \ldots, U_k, U_\infty$ and $V_1, \ldots, V_k, V_\infty$ be the unisigned partition of A_{p}. If $g \in V \setminus V_\infty$, then $\mathrm{LP}(A, b, c)$ is infeasible, and if $g \in V_\infty$, then it is feasible.

Suppose that $\mathrm{LP}(A, b, c)$ is feasible. Then any feasible solution x satisfies that $x[V \setminus V_\infty] = 0$. By Lemma 7.1, $A_{\mathrm{p}}[U_\infty, V_\infty]$ is totally sign-nonsingular, and $A[U_\infty, V_\infty \setminus \{g\}]$ has row-full term-rank. Since $A_{\mathrm{d}}[U \setminus U_\infty, V \setminus V_\infty]$ has row-full term-rank, the total sign-nonsingularity of A_{d} implies that $A_{\mathrm{d}}[U_\infty \cup \{r\}, V_\infty \setminus \{g\}]$ is also totally sign-nonsingular. Therefore, we can reduce the totally sign-nonsingular linear program $\mathrm{LP}(A, b, c)$ to the smaller one $\mathrm{LP}(A[U_\infty, V_\infty \setminus \{g\}], b[U_\infty], c[V_\infty])$ such that $A_{\mathrm{p}}[U_\infty, V_\infty]$ is row-mixed.

For a totally sign-nonsingular linear program $\mathrm{LP}(A, b, c)$ with row-mixed A_{p}, Procedure $\mathsf{OptBasis}(A, b, c)$ is defined to be a procedure which returns an optimal basis of $\mathrm{LP}(A, b, c)$ or that $\mathrm{LP}(A, b, c)$ is unbounded. Note that, since A_{p} is row-mixed, $\mathrm{LP}(A, b, c)$ is feasible by Lemma 7.2. Using Procedure $\mathsf{OptBasis}$, Algorithm $\mathsf{OptSign}(A, b, c)$ finds the sign pattern of an optimal solution of a totally sign-nonsingular linear program $\mathrm{LP}(A, b, c)$ as follows.

Algorithm: $\mathsf{OptSign}(A, b, c)$

Step 1: (Testing feasibility) Obtain the unisigned partition of A_{p} into $U_1, \ldots, U_k, U_\infty$ and $V_1, \ldots, V_k, V_\infty$. If $g \notin V_\infty$, then return that $\mathrm{LP}(A, b, c)$ is not feasible. Otherwise, put $A' = A[U_\infty, V_\infty \setminus \{g\}]$, $b' = b[U_\infty]$, and $c' = c[V_\infty]$, and set $x[V \setminus V_\infty] = 0$.

Step 2: Call Procedure $\mathsf{OptBasis}(A', b', c')$ to obtain an optimal basis B of $\mathrm{LP}(A', b', c')$. If $\mathrm{LP}(A', b', c')$ turns out to be unbounded, then return that $\mathrm{LP}(A, b, c)$ is unbounded.

Step 3: Obtain the sign pattern $x[V_\infty \setminus \{g\}]$ of the optimal basic solution for B of $\mathrm{LP}(A', b', c')$, and return x.

Even if $\mathrm{LP}(A,b,c)$ is not totally sign-nonsingular, any feasible solution x satisfies that $x[V\setminus V_\infty]=0$. Therefore, for a linear program $\mathrm{LP}(A,b,c)$ that is not totally sign-nonsingular, if $\mathrm{LP}(A',b',c')$ in Step 2 is totally sign-nonsingular, then this algorithm returns the sign pattern of an optimal solution of $\mathrm{LP}(A,b,c)$.

Applying Algorithm $\mathsf{OptSign}(A,b,c)$, we can compute an optimal solution of $\mathrm{LP}(A,b,c)$ as well as the sign pattern of an optimal solution. Indeed, since we have an optimal basis B of $\mathrm{LP}(A',b',c')$, we can compute the optimal basic solution for B of $\mathrm{LP}(A',b',c')$ by performing Gaussian elimination only once. By extending this optimal basic solution with zero entries in $V\setminus V_\infty$, we obtain an optimal solution of $\mathrm{LP}(A,b,c)$.

In Section 8.4, we will present Procedure $\mathsf{OptBasis}(A,b,c)$ for finding an optimal basis of a totally sign-nonsingular linear program $\mathrm{LP}(A,b,c)$ with row-mixed A_{p}.

8.2 Dual Contraction

In this section, we introduce another type of operations on totally sign-nonsingular matrices, called *dual contraction*. Let $H=A[I,J]$ be a term-nonsingular submatrix in A. Suppose that A_{d} is in the form of

$$A_{\mathrm{d}}=\begin{pmatrix} -c[J] & -c[V'] \\ H & A_{\mathrm{d}}[I,V'] \\ A_{\mathrm{d}}[U',J] & A' \end{pmatrix},$$

where $U'=U\setminus I$, $V'=V\setminus J$, and $A'=A_{\mathrm{d}}[U',V']$. Then we construct a new matrix

$$A'_{\mathrm{d}}=\begin{pmatrix} -c' \\ A' \end{pmatrix} \tag{7}$$

with row set $\{r\}\cup U'$ and column set V', where the first row $-c'$ is indexed by r. The vector $-c'$ is defined as follows. Let K be the matrix in the form of $K=\begin{pmatrix} 1 & -c \\ 0 & A\end{pmatrix}$, where the first column is indexed by f. We denote by M_f the matching in $G(K)$ consisting of a perfect matching in $G(H)$ and the edge (r,f). Each entry $-c'_j$ for $j\in V'$ is defined by

$$-c'_j=\begin{cases} +1 & \text{if } \textit{some } f\text{-}j\ M_f\text{-alternating path in } G(K) \text{ is oddly oriented,}\\ -1 & \text{if } \textit{all } f\text{-}j\ M_f\text{-alternating paths in } G(K) \text{ are evenly oriented,}\\ 0 & \text{if there are no } f\text{-}j\ M_f\text{-alternating paths in } G(K).\end{cases}$$

The matrix A'_{d} obtained by this operation is called the *dual contraction of A_{d} by H*.

Theorem 8.1. *Let A_{d} be a totally sign-nonsingular matrix, and $H=A[I,J]$ a term-nonsingular submatrix in A. Then the dual contraction A'_{d} by H is totally sign-nonsingular.*

Proof. If A'_{d} does not have row-full term-rank, then A'_{d} is totally sign-nonsingular. Hence we may assume that A'_{d} has row-full term-rank.

Assume that A'_{d} is not totally sign-nonsingular. This implies that there exists $B'\subseteq V'$ such that $A'_{\mathrm{d}}[B']$ is term-nonsingular but not sign-nonsingular. Then there exists a central cycle C' in $G(A'_{\mathrm{d}}[B'])$ which is evenly oriented. Let M' be a perfect matching in $G(A'_{\mathrm{d}}[B'])$ such that C' is M'-alternating, and let $s\in V'$ denote the vertex with $(r,s)\in M'$. The existence of the edge (r,s) implies that $G(K)$ has an f-s M_f-alternating path P_s whose sign equals to the sign of the path $\{(r,f),(r,s)\}$ in $G(K')$, where $K'=\begin{pmatrix} 1 & -c' \\ 0 & A'\end{pmatrix}$. Then $M=M'\setminus\{(r,s)\}\cup(M_f\triangle P_s)$ is a perfect matching in $G(A_{\mathrm{d}}[B'\cup J])$.

If C' is contained in $G(A')$, then C' is an M-alternating cycle in $G(A_{\mathrm{d}})$ which is evenly oriented. This contradicts the total sign-nonsingularity of A_{d}. Next suppose that C' has the edges $(r,s)\in M'\cap C'$ and $(r,t)\in C'\setminus M'$ in $G(A'_{\mathrm{d}})$. We denote by P_t an f-t M_f-alternating path in $G(K)$ with the same sign as the path $\{(r,f),(r,t)\}$ in $G(K')$. Then the symmetric difference $P_s\triangle P_t$ includes the even path P in

$G(A_{\mathrm{d}})$ with the same sign as the path $\{(r,s),(r,t)\}$ in $G(A'_{\mathrm{d}})$. Hence $G(A_{\mathrm{d}})$ has an evenly oriented cycle $C = (C' \setminus \{(r,s),(r,t)\}) \cup P$, and C is M-alternating. This contradicts the total sign-nonsingularity of A_{d}. Thus A'_{d} is totally sign-nonsingular. □

Notice that it is not easy to obtain the dual contraction, that is, to decide whether all f-j M_f-alternating paths in $G(K)$ are evenly oriented or not for $j \in V \setminus J$. However, for a term-nonsingular submatrix H that arises in Procedure OptBasis, we can obtain the dual contraction in polynomial time.

8.3 Contraction of Totally Sign-Nonsingular Linear Programs

We introduce an operation on a totally sign-nonsingular bounded linear program $\mathrm{LP}(A,b,c)$ with row-mixed A_{p}, using primal and dual contractions.

Let H be a primal contractible submatrix in A_{p}. Note that H is term-nonsingular. Consider the primal contraction A'_{p} of A_{p} by H in the form of (6), and the dual contraction A'_{d} of A_{d} by H in the form of (7). Then we define a new linear program $\mathrm{LP}(A',b',c')$, called the *LP-contraction of* $\mathrm{LP}(A,b,c)$ *by* H. We denote by L' the matrix in the form of

$$L' = \begin{array}{c c} & \begin{array}{ccc} f & V' & g \end{array} \\ \begin{array}{c} r \\ U' \end{array} & \begin{pmatrix} 1 & -c' & 0 \\ 0 & A' & -b' \end{pmatrix} \end{array}, \tag{8}$$

where the row and column sets are indexed by $\{r\} \cup U'$ and $\{f\} \cup V' \cup \{g\}$ as above. Theorems 7.7 and 8.1 imply that the LP-contraction is a totally sign-nonsingular linear program. Moreover, since A'_{p} is row-mixed, the LP-contraction has feasible solutions by Lemma 7.2. However, it is not necessarily bounded.

We say that a submatrix H in A is *LP-contractible* if H is primal contractible in A_{p} and the LP-contraction is bounded. This implies that the LP-contraction by an LP-contractible submatrix has an optimal basis. Moreover, the following lemma guarantees that, if we have an LP-contractible submatrix, then the totally sign-nonsingular linear program $\mathrm{LP}(A,b,c)$ can be reduced to the LP-contraction.

Lemma 8.2. *Suppose that a totally sign-nonsingular linear program* $\mathrm{LP}(A,b,c)$ *is bounded, and that* A_{p} *is row-mixed. Let* $H = A[I,J]$ *be an LP-contractible submatrix, and* B' *an optimal basis of the LP-contraction. Then* $B = B' \cup J$ *is an optimal basis of* $\mathrm{LP}(A,b,c)$.

Proof. Let M' be a perfect matching in $G(L'[B' \cup \{f\}])$, and M_H a perfect matching in $G(H)$. Then $M = M' \cup M_H$ is a perfect matching in $G(L)$, and hence B is a basis of $\mathrm{LP}(A,b,c)$. It follows from Lemma 7.8 that B is feasible.

We will show that B is optimal. Let j be a vertex in $V \setminus B$, and P an f-j M-alternating path. Suppose that P uses no vertices in B'. This means that c'_j is nonzero. Since B' is an optimal basis of the LP-contraction, $-c'_j$ is positive by Theorem 6.1. Hence there exists an f-j M_H-alternating path in $G(L)$ which is oddly oriented, which implies that P is also oddly oriented by Corollary 6.4. Next suppose that P has a vertex v in B'. Then P is partitioned into the f-v M_H-alternating path and the v-j M'-alternating path P_j. Since B' is an optimal basis of the LP-contraction, Theorem 6.1 implies that the f-j M'-alternating path $\{(r,f),(r,v)\} \cup P_j$ in $G(L')$ is oddly oriented. The bipartite graph $G(L)$ has an f-v M_H-alternating path P_v with the same sign as the path $\{(r,f),(r,v)\}$ in $G(L')$. Hence the f-j M-alternating path $P_v \cup P_j$ in $G(L)$ is oddly oriented, which implies that so is P by Corollary 6.4. Thus all f-j M-alternating paths are oddly oriented, and hence B is optimal by Theorem 6.1. □

8.4 Finding an Optimal Basis

In this section, we present Procedure $\mathsf{OptBasis}(A,b,c)$ for a totally sign-nonsingular linear program $\mathrm{LP}(A,b,c)$ with row-mixed A_{p}. Lemma 7.2 implies that $\mathrm{LP}(A,b,c)$ is feasible. It follows from Lemma

8.2 that, if $\mathrm{LP}(A,b,c)$ is bounded, then we can reduce $\mathrm{LP}(A,b,c)$ to a smaller one by finding an LP-contractible submatrix. We now describe how to test the boundedness of $\mathrm{LP}(A,b,c)$ and how to find an LP-contractible submatrix if $\mathrm{LP}(A,b,c)$ is bounded. We consider two cases, that is, the case where c has a positive entry and the case where c is nonpositive.

In the Case of $c \not\leq 0$

Suppose that c has a positive entry. We first test whether $\mathrm{LP}(A,b,c)$ is bounded or not. Proposition 3.1 implies that testing boundedness is equivalent to deciding whether $A_{\mathrm{d}}x = -\binom{1}{0}$ has a nonnegative solution or not. Since $L[V \cup \{f\}]$ is a totally sign-nonsingular matrix with row-full term-rank, Theorem 7.3 implies that we can test the boundedness by the unisigned partition of $L[V \cup \{f\}]$, denoted by $U_1,\ldots,U_k,U_\infty$ and $V_1,\ldots,V_k,V_\infty$. Let $U^h = \bigcup_{l=1}^h U_l$ and $V^h = \bigcup_{l=1}^h V_l$ for $h = 1,\ldots,k$.

If $f \in V_\infty$, then the linear system $A_{\mathrm{d}}x = -\binom{1}{0}$ has a nonnegative solution, and hence $\mathrm{LP}(A,b,c)$ is unbounded by Proposition 3.1.

Next suppose that $f \notin V_\infty$. Then $A_{\mathrm{d}}x = -\binom{1}{0}$ has no nonnegative solutions, which implies that the linear program is bounded. We denote by F the union of the edge sets of $G(L[U_l,V_l])$ for all $l = 1,\ldots,k$. A path P is *admissible* if P is mixed and F-alternating. Let P_{fg} be an f-g admissible path in $G(L)$. As we shall show in Lemma 8.4, the submatrix $H = A[U(P_{fg}) \setminus \{r\}, V(P_{fg}) \setminus \{f,g\}]$ is LP-contractible. Therefore, by Lemma 8.2, we can reduce the linear program $\mathrm{LP}(A,b,c)$ to the LP-contraction by H that is feasible and bounded.

We can find an f-g admissible path P_{fg} in $G(L)$ as follows. Since c has a positive entry, the vertex f is not in V_1. Let f be in V_p for some $1 < p \leq k$, and set $v_p = f$. Repeat the following for $l = p,\ldots,1$. As $v_l \in V_l$, there exists $u_l \in U_l$ with $e_{2l} = (u_l,v_l) \in F$. Suppose that $l = 1$. Since A_{p} is row-mixed, $e_1 = (u_1,g)$ has the different sign from e_2. Next suppose that $l \geq 2$. Then the submatrix $L[(U \cup \{r\}) \setminus U^{l-1}, (V \cup \{f\}) \setminus V^{l-2}]$ is row-mixed, where $V^0 = \emptyset$. Hence there exists $v_{l-1} \in V_{l-1}$ such that $e_{2l-1} = (u_l, v_{l-1})$ is not in F and e_{2l-1} has the different sign from e_{2l}. At the end of the repetition, we obtain the f-g admissible path $P_{fg} = \{e_{2p}, e_{2p-1},\ldots,e_1\}$ in $G(L)$.

In the Case of $c \leq 0$

We next suppose that c is nonpositive. Let $W = \{j \in V \mid c_j < 0\}$. As $c \leq 0$, the objective value of any feasible solution is not greater than zero, and hence $\mathrm{LP}(A,b,c)$ is bounded. We then decide whether the optimal value is equal to zero or less than zero. This is equivalent to testing whether the linear system $A[V \setminus W]x = b$ has nonnegative solutions or not. Consider the unisigned partition of $L[V \cup \{g\}]$ into $U_1,\ldots,U_k,U_\infty$ and $V_1,\ldots,V_k,V_\infty$. Let $U^h = \bigcup_{l=1}^h U_l$ and $V^h = \bigcup_{l=1}^h V_l$ for $h = 1,\ldots,k$. Note that $U_1 = \{r\}$ and $V_1 = W$. Hence the linear system $L[V]x = \binom{0}{b}$ has nonnegative solutions if and only if so does $A[V \setminus W]x = b$.

Suppose that $g \in V_\infty$. The submatrix $A_{\mathrm{p}}[U_\infty, V_\infty]$ is row-mixed, and $G(A_{\mathrm{p}}[U_\infty,V_\infty])$ has no mixed cycles. Hence it follows from Lemma 7.5 that $A_{\mathrm{p}}[U_\infty,V_\infty]$ has row-full term-rank. Since $A_{\mathrm{p}}[U^k \setminus \{r\}, V^k]$ has row-full term-rank, the total sign-nonsingularity of A_{p} implies that $A_{\mathrm{p}}[U_\infty,V_\infty]$ is totally sign-nonsingular. Therefore, it follows from Lemma 7.2 that the linear system $A[U_\infty, V_\infty \setminus \{g\}]x = b[U_\infty]$ has nonnegative solutions, and hence so does $A[V \setminus W]x = b$. Thus the optimal value equals to zero.

An optimal basis can be found as follows. We first obtain a feasible basis B_∞ of $A[U_\infty, V_\infty \setminus \{g\}]x = b[U_\infty]$. We denote by M_∞ a matching in $G(A[U_\infty, V_\infty \setminus \{g\}])$ corresponding to B_∞. Obtain a basis B_* of $A[U^k \setminus \{r\}, V^k]$ such that $|B_* \cap W|$ is minimum. Let M_* be the matching consisting of a perfect matching in $G(A[U^k \setminus \{r\}, B_*])$ and the edge (r,f). Then the following lemma holds.

Lemma 8.3. *The column subset $B = B_\infty \cup B_*$ is an optimal basis of* $\mathrm{LP}(A,b,c)$.

Proof. We denote $M = M_* \cup M_\infty$. Since $G(A[B])$ has the perfect matching $M \setminus \{(r,f)\}$, B is a basis. For a vertex i in B_*, there exist no M-alternating paths from g to i. Since B_∞ is a feasible basis of

$A[U_\infty, V_\infty \setminus \{g\}]x = b[U_\infty]$, Theorem 6.1 implies that all M_∞-alternating paths from g to i are evenly oriented for any $i \in B_\infty$. Thus B is a feasible basis of $\mathrm{LP}(A, b, c)$ by Theorem 6.1.

We next show that B is optimal. Since $c[W] < 0$, all f-j M-alternating paths are oddly oriented for any $j \in W \setminus B$ by Corollary 6.4. Suppose that j is in $V \setminus (B \cup W)$. We claim that there exist no f-j M-alternating paths. Indeed, assume that there exists an f-j M-alternating path P. As $L[U^k, V_\infty] = O$, the path P uses no edges in M_∞. The vertex set $V(P)$ contains the vertex v in W. Let P' be the v-j M_*-alternating path along P. Then $M' = M_* \triangle P'$ is a perfect matching in $G(L[U^k, V^k \cup \{f\}])$. However, it holds that $|V(M') \cap W| < |V(M_*) \cap W|$, which contradicts that $|B_* \cap W|$ is minimum. Hence there exist no f-j M-alternating paths for any $j \in V \setminus (B \cup W)$. Thus B is an optimal basis by Theorem 6.1. $\square$

Next suppose that $g \in V^k$, that is, $A[V \setminus W]x = b$ has no nonnegative solutions. This means that the optimal value is less than zero. We then find an LP-contractible submatrix in a similar way to the case where c has a positive entry.

We denote by F the union of the edge sets of $G(L[U_l, V_l])$ for all $l = 1, \dots, k$. A path P is said to be *admissible* if P is F-alternating and, for each inner vertex $u \in U(P) \setminus \{r\}$, the two edges incident to u in P have the different signs. Notice that, since $c \leq 0$, the two edges incident to r have the same signs. We find an f-g admissible path in $G(L)$ as follows. As $V_1 = W$, let g be in V_p for some $1 < p \leq k$. Set $v_p = g$. Repeat the following for $l = p, \dots, 1$. If $l = 1$, then set $u_1 = r$, $e_2 = (r, v_1)$, and $e_1 = (r, f)$. Suppose that $l \geq 2$. As $v_l \in V_l$, there exists $u_l \in U_l$ with $e_{2l} = (u_l, v_l) \in F$. The submatrix $L[(U \cup \{r\}) \setminus U^{l-1}, (V \cup \{g\}) \setminus V^{l-2}]$ is row-mixed, where $V^0 = \emptyset$. Hence there exists $v_{l-1} \in V_{l-1}$ such that $e_{2l-1} = (u_l, v_{l-1})$ is not in F and e_{2l-1} has the different sign from e_{2l}. Then the path $P_{fg} = \{e_1, e_2, \dots, e_{2p}\}$ is an f-g admissible path in $G(L)$. By Lemma 8.4 below, the submatrix $H = A[U(P_{fg}) \setminus \{r\}, V(P_{fg}) \setminus \{f, g\}]$ is LP-contractible. Therefore, Lemma 8.2 implies that we can reduce $\mathrm{LP}(A, b, c)$ to the LP-contraction by H that is feasible and bounded.

Contractibility

Let P_{fg} be an f-g admissible path in either of the above two cases, and let us denote $I = U(P_{fg})$ and $J = V(P_{fg})$. We will show that the submatrix $H = A[I \setminus \{r\}, J \setminus \{f, g\}]$ is LP-contractible. In either case, an f-g admissible path P_{fg} satisfies that, for any edge $e = (u, v) \in F$ with $u \in I$ and $v \in \Gamma(I) \setminus J$, the f-v path along P_{fg} and e is oddly oriented. We denote by M_f a perfect matching in $G(L[I, J \setminus \{g\}])$ such that $M_f \subseteq P_{fg}$.

Lemma 8.4. *For an f-g admissible path P_{fg}, the submatrix $H = A[I \setminus \{r\}, J \setminus \{f, g\}]$ in A is LP-contractible, where $I = U(P_{fg})$ and $J = V(P_{fg})$.*

Proof. Since P_{fg} is admissible, $A_{\mathrm{p}}[I \setminus \{r\}, J \setminus \{f\}]$ is row-mixed, and hence H is primal contractible. Let $\mathrm{LP}(A', b', c')$ be the LP-contraction by the submatrix H. Suppose that A'_{p}, A'_{d}, and L' are in the forms of (6), (7), and (8), respectively.

Assume that $\mathrm{LP}(A', b', c')$ is unbounded. Then, by Proposition 3.1, $L'[V']x = -\binom{1}{0}$ has nonnegative solutions. Let $U'_1, \dots, U'_{k'}, U'_\infty$ and $V'_1, \dots, V'_{k'}, V'_\infty$ be the unisigned partition of $L'[V' \cup \{f\}]$. The unboundedness implies that $f \in V'_\infty$ and $r \in U'_\infty$. Since $L'[V' \cup \{f\}]$ is a totally sign-nonsingular matrix with row-full term-rank, Lemma 7.1 implies that the row-mixed submatrix $L'[U'_\infty, V'_\infty]$ is also a totally sign-nonsingular matrix with row-full term-rank.

We show that $G(L'[U'_\infty, V'_\infty])$ has a mixed cycle. Consider the submatrix $T = L[U_T, V_T]$ in L, where $U_T = U^k \cap (U'_\infty \setminus \{r\})$ and $V_T = V^k \cap (V'_\infty \setminus \{f\})$. Since $L'[U'_\infty, V'_\infty]$ is row-mixed and $L[U^k, V_\infty] = O$, the submatrix T is row-mixed. As $r \in U'_\infty$, there exists $s \in V'_\infty$ such that (r, s) in $G(L')$ is negative. This implies that $G(L)$ has an f-s M_f-alternating path. Since no M_f-alternating paths from f use a vertex in V_∞, the vertex s is in V^k, and hence $s \in V_T$ holds. Let P be the longest path from s in $G(T)$ which is F-alternating and mixed. Since T is row-mixed, we may assume that P is even. We denote by $t \in V_T$ the end vertex of P. As $t \in V^k$, there exists a vertex $u \in U^k$ with $(u, t) \in F$. By $t \in V'_\infty$, the vertex u

is in I or U_∞. Since P is the longest path in $G(T)$, u is not in U_T, and hence $u \in I$ holds. Since P_{fg} is admissible, the f-t M_f-alternating path along P_{fg} and (u,t) is oddly oriented, which implies that (r,t) in $G(L')$ is positive. Hence $P \cup \{(r,s),(r,t)\}$ is a mixed cycle in $G(L'[U'_\infty, V'_\infty])$.

Therefore, $L'[U'_\infty, V'_\infty]$ is row-mixed and totally sign-nonsingular, but $G(L'[U'_\infty, V'_\infty])$ has a mixed cycle, which contradicts Theorem 7.4. Thus $\mathrm{LP}(A',b',c')$ is bounded, which means that H is LP-contractible. $\square$

The dual contraction by the submatrix H in either case can be obtained as follows. Find the vertex $u \in J$ closest to f along P_{fg} such that the f-u path P along P_{fg} and some edge incident to u is oddly oriented. We denote by P_u the u-g path along P_{fg}. Since P_u is mixed, the f-i M_f-alternating path along P and P_u is oddly oriented for any $i \in V(P_u)$. Therefore, for $j \in V \setminus J$, if there exists $s \in U(P_u)$ such that the path $\{(s,t),(s,j)\}$ with $(s,t) \in M_f$ is evenly oriented, then $G(L)$ has an f-j M_f-alternating path which is oddly oriented, which implies that $-c'_j$ is positive. Similarly, let $v \in J$ be the vertex closest to f such that the f-v path along P_{fg} and some edge incident to v is evenly oriented, and let P_v be the v-g path along P_{fg}. Then, for $j \in V \setminus J$, if there exists $s \in U(P_v)$ such that the path $\{(s,t),(s,j)\}$ with $(s,t) \in M_f$ is oddly oriented, then $-c'_j$ is positive. The submatrix $L[I, J \setminus \{g\}]$ forms an upper triangular matrix. Hence, if j has a nonzero entry but satisfies neither the above two conditions, then $-c'_j$ is negative.

Procedure Description

We now describe Procedure $\mathsf{OptBasis}(A,b,c)$ for finding an optimal basis.

Procedure: $\mathsf{OptBasis}(A,b,c)$

Input: A totally sign-nonsingular linear program $\mathrm{LP}(A,b,c)$ with row-mixed A_{p}.

Output: An optimal basis or that $\mathrm{LP}(A,b,c)$ is unbounded.

In the case of $c \not\leq 0$: Obtain the unisigned partition of $L[V \cup \{f\}]$ into $U_1,\dots,U_k,U_\infty$ and $V_1,\dots,V_k,V_\infty$.

(Unboundedness) If $f \in V_\infty$, then return that $\mathrm{LP}(A,b,c)$ is unbounded.

(Contraction) If $f \notin V_\infty$, then find an f-g admissible path P_{fg}. Let $\mathrm{LP}(A',b',c')$ be the LP-contraction by $A[U(P_{fg}) \setminus \{r\}, V(P_{fg}) \setminus \{f,g\}]$. Call $\mathsf{OptBasis}(A',b',c')$ to obtain an optimal basis B' of $\mathrm{LP}(A',b',c')$. Return $B' \cup V(P_{fg}) \setminus \{f,g\}$.

In the case of $c \leq 0$: Obtain the unisigned partition of $L[V \cup \{g\}]$ into $U_1,\dots,U_k,U_\infty$ and $V_1,\dots,V_k,V_\infty$. Let $W = \{j \mid c_j < 0\}$.

(The optimal value is zero) If $g \in V_\infty$, then find a feasible basis B_∞ of the linear system $A[U_\infty, V_\infty \setminus \{g\}]x = b[U_\infty]$ by Theorem 7.9, and find a basis B_* of $A[U^k \setminus \{r\}, V^k]$ such that $|B_* \cap W|$ is minimum. Return $B_* \cup B_\infty$.

(Contraction) If $g \notin V_\infty$, then find an f-g admissible path P_{fg}. Let $\mathrm{LP}(A',b',c')$ be the LP-contraction by $A[U(P_{fg}) \setminus \{r\}, V(P_{fg}) \setminus \{f,g\}]$. Call $\mathsf{OptBasis}(A',b',c')$ to obtain an optimal basis B' of $\mathrm{LP}(A',b',c')$. Return $B' \cup V(P_{fg}) \setminus \{f,g\}$.

8.5 Complexity Analysis

In this section, we analyze the complexity of Procedure $\mathsf{OptBasis}$ and Algorithm $\mathsf{OptSign}$. For a linear program $\mathrm{LP}(A,b,c)$, let m denote the number of rows of A, and γ the number of all the nonzero entries in A, b, and c.

Lemma 8.5. *For a totally sign-nonsingular linear program* $\mathrm{LP}(A, b, c)$ *with row-mixed* A_{p}*, Procedure* $\mathsf{OptBasis}(A, b, c)$ *finds an optimal basis in* $\mathrm{O}(m\gamma)$ *time.*

Proof. In each recursive step, it takes $\mathrm{O}(\gamma)$ time to obtain the unisigned partition, and $\mathrm{O}(m)$ time to find an f-g admissible path P_{fg}. We can obtain the LP-contraction in $\mathrm{O}(\gamma)$ time. Since the linear program is reduced to the LP-contraction smaller by at least one, the number of iterations is at most m times. Hence it takes $\mathrm{O}(m\gamma)$ time until the recursion terminates.

Suppose that $c \leq 0$ and the optimal value is zero. Then we can find an optimal basis with the aid of an efficient maximum weight bipartite matching algorithm. The column subset B_∞ can be found in $\mathrm{O}(\sqrt{m}\gamma)$ time by Theorem 7.9. In order to find the basis B_*, define the $\{0, 1\}$-weight function w on the edge set of $G(L[U^k, V^k])$ by $w_e = 0$ if the end vertex of e is in W and $w_e = 1$ otherwise. Then a maximum weight perfect matching M_* in $G(L[U^k, V^k])$ with respect to w corresponds to the basis B_*, namely, $B_* = V(M_*)$. Since the weight is 0 or 1, the weighted bipartite matching algorithm due to Gabow and Tarjan [4] runs in $\mathrm{O}(\sqrt{m}\gamma \log m)$ time. Thus we can find an optimal basis in $\mathrm{O}(\sqrt{m}\gamma \log m)$.

Therefore, the complexity of Procedure $\mathsf{OptBasis}$ is $\mathrm{O}(m\gamma)$. □

The running time bound of Algorithm $\mathsf{OptSign}(A, b, c)$ is now given as follows.

Theorem 8.6. *Algorithm* $\mathsf{OptSign}(A, b, c)$ *finds the sign pattern of an optimal solution of a totally sign-nonsingular linear program* $\mathrm{LP}(A, b, c)$ *in* $\mathrm{O}(m\gamma)$ *time.*

Proof. Step 1 requires $\mathrm{O}(\gamma)$ time by Theorem 7.3. In Step 2, Procedure $\mathsf{OptBasis}$ finds an optimal basis B of $\mathrm{LP}(A', b', c')$ in $\mathrm{O}(m\gamma)$ time by Lemma 8.5. We have a perfect matching M in $G(L[B \cup \{f\}])$ by Procedure $\mathsf{OptBasis}$. In Step 3, we can obtain the sign pattern of the basic solution by finding M-alternating paths from g to vertices in B by Lemma 6.3 and (3). Hence Step 3 takes $\mathrm{O}(\gamma)$ time. Thus the complexity of Algorithm $\mathsf{OptSign}$ is $\mathrm{O}(m\gamma)$. □

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References

[1] T. Ando and R. A. Brualdi: Sign-central matrices, *Linear Algebra and Its Applications*, 208/209, pp. 283-295, 1994.

[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G.M. Ziegler: *Oriented Matroids*, Cambridge University Press, 1993; 2nd ed., 1999.

[3] R. A. Brualdi and B. L. Shader: *Matrices of Sign-solvable Linear Systems*, Cambridge University Press, Cambridge, 1995.

[4] H. N. Gabow and R. E. Tarjan: Faster scaling algorithms for network problems, *SIAM Journal on Computing*, 18, pp. 1013–1036, 1989.

[5] J. E. Hopcroft and R. M. Karp: An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs, *SIAM Journal on Computing*, 2, pp. 225–231, 1973.

[6] N. Karmarkar: A new polynomial-time algorithm in linear programming, *Combinatorica*, 4, pp. 373–395, 1984.

[7] L. G. Khachiyan: A polynomial time algorithm in linear programming (in Russian), *Doklady Akademii Nauk SSSR*, 244, pp. 1093–1096, 1979 (English translation: *Soviet Mathematics Doklady*, 20, pp. 191–194, 1979).

[8] S.-J. Kim and B. L. Shader: Linear systems with signed solutions, *Linear Algebra and Its Applications*, 313, pp. 21–40, 2000.

[9] S.-J. Kim and B. L. Shader: On matrices which have signed null-spaces, *Linear Algebra and Its Applications*, 353, pp. 245–255, 2002.

[10] V. Klee, R. Ladner and R. Manber: Sign-solvability revisited, *Linear Algebra and Its Applications*, 59, pp. 131–158, 1984.

[11] G.-Y. Lee and B. L. Shader: Sign-consistency and solvability of constrained linear systems, *The Electronic Journal of Linear Algebra*, 4, pp. 1–18, 1998.

[12] L. Lovász and M. D. Plummer: *Matching Theory*, *Annals of Discrete Mathematics*, 29, North-Holland, Amsterdam, 1986.

[13] W. McCuaig: Brace decomposition, *Journal of Graph Theory*, 38, pp. 124–169, 2001.

[14] W. McCuaig: Pólya's permanent problem, *The Electronic Journal of Combinatorics*, 11, R79, 2004.

[15] G. Pólya: Aufgabe 424. *Archiv der Mathematik und Physik*, 20, p. 271, 1913.

[16] N. Robertson, P. D. Seymour and R. Thomas: Permanents, Pfaffian orientations, and even directed circuits, *Annals of Mathematics*, 150, pp. 929–975, 1999.

[17] P. A. Samuelson: *Foundations of Economics Analysis*, Harvard University Press, 1947; Atheneum, New York, 1971.

[18] J.-Y. Shao and L.-Z. Ren: Some properties of matrices with signed null spaces, *Discrete Mathematics*, 279, pp. 423–435, 2004.

[19] P. Seymour and C. Thomassen: Characterization of even directed graphs, *Journal of Combinatorial Theory, Series B*, 42, pp. 36–45, 1987.

[20] V. V. Vazirani and M. Yannakakis: Pfaffian orientations, 0-1 permanents, and even cycles in directed graphs, *Discrete Applied Mathematics*, 25, pp. 179–190, 1989.