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Abstract

This paper is concerned with inherent $\mathcal{H}_2$ tracking performance limitations in feedback control systems. We deal first with the single-input and multiple-output (SIMO) linear time-invariant (LTI) discrete-time system and provide the analytical closed-form expression of the best achievable performance. Then we reformulate and resolve the problem in delta domain by means the delta operator, from which we can completely recover the counterpart expression for the continuous-time case by approaching the sampling time to zero. In addition, we provide a similar result in sampled-data feedback control systems by using the fast sampling.

Key words: Control performance limits, $\mathcal{H}_2$ optimal control, sampled-data systems, fast sampling.

1 Introduction

Problems concerning the fundamental performance limitation and trade-off in feedback control systems have been intensively studied for decades, beginning with the work of Bode on logarithmic sensitivity integrals [3]. There are two main research directions in the area. First direction lies in the extensions of the Bode's integral theorem to assess design constraints and performance limitations via logarithmic type integrals (see e.g., [4, 9]). Second direction focuses on the formulations of optimal control problems to quantify and characterize the fundamental performance limits in terms of plant properties.

This kind of researches relates to the plant/controller design integration, where the main attention is not to design a robust or optimal controller but to design a plant which is easily controllable in practice. Therefore, study on control performance limitations achievable by feedback has been paid much attention in the recent years as seen in a special issue of the IEEE Transactions on Automatic Control in August 2003 and a book [13].

Especially, the $\mathcal{H}_2$ tracking performance limitation achievable by feedback control has been intensively investigated [1, 5, 6, 7, 8, 11, 14], which led to some complete results for single-input and single-output (SISO) continuous/discrete-time/sampled-data systems. Beyond the SISO case, existing results on the optimal tracking performance problem include the single-input and multiple-output (SIMO) and multiple-input and multiple-output (MIMO) cases [1, 2, 5, 6, 8, 11, 14].

The existing results show that, in general, unstable poles and non-minimum phase zeros of the plant to be controlled impose inevitable limitations on tracking performance. However, all the results except one in [6] for SIMO or MIMO cases are not practically useful, since the problems without control input constraint were only treated. Note that the paper [6] only considers the marginally stable case. Moreover, the result for the SIMO case in [5] is not completely correct as will be shown in Section 3. The result in [8] is only valid for the MIMO right invertible case, where the number of inputs is greater than or equal to that of outputs. In other words, the result can not be applied to the SIMO case.

This paper focuses on the $\mathcal{H}_2$ optimal tracking problems with control input penalty for possibly unstable, non-minimum phase, SIMO LTI plants. The tracking performance is measured by the tracking error between measurement output and a step reference input under control input constraint, and is minimized over all possible stabilizing controllers.

The problem formulation is more realistic than the problem without penalty on the control input, since the controller could not produce an input beyond the capability of the actuator. The treatment of the SIMO case is practically meaningful, since the plant to be controlled has only one actuator with two or more sensors, which commonly appear in real control applications to get the better control performance by putting extra sensors. The class of feedback systems investigated here is fairly wide which covers continuous-time, discrete-time, and sampled-data systems, and we provide comprehensive complete results for the analytical closed-form expressions on the performance limitations by a unified approach.

The contribution of the paper is threefold. Firstly, we derive an analytical closed-form expression of the $\mathcal{H}_2$ optimal tracking performance for discrete-time SIMO LTI systems. The idea of the derivation is to introduce an augmented plant which enables us to apply the result for the non-penalty case directly. Then, the parallel discussions with the continuous-time case can be carried out for the discrete-time case, where we corrects an error in the expression in [5] for unstable plants. Secondly, we appropriately reformulate and solve the problem in terms of delta operator (see [12]), and show its continuity properties. In other words, we can completely recover the continuous-time solution by taking the sampling time tends to zero. Thirdly, we employ an approximation approach by implementing fast sampling technique to derive the similar result for SISO sampled-data feedback control systems, where the idea of plant augmentation plays a key role to derive the result.

In general, the results show that the plant gain as well as the plant's non-minimum phase zeros, unstable poles, and their relations impose inevitable limitation on the tracking performance, and they are confirmed by several numerical examples.

The remainder of this paper is organized as follows. In Section 2, we describe the problem formulation including the description of the standard unity feedback control system and a brief explanation about plant augmentation strategy. Section 3 provides the analytical closed-form expressions of the optimal performance in discrete-time case. Section 4 is devoted to the delta domain results. We provide the results for sampled-data systems in Section 5. We then conclude the paper in Section 6.

The notation used throughout this paper is fairly standard. We denote the real set by $\mathbb{R}$ and the complex set by $\mathbb{C}$. For any $z \in \mathbb{C}$, its complex conjugate is denoted by $\bar{z}$. For any vector v we shall use v^{T}, v^{H}, and $\|v\|$ as its transpose, conjugate transpose, and Euclidean norm, respectively. For any matrix $A \in \mathbb{C}^{m\times n}$, we denote its conjugate transpose by A^{H} and its column space by $\mathbb{R}[A]$. Several subsets in the complex plane are defined as follows: $\mathbb{C}_- := \{s \in \mathbb{C} : \mathrm{Re}\, s < 0\}$, $\mathbb{C}_+ := \{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$, $\bar{\mathbb{C}}_+ := \{s \in \mathbb{C} : \mathrm{Re}\, s \geq 0\}$, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{D}^{\mathrm{c}} := \{z \in \mathbb{C} : |z| \geq 1\}$, $\bar{\mathbb{D}}^{\mathrm{c}} := \{z \in \mathbb{C} : |z| > 1\}$. We denote by $\mathbb{R}\mathcal{H}_\infty$ the set of all rational matrix functions which are bounded and analytic in $\mathbb{D}^{\mathrm{c}}$ and for any matrix function $f \in \mathbb{C}^{m\times n}$ we denote $f^{\sim}(z) = f^{\mathrm{T}}(z^{-1})$. We define by $\hat{x}(z)$ the $\mathcal{Z}$-transform of sequence $x(k)$. The cardinality of a set S is denoted by $\#S$.

2 Problem Formulation

2.1 Feedback Control Systems

We consider the LTI unity feedback control system depicted in Fig. 1, where P denotes a SIMO LTI plant to be controlled and K is a stabilizing controller. The plant P can be written as

$$P = \begin{pmatrix} P_1, & P_2, & \ldots, & P_m \end{pmatrix}^{\mathrm{T}}, \tag{1}$$

where P_i $(i = 1, \ldots, m)$ are scalar transfer functions. The signals $r \in \mathbb{R}^m$, $u \in \mathbb{R}$, $y \in \mathbb{R}^m$, and $e := r - y \in \mathbb{R}^m$ are the reference input, the control

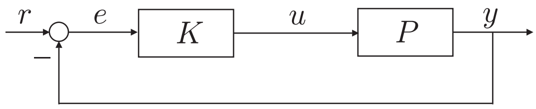

Figure 1: Unity feedback control system

input, the measurable output, and the error signals, respectively. Hereafter, it will be assumed that all the vectors and matrices involved in the sequel have compatible dimensions.

The plant rational transfer function P admits right and left coprime factorizations

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \tag{2}$$

where $N, M, \tilde{N}, \tilde{M} \in \mathbb{R}\mathcal{H}_\infty$, and there exist $X, Y, \tilde{X}, \tilde{Y} \in \mathbb{R}\mathcal{H}_\infty$ that satisfy the double Bezout identity

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I. \tag{3}$$

The set of all the stabilizing compensators K is then characterized by the Youla parameterization

$$\begin{aligned} \mathcal{K} \; := \; \{K \, : \, K &= (Y - MQ)(NQ - X)^{-1} \\ &= (Q\tilde{N} - \tilde{X})^{-1}(\tilde{Y} - Q\tilde{M}); \; Q \in \mathbb{R}\mathcal{H}_\infty\}. \end{aligned} \tag{4}$$

A number $\eta \in \mathbb{C}$ is said to be *zero* of P if $P_i(\eta) = 0$ holds for some $i = 1, \ldots, m$. In addition, if η is lying in $\bar{\mathbb{D}}^{\mathrm{c}}$, then η is said to be *a non-minimum phase zero*. P is said to be minimum phase if it has no non-minimum phase zero; otherwise, it is said to be non-minimum phase. A number $\lambda \in \mathbb{C}$ is said to be a *pole* of P if $P(\lambda)$ is unbounded. If λ is lying in $\bar{\mathbb{D}}^{\mathrm{c}}$, then λ is *an unstable pole* of P. We say P is stable if it has no unstable pole; otherwise, unstable. An equivalent statement for pole λ is that $\tilde{M}(\lambda)w = 0$ for some unitary vector w. And w is called *a pole direction* vector associated with λ. For technical reasons, it is assumed that the plant does not have non-minimum phase zeros and unstable poles at the same location.

A transfer function N, not necessarily square, is called an *inner* if N is in $\mathbb{R}\mathcal{H}_\infty$ and $N^{\sim}(z)N(z) = I$ for all $z = \mathrm{e}^{\mathrm{j}\theta}$. A transfer function M is called *outer* if M is in $\mathbb{R}\mathcal{H}_\infty$ and has a right inverse which is analytic in $\bar{\mathbb{D}}^{\mathrm{c}}$. For an arbitrary $P \in \mathbb{R}\mathcal{H}_\infty$,

$$P(z) = \Theta(z)\Phi(z), \tag{5}$$

where Θ is inner and Φ is outer, is defined as an *inner-outer factorization* of P. We call Θ the *inner factor* and Φ the *outer factor.*

2.2 $\mathcal{H}_2$ Optimal Tracking Problem

The problem to be investigated in this paper is the standard $\mathcal{H}_2$ optimal tracking problem. For discrete-time case, the reference input signal r is a unit step function defined as

$$r(k) = \begin{cases} \nu, & k \geq 0 \\ 0, & k < 0 \end{cases}, \quad \hat{r}(z) = \frac{z\nu}{z-1}, \tag{6}$$

where $\nu = (\nu_1, \nu_2, \ldots, \nu_m)^{\mathrm{T}}$ is a constant vector of unit length and specifies the direction of the reference input. The performance index to be minimized is given by

$$J_{\mathrm{d}} := \sum_{k=0}^{\infty} \left(\|e(k)\|^2 + |u_{\mathrm{w}}(k)|^2 \right), \tag{7}$$

where u_{w} is the weighted control input, i.e., $u_{\mathrm{w}}(k) = \mathcal{Z}^{-1}\{W(z)\hat{u}(z)\}$, with proper, stable, and minimum phase weighting function $W(z)$. Note that, if $W = 0$, the problem then reduces to an $\mathcal{H}_2$ tracking error minimization problem (i.e., the $\mathcal{H}_2$ optimal tracking problem without control input penalty), which has been discussed in [2, 5].

It follows from the well-known Parseval's identity that

$$\begin{aligned} J_{\mathrm{d}} &= \|\hat{e}(z)\|_2^2 + |\hat{u}_w(z)|_2^2 \\ &= \|S_{\mathrm{o}}(z)\hat{r}(z)\|_2^2 + |W(z)K(z)S_{\mathrm{o}}(z)\hat{r}(z)|_2^2, \end{aligned}$$

where $S_{\mathrm{o}} := (I + PK)^{-1}$ is the output sensitivity function. Using (2)–(4), the optimal performance then can be represented by

$$J_{\mathrm{d}}^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \left\{ \begin{bmatrix} WY \\ X \end{bmatrix} - \begin{bmatrix} WM \\ N \end{bmatrix} Q \right\} \tilde{M}\hat{r} \right\|_2^2. \tag{8}$$

We make the following standard assumptions to guarantee the finiteness of J_{d}:

Assumption 1. $N(1) \neq 0$.

Assumption 2. *For $r(k)$ in (6), $\nu \in \mathbb{R}[N(1)]$.*

Assumption 3. *$P(z)$ has a pole at $z = 1$.*

In order for J_{d} to be finite, it is obvious the output sensitivity function S_{o} must have a zero at $z = 1$ with input zero direction ν, i.e. $S_{\mathrm{o}}(1)\nu = 0$. Condition $N(1) \neq 0$ is then required to avoid any hidden pole-zero cancellation at $z = 1$ so that the open loop system has an integrator. Condition $\nu \in \mathbb{R}[N(1)]$ requires that the input signal must enter from direction lying in the column space of $N(1)$ and gives the condition of step reference signal r that a non-right invertible plant P may track. In order to make the steady state zero, the open-loop transfer function PK must contain an integrator. Consequently, plant P must have an integrator instead of compensator K, which should have no integrator to maintain a finite control energy cost. Assumption 3 is then necessary.

2.3 Plant Augmentation

To solve the tracking error problem under control penalty, we adopt the key idea of plant augmentation initially introduced in one of the authors' conference papers [11]. An augmented plant $P_{\rm a}$ is defined as

$$P_{\rm a} := \begin{pmatrix} W \\ P \end{pmatrix}, \tag{9}$$

from which we obtain the corresponding step input signal $r_{\rm a} := (\ 0,\ r^{\rm T}\ )^{\rm T}$ with direction $\nu_{\rm a} := (\ 0,\ \nu^{\rm T}\ )^{\rm T}$ and the tracking measure

$$J_{\rm da} := \sum_{k=0}^{\infty} \|e_{\rm a}(k)\|^2, \tag{10}$$

where

$$e_{\rm a} := \begin{pmatrix} 0 \\ r \end{pmatrix} - \begin{pmatrix} u_{\rm w} \\ y \end{pmatrix}.$$

One of the key points addressed by this strategy is that the tracking measure does not explicitly include the control input penalty u.

Furthermore, the corresponding right and left coprime factorizations of $P_{\rm a}$ are provided as

$$P_{\rm a} = N_{\rm a} M_{\rm a}^{-1} = \tilde{M}_{\rm a}^{-1} \tilde{N}_{\rm a}, \tag{11}$$

where

$$N_{\rm a} = \begin{pmatrix} WM \\ N \end{pmatrix},\ M_{\rm a} = M,\ \tilde{M}_{\rm a} = \begin{pmatrix} 1 & 0 \\ 0 & \tilde{M} \end{pmatrix},\ \tilde{N}_{\rm a} = \begin{pmatrix} W \\ \tilde{N} \end{pmatrix},$$

and the corresponding double Bezout identity is written as

$$\begin{pmatrix} \tilde{X}_{\rm a} & -\tilde{Y}_{\rm a} \\ -\tilde{N}_{\rm a} & \tilde{M}_{\rm a} \end{pmatrix} \begin{pmatrix} M_{\rm a} & Y_{\rm a} \\ N_{\rm a} & X_{\rm a} \end{pmatrix} = I, \tag{12}$$

where

$$Y_{\rm a} = (0,\ Y),\ \tilde{X}_{\rm a} = \tilde{X},\ \tilde{Y}_{\rm a} = (0,\ \tilde{Y}),\ X_{\rm a} = \begin{pmatrix} 1 & WY \\ 0 & X \end{pmatrix}.$$

For a free parameter $Q_{\rm a} = (Q_1,\ Q_2) \in \mathbb{R}\mathcal{H}_\infty$, the optimal tracking performance $J_{\rm da}^*$ can be expressed as

$$J_{\rm da}^* = \inf_{Q_{\rm a} \in \mathbb{R}\mathcal{H}_\infty} \|(X_{\rm a} - N_{\rm a} Q_{\rm a}) \tilde{M}_{\rm a} \hat{r}_{\rm a}\|_2^2, \tag{13}$$

and subsequently,

$$J_{\rm da}^* = \inf_{Q_2 \in \mathbb{R}\mathcal{H}_\infty} \left\| \left\{ \begin{bmatrix} WY \\ X \end{bmatrix} - \begin{bmatrix} WM \\ N \end{bmatrix} Q_2 \right\} \tilde{M} \hat{r} \right\|_2^2. \tag{14}$$

The expression of J_{da}^* in (14) is exactly equivalent with that of J_{d}^* in (8) for the original plant P, i.e., $J_{\mathrm{da}}^* = J_{\mathrm{d}}^*$ holds. By taking into account that there is no penalty to the control input to be imposed for computing J_{da}^*, we can immediately follow the approach of the tracking error problem in [1, 5] to derive the analytical closed-form expression of J_{d}^*. Note that the results for the non-penalty case in [1, 5] are not completely correct. Hence we need a small modification to derive the complete expressions which will be shown in the next two sections.

3 Discrete-time Case

3.1 Closed-form Expression

This section provides an analytical closed-form expression of the optimal tracking performance for the discrete-time case. The derivation is parallel to the continuous-time case [5, 11], but we will clarify a missing term in the expressions in [5, 11] and give the complete expression for the dicrete-time case.

Note first that (13) can be expressed as

$$J_{\mathrm{d}}^* = \inf_{Q_{\mathrm{a}} \in \mathbb{R}\mathcal{H}_\infty} \left\| [I + N_{\mathrm{a}}(\tilde{Y}_{\mathrm{a}} - Q_{\mathrm{a}}\tilde{M}_{\mathrm{a}})] \frac{\nu_{\mathrm{a}}}{z-1} \right\|_2^2 . \tag{15}$$

Write

$$N_{\mathrm{a}} = \begin{pmatrix} N_0, & N_1, & \ldots, & N_m \end{pmatrix}^{\mathrm{T}},$$

where N_i $(i = 0, 1, \ldots, m)$ are scalar transfer functions and $N_0 = WM$. We denote by $\lambda_k \in \bar{\mathbb{D}}^{\mathrm{c}}$ $(k = 1, \ldots, n_\lambda)$ the unstable poles of $P(z)$ and by $\eta_{ij} \in \bar{\mathbb{D}}^{\mathrm{c}}$ $(i = 1, \ldots, m,\ j = 1, \ldots, n_i)$ the non-minimum phase zeros of $P_i(z)$.

We further define the following index sets:

$$\begin{aligned} \mathbb{J}_{\mathrm{z}} &:= \{i : N_i(1) \neq 0\}, \\ \mathbb{J}_{\mathrm{p}} &:= \{k : \tilde{M}(\lambda_k)\nu = 0\}, \\ \mathbb{J}_{\mathrm{p}i} &:= \{k : N_i(\lambda_k) = 0\}\ (i = 0, 1, \ldots, m). \end{aligned}$$

Note that $\mathbb{J}_{\mathrm{p}}$ contains the index of unstable poles whose direction is coincident with that of step input signal r. While, due to the relation $N = PM$, $\mathbb{J}_{\mathrm{p}i}$ contains the index of unstable poles of P but not those of P_i. The index set $\mathbb{J}_{\mathrm{p}i}$ will play a key role for collecting an error in existing results shown in [1, 5]. To facilitate our derivation, we introduce the inner-outer factorization of $N_{\mathrm{a}}(z)$ as follows,

$$N_{\mathrm{a}}(z) = \Theta(z)\Phi(z), \tag{16}$$

where Θ is an inner factor and Φ is an outer factor.

Theorem 1. *Suppose that the SIMO plant $P(z)$ given in (1) has unstable poles λ_k ($k = 1, \ldots, n_\lambda$) and $P_i(z)$ has non-minimum phase zeros η_{ij} ($i = 1, \ldots, m$, $j = 1, \ldots, n_i$). Then, under Assumptions 1–3, the optimal tracking performance J_d^* is given by*

$$J_\mathrm{d}^* = J_\mathrm{ds} + J_\mathrm{du}, \tag{17}$$

where

$$J_\mathrm{ds} = J_\mathrm{ds1} + J_\mathrm{ds2}$$

with

$$\begin{aligned} J_\mathrm{ds1} &:= \sum_{i \in \mathbb{J}_\mathrm{z}} \nu_i^2 \sum_{j=1}^{n_i} \frac{|\eta_{ij}|^2 - 1}{|\eta_{ij} - 1|^2}, \\ J_\mathrm{ds2} &:= \frac{1}{2\pi} \sum_{i \in \mathbb{J}_\mathrm{z}} \nu_i^2 \int_0^\pi \log \left(\frac{|P_i(1)|^2}{\|P(1)\|^2} \frac{\|P(\mathrm{e}^{\mathrm{j}\theta})\|^2 + |W(\mathrm{e}^{\mathrm{j}\theta})|^2}{|P_i(\mathrm{e}^{\mathrm{j}\theta})|^2} \right) \frac{\mathrm{d}\theta}{1 - \cos\theta}, \end{aligned}$$

and

$$J_\mathrm{du} = J_\mathrm{du1} + J_\mathrm{du2}$$

with

$$\begin{aligned} J_\mathrm{du1} &:= \sum_{i \in \mathbb{J}_\mathrm{z}} \nu_i^2 \sum_{k \in \mathbb{J}_{\mathrm{p}i}} \frac{|\lambda_k|^2 - 1}{|\lambda_k - 1|^2}, \\ J_\mathrm{du2} &:= \sum_{k, \ell \in \mathbb{J}_\mathrm{p}} \frac{(|\lambda_k|^2 - 1)(|\lambda_\ell|^2 - 1)(1 - \Theta^\sim(\bar\lambda_k)\Theta(1))(1 - \Theta^\sim(\lambda_\ell)\Theta(1))}{\bar h_k h_\ell (\bar\lambda_k - 1)(\lambda_\ell - 1)(\bar\lambda_k \lambda_\ell - 1)}, \end{aligned}$$

and

$$h_k := \begin{cases} 1 & ;\ \#\mathbb{J}_\mathrm{p} = 1 \\ \displaystyle\prod_{\ell \in \mathbb{J}_\mathrm{p}, \ell \neq k} \frac{\lambda_k - \lambda_\ell}{1 - \bar\lambda_\ell \lambda_k} & ;\ \#\mathbb{J}_\mathrm{p} \geq 2 \end{cases}$$

Proof. See the Appendix B for the proof. □

Theorem 1 reveals that the optimal performance in tracking a step reference signal is explicitly characterized by the plant's non-minimum phase zeros η_{ij} and unstable poles λ_k, the plant direction which mostly determined by the plant gain, and the reference input direction ν. Furthermore, problem of minimizing the tracking error under control input penalty generally provides additional limits imposed by W, which appears in the logarithmic term in J_ds2 and the inner factor Θ in J_du2.

If we set $W = 0$ then we can easily obtain the non-penalty result. If the plant is marginally stable, we can see $J_\mathrm{d}^* = J_\mathrm{ds}$ (or $J_\mathrm{du} = 0$). In this theorem we also provide a clearer expression by accounting explicitly additional effects caused by unstable poles λ_k in J_cu1. This term was missing and not properly recognized in [1, 5, 11].

3.2 Remarks and Corollaries

We have a couple of further remarks on Theorem 1.

- The expression in the theorem is complete for SIMO marginally stable plants in a sense that the best achievable tracking performance with control input penalty is characterized by non-minimum phase zeros and gain of the plant without using any inner-outer factorization or solving any Riccati equation. See Corollary 1 below for the SISO case.
- The expression for the general unstable case is not complete, because it includes an inner factor $\Theta(z)$ in the last term J_{du2}. We can only obtain the closed-form expression of $\Theta(z)$ for the SISO without control input penalty case (See Corollary 2 below.) and some special cases.
- However, fortunately, there exists a special case where we can see the terms J_{du2} caused by unstable poles is zero even if the plant is unstable. See Corollary 3 below.
- We can also show that $J_{\mathrm{du}} = J_{\mathrm{du1}} + J_{\mathrm{du2}}$ is zero when the sets of all unstable poles of $P_i(z)$ $(i = 1, \ldots, m)$ are completely same as seen in Corollary 4. The case often happens for practical applications where we have only one actuator but we may add one or more extra sensors. The extra sensor can dramatically improve the tracking performance for unstable and non-minimum phase plants as seen in an example of inverted pendulum in [2].

We now consider four specific cases for illustrating the implication of Theorem 1. The first case is the simplest case, where we consider a (marginally) stable scalar system.

Corollary 1. *Suppose that the SISO plant $P(z)$ is marginally stable and has non-minimum phase zeros η_i $(i = 1, \ldots, n_\eta)$. Under Assumptions 1 and 3, then*

$$J_{\mathrm{d}}^* = \sum_{i=1}^{n_\eta} \frac{|\eta_i|^2 - 1}{|\eta_i - 1|^2} + \frac{1}{2\pi} \int_0^\pi \log\left(1 + \frac{|W(\mathrm{e}^{\mathrm{j}\theta})|^2}{|P(\mathrm{e}^{\mathrm{j}\theta})|^2}\right) \frac{\mathrm{d}\theta}{1 - \cos\theta}.$$

The second one is for the SISO without control input penalty case, i.e., $W(z) = 0$. Suppose the plant has non-minimum phase zeros η_i $(i = 1, \ldots, n_\eta)$, then the inner factor in (16), without loss of generality, can be fixed as

$$\Theta(z) = \prod_{i=1}^{n_\eta} \frac{z - \eta_i}{1 - \bar{\eta}_i z},$$

from which we get $\Theta(1) = 1$. Let define $\phi(z) := \Theta^{\sim}(z)\Theta(1)$, i.e.,

$$\phi(z) = \prod_{i=1}^{n_\eta} \frac{1 - \eta_i z}{z - \bar{\eta}_i},$$

then we state the tracking performance limitations for SISO without control input penalty case in the following result.

Corollary 2. *Let consider the non-penalty case, i.e., $W(z) = 0$, for the SISO plant $P(z)$ which has non-minimum phase zeros η_i $(i = 1, \dots, n_\eta)$ and unstable poles λ_k $(k = 1, \dots, n_\lambda)$. Then,*

$$J_{\mathrm{d}}^* = \sum_{i=1}^{n_\eta} \frac{|\eta_i|^2 - 1}{|\eta_i - 1|^2} + \sum_{k,\ell \in \mathbb{J}_{\mathrm{p}}} \frac{(|\lambda_k|^2 - 1)(|\lambda_\ell|^2 - 1)(1 - \phi(\bar{\lambda}_k))(1 - \phi(\lambda_\ell))}{\bar{h}_k h_\ell (\bar{\lambda}_k - 1)(\lambda_\ell - 1)(\bar{\lambda}_k \lambda_\ell - 1)}.$$

The third corollary is for a special class of SIMO unstable systems, where the tracking performance limit is explicitly given in terms of plant characteristics.

Corollary 3. *Let the SIMO plant P satisfies $P(1) = [P_1(1), 0, \dots, 0]^{\mathrm{T}}$, and the input signal r be given by (6) with $\nu = [1, 0, \dots, 0]^{\mathrm{T}}$. Suppose that $P_1(z)$ is stable and has non-minimum phase zeros at η_{1j} $(j = 1, \dots, n_1)$ and P has unstable poles λ_k $(k = 1, \dots, n_\lambda)$. Then*

$$J_{\mathrm{d}}^* = \sum_{j=1}^{n_1} \frac{|\eta_{1j}|^2 - 1}{|\eta_{1j} - 1|^2} + \sum_{k \in \mathbb{J}_{\mathrm{p1}}} \frac{|\lambda_k|^2 - 1}{|\lambda_k - 1|^2} + \frac{1}{2\pi} \int_0^\pi \frac{\log \left[\frac{\|P(\mathrm{e}^{\mathrm{j}\theta})\|^2 + |W(\mathrm{e}^{\mathrm{j}\theta})|^2}{|P_1(\mathrm{e}^{\mathrm{j}\theta})|^2} \right]}{1 - \cos\theta} \mathrm{d}\theta.$$

The last corollary deals with SIMO plant $P(z)$, in which the set of unstable poles of $P_i(z)$ $(i = 1, \dots, m)$ are completely same.

Corollary 4. *Consider the SIMO plant $P(z)$ given in (1). Suppose that $P_i(z)$ has non-minimum phase zeros η_{ij} $(i = 1, \dots, m,\ j = 1, \dots, n_i)$ and has unstable poles λ_k $(k = 1, \dots, n_\lambda)$ for all $i = 1, \dots, m$, i.e., the set of unstable poles of $P_i(z)$ are same. Then,*

$$J_{\mathrm{d}}^* = J_{\mathrm{ds1}} + J_{\mathrm{ds2}}. \tag{18}$$

Proof. If the set of unstable poles of $P_i(z)$ $(i = 1, \dots, m)$ are completely same, then it is not difficult to verify that $\mathbb{J}_{\mathrm{p}i}$ is empty for $i = 1, \dots, m$. Note that $\mathbb{J}_{\mathrm{p0}}$ may not be empty, but this will not give an effect since the first element of ν_{a} is zero. Furthermore, we know that $\mathbb{J}_{\mathrm{p}}$ is also empty for the case, since $\tilde{M}(\lambda_k)\nu \neq 0$ for all $k = 1, \dots, n_\lambda$. These two facts make $J_{\mathrm{du1}} = 0$ and $J_{\mathrm{du2}} = 0$. Hence, $J_{\mathrm{du}} = 0$. □

3.3 Numerical Examples

We demonstrate simple numerical examples to clarify the correctness of the derived expressions. We consider the following SISO plant:

$$P(z) = \frac{z - \eta}{(z - 1)(z - \lambda)}.$$

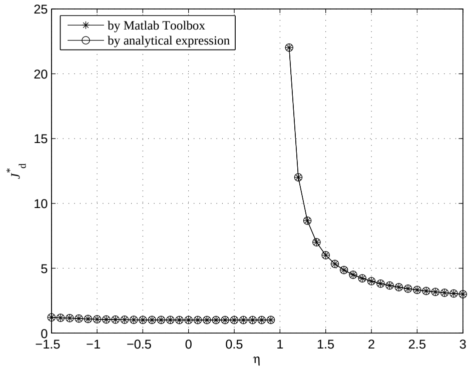

Figure 2: The optimal performance for (marginally) stable case

First, we calculate the optimal tracking performance for $\lambda = \frac{1}{2}$, i.e., we consider a (marginally) stable plant. We can see from Corollary 1 whenever $|\eta| > 1$ the optimal performance obeys

$$J^*_\mathrm{d} = 1 + \frac{\eta+1}{\eta-1} + \frac{1}{2\pi}\int_0^\pi \frac{\log\left(1 + \frac{W^2}{|P(\mathrm{e}^{\mathrm{j}\theta})|^2}\right)}{1-\cos\theta}\,\mathrm{d}\theta.$$

Fig. 2 depicts the optimal performance for $W = 0.1$ and η from -1.5 to 3, where the correctness of the derived expression is confirmed by comparing with the numerical computations by the Matlab toolbox.

Second, we take $\lambda = 2$, i.e., we consider an unstable plant. This unstable pole gives an additional term

$$J_{\mathrm{du}2} = \frac{\lambda+1}{\lambda-1}(1 - \Theta^\mathrm{T}(1/\lambda)\Theta(1))^2,$$

where Θ is defined by (16). Note that for scalar case $J_{\mathrm{du}1} = 0$. Fig. 3 shows that the optimal performance is unbounded not only at $\eta = 1$ but also at $\eta = 2$ when it happens an unstable pole-zero cancelation. In this case we take $W = 0.01$.

4 Delta Domain Case

This section is devoted to the investigation for the delta-domain case and we shows the continuity property which leads to a correct version of the continuous-time result.

4.1 Delta Transform

The preliminary results on the delta transform presented here almost follows from [12].

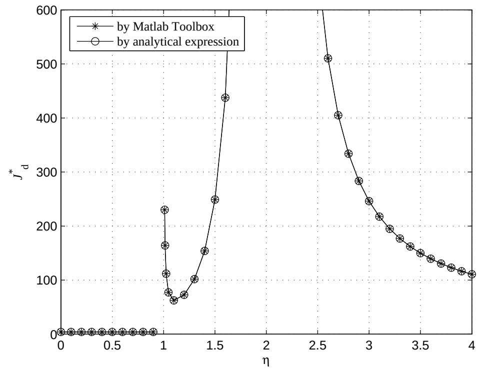

Figure 3: The optimal performance for unstable case

For any sequence $x(k)$, $k = 1, 2, \ldots$, the delta operator δ is defined by

$$\delta x(k) = \frac{x(k+1) - x(k)}{T},$$

where $T > 0$ is the sampling time. By taking the $\mathcal{Z}$-transform of above equation we obtain

$$\delta \hat{x}(z) = \frac{z-1}{T}\hat{x}(z).$$

Later, the variable δ is used as the delta operator variable and is analogous to the Laplace variable s for continuous-time systems and the $\mathcal{Z}$-transform variable z for discrete-time systems. We then obtain the relationship

$$\delta = \frac{z-1}{T} \text{ or } z = T\delta + 1. \tag{19}$$

For any sequence $x(k)$ we define its delta transform by

$$\mathcal{D}\{x(k)\} = \hat{x}_T(\delta) := T\sum_{k=0}^{\infty} x(k)(T\delta+1)^{-k}, \tag{20}$$

or equivalently,

$$\hat{x}_T(\delta) = T\hat{x}(z)|_{z=T\delta+1}.$$

The Hilbert space $\mathcal{L}_2$ is then equipped with an inner product defined by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} f^{\mathrm{H}}\left(\frac{\mathrm{e}^{\mathrm{j}\omega T}-1}{T}\right) g\left(\frac{\mathrm{e}^{\mathrm{j}\omega T}-1}{T}\right)\mathrm{d}\omega.$$

Let $F(z)$ be given and define $G_T(\delta) := F(T\delta+1)$. Then by setting $\theta = \omega T$ we have the following norms relationship:

$$\|G_T(\delta)\|_2^2 = \|F(z)\|_2^2/T. \tag{21}$$

In subsequent analysis, we define the following sets: $\mathbb{D}_T = \{\delta \in \mathbb{C} : |T\delta+1| < 1\}$, $\mathbb{D}_T^{\mathrm{c}} = \{\delta \in \mathbb{C} : |T\delta+1| \geq 1\}$, $\bar{\mathbb{D}}_T^{\mathrm{c}} = \{\delta \in \mathbb{C} : |T\delta+1| > 1\}$, and $\partial\mathbb{D}_T = \{\delta \in \mathbb{C} : |T\delta+1| = 1\}$.

4.2 Closed-form Expression

We here reformulate and solve the tracking performance problem in term of delta operator. We consider the following tracking measure:

$$J_\delta := T\sum_{k=0}^{\infty}\left(\|e(k)\|^2 + |u_{\rm w}(k)|^2\right), \tag{22}$$

where $u_{\rm w}(k) = \mathcal{D}^{-1}\{W_T(\delta)\hat{u}_T(\delta)\}$. Note that the factor T is introduced to have a consistency with the continuous-time case. As the reference input we consider the step function (6) whose delta transform is given by

$$\hat{r}_T(\delta) = \frac{T\delta+1}{\delta}\nu. \tag{23}$$

To avoid ambiguity, in this section we denote by

$$P_T = \begin{pmatrix} P_{T1}, & P_{T2}, & \ldots, & P_{Tm} \end{pmatrix}^{\rm T}, \tag{24}$$

the respecting plant in delta domain. All the transfer function matrices in delta domain should follow the such kind of notation. For instance, the coprime factorizations of P_T are given by

$$P_T = N_T M_T^{-1} = \tilde{M}_T^{-1}\tilde{N}_T. \tag{25}$$

For the finiteness of J_δ we impose the following assumptions.

Assumption 4. $N_T(0) \neq 0$.

Assumption 5. *For $r(k)$ in (6), $\nu \in \mathbb{R}[N_T(0)]$.*

Assumption 6. *$P_T(\delta)$ has a pole at $\delta = 0$.*

The optimal performance J_δ^* then can be deduce as

$$J_\delta^* = \inf_{Q_{T{\rm a}}\in\mathbb{R}\mathcal{H}_\infty}\left\|\left[I + N_{T{\rm a}}(\tilde{Y}_{T{\rm a}} - Q_{T{\rm a}}\tilde{M}_{T{\rm a}})\right]\frac{\nu_{\rm a}}{\delta}\right\|_2^2.$$

We denote by $\rho_k \in \bar{\mathbb{D}}_T^{\rm c}$ $(k = 1,\ldots,n_\rho)$ the unstable poles of $P_T(\delta)$ and by $\zeta_{ij} \in \bar{\mathbb{D}}_T^{\rm c}$ $(i = 1,\ldots,m,\ j = 1,\ldots,n_i)$ the non-minimum phase zeros zeros of $P_{Ti}(\delta)$. We further introduce the following index sets:

$$\begin{aligned}
\mathbb{I}_{\rm z} &:= \{i : N_{Ti}(0) \neq 0\},\\
\mathbb{I}_{\rm p} &:= \{k : \tilde{M}_T(\rho_k)\nu = 0\},\\
\mathbb{I}_{{\rm p}i} &:= \{k : N_{Ti}(\rho_k) = 0\}\ (i = 0,1,\ldots,m).
\end{aligned}$$

We define an inner-outer factorization such that

$$N_{T{\rm a}}(\delta) = \Theta_T(\delta)\Phi_T(\delta). \tag{26}$$

Theorem 2. *Suppose that the SIMO plant $P_T(\delta)$ given in (24) has unstable poles ρ_k ($k = 1, \dots, n_\rho$) and $P_{Ti}(\delta)$ has non-minimum phase ζ_{ij} ($i = 1, \dots, m$, $j = 1, \dots, n_i$). Let denote $\eta_{ij} = T\zeta_{ij} + 1$ and $\lambda_k = T\rho_k + 1$. Then, under Assumptions 4–6, the optimal tracking performance J_δ^* is given by*

$$J_\delta^* = J_{\delta\mathrm{s}} + J_{\delta\mathrm{u}}, \tag{27}$$

where

$$J_{\delta\mathrm{s}} = J_{\delta\mathrm{s}1} + J_{\delta\mathrm{s}2}$$

with

$$J_{\delta\mathrm{s}1} := \sum_{i\in\mathbb{I}_\mathrm{z}} \nu_i^2 \sum_{j=1}^{n_i} \left(\frac{2\,\mathrm{Re}\,\zeta_{ij}}{|\zeta_{ij}|^2} + T \right),$$

$$J_{\delta\mathrm{s}2} := \frac{T^2}{2\pi} \sum_{i\in\mathbb{I}_\mathrm{z}} \nu_i^2 \int_0^{\pi/T} \frac{\log\left(\frac{|P_{Ti}(0)|^2}{\|P_T(0)\|^2} \frac{\|P_T(\frac{e^{j\omega T}-1}{T})\|^2 + |W_T(\frac{e^{j\omega T}-1}{T})|^2}{|P_{Ti}(\frac{e^{j\omega T}-1}{T})|^2} \right)}{1-\cos\omega T}\,\mathrm{d}\omega,$$

and

$$J_{\delta\mathrm{u}} = J_{\delta\mathrm{u}1} + J_{\delta\mathrm{u}2}$$

with

$$J_{\delta\mathrm{u}1} := \sum_{i\in\mathbb{I}_\mathrm{z}} \nu_i^2 \sum_{k\in\mathbb{I}_{\mathrm{p}i}} \left(\frac{2\,\mathrm{Re}\,\rho_k}{|\rho_k|^2} + T \right),$$

$$J_{\delta\mathrm{u}2} := T \sum_{k,\ell\in\mathbb{I}_\mathrm{p}} \frac{(|\lambda_k|^2-1)(|\lambda_\ell|^2-1)(1-\Theta_T^{\sim}(\bar{\rho}_k)\Theta_T(0))(1-\Theta_T^{\sim}(\rho_\ell)\Theta_T(0))}{\bar{q}_k q_\ell(\bar{\lambda}_k-1)(\lambda_\ell-1)(\bar{\lambda}_k\lambda_\ell-1)},$$

and

$$q_k := \begin{cases} 1 & ;\ \#\mathbb{I}_\mathrm{p} = 1 \\ \displaystyle\prod_{\ell\in\mathbb{I}_\mathrm{p},\ell\neq k} \frac{\lambda_k-\lambda_\ell}{1-\bar{\lambda}_\ell\lambda_k} & ;\ \#\mathbb{I}_\mathrm{p} \geq 2 \end{cases}$$

Proof. Follow the similar way as in the proof of Theorem 1. Use Lemmas 3 and 4 to derive $J_{\delta\mathrm{s}1}$, $J_{\delta\mathrm{s}2}$, and $J_{\delta\mathrm{u}1}$. In the partial fraction expansion to derive $J_{\delta\mathrm{u}2}$ we may obtain

$$\left[\frac{1-\bar{\lambda}_k(T\delta+1)}{T\delta+1-\lambda_k} - \frac{1-\bar{\lambda}_k}{1-\lambda_k} \right] \frac{1}{\delta} = \frac{T(|\lambda_k|^2-1)}{(1-\lambda_k)(T\delta+1-\lambda_k)}.$$

Furthermore, we have

$$\left\| \frac{1}{T\delta+1-\lambda_k} \right\|_2^2 = \frac{1}{T} \left\| \frac{1}{z-\lambda_k} \right\|_2^2 = \frac{1}{T} \frac{1}{|\lambda_k|^2-1}$$

by taking into account the norms relation (21). Note that in $J_{\delta\mathrm{u}2}$ we define $\Theta_T^{\sim}(\rho_k) := \Theta_T^{\mathrm{T}}(\frac{-\rho_k}{T\rho_k+1})$. □

4.3 Continuity Property

In this subsection we shall show the continuity properties of the delta domain expressions. Let consider a continuous-time plant $P_{\rm c}(s)$ given by

$$P_{\rm c} = \left(\begin{array}{cccc} P_{\rm c1}, & P_{\rm c2}, & \ldots, & P_{{\rm c}m} \end{array}\right)^{\rm T}. \tag{28}$$

The coprime factorizations of $P_{\rm c}$ are provided by

$$P_{\rm c} = N_{\rm c} M_{\rm c}^{-1} = \tilde{M}_{\rm c}^{-1} \tilde{N}_{\rm c}. \tag{29}$$

The corresponding $\mathcal{H}_2$ tracking performance index to be minimized is

$$J_{\rm c} := \int_0^\infty \left(\|e(t)\|^2 + |u_{\rm w}(t)|^2 \right) \,{\rm d}t, \tag{30}$$

where $u_{\rm w}(t) = \mathcal{L}^{-1}\{W_{\rm c}(s)\hat{u}_{\rm c}(s)\}$ and the optimal value is denoted by $J_{\rm c}^*$. Suppose $P_{\rm c}$ has non-minimum phase zeros z_i and unstable poles p_k. Under the zero-order hold operation we obtain the corresponding delta domain plant $P_T(\delta)$ which has those of ζ_i and ρ_k. We also define the corresponding index sets $\mathbb{K}_{\rm z}$, $\mathbb{K}_{\rm p}$, and $\mathbb{K}_{{\rm p}i}$ in similar manner.

Now we show the convergence of the delta domain expression J_δ^* given in Theorem 2. It is well-known that the zeros and poles of $P_{\rm c}(s)$ and $P_T(\delta)$ are determined by $\zeta_{ij} = ({\rm e}^{z_{ij}T} - 1)/T$ and $\rho_k = ({\rm e}^{p_k T} - 1)/T$. Obviously,

$$\lim_{T\to 0} J_{\delta {\rm s}1} = \sum_{i\in\mathbb{K}_{\rm z}} \nu_i^2 \sum_{j=1}^{n_i} \frac{2\,{\rm Re}\, z_{ij}}{|z_{ij}|^2} =: J_{\rm cs1}, \tag{31}$$

$$\lim_{T\to 0} J_{\delta {\rm u}1} = \sum_{i\in\mathbb{K}_{\rm z}} \nu_i^2 \sum_{k\in\mathbb{K}_{{\rm p}i}} \frac{2\,{\rm Re}\, p_k}{|p_k|^2} =: J_{\rm cu1}. \tag{32}$$

Next, since

$$\lim_{T\to 0} \frac{T^2}{2(1-\cos\omega T)} = \frac{1}{\omega^2},$$

we have

$$\lim_{T\to 0} J_{\delta {\rm s}2} = \frac{1}{\pi} \sum_{i\in\mathbb{K}_z} \nu_i^2 \int_0^\infty \log\left[\frac{|P_{{\rm c}i}(0)|^2}{\|P_{\rm c}(0)\|^2} \frac{\|P_{\rm c}({\rm j}\omega)\|^2 + |W_{\rm c}({\rm j}\omega)|^2}{|P_{{\rm c}i}({\rm j}\omega)|^2} \right] \frac{{\rm d}\omega}{\omega^2} =: J_{\rm cs2}. \tag{33}$$

We show the convergence of $J_{\delta {\rm u}2}$ part by part. Let define

$$J_\lambda = T \sum_{k,\ell\in\mathbb{I}_{\rm p}} \frac{(|\lambda_k|^2 - 1)(|\lambda_\ell|^2 - 1)}{(\bar{\lambda}_k - 1)(\lambda_\ell - 1)(\bar{\lambda}_k \lambda_\ell - 1)}.$$

Noting that $\lambda_k = {\rm e}^{p_k T}$, we have

$$J_\lambda = \sum_{k,\ell\in\mathbb{I}_{\rm p}} \frac{T({\rm e}^{2T\,{\rm Re}\, p_k} - 1)({\rm e}^{2T\,{\rm Re}\, p_\ell} - 1)}{({\rm e}^{\bar{p}_k T} - 1)({\rm e}^{p_\ell T} - 1)({\rm e}^{(\bar{p}_k + p_\ell)T} - 1)}.$$

Hence,

$$\lim_{T\to 0} J_\lambda = \sum_{k,\ell\in\mathbb{K}_{\rm p}} \frac{4\,\mathrm{Re}\,p_k\,\mathrm{Re}\,p_\ell}{(\bar{p}_k+p_\ell)\bar{p}_k p_\ell}.$$

Furthermore, for $\#\mathbb{I}_{\rm p} \geq 2$, we obtain

$$\lim_{T\to 0} q_k = \prod_{\ell\in\mathbb{K}_{\rm p},\ell\neq k} \lim_{T\to 0} \frac{\mathrm{e}^{p_k T}-\mathrm{e}^{p_\ell T}}{1-\mathrm{e}^{(p_k+\bar{p}_\ell)T}} = \prod_{\ell\in\mathbb{K}_{\rm p},\ell\neq k} \frac{p_\ell - p_k}{\bar{p}_\ell + p_k} \;\; =: \sigma_k.$$

Finally, collecting all the parts gives

$$\lim_{T\to 0} J_{\delta{\rm u}2} = \sum_{k,\ell\in\mathbb{K}_{\rm p}} \frac{4\,\mathrm{Re}\,p_k\,\mathrm{Re}\,p_\ell\,(1-\Theta_{\rm c}^{\sim}(\bar{p}_k)\Theta_{\rm c}(0))(1-\Theta_{\rm c}^{\sim}(p_\ell)\Theta_{\rm c}(0))}{(\bar{p}_k+p_\ell)\bar{p}_k p_\ell \bar{\sigma}_k \sigma_\ell} =: J_{\rm cu2} \tag{34}$$

where $\Theta_{\rm c}^{\sim}(p_k) := \Theta_{\rm c}^{\rm T}(-p_k)$ and

$$\sigma_k := \begin{cases} 1 & ;\ \#\mathbb{K}_{\rm p} = 1 \\ \displaystyle\prod_{\ell\in\mathbb{K}_{\rm p},\ell\neq k} \frac{p_\ell - p_k}{\bar{p}_\ell + p_k} & ;\ \#\mathbb{K}_{\rm p} \geq 2. \end{cases}$$

Note that $\Theta_{\rm c}(s)$ is the inner factor of

$$N_{\rm ca}(s) := \begin{pmatrix} W_{\rm c}(s)M_{\rm c}(s) \\ N_{\rm c}(s) \end{pmatrix}.$$

We can see that (31)–(34) show the continuity property:

$$\lim_{T\to 0} J_\delta^* = J_{\rm c}^*, \tag{35}$$

where $J_{\rm c}^*$ is the corresponding optimal performance in continuous-time case, see [5, 11]. It means that we completely recover the continuous-time expression from the delta domain expression stand point by making the sampling time approaches zero.

We summarize this continuity property in the following theorem. For continuous-time system, we make the following assumptions: $N_{\rm c}(0) \neq 0$, $\nu \in \mathbb{R}[N_{\rm c}(0)]$, and $P_{\rm c}(s)$ has a pole at $s = 0$.

Theorem 3. *Suppose that the SIMO plant $P_{\rm c}(s)$ given in (28) has unstable poles p_k $(k = 1, \ldots, n_p)$ and $P_{{\rm c}i}(s)$ has non-minimum phase zeros z_{ij} $(i = 1, \ldots, m,\ j = 1, \ldots, n_i)$. Then, the optimal tracking performance $J_{\rm c}^*$ is given by*

$$J_{\rm c}^* = J_{\rm cs1} + J_{\rm cs2} + J_{\rm cu1} + J_{\rm cu2}. \tag{36}$$

We pick one example to verify the derived expressions. We consider the following SISO continuous-time plant:

$$P(s) = \frac{s-1}{s(s-p)}.$$

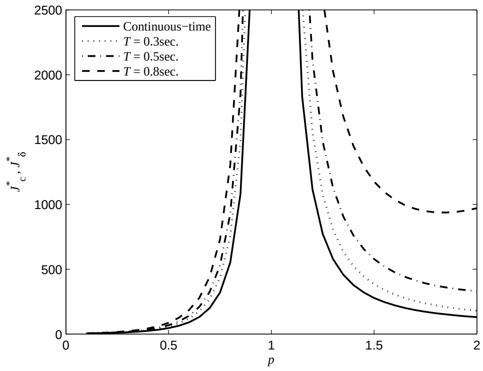

Figure 4: The convergence of the delta domain solution to its continuous-time counterpart.

We can see that $P(s)$ has one non-minimum phase zero at $s = 1$ and one unstable pole at $s = p$, provided $p > 0$. By using zero-order hold with sampling time T we obtain the corresponding delta domain plant $P_T(\delta)$, which has one non-minimum phase zero at $\delta = (\mathrm{e}^T - 1)/T$ and one unstable pole at $\delta = (\mathrm{e}^{pT} - 1)/T$.

We compute the optimal tracking performance of $P(s)$, i.e., J_{c}^*, by using Theorem 3. We fix $W(s) = 1$ for simplicity. We then compute that of $P_T(\delta)$, i.e., J_δ^*, by using Theorem 2 for $T = 0.3, 0.5, 0.8$ seconds. Fig. 4, which plots the optimal performances for p from 0 to 2, confirms that the delta domain solution (dashed/dotted line) converges to its continuous-time counterpart (solid line) when the sampling time T gets closer to zero.

5 Sampled-data Case

This section addresses the formulation of the tracking performance problem for sampled-data systems, where we evaluate the tracking measure in the continuous-time setting rather than the discrete-time setting. In other words, we take the inter-sample behavior into account to evaluate the tracking performance.

We consider a standard setup of a single-input single-output (SISO) sampled-data feedback control system depicted in Fig. 5, where $P_{\mathrm{c}}(s)$ represents the continuous-time plant and $K_{\mathrm{d}}(z)$ the discrete-time stabilizing controller. Note that e_k and u_k represent digital signals relate to $e(t)$ and $u(t)$ conduced by the sampler S and the zero-order hold H with sampling time T.

We want to minimize the tracking measure

$$J_{\mathrm{sd}} = \int_0^\infty \left(|e(t)|^2 + |u_{\mathrm{w}}(t)|^2\right) \mathrm{d}t, \tag{37}$$

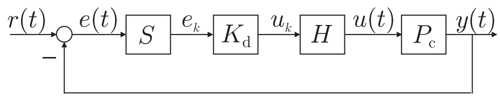

Figure 5: Sampled-data feedback control system.

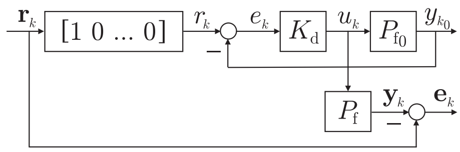

Figure 6: Approximation of the sampled-data feedback control system.

where $u_{\mathrm{w}}(t) = W\mathcal{L}^{-1}\{\hat{u}(s)\}$. Here we consider a real constant weighting function W for simplicity since it will also give a constant under sampling. We assume that $P_{\mathrm{c}}(0) \neq 0$ and $P_{\mathrm{c}}(s)$ has a pole at $s = 0$. Note that the tracking problem without control input penalty for stable SISO systems has been investigated in [7].

5.1 Fast Sampling

Under the fast sampling procedure, a fast sampler S_{f} with sampling time T/N is embedded at the reference input and the plant output, from which we subdivide the k-th sampling interval $[kT, (k+1)T)$ into N subintervals $[kT + \frac{i}{N}T, kT + \frac{i+1}{N}T)$, $i = 0, 1, \ldots, N-1$. Then the feedback control setup of Fig. 5 can be approximated by that of Fig. 6. We denote

$$\begin{aligned} \mathbf{r}_k &:= \begin{pmatrix} 1, & 1, & \ldots, & 1 \end{pmatrix}^{\mathrm{T}} r_k, \\ \mathbf{y}_k &:= \begin{pmatrix} y_{k_0}, & y_{k_1}, & \ldots, & y_{k_{N-1}} \end{pmatrix}^{\mathrm{T}}, \end{aligned}$$

where r_k is a discrete-time unit step function and $y_{k_i} = y(kT + \frac{i}{N}T)$, for $i = 0, 1, \ldots, N-1$.

Suppose that the transfer functions of the continuous-time plant $P_{\mathrm{c}}(s)$ and its discretized plant $P_{\mathrm{d}}(z)$ are determined by

$$P_{\mathrm{c}}(s) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right), \quad P_{\mathrm{d}}(z) = \left(\begin{array}{c|c} A_{\mathrm{d}} & B_{\mathrm{d}} \\ \hline C_{\mathrm{d}} & D_{\mathrm{d}} \end{array}\right),$$

where

$$A_{\mathrm{d}} = \mathrm{e}^{AT},\ B_{\mathrm{d}} = \int_0^T \mathrm{e}^{At} B\,\mathrm{d}t,\ C_{\mathrm{d}} = C,\ D_{\mathrm{d}} = D.$$

The transfer functions from u_k to y_{k_i}, denoted by P_{f_i}, are then determined by

$$P_{\mathrm{f}_i}(z) = \left(\begin{array}{c|c} A_{\mathrm{d}} & B_{\mathrm{d}} \\ \hline C_{\mathrm{f}_i} & D_{\mathrm{f}_i} \end{array}\right), \tag{38}$$

where

$$C_{\mathrm{f}_i} = C_{\mathrm{d}} \mathrm{e}^{A \frac{i}{N} T}, D_{\mathrm{f}_i} = C_{\mathrm{d}} \int_0^{\frac{i}{N} T} \mathrm{e}^{At} B \, \mathrm{d}t + D_{\mathrm{d}}.$$

Obviously, $P_{\mathrm{f}_0}(z) = P_{\mathrm{d}}(z)$. Furthermore, we define

$$P_{\mathrm{f}}(z) = \left(P_{\mathrm{f}_0}(z),\ P_{\mathrm{f}_1}(z),\ \ldots,\ P_{\mathrm{f}_{N-1}}(z) \right)^{\mathrm{T}}. \tag{39}$$

To solve the problem we implement the plant augmentation strategy described in Subsection 2.3, from which by fast sampling procedure we obtain the following sequences:

$$\begin{aligned} \mathbf{r}_{\mathrm{a}k} &:= \left(\begin{array}{cc} 0, & \mathbf{r}_k^{\mathrm{T}} \end{array} \right)^{\mathrm{T}}, \\ \mathbf{y}_{\mathrm{a}k} &:= \left(\begin{array}{cc} \sqrt{N} W, & \mathbf{y}_k^{\mathrm{T}} \end{array} \right)^{\mathrm{T}}, \end{aligned}$$

Note that originally we fast-sample the constant signal W such that we obtain $(W,\ \ldots,\ W)^{\mathrm{T}}$ of N-tuple. Since the sampling points are all constant we represent them only by single point $\sqrt{N} W$. Then it is possible to approximate the performance index (37) by

$$J_{\mathrm{f}} := \frac{T}{N} \sum_{k=0}^{\infty} \|\mathbf{r}_{\mathrm{a}k} - \mathbf{y}_{\mathrm{a}k}\|^2. \tag{40}$$

We put a factor of $\frac{T}{N}$ as implication of the sampling and hold operations.

5.2 Closed-form Expression

Let the coprime factorization of P_{f_0} is given by

$$P_{\mathrm{f}_0}(z) = N_{\mathrm{f}_0}(z) M_{\mathrm{f}_0}^{-1}(z).$$

Since P_{f_i} $(i = 0, \ldots, N-1)$ have only common unstable poles then the coprime factorization of P_{f} is given by

$$P_{\mathrm{f}}(z) = N_{\mathrm{f}}(z) M_{\mathrm{f}_0}^{-1}(z),$$

where $N_{\mathrm{f}} = (N_{\mathrm{f}_0},\ N_{\mathrm{f}_1},\ \ldots,\ N_{\mathrm{f}_{N-1}})^{\mathrm{T}}$. Youla parameterization (4) tells that the stabilizing digital controller is parameterized by

$$K_{\mathrm{d}} = \frac{Y_{\mathrm{f}_0} - M_{\mathrm{f}_0} Q_{\mathrm{f}}}{N_{\mathrm{f}_0} Q_{\mathrm{f}} - X_{\mathrm{f}_0}},$$

where $Q_{\mathrm{f}} \in \mathbb{R}\mathcal{H}_\infty$ is a scalar free parameter. Since $e_k = (X_{\mathrm{f}_0} - N_{\mathrm{f}_0} Q_{\mathrm{f}}) M_{\mathrm{f}_0} r_k$, it yields $\mathbf{y}_{\mathrm{a}k} = -N_{\mathrm{f}}(Y_{\mathrm{f}_0} - M_{\mathrm{f}_0} Q_{\mathrm{f}}) r_k$. Consequently the minimum value of (40) is given by

$$J_{\mathrm{f}}^* = \frac{T}{N} \inf_{Q_{\mathrm{f}} \in \mathbb{R}\mathcal{H}_\infty} \left\| \frac{\nu_{\mathrm{f}} + N_{\mathrm{fa}}(Y_{\mathrm{f}_0} - Q_{\mathrm{f}} M_{\mathrm{f}_0})}{z-1} \right\|_2^2, \tag{41}$$

where $\nu_{\rm f} = (\,0,\ 1,\ \ldots,\ 1\,)^{\rm T} \in \mathbb{R}^{N+1}$ and

$$N_{\rm fa}(z) = \begin{pmatrix} \sqrt{N} W M_{{\rm f}_0}(z) \\ N_{\rm f}(z) \end{pmatrix}.$$

Fortunately, expression (41) is coincident with that of the optimal tracking performance for SIMO discrete-time case (15) whenever $\nu_{\rm a} = \nu_{\rm f}$. In other words, we can write (41) as

$$J_{\rm f}^* = \frac{T}{N} \inf_{Q_{\rm fa} \in \mathbb{R}\mathcal{H}_\infty} \left\| \frac{[I + N_{\rm fa}(Y_{{\rm f}_0{\rm a}} - Q_{\rm fa} M_{{\rm f}_0})]\nu_{\rm f}}{z-1} \right\|_2^2,$$

where $Q_{\rm fa} = (0_m,\ Q_{\rm f})$ and $Y_{{\rm f}_0{\rm a}} = (0_m,\ Y_{{\rm f}_0})$, where 0_m denotes the row vector of size m whose elements are 0. Hence, by defining an inner-outer factorization such that $N_{\rm fa} = \Theta_{\rm f}\Phi_{\rm f}$, we are ready to state our result.

Theorem 4. *Consider the sampled-data system depicted in Fig. 5 with an SISO plant $P_{\rm c}(s)$. Let η_{ij} $(i = 0, \ldots, N-1,\ j = 1, \ldots, n_i)$ be the NMP zeros of $P_{{\rm f}_i}(z)$ and λ_k $(k = 1, \ldots, n_\lambda)$ be the unstable poles of $P_{\rm f}(z)$. Then the approximation value of the optimal tracking error performance is given by*

$$J_{\rm f}^* = J_{\rm fs1} + J_{\rm fs2} + J_{\rm fu2}, \tag{42}$$

where

$$J_{\rm fs1} := \frac{T}{N} \sum_{i=0}^{N-1} \sum_{j=1}^{n_i} \frac{|\eta_{ij}|^2 - 1}{|\eta_{ij} - 1|^2},$$

$$J_{\rm fs2} := \frac{T}{2\pi N} \sum_{i=0}^{N-1} \int_0^\pi \log\left(\frac{|P_{{\rm f}_i}(1)|^2}{\|P_{\rm f}(1)\|^2} \frac{\|P_{\rm f}(e^{j\theta})\|^2 + NW^2}{|P_{{\rm f}_i}(e^{j\theta})|^2} \right) \frac{d\theta}{1 - \cos\theta},$$

$$J_{\rm fu2} := \frac{T}{N} \sum_{k,\ell=1}^{n_\lambda} \frac{(|\lambda_k|^2 - 1)(|\lambda_\ell|^2 - 1)(1 - \Theta_{\rm f}^\sim(\bar\lambda_k)\Theta_{\rm f}(1))(1 - \Theta_{\rm f}^\sim(\lambda_\ell)\Theta_{\rm f}(1))}{\bar h_k h_\ell (\bar\lambda_k - 1)(\lambda_\ell - 1)(\bar\lambda_k \lambda_\ell - 1)},$$

with

$$h_k := \begin{cases} 1 & ;\ n_\lambda = 1, \\ \displaystyle\prod_{\ell \neq k} \frac{\lambda_k - \lambda_\ell}{1 - \bar\lambda_\ell \lambda_k} & ;\ n_\lambda \geq 2. \end{cases}$$

Remark 1. *If $P_{\rm c}(s)$ has unstable poles p_k $(k = 1, \ldots, n_p)$ then the discretized plant $P_{{\rm f}_i}$ will have only common unstable poles λ_k $(k = 1, \ldots, n_\lambda)$, where $\lambda_k = e^{p_k T}$ and $n_p = n_\lambda$. Consequently, $J_{\rm fu2}$ is non-negative since $M_{{\rm f}_0}(\lambda_k)\nu_{\rm f} = 0$, but $J_{\rm fu1} = 0$. Note that if $P_{\rm c}(s)$ is marginally stable then $J_{\rm fu2} = 0$, and hence we can compute $J_{\rm f}^*$ without using $\Theta_{\rm f}^\sim$.*

5.3 Numerical Example

Having a sampled-data feedback control system in Fig. 5, we consider the following SISO continuous-time plant:

$$P_{\mathrm{c}}(s) = \frac{s-x}{s(s+1)},\ x > 0.$$

Note that $P_{\mathrm{c}}(s)$ is marginally stable and has a non-minimum phase zero at $s = x$. It is not difficult to verify that

$$\begin{aligned}
A_{\mathrm{d}} &= \begin{pmatrix} \mathrm{e}^{-T} & 0 \\ 1-\mathrm{e}^{-T} & 1 \end{pmatrix}, \\
B_{\mathrm{d}} &= \begin{pmatrix} 1-\mathrm{e}^{-T} \\ T+\mathrm{e}^{-T}-1 \end{pmatrix}, \\
C_{\mathrm{f}_i} &= \begin{pmatrix} (1+x)\mathrm{e}^{-\frac{i}{N}T} - x & -x \end{pmatrix}, \\
D_{\mathrm{f}_i} &= 1 + x(1 - iT/N) - (1+x)\mathrm{e}^{-\frac{i}{N}T}.
\end{aligned}$$

Suppose that $n_{\mathrm{f}_i}(z)$ is the numerator of $P_{\mathrm{f}_i}(z)$. Then, $n_{\mathrm{f}_i}(1) = x(1-\mathrm{e}^{-T})(2-2\mathrm{e}^{-T} - T)$ for all $i = 0, \ldots, N-1$. Consequently,

$$\frac{|P_{\mathrm{f}_i}(1)|^2}{\|P_{\mathrm{f}}(1)\|^2} = \frac{|n_{\mathrm{f}_i}(1)|^2}{\sum_{i=0}^{N-1} |n_{\mathrm{f}_i}(1)|^2} = \frac{1}{N},$$

from which we simplify

$$J_{\mathrm{fs2}} = \frac{T}{2\pi N} \sum_{i=0}^{N-1} \int_0^{\pi} \log\left(\frac{1}{N} \frac{\|P_{\mathrm{f}}(\mathrm{e}^{\mathrm{j}\theta})\|^2 + NW^2}{|P_{\mathrm{f}_i}(\mathrm{e}^{\mathrm{j}\theta})|^2}\right) \frac{\mathrm{d}\theta}{1-\cos\theta}.$$

First we consider a case without input penalty, i.e., $W = 0$. We compute the optimal tracking performance for different pairs of $\{T, N\}$: $\{0.1\text{sec.}, 30\}$ and $\{0.01\text{sec.}, 3\}$ by using Theorem 4. We also compute the exact value by using [7, Theorem 1]. Fig. 7, which plots the optimal performance for x from 1 to 3, shows that we approximate the exact results well. Particularly if the sampling time T is small, N can be made small. Second we consider nonzero W. We select $W = \{5, 3, 1\} \times 10^{-5}$ and compute the optimal performance for $T = 0.01$sec. and $N = 3$. Fig. 8 shows that the results converge to those of the first case as W gets smaller.

6 Conclusion

We have examined the $\mathcal{H}_2$ tracking performance problem in SIMO LTI feedback control systems, where the tracking performance is quantified by the error response under control input constraint. We provide a comprehensive

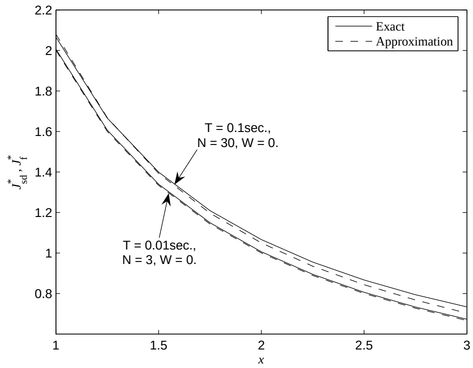

Figure 7: The exact and approximation values for $W = 0$.

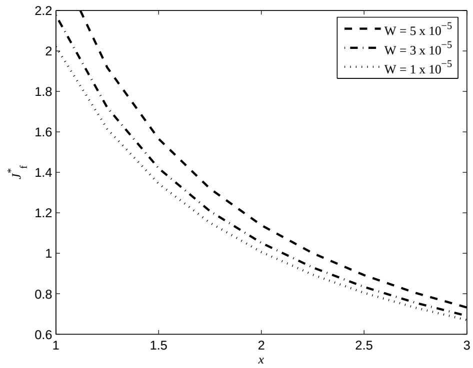

Figure 8: The approximation values for $W \neq 0$.

and unified results since we have derived the analytical closed-form expressions of the optimal tracking performance for discrete-time system and subsequently reformulate the results in delta domain and show the continuity property. This means that we can recover the continuous-time expressions as the sampling time tends to zero. Additionally, by invoking the discrete-time expression we can also derive the corresponding expression for the optimal tracking performance of SISO sampled-data systems. In this case, implementation of fast sampling technique is proposed.

In general, our results show that the non-minimum phase zeros and unstable poles of the plant as well as the plant gain impose the limits.

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A Two Key Lemmas for Discrete-time Case

We introduce two lemmas which play important roles in our subsequent analysis. These lemmas serve as the discrete-time counterparts of Lemmas 1 and 2 in [5]. The proofs are immediate by application of bilinear transformation. Consider the class of functions in

$$\mathbb{F} := \left\{ f : \lim_{R\to\infty} \max_{\theta\in[-\pi/2,\pi/2]} \frac{|f(R\mathrm{e}^{\mathrm{j}\theta})|}{R} = 0 \right\}.$$

The above class consists of functions with restricted behavior at infinity. By this, we intend to deal with integration over a contour that becomes arbitrarily long. Generally speaking, if f is analytic and bounded magnitude in $\mathbb{D}^{\mathrm{c}}$, then f is of class $\mathbb{F}$.

Lemma 1. *Let $f(z) \in \mathbb{F}$ and analytic in $\mathbb{D}^{\mathrm{c}}$. Denote that $f(\mathrm{e}^{\mathrm{j}\theta}) = f_1(\theta) + \mathrm{j}f_2(\theta)$. Suppose that $f(z)$ is conjugate symmetric, i.e., $f(z) = \overline{f(\bar{z})}$. Then*

$$f'(1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_1(\theta) - f_1(0)}{1-\cos\theta}\,\mathrm{d}\theta.$$

Lemma 2. *Let $f(z)$ be a meromorphic function in $\mathbb{D}^{\mathrm{c}}$ and has no zero or pole on unit circle. Suppose that $f(z)$ is conjugate symmetric and $\log f(z) \in \mathbb{F}$. Also, suppose that $\eta_i \in \bar{\mathbb{D}}^{\mathrm{c}}$ $(i = 1,\ldots,n_\eta)$ and $\lambda_k \in \bar{\mathbb{D}}^{\mathrm{c}}$ $(k = 1,\ldots,n_\lambda)$ are, respectively, zeros and poles of $f(z)$, all counting multiplicities. Provided that $f(1) \neq 0$, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\frac{f(\mathrm{e}^{\mathrm{j}\theta})}{f(1)}\right| \frac{\mathrm{d}\theta}{1-\cos\theta} = \sum_{i=1}^{n_\eta}\frac{|\eta_i|^2-1}{|\eta_i-1|^2} - \sum_{k=1}^{n_\lambda}\frac{|\lambda_k|^2-1}{|\lambda_k-1|^2} + \frac{f'(1)}{f(1)}.$$

B Proof of Theorem 1

Define

$$\Psi(z) := \begin{bmatrix} \Theta^{\sim}(z) \\ I - \Theta(z)\Theta^{\sim}(z) \end{bmatrix}.$$

It is easy to show that $\Psi(z)$ is a norm preserving function, i.e., $\Psi^{\sim}(\mathrm{e}^{\mathrm{j}\theta})\Psi(\mathrm{e}^{\mathrm{j}\theta}) = I$. By pre-multiplying Ψ to (15) we get

$$J_{\mathrm{d}}^* = \inf_{Q_{\mathrm{a}}\in\mathbb{R}\mathcal{H}_\infty} \left\| \frac{[\Theta^{\sim} + \Phi(\tilde{Y}_{\mathrm{a}} - Q_{\mathrm{a}}\tilde{M}_{\mathrm{a}})]\nu_{\mathrm{a}}}{z-1} \right\|_2^2 + \left\| \frac{(I-\Theta\Theta^{\sim})\nu_{\mathrm{a}}}{z-1} \right\|_2^2.$$

Noting that

$$\frac{(\Theta^{\sim} - \Theta^{\sim}(1))\nu_{\mathrm{a}}}{z-1} \in \mathcal{H}_2^{\perp},$$

we may select $Q_{\rm a} \in \mathbb{R}\mathcal{H}_\infty$ such that $\Theta^\sim(1) - \Phi(1)(\tilde{Y}_{\rm a}(1) - Q_{\rm a}(1)\tilde{M}_{\rm a}(1)) = 0$, and therefore

$$\frac{[\Theta^\sim(1) - \Phi(\tilde{Y}_{\rm a} - Q_{\rm a}\tilde{M}_{\rm a})]\nu_{\rm a}}{z-1} \in \mathcal{H}_2.$$

As a result, we can write $J_{\rm d}^* = J_1 + J_2$, where

$$\begin{aligned} J_1 &:= \left\| \frac{(\Theta^\sim - \Theta^\sim(1))\nu_{\rm a}}{z-1} \right\|_2^2 + \left\| \frac{(I - \Theta\Theta^\sim)\nu_{\rm a}}{z-1} \right\|_2^2, \\ J_2 &:= \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} \left\| \frac{[\Theta^\sim(1) + \Phi(\tilde{Y}_{\rm a} - Q_{\rm a}\tilde{M}_{\rm a})]\nu_{\rm a}}{z-1} \right\|_2^2. \end{aligned}$$

Next we will show that $J_1 = J_{\rm ds1}^* + J_{\rm ds2}^* + J_{\rm du1}$ and $J_2 = J_{\rm du2}$. A direct calculation leads to

$$J_1 = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{{\rm Re}(\nu_{\rm a}^{\rm H}\Theta({\rm e}^{{\rm j}\theta})\Theta^\sim(1)\nu_{\rm a}) - 1}{1 - \cos\theta}\, {\rm d}\theta.$$

Let define $f(z) := \nu_{\rm a}^{\rm H}\Theta(z)\Theta^\sim(1)\nu_{\rm a}$. Under Assumption 2, we obtain $f(1) = 1$. Applying Lemma 1 yields $J_1 = -f'(1) = -\nu_{\rm a}^{\rm H}\Theta(1)\Theta^\sim(1)\nu_{\rm a}$. Denote the inner factor $\Theta(z)$ in (16) as $\Theta(z) = [w_0(z), w_1(z), \dots, w_m(z)]^{\rm T}$. According to Assumption 2, we may select $\nu_{\rm a} = \Theta(1)$ without loss of generality. Noting that the first element of $\nu_{\rm a}$ is zero, we have

$$J_1 = -\sum_{i=0}^{m} w_i(1)w_i'(1) = -\sum_{i \in \mathbb{J}_{\rm z}} \nu_i^2 \frac{w_i'(1)}{w_i(1)}.$$

Condition $i \in \mathbb{J}_{\rm z}$ guarantees that $w_i(1) \neq 0$. Since $w_i(z)$ is element of inner factor $\Theta(z)$, it has the same set of non-minimum phase zeros as $N_i(z)$, which includes the set of unstable poles of P but not those of P_i as well as the set of non-minimum phase zeros of $P_i(z)$. Hence, by invoking Lemma 2, we have [1]

$$\frac{w_i'(1)}{w_i(1)} = -\sum_{j=1}^{n_i} \frac{|\eta_{ij}|^2 - 1}{|\eta_{ij} - 1|^2} - \sum_{k \in \mathbb{J}_{{\rm p}i}} \frac{|\lambda_k|^2 - 1}{|\lambda_k - 1|^2} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left| \frac{w_i({\rm e}^{{\rm j}\theta})}{w_i(1)} \right| \frac{{\rm d}\theta}{1 - \cos\theta}.$$

Note here that $|w_i({\rm e}^{{\rm j}\theta})| = |P_i({\rm e}^{{\rm j}\theta})|/\|P_{\rm a}({\rm e}^{{\rm j}\theta})\|$, we obtain

$$\log\left| \frac{w_i({\rm e}^{{\rm j}\theta})}{w_i(1)} \right| = -\frac{1}{2}\log\left[\frac{|P_i(1)|^2}{\|P_{\rm a}(1)\|^2} \frac{\|P_{\rm a}({\rm e}^{{\rm j}\theta})\|^2}{|P_i({\rm e}^{{\rm j}\theta})|^2} \right].$$

We can see from Assumption 3 that $|P_i(1)|$ and $\|P(1)\|$ are infinite but $|W(1)|$ is finite, then

$$\frac{|P_i(1)|^2}{\|P_{\rm a}(1)\|^2} = \frac{|P_i(1)|^2}{\|P(1)\|^2 + |W(1)|^2} = \frac{|P_i(1)|^2}{\|P(1)\|^2}$$

[1] The second term in the right hand side is missing in the expression in [1, 5].

holds. Also note that

$$\frac{\|P_{\mathrm{a}}(\mathrm{e}^{\mathrm{j}\theta})\|^2}{|P_i(\mathrm{e}^{\mathrm{j}\theta})|^2} = \frac{\|P(\mathrm{e}^{\mathrm{j}\theta})\|^2 + |W(\mathrm{e}^{\mathrm{j}\theta})|^2}{|P_i(\mathrm{e}^{\mathrm{j}\theta})|^2}.$$

This completes the proof of $J_1 = J^*_{\mathrm{ds1}} + J^*_{\mathrm{ds2}} + J_{\mathrm{du1}}$. Next, by factorizing $\tilde{M}_{\mathrm{a}}(z)\nu_{\mathrm{a}} = g_m(z)h(z)$, where $g_m(z)$ is left invertible in $\mathbb{R}\mathcal{H}_\infty$ and $h(z)$ is defined by

$$h(z) = \prod_{k\in\mathbb{J}_{\mathrm{p}}} \frac{z-\lambda_k}{1-\bar{\lambda}_k z},$$

we can show that $J_2 = J_{\mathrm{du2}}$ by following the standard partial fraction expansion using in the proof of [5, Theorem 3.3].

C Two Key Lemmas for Delta Domain Case

We introduce two key lemmas which are counterparts with Lemmas 1 and 2. The proofs can be easily done by variable changes.

Lemma 3. *Let $h \in \mathbb{F}$ and analytic in $\mathbb{D}^{\mathrm{c}}_T$. Denote that $h(\frac{\mathrm{e}^{\mathrm{j}\omega T}-1}{T}) = h_1(\omega) + \mathrm{j}h_2(\omega)$. Suppose that h is conjugate symmetric, i.e. $h(\delta) = \overline{h(\bar{\delta})}$. Then*

$$\frac{h'(0)}{T} = \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \frac{h_1(\omega)-h_1(0)}{1-\cos\omega T}\,\mathrm{d}\omega.$$

Lemma 4. *Let h be a meromorphic function in $\mathbb{D}^{\mathrm{c}}_T$ and has no zero or pole on $\partial\mathbb{D}_T$. Suppose that h is conjugate symmetric and $\log h \in \mathbb{F}$. Also, suppose that $\zeta_i \in \bar{\mathbb{D}}^{\mathrm{c}}_T$ $(i = 1,\ldots,n_\zeta)$ and $\rho_k \in \bar{\mathbb{D}}^{\mathrm{c}}_T$ $(k = 1,\ldots,n_\rho)$ are, respectively zeros and poles of h, all counting multiplicities. Provided that $h(0) \neq 0$, then*

$$\frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \log\left|\frac{h(\frac{\mathrm{e}^{\mathrm{j}\omega T}-1}{T})}{h(0)}\right| \frac{\mathrm{d}\omega}{1-\cos\omega T} = \sum_{i=1}^{n_\zeta}\left[\frac{2\,\mathrm{Re}\,\zeta_i}{T|\zeta_i|^2}+1\right] - \sum_{k=1}^{n_\rho}\left[\frac{2\,\mathrm{Re}\,\rho_k}{T|\rho_k|^2}+1\right] + \frac{1}{T}\frac{h'(0)}{h(0)}.$$