

**MATHEMATICAL ENGINEERING
TECHNICAL REPORTS**

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METR 2006-37

June 2006

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WWW page: <http://www.i.u-tokyo.ac.jp/mi/mi-e.htm>

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Complex Analytic Approach to the Sinc-Gauss Sampling Formula

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June 2006

Abstract

For a sampling formula with the sinc-Gaussian kernel Qian et al. have recently given an error estimate for the class of band-limited functions with the aid of tools from Fourier analysis. By complex analytic approach, we give in this paper an error estimate for a wider class of functions including unbounded ones on \mathbb{R} . Part of the result of Qian et al. can be derived from ours as an immediate corollary. Computational results show a fairly good agreement with the theoretical convergence rates.

1 Introduction

Shannon's sampling theorem [5] is fundamental in the field of information processing. Let

$$B_\sigma = \{f \in L^2(\mathbb{R}) \mid |\omega| > \sigma \Rightarrow \hat{f}(\omega) = 0\}, \quad (1.1)$$

which denotes a set of band-limited functions, where $\sigma > 0$ and \hat{f} is the Fourier transform of f . The sampling theorem states that an identity

$$f(x) = \sum_{k=-\infty}^{\infty} f(kh) \operatorname{sinc}(x/h - k) \quad (x \in \mathbb{R}) \quad (1.2)$$

is valid for $f \in B_\sigma$, where $h = \pi/\sigma$ and

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x} & (x \neq 0), \\ 1 & (x = 0). \end{cases} \quad (1.3)$$

The sampling formula (1.2) shows that the function f can be reconstructed from the sampled values $f(kh)$ ($k \in \mathbb{Z}$).

Recently, this sampling formula has been put to use for numerical computation. A naive formula

$$f(x) \approx \sum_{k=\lfloor x/h \rfloor - N}^{\lceil x/h \rceil + N} f(kh) \operatorname{sinc}(x/h - k), \quad (1.4)$$

however, requires a prohibitively large number of sampling points due to the fact that the sinc function does not decrease rapidly at infinity. To overcome this difficulty, the following two methods are contrived.

The first is to transform $f(x)$ so that the transformed function $f(\varphi(t))$ may decrease rapidly at infinity through an appropriate change of variable $x = \varphi(t)$. Then a simple formula

$$f(x) \approx \sum_{k=-N}^N f(kh) \operatorname{sinc}(x/h - k) \quad (1.5)$$

is applied to the transformed function. The truncation error incurred in this approximation is bounded by $\sum_{|k|>N} |f(\varphi(kh))|$. If $f(\varphi(t))$ decreases rapidly at infinity, say, exponentially, the truncation error decreases exponentially with respect to the number of sampled points $2N + 1$. Numerical methods based on such function approximation are often referred to as ‘‘Sinc numerical methods’’ [6, 7, 8].

The second is to use rapidly decreasing kernel functions. A typical formula under this category is

$$f(x) \approx (\mathcal{T}_{N,h}f)(x) := \sum_{k=\lfloor x/h \rfloor - N}^{\lceil x/h \rceil + N} f(kh) \operatorname{sinc}(x/h - k) \exp\left[-\frac{(x - kh)^2}{2r^2h^2}\right], \quad (1.6)$$

where r is a positive constant. Seeing that no standard name of this formula is found in the literature, we call this formula the sinc-Gauss sampling formula. This formula is used by Wei et al. in numerical solution of partial differential equations [11, 12]¹. Qian et al. show that the error $\|f - \mathcal{T}_{N,h}f\|_\infty$ of the sinc-Gauss sampling formula decreases exponentially with respect to N for $f \in B_\sigma$, and also demonstrate similar results about the approximation of the derivatives of f [1, 2, 3, 4]. In estimating the discretization error, they make use of the Fourier transform and the Parseval identity to exploit the band-limited condition. In Japan, as early as in 1975, H. Takahasi [9] proposed the sinc-Gauss sampling formula above to apply Shannon’s sampling formula to numerical analysis. He made an error analysis for holomorphic functions by using complex analysis. His analysis lacks, however, in mathematical rigor, although it captures the essential feature.

The objective of this paper is to provide a mathematically rigorous version of Takahasi’s error analysis for the sinc-Gauss sampling formula. Furthermore, we point out that the formula is applicable to a wider class of functions including unbounded ones on \mathbb{R} . Specifically, we estimate the error of the formula for those functions which are holomorphic on a band-shaped region on the complex plane

$$\mathcal{D}_d := \{z \in \mathbb{C} \mid |\operatorname{Im}z| \leq d\} \quad (1.7)$$

¹To be precise, Wei et al. set $\lfloor x/h \rfloor + N$ as the upper bound of the sum, whereas we use $\lceil x/h \rceil + N$ for symmetry. This does not affect the following argument.

and satisfy

$$|f(z)| \leq A + B|z|^\alpha \quad (\forall z \in \mathcal{D}_d),$$

where $d > 0$, $A \geq 0$, $B \geq 0$, and $\alpha \geq 0$. Furthermore, we show that part of Qian et al.'s result for $f \in B_\sigma$ can be derived from ours as an immediate corollary. It is mentioned that a preliminary result for bounded functions (i.e., for the case of $B = 0$) is discussed in [10].

The organization of this paper is as follows. In Section 2, we present our main results. In Section 3, we specialize our results to bounded functions, and explain the relationship to some results of Qian-Ogawa [4]. In Section 4 we show computational results. Proofs of theorems and lemmas are given in Section 5.

2 Main Results

For nonnegative integer m and $r, h > 0$, we define operators $\mathcal{G}_h^{(m)}$, $\mathcal{T}_{N,h}^{(m)}$ approximating the m -th order derivative $f^{(m)}$ of a function f as

$$\left(\mathcal{G}_h^{(m)} f\right)(x) := \sum_{k=-\infty}^{\infty} f(kh) \frac{d^m}{dx^m} \left[\text{sinc}(x/h - k) \exp \left[-\frac{(x - kh)^2}{2r^2h^2} \right] \right], \quad (2.1)$$

$$\left(\mathcal{T}_{N,h}^{(m)} f\right)(x) := \sum_{k=\lfloor x/h \rfloor - N}^{\lfloor x/h \rfloor + N} f(kh) \frac{d^m}{dx^m} \left[\text{sinc}(x/h - k) \exp \left[-\frac{(x - kh)^2}{2r^2h^2} \right] \right], \quad (2.2)$$

where sinc is the function defined in (1.3). Note that (2.2) with $m = 0$ coincides with (1.6). We call the formula given by $\mathcal{T}_{N,h}^{(m)} f$ the sinc-Gauss sampling formula.

Let \mathcal{D}_d be the band-shaped region defined in (1.7). In this section, we assume that $f : \mathcal{D}_d \rightarrow \mathbb{C}$ is a holomorphic function on \mathcal{D}_d with $|f(z)| \leq A + B|z|^\alpha$ ($\forall z \in \mathcal{D}_d$) for constants $A \geq 0$, $B \geq 0$ and $\alpha \geq 0$. The error of the formula will be measured by the supremum of the absolute value of $f(x) - \left(\mathcal{T}_{N,h}^{(m)} f\right)(x)$ over a finite interval $[-L, L]$ for $L > 0$. The proofs of Lemma 2.1 and Lemma 2.2 below are given in Section 5.1 and Section 5.2, respectively.

First, the discretization error of the sinc-Gauss sampling formula is estimated as follows.

Lemma 2.1 (Discretization error). *Let $d > 0$. Let $f : \mathcal{D}_d \rightarrow \mathbb{C}$ be a holomorphic function on \mathcal{D}_d with $|f(z)| \leq A + B|z|^\alpha$ ($\forall z \in \mathcal{D}_d$) for constants $A \geq 0$, $B \geq 0$ and $\alpha \geq 0$. Let $m \in \mathbb{Z}_+$, $L > 0$, $r > 0$, and $h > 0$ with $h \leq 2\pi d / \log 2$. Then we have*

$$\begin{aligned} \sup_{-L \leq x \leq L} \left| f^{(m)}(x) - \left(\mathcal{G}_h^{(m)} f\right)(x) \right| &\leq \exp \left(-\frac{\pi d}{h} + \frac{d^2}{2r^2h^2} \right) \\ &\cdot C_0 \left[C_1 C_3 \sqrt{2\pi} + C_2 C_3 2^{\frac{\alpha+1}{2}} \Gamma \left(\frac{\alpha+1}{2} \right) \right. \\ &\left. + C_1 2^{\frac{2m+1}{2}} \Gamma \left(\frac{m+1}{2} \right) + C_2 2^{\frac{\alpha+2m+1}{2}} \Gamma \left(\frac{\alpha+m+1}{2} \right) \right], \end{aligned}$$

where

$$C_0 = \frac{2(m+3)! rh}{\pi} \left(1 + \left(\frac{\sqrt{2}}{rh} \right)^m \right) \left(\frac{1}{d} + \frac{1}{d^{m+1}} \right), \quad (2.3)$$

$$C_1 = A + 2^\alpha B(L+d)^\alpha, \quad (2.4)$$

$$C_2 = 2^\alpha B(rh)^\alpha, \quad (2.5)$$

$$C_3 = 2 + \left(\frac{\sqrt{2}d}{rh} \right)^m. \quad (2.6)$$

Second, the truncation error of the sinc-Gauss sampling formula is estimated as follows.

Lemma 2.2 (Truncation error). *Let $d > 0$. Let $f : \mathcal{D}_d \rightarrow \mathbb{C}$ be a holomorphic function on \mathcal{D}_d with $|f(z)| \leq A + B|z|^\alpha$ ($\forall z \in \mathcal{D}_d$) for constants $A \geq 0$, $B \geq 0$ and $\alpha \geq 0$. Let $m \in \mathbb{Z}_+$, $L > 0$, $r > 0$, and $h > 0$. If $N \geq \max \left\{ 2, mr/\sqrt{2}, \sqrt{[\alpha]r} + 1 \right\}$, we have*

$$\sup_{-L \leq x \leq L} \left| \left(\mathcal{G}_h^{(m)} f \right) (x) - \left(\mathcal{T}_{N,h}^{(m)} f \right) (x) \right| \leq C'_0 (C'_1 + C'_2) \exp \left[-\frac{(N-1)^2}{2r^2} \right],$$

where

$$C'_0 = \frac{2(m+1)! e^\pi e^{\frac{3}{2r^2}} r^2}{N(N-1)h^m \pi}, \quad (2.7)$$

$$C'_1 = A + 2^\alpha B[(L+h)^\alpha + 2^\alpha h^\alpha], \quad (2.8)$$

$$C'_2 = 2^{2\alpha} B h^\alpha ([\alpha] + 1)!! \max \{ (N-1)^{[\alpha]}, r^{[\alpha]} \}. \quad (2.9)$$

From the lemmas above, we can derive the following error estimate by setting h and r appropriately for a given N .

Theorem 2.3 (Error of the sinc-Gauss sampling formula). *Let $d > 0$. Let $f : \mathcal{D}_d \rightarrow \mathbb{C}$ be a holomorphic function on \mathcal{D}_d with $|f(z)| \leq A + B|z|^\alpha$ ($\forall z \in \mathcal{D}_d$) for constants $A \geq 0$, $B \geq 0$ and $\alpha \geq 0$. Let $m \in \mathbb{Z}_+$ and $L > 0$. For a positive integer N , define h and r as*

$$h = \frac{d'}{N}, \quad r = \sqrt{\frac{N}{\pi}} \quad (2.10)$$

with an arbitrary constant d' satisfying $0 < d' \leq d$. Then we have

$$\begin{aligned} & \sup_{-L \leq x \leq L} \left| f^{(m)}(x) - \left(\mathcal{T}_{N,h}^{(m)} f \right) (x) \right| \\ &= O \left(N^{m - \min \{ 1/2, 1 - [\alpha] + \alpha \}} \exp \left(-\frac{\pi N}{2} \right) \right) \quad (N \rightarrow \infty). \end{aligned}$$

Proof. If N is sufficiently large, the assumptions in Lemmas 2.1 and 2.2,

$$h \leq 2\pi d / \log 2, \quad N \geq \max \left\{ 2, mr/\sqrt{2}, \sqrt{[\alpha]r} + 1 \right\}$$

are satisfied under (2.10). We apply the lemmas to the right hand side of the inequality

$$\begin{aligned} & \sup_{-L \leq x \leq L} \left| f^{(m)}(x) - \left(\mathcal{T}_{N,h}^{(m)} f \right) (x) \right| \\ & \leq \sup_{-L \leq x \leq L} \left| f^{(m)}(x) - \left(\mathcal{G}_h^{(m)} f \right) (x) \right| + \sup_{-L \leq x \leq L} \left| \left(\mathcal{G}_h^{(m)} f \right) (x) - \left(\mathcal{T}_{N,h}^{(m)} f \right) (x) \right|. \end{aligned}$$

The estimate in Lemma 2.1 remains valid when d is replaced by d' . With h and r in (2.10) we have

$$\begin{aligned} \exp \left(-\frac{\pi d'}{h} + \frac{d'^2}{2r^2 h^2} \right) &= \exp \left(-\frac{\pi N}{2} \right), \\ \exp \left[-\frac{(N-1)^2}{2r^2} \right] &= O \left(\exp \left(-\frac{\pi N}{2} \right) \right) \quad (N \rightarrow \infty). \end{aligned}$$

Furthermore, the orders of C_0, \dots, C_3 in Lemma 2.1 and C'_0, C'_1, C'_2 in Lemma 2.2 as $N \rightarrow \infty$ are estimated as follows:

$$\begin{aligned} C_0 &= O \left(N^{\frac{m-1}{2}} \right), \quad C_1 = O(1), \quad C_2 = O \left(N^{-\frac{\alpha}{2}} \right), \quad C_3 = O \left(N^{\frac{m}{2}} \right), \\ C'_0 &= O(N^{m-1}), \quad C'_1 = O(1), \quad C'_2 = O \left(N^{[\alpha]-\alpha} \right). \end{aligned}$$

Thus we obtain the claim of the theorem. \square

The error estimate in Theorem 2.3 presupposes approximation of $f(x)$ at a single point x and, accordingly, expresses the error bound in terms of the number $2N + 1$ of the sampling points required for a single point. In some situations, however, it is more natural to consider approximation over a finite interval $[-L, L]$ with $L > 0$. This is the case, for instance, in applications to differential equations. In such a case it is more appropriate to express the error bound in terms of the number

$$M = 2 \left(\frac{L}{d'} + 1 \right) N \tag{2.11}$$

of the sampling points needed for the approximation over the entire interval, rather than at a single point, where d' is in (2.10). In accordance with this, Theorem 2.3 can be recast into the following form.

Corollary 2.4. *Let $d > 0$. Let $f : \mathcal{D}_d \rightarrow \mathbb{C}$ be a holomorphic function on \mathcal{D}_d with $|f(z)| \leq A + B|z|^\alpha$ ($\forall z \in \mathcal{D}_d$) for constants $A \geq 0$, $B \geq 0$ and $\alpha \geq 0$. Let $m \in \mathbb{Z}_+$ and $L > 0$. For a positive integer N , define h and r as (2.10), and M as (2.11). Then we have*

$$\begin{aligned} & \sup_{-L \leq x \leq L} \left| f^{(m)}(x) - \left(\mathcal{T}_{N,h}^{(m)} f \right) (x) \right| \\ & = O \left(M^{m - \min\{1/2, 1 - [\alpha] + \alpha\}} \exp \left(-\frac{\pi d'}{4(d' + L)} M \right) \right) \quad (M \rightarrow \infty). \end{aligned}$$

Remark 2.5. The upper bound of h and the value r in (2.10) is “optimal” in the sense that they attain

$$\min_{h,r>0} \max \left\{ \exp \left(-\frac{\pi d}{h} + \frac{d^2}{2r^2 h^2} \right), \exp \left[-\frac{(N-1)^2}{2r^2} \right] \right\}$$

asymptotically as $N \rightarrow \infty$. Noting that replacing $N - 1$ with N does not change the order of the expression above, we have

$$-\frac{\pi d}{h} + \frac{d^2}{2r^2h^2} = \frac{1}{2r^2} \left(\frac{d}{h} - \pi r^2 \right)^2 - \frac{\pi^2 r^2}{2} \geq -\frac{\pi^2 r^2}{2},$$

where the equality holds when $d/h = \pi r^2$, and

$$-\frac{\pi^2 r^2}{2} = -\frac{N^2}{2r^2} \iff r = \sqrt{\frac{N}{\pi}}.$$

Therefore

$$\begin{aligned} & \min_{h,r>0} \max \left\{ \exp \left(-\frac{\pi d}{h} + \frac{d^2}{2r^2h^2} \right), \exp \left(-\frac{N^2}{2r^2} \right) \right\} \\ & \geq \min_{r>0} \max \left\{ \exp \left(-\frac{\pi^2 r^2}{2} \right), \exp \left(-\frac{N^2}{2r^2} \right) \right\} \\ & \geq \exp \left(-\frac{\pi N}{2} \right). \end{aligned}$$

The equality holds for h and r in (2.10).

3 Error Estimates for Bounded Functions

In this section, we present the error estimate for holomorphic functions $f : \mathcal{D}_d \rightarrow \mathbb{C}$ on \mathcal{D}_d with $|f(z)| \leq A$ ($\forall z \in \mathcal{D}_d$) for a constant $A \geq 0$, and discuss its relationship to the estimate of Qian-Ogawa [4]. For bounded functions it is possible to consider supremum error bounds over the entire real number \mathbb{R} . The error estimates over \mathbb{R} can be obtained easily from our results in Section 2 by setting $B = 0$, $\alpha = 0$ and letting $L \rightarrow \infty$. We set $\|g\|_\infty := \sup_{-\infty < x < \infty} |g(x)|$ for a function g on \mathbb{R} .

3.1 Error Estimates

Letting $B = 0$, $\alpha = 0$, $L \rightarrow \infty$ in Section 2, we obtain the following lemmas and theorem.

Lemma 3.1 (Discretization error). *Let $d > 0$. Let $f : \mathcal{D}_d \rightarrow \mathbb{C}$ be a holomorphic function on \mathcal{D}_d with $|f(z)| \leq A$ ($\forall z \in \mathcal{D}_d$) for a constant $A \geq 0$. Let $m \in \mathbb{Z}_+$, $r > 0$, and $h > 0$ with $h \leq 2\pi d / \log 2$. Then we have*

$$\begin{aligned} \left\| f^{(m)} - \mathcal{G}_h^{(m)} f \right\|_\infty & \leq \exp \left(-\frac{\pi d}{h} + \frac{d^2}{2r^2h^2} \right) \\ & \cdot A \left[\frac{2(m+3)! rh}{\pi} \left(1 + \left(\frac{\sqrt{2}}{rh} \right)^m \right) \left(\frac{1}{d} + \frac{1}{d^{m+1}} \right) \right] \\ & \cdot \left[\sqrt{2\pi} \left(2 + \left(\frac{\sqrt{2}d}{rh} \right)^m \right) + 2^{\frac{2m+1}{2}} \Gamma \left(\frac{m+1}{2} \right) \right]. \end{aligned}$$

Lemma 3.2 (Truncation error). *Let $d > 0$. Let $f : \mathcal{D}_d \rightarrow \mathbb{C}$ be a holomorphic function on \mathcal{D}_d with $|f(z)| \leq A$ ($\forall z \in \mathcal{D}_d$) for a constant $A \geq 0$. Let $m \in \mathbb{Z}_+$, $r > 0$, and $h > 0$. If $N \geq \max\{2, mr/\sqrt{2}\}$, we have*

$$\left\| \mathcal{G}_h^{(m)} f - \mathcal{T}_{N,h}^{(m)} f \right\|_{\infty} \leq \frac{2A(m+1)! e^{\pi} e^{\frac{3}{2r^2}} r^2}{N(N-1)h^m \pi} \exp\left[-\frac{(N-1)^2}{2r^2}\right].$$

Theorem 3.3 (Error of the sinc-Gauss sampling formula). *Let $d > 0$. Let $f : \mathcal{D}_d \rightarrow \mathbb{C}$ be a holomorphic function on \mathcal{D}_d with $|f(z)| \leq A$ ($\forall z \in \mathcal{D}_d$) for a constant $A \geq 0$. Let $m \in \mathbb{Z}_+$. For a positive integer N , define h and r as*

$$h = \frac{d'}{N}, \quad r = \sqrt{\frac{N}{\pi}} \tag{3.1}$$

with an arbitrary constant d' satisfying $0 < d' \leq d$. Then we have

$$\left\| f^{(m)} - \mathcal{T}_{N,h}^{(m)} f \right\|_{\infty} = O\left(N^{m-1/2} \exp\left(-\frac{\pi N}{2}\right)\right) \quad (N \rightarrow \infty).$$

3.2 Relationship to Qian-Ogawa's Result

We investigate the relationship between the result of Qian-Ogawa [4] and our Theorem 3.3 in Section 3.1. The following theorem is an immediate corollary of Corollary 3.1 of [4], where B_{σ} is defined as (1.1).

Theorem 3.4 ([4]). *Let $f \in B_{\sigma}$ and $0 < h < \pi/\sigma$. For $N > 2$, define $r = \sqrt{(N-2)/(\pi-h\sigma)}$. Then we have*

$$\begin{aligned} & \left\| f^{(m)} - \mathcal{T}_{N,h}^{(m)} f \right\|_{\infty} \\ &= O\left(\frac{1}{\sqrt{N-2}} \exp\left[-\frac{(\pi-h\sigma)(N-2)}{2}\right]\right) \quad (N \rightarrow \infty). \end{aligned} \tag{3.2}$$

The objective of this section is to demonstrate how (3.2) with $m = 0$ can be derived from our result of Section 3.1. In the case of $m \geq 1$ we also derive a weaker result²

$$\begin{aligned} & \left\| f^{(m)} - \mathcal{T}_{N,h}^{(m)} f \right\|_{\infty} \\ &= O\left((N-2)^{(m-1)/2} \exp\left[-\frac{(\pi-h\sigma)(N-2)}{2}\right]\right) \quad (N \rightarrow \infty). \end{aligned} \tag{3.3}$$

First we note the following fact, which may be regarded as a part of the Paley-Wiener theorem.

Lemma 3.5. *If $f \in B_{\sigma}$, then f is holomorphic on \mathbb{C} and there exists a constant $A' \geq 0$ such that*

$$|f(z)| \leq A' \exp(\sigma|\operatorname{Im}z|) \quad (z \in \mathbb{C}). \tag{3.4}$$

²The estimate (3.2) does not seem to be derived in the case of $m \geq 1$ from our results. This is because our estimate of the discretization error is considered under a more general condition, and is necessarily weaker.

Proof. Denote the Fourier transform of f by \hat{f} . By $f \in B_\sigma$, we have $\hat{f} \in L^2(\mathbb{R})$ and

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} \hat{f}(\omega) \exp(iz\omega) d\omega. \quad (3.5)$$

Since the interval of integration is finite, we can exchange the differentiation and integration. Therefore f is holomorphic on \mathbb{C} .

Next, again by (3.5), we have

$$\begin{aligned} |f(\xi + i\eta)| &\leq \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)| |\exp(i(\xi + i\eta)\omega)| d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)| \exp(-\eta\omega) d\omega \\ &\leq \exp(|\eta|\sigma) \cdot \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)| d\omega \end{aligned}$$

for $\xi, \eta \in \mathbb{R}$, and therefore (3.4) by setting $A' = (2\pi)^{-1/2} \int_{-\sigma}^{\sigma} |\hat{f}(\omega)| d\omega$, which is finite since

$$\left(\int_{-\sigma}^{\sigma} |\hat{f}(\omega)| d\omega \right)^2 \leq \int_{-\sigma}^{\sigma} d\omega \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^2 d\omega < \infty.$$

□

Lemma 3.5 above implies the following, which states that our function class contains band-limited functions.

Lemma 3.6. *Let $f \in B_\sigma$. For any $d > 0$, $f : \mathcal{D}_d \rightarrow \mathbb{C}$ is a holomorphic function on \mathcal{D}_d with $|f(z)| \leq A$ ($\forall z \in \mathcal{D}_d$), where*

$$A = A' \exp(\sigma d) \quad (3.6)$$

with A' in (3.4).

This lemma enables us to apply Lemmas 3.1 and 3.2 to $f \in B_\sigma$. We take r as in Theorem 3.4 and assume that N is sufficiently large.

To estimate the discretization error, we set

$$d = h(N - 2)$$

and take A as (3.6). Then, by Lemma 3.1, we have

$$\begin{aligned} \left\| f^{(m)} - \mathcal{G}_h^{(m)} f \right\|_\infty &\leq A' \exp \left(\sigma d - \frac{\pi d}{h} + \frac{d^2}{2r^2 h^2} \right) \\ &\cdot \left[\frac{2(m+3)! rh}{\pi} \left(1 + \left(\frac{\sqrt{2}}{rh} \right)^m \right) \left(\frac{1}{d} + \frac{1}{d^{m+1}} \right) \right] \\ &\cdot \left[\sqrt{2\pi} \left(2 + \left(\frac{\sqrt{2}d}{rh} \right)^m \right) + 2^{\frac{2m+1}{2}} \Gamma \left(\frac{m+1}{2} \right) \right]. \end{aligned}$$

The degree of the underlined part with respect to $N - 2$ is $(m - 1)/2$. The exponent of the remaining part is

$$\begin{aligned} \sigma d - \frac{\pi d}{h} + \frac{d^2}{2r^2h^2} &= -\frac{(\pi - h\sigma)d}{h} + \frac{d^2}{2r^2h^2} \\ &= -(\pi - h\sigma)(N - 2) + \frac{(\pi - h\sigma)h^2(N - 2)^2}{2(N - 2)h^2} \\ &= -\frac{(\pi - h\sigma)(N - 2)}{2}. \end{aligned}$$

Thus we obtain the following estimate:

$$\begin{aligned} &\left\| f^{(m)} - \mathcal{G}_h^{(m)} f \right\|_\infty \\ &= O \left((N - 2)^{(m-1)/2} \exp \left[-\frac{(\pi - h\sigma)(N - 2)}{2} \right] \right) \quad (N \rightarrow \infty). \end{aligned} \quad (3.7)$$

To estimate the truncation error, we set $d = 1$ and $A = A'e^\sigma$ according to (3.6). Then, by Lemma 3.2, we have

$$\begin{aligned} \left\| \mathcal{G}_h^{(m)} f - \mathcal{T}_{N,h}^{(m)} f \right\|_\infty &\leq \frac{2A'e^\sigma(m+1)!e^\pi e^{\frac{3}{2r^2}} r^2}{N(N-1)h^m\pi} \exp \left[-\frac{(N-1)^2}{2r^2} \right] \\ &\leq \frac{2A'e^\sigma(m+1)!e^\pi e^{\frac{3}{2r^2}}}{h^m\pi} \frac{r^2}{(N-2)^2} \exp \left[-\frac{(N-2)^2}{2r^2} \right]. \end{aligned}$$

The degree of $r^2/(N-2)^2$ with respect to $N-2$ is -1 . Furthermore, $e^{\frac{3}{2r^2}} \rightarrow 1$ as $N \rightarrow \infty$. The exponent of the remaining part is

$$-\frac{(N-2)^2}{2r^2} = -\frac{(\pi - h\sigma)(N-2)}{2}.$$

Thus we obtain the following estimate:

$$\begin{aligned} &\left\| \mathcal{G}_h^{(m)} f - \mathcal{T}_{N,h}^{(m)} f \right\|_\infty \\ &= O \left(\frac{1}{N-2} \exp \left[-\frac{(\pi - h\sigma)(N-2)}{2} \right] \right) \quad (N \rightarrow \infty). \end{aligned} \quad (3.8)$$

By (3.7) and (3.8), we have (3.3) in the case of $m \geq 0$ and (3.2) in the case of $m = 0$.

Remark 3.7. It is worth noting that h depends on N in Theorem 3.3, whereas it is fixed in Theorem 3.4. To realize exponential decrease of errors for holomorphic functions on \mathcal{D}_d with a finite d , it seems inevitable to make h depend on N . On the other hand, what we have shown above is that exponential decrease of errors can be obtained with a fixed h if d can be taken arbitrarily large, and that this is true of band-limited functions.

4 Numerical Experiments

In this section, we present computational results on sinc-Gauss sampling formula for two types of functions: (i) rational functions

$$f_{\beta,d}(z) = \frac{z^{\beta+2}}{z^2 + d^2}$$

with $\beta \in \{-2, -1, 0, 1, 2\}$ and $d > 0$, and (ii) band-limited functions

$$f_l(z) = (\text{sinc}(z))^l$$

with a positive integer l . The former is not band-limited, and the latter is included to confirm that the performance of the sinc-Gauss sampling formula is essentially independent of the band-limited property of the functions to be approximated.

We consider errors on a finite interval $[-3, 3]$ (i.e., $L = 3$), which we evaluate numerically as the maximum of the errors at 6000 equally-spaced points in the interval. The relationship of the error against the number of sampling points will be presented in graphs. Specifically, the ordinates are the errors in logarithm,

$$\log_{10} \left(\sup_{-3 \leq x \leq 3} \left| f(x) - \left(\mathcal{T}_{N,h}^{(m)} f \right) (x) \right| \right), \quad (4.1)$$

and the abscissae are N as well as $M = 2(3/d' + 1)N$ (with $L = 3$ in (2.11)), where M is indicated at the top.

According to our theoretical analysis summarized in Theorem 2.3, the error curves are expected to be almost linear, with the slope against N being

$$-\frac{\pi}{2} \log_{10} e = -0.682 \dots \quad (4.2)$$

This theoretical slope will be compared with the observed values, which we obtain from the computational results by the least square method.

The program for the computation is written in C. Our computer is SUN Blade 2000, whose environment is as follows: the operating system is Solaris 9, the CPU is UltraSPARC-III+(900MHz, 64bit) with 3 GB memory, the compiler is Sun Studio 11, in which “long double” is 128 bits wide.

4.1 Rational Functions

For $\beta \in \{-2, -1, 0, 1, 2\}$ and $d > 0$, define $f_{\beta,d}$ as

$$f_{\beta,d}(z) = \frac{z^{\beta+2}}{z^2 + d^2} \quad (z \in \mathbb{C}). \quad (4.3)$$

Then $f_{\beta,d}$ is holomorphic on $\mathcal{D}_{d-\varepsilon}$ for ε with $0 < \varepsilon \ll d$, and satisfies

$$|f_{\beta,d}(z)| \leq \frac{\max\{d, d^{-1}\}}{\varepsilon} |z|^\alpha \quad (\forall z \in \mathcal{D}_{d-\varepsilon}),$$

where $\alpha = \max\{\beta, 0\}$. The sinc-Gauss sampling formula is applied to $f_{\beta,d}$ for $\beta = -2, -1, 0, 1, 2$, $d = 10^{-i}$ ($i = 0, 1, 2$), and $m = 0, 1, 2$. We set $\varepsilon = d/100$ and $h = (d - \varepsilon)/N$. Furthermore, in computing the slopes, we exclude the data for $N = 45$ and 50 to avoid the effect of rounding errors.

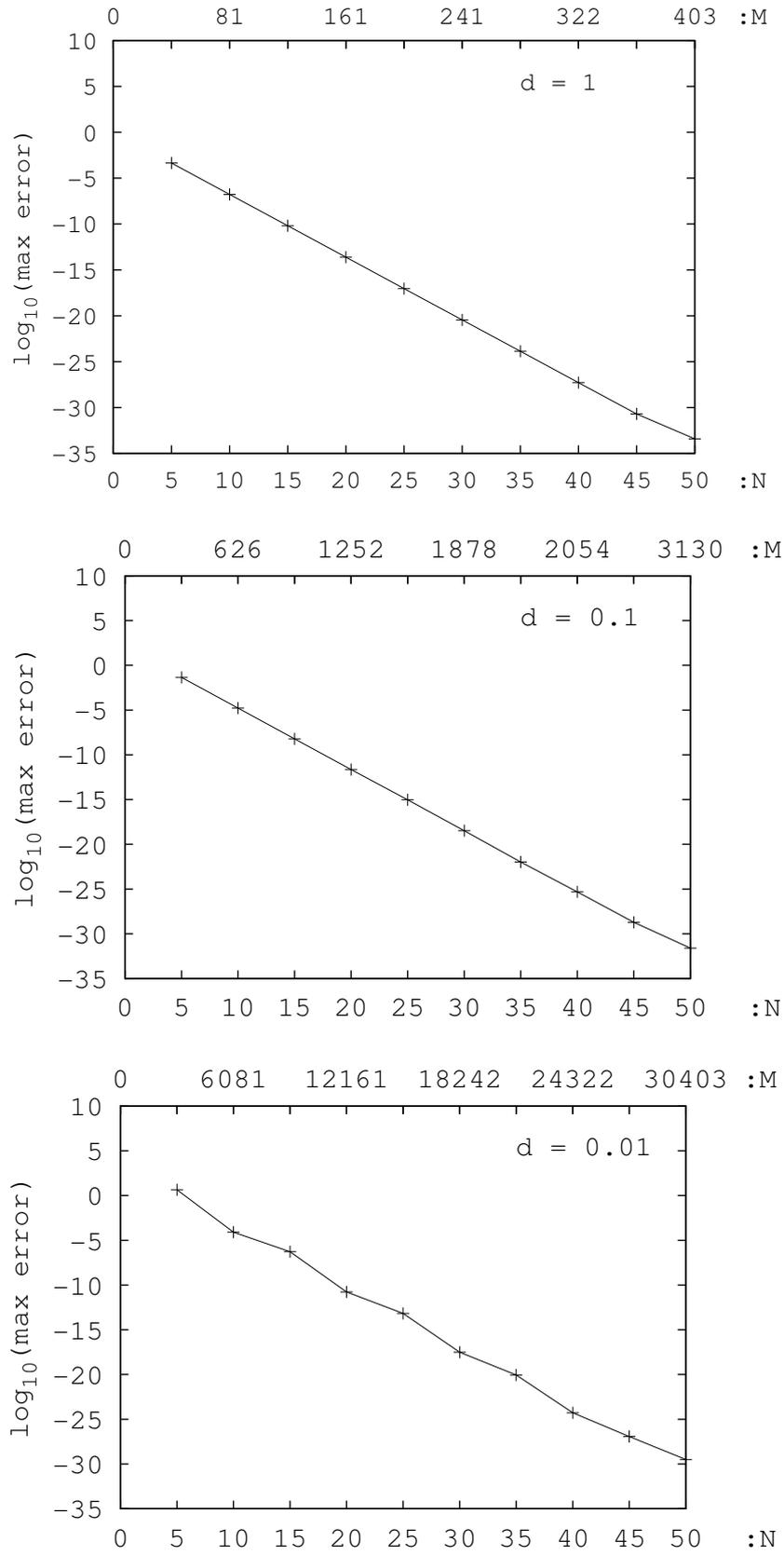


Figure 1: Errors for $f_{\beta,d}$ of (4.3) with $\beta = -2$ and for $m = 0$

Table 1: \log_{10} (max error) for $f_{\beta,d}$ of (4.3) with $\beta = -2$

	$m = 0$		$m = 1$		$m = 2$	
	$N = 10$	$N = 40$	$N = 10$	$N = 40$	$N = 10$	$N = 40$
$d = 1.00$	-6.77	-27.28	-5.29	-25.19	-3.52	-22.80
$d = 0.10$	-4.77	-25.32	-2.29	-22.20	0.48	-18.84
$d = 0.01$	-4.09	-24.29	0.45	-19.55	4.43	-14.96

Table 2: \log_{10} (max error) for $f_{\beta,d}$ of (4.3) with $\beta = -1$

	$m = 0$		$m = 1$		$m = 2$	
	$N = 10$	$N = 40$	$N = 10$	$N = 40$	$N = 10$	$N = 40$
$d = 1.00$	-6.94	-27.45	-5.31	-25.21	-3.58	-22.84
$d = 0.10$	-5.96	-26.47	-3.31	-23.21	-0.59	-19.84
$d = 0.01$	-6.09	-26.42	-1.31	-21.21	2.18	-17.18

Table 3: \log_{10} (max error) for $f_{\beta,d}$ of (4.3) with $\beta = 0$

	$m = 0$		$m = 1$		$m = 2$	
	$N = 10$	$N = 40$	$N = 10$	$N = 40$	$N = 10$	$N = 40$
$d = 1.00$	-6.89	-27.35	-5.40	-25.26	-3.65	-22.87
$d = 0.10$	-6.90	-27.39	-4.41	-24.26	-1.66	-20.87
$d = 0.01$	-7.38	-28.14	-3.53	-23.52	0.34	-18.87

Table 4: \log_{10} (max error) for $f_{\beta,d}$ of (4.3) with $\beta = 1$

	$m = 0$		$m = 1$		$m = 2$	
	$N = 10$	$N = 40$	$N = 10$	$N = 40$	$N = 10$	$N = 40$
$d = 1.00$	-6.94	-27.49	-5.41	-25.28	-3.61	-22.91
$d = 0.10$	-6.91	-27.66	-4.41	-24.56	-1.60	-21.15
$d = 0.01$	-6.92	-27.66	-3.40	-23.56	0.40	-19.15

Table 5: \log_{10} (max error) for $f_{\beta,d}$ of (4.3) with $\beta = 2$

	$m = 0$		$m = 1$		$m = 2$	
	$N = 10$	$N = 40$	$N = 10$	$N = 40$	$N = 10$	$N = 40$
$d = 1.00$	-6.50	-27.23	-4.91	-25.07	-3.12	-22.67
$d = 0.10$	-6.43	-27.19	-3.93	-24.08	-1.13	-20.68
$d = 0.01$	-6.45	-27.19	-2.93	-23.09	0.87	-18.68

Table 6: \log_{10} (max error)/ N for $f_{\beta,d}$ of (4.3) with $\beta = -2$

	$m = 0$	$m = 1$	$m = 2$
$d = 1.00$	-0.684	-0.660	-0.635
$d = 0.10$	-0.686	-0.660	-0.636
$d = 0.01$	-0.692	-0.664	-0.639

Table 7: \log_{10} (max error)/ N for $f_{\beta,d}$ of (4.3) with $\beta = -1$

	$m = 0$	$m = 1$	$m = 2$
$d = 1.00$	-0.684	-0.660	-0.634
$d = 0.10$	-0.685	-0.660	-0.634
$d = 0.01$	-0.691	-0.660	-0.638

Table 8: \log_{10} (max error)/ N for $f_{\beta,d}$ of (4.3) with $\beta = 0$

	$m = 0$	$m = 1$	$m = 2$
$d = 1.00$	-0.681	-0.658	-0.632
$d = 0.10$	-0.683	-0.658	-0.632
$d = 0.01$	-0.694	-0.663	-0.632

Table 9: \log_{10} (max error)/ N for $f_{\beta,d}$ of (4.3) with $\beta = 1$

	$m = 0$	$m = 1$	$m = 2$
$d = 1.00$	-0.687	-0.660	-0.637
$d = 0.10$	-0.694	-0.670	-0.646
$d = 0.01$	-0.694	-0.670	-0.646

Table 10: $\log_{10}(\text{max error})/N$ for $f_{\beta,d}$ of (4.3) with $\beta = 2$

	$m = 0$	$m = 1$	$m = 2$
$d = 1.00$	-0.693	-0.670	-0.646
$d = 0.10$	-0.694	-0.670	-0.646
$d = 0.01$	-0.693	-0.669	-0.646

From Table 6–Table 10, we see that the experimental values of the slopes are close to the theoretical ones in (4.2). As m becomes larger, the slope tends to be larger than the theoretical value. This may be because we only use $\exp(-(\pi/2)N)$ in Theorem 3.3 in computing the theoretical value, with the secondary factor $N^{m-\min\{1/2, 1-[\alpha]+\alpha\}}$ disregarded.

Next, we consider the effect of m , the order of differentiation. By Theorem 3.3, we expect that $\log_{10}(\text{max error})$ will increase approximately by $\log_{10} N$ if m increases by one. The results of Table 6–Table 10 agree with this expectation, whereas m is also included in the constant part independent of N in the estimate.

Next, we consider the effect of d , representing the location of the singular points. Noting the order with respect to N , we conclude that d does not affect the error. It is expected, however, that $\log_{10}(\text{max error})$ will increase approximately by $m + 1$ if d is multiplied by $1/10$, due to the term $1/d^{m+1}$ in the estimate of Lemma 3.1. Computational results appear to support this observation.

Finally, we consider the effect of β . From the results, we see that β does not affect the errors substantially, which is theoretically appropriate.

4.2 Band-limited Functions

For a positive integer l , we define f_l as

$$f_l(z) = (\text{sinc}(z))^l \quad (z \in \mathbb{C}). \quad (4.4)$$

Then we have $f_l \in B_{\pi l}$. The function f_l is holomorphic on \mathbb{C} and satisfies

$$|f_l(z)| \leq \max \left\{ \left(\frac{e^{\pi d}}{\pi} \right)^l, e^{\pi l} \right\} \quad (z \in \mathcal{D}_d) \quad (4.5)$$

for arbitrary $d > 0$. Setting $h = 1/N$, we apply the sinc-Gauss sampling formula to f_l for $l = 5, 10, 15, 20$ and $m = 0, 1, 2$. In computing the slopes, we exclude the data for $N = 5, 10, 45, 50$.

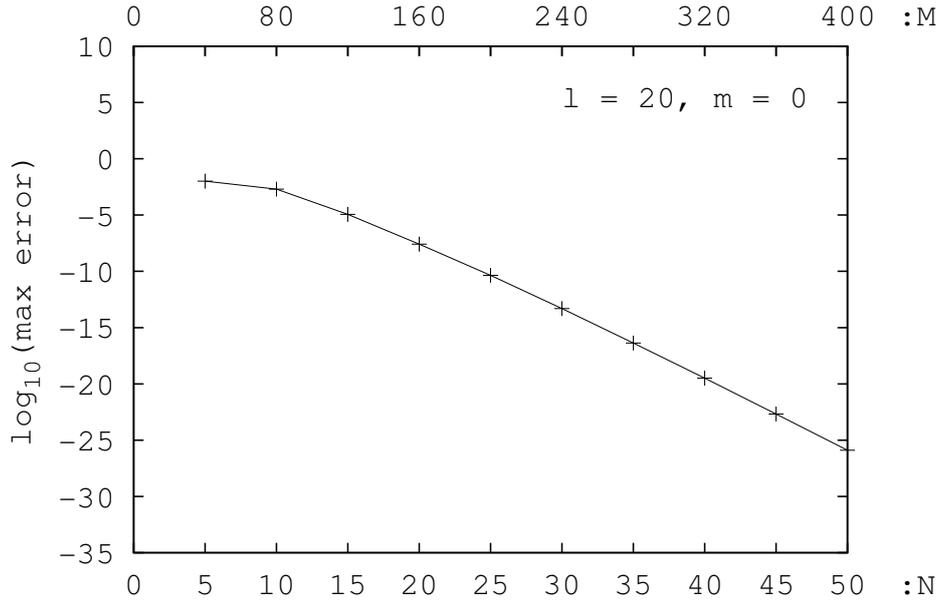


Figure 2: Errors for $f = \text{sinc}^{20}$ with $m = 0$ and $d = 1$

Table 11: $\log_{10}(\max \text{ error})$ for $f = \text{sinc}^l$ with $d = 1$

	$m = 0$		$m = 1$		$m = 2$	
	$N = 10$	$N = 40$	$N = 10$	$N = 40$	$N = 10$	$N = 40$
$l = 5$	-5.18	-25.48	-3.67	-23.39	-1.98	-21.01
$l = 10$	-3.99	-23.18	-2.39	-21.09	-0.81	-18.74
$l = 15$	-3.18	-21.20	-1.66	-19.09	-0.11	-16.78
$l = 20$	-2.70	-19.49	-1.23	-17.38	0.34	-15.09

Table 12: $\log_{10}(\max \text{ error})/N$ for $f = \text{sinc}^l$ with $d = 1$

	$m = 0$	$m = 1$	$m = 2$
$l = 5$	-0.680	-0.664	-0.645
$l = 10$	-0.653	-0.638	-0.618
$l = 15$	-0.618	-0.605	-0.583
$l = 20$	-0.583	-0.568	-0.546

As to the effect of m on the errors, we see the same as in Section 4.1.

Next, we consider the slopes of the error curves in the graphs. The experimental values of the slopes are close to the theoretical ones in (4.2) when l is small (Table 12). In the case where l is large, however, this is not the case. This may be because the constant on the right hand

side of (4.5) is large when l is large (note that $d = 1$), and the effect of the constant cannot be ignored.

Taking this fact into consideration, we apply the formula in the case of $d = \pi^{-1} \log \pi$, i.e., $h = (\pi N)^{-1} \log \pi$. The results of the experiments are presented in Fig. 3 and Table 14, which justify the above observation. The numerical results support our expectation that smaller width between neighboring sampling points yields better approximation for functions with strong vibration.

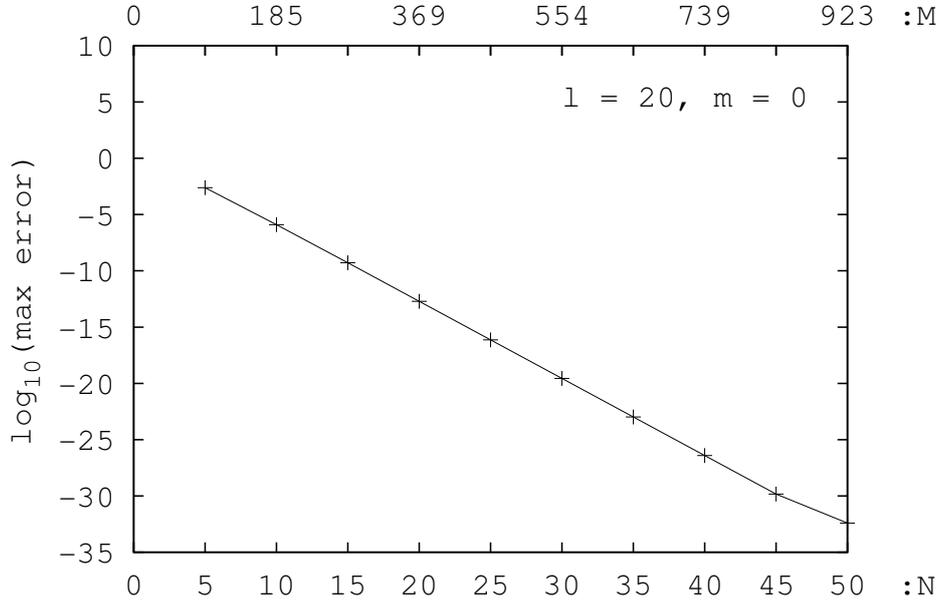


Figure 3: Errors for $f = \text{sinc}^{20}$ with $m = 0$ and $d = \pi^{-1} \log \pi$

Table 13: $\log_{10}(\max \text{ error})$ for $f = \text{sinc}^l$ with $d = \pi^{-1} \log \pi$

	$m = 0$		$m = 1$		$m = 2$	
	$N = 10$	$N = 40$	$N = 10$	$N = 40$	$N = 10$	$N = 40$
$l = 5$	-6.94	-27.69	-5.01	-25.15	-2.79	-22.32
$l = 10$	-6.55	-27.25	-4.63	-24.72	-2.42	-21.49
$l = 15$	-6.20	-26.83	-4.28	-24.29	-2.09	-21.16
$l = 20$	-5.89	-26.41	-3.97	-23.88	-1.80	-20.62

Table 14: $\log_{10}(\max \text{error})/N$ for $f = \text{sinc}^l$ with $d = \pi^{-1} \log \pi$

	$m = 0$	$m = 1$	$m = 2$
$l = 5$	-0.690	-0.673	-0.656
$l = 10$	-0.689	-0.672	-0.643
$l = 15$	-0.688	-0.670	-0.644
$l = 20$	-0.685	-0.668	-0.638

5 Proofs

In this section, the following fact is sometimes used, whose proof is omitted.

Lemma 5.1. *For arbitrary nonnegative real numbers a , b , and τ , we have $(a+b)^\tau \leq 2^\tau (a^\tau + b^\tau)$, where $0^0 = 1$ by convention.*

5.1 Proof of Lemma 2.1

For $n \in \mathbb{Z}_+$, we set $V_n := \{x + yi \mid |x| = (n + 1/2)h, |y| \leq d\}$ and $U_n := \{x + yi \mid |x| \leq (n + 1/2)h, |y| = d\}$. By the residue theorem we have

$$\begin{aligned} f(x) - \sum_{k=-n}^n f(kh) \left(\frac{\sin(\pi(x - kh)/h)}{\pi(x - kh)/h} \exp \left[-\frac{(x - kh)^2}{2r^2h^2} \right] \right) \\ = \frac{1}{2\pi i} \int_{V_n \cup U_n} \frac{\sin(\pi x/h)}{z - x} \exp \left[-\frac{(x - z)^2}{2r^2h^2} \right] \frac{f(z)}{\sin(\pi z/h)} dz, \end{aligned}$$

and hence it suffices to estimate

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{V_n \cup U_n} \frac{d^m}{dx^m} \left(\frac{\sin(\pi x/h)}{z - x} \exp \left[-\frac{(x - z)^2}{2r^2h^2} \right] \right) \frac{f(z)}{\sin(\pi z/h)} dz. \quad (5.1)$$

Note that

$$\begin{aligned} & \frac{d^m}{dx^m} \left(\frac{\sin(\pi x/h)}{z - x} \exp \left[-\frac{(x - z)^2}{2r^2h^2} \right] \right) \\ &= \sum_{k=0}^m \sum_{l=0}^k \frac{m!}{l!(m-k)!} \left(\frac{-1}{\sqrt{2rh}} \right)^{m-k} \\ & \quad \cdot \frac{\sin(\pi x/h + \pi l/2)}{(z - x)^{k-l+1}} H_{m-k} \left(\frac{x - z}{\sqrt{2rh}} \right) \exp \left[-\frac{(x - z)^2}{2r^2h^2} \right], \end{aligned} \quad (5.2)$$

where $H_j(x)$ is the j -th degree Hermite polynomial

$$H_j(x) := (-1)^j \exp(x^2) \left(\frac{d}{dx} \right)^j \exp(-x^2) = \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \frac{(-1)^i j! (2x)^{j-2i}}{i! (j-2i)!}. \quad (5.3)$$

We assume that $-L \leq x \leq L$ in the following.

Estimate of the integral on V_n . By (5.2), it follows from

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{V_n} \frac{1}{|z-x|^{k-l+1}} \left| H_{m-k} \left(\frac{x-z}{\sqrt{2rh}} \right) \right| \\ & \cdot \left| \exp \left[-\frac{(x-z)^2}{2r^2h^2} \right] \right| \frac{|f(z)|}{|\sin(\pi z/h)|} |dz| = 0 \end{aligned} \quad (5.4)$$

that the integral on V_n of (5.1) converges to 0 as $n \rightarrow \infty$. We assume that n is sufficiently large. For $z \in V_n$ we have

$$\begin{aligned} & (n+1/2)h - L \leq |z-x| \leq (n+1/2)h + L + d, \\ & \operatorname{Re}(x-z)^2 = [\pm(n+1/2)h - x]^2 - (\operatorname{Im}z)^2 \geq [(n+1/2)h - L]^2 - d^2, \\ & |z| \leq (n+1/2)h + d, \\ & |\sin(\pi z/h)| = |\cosh(\pi \operatorname{Im}z/h)| \geq 1, \end{aligned}$$

and therefore

$$\begin{aligned} & \int_{V_n} \frac{1}{|z-x|^{k-l+1}} \left| H_{m-k} \left(\frac{x-z}{\sqrt{2rh}} \right) \right| \left| \exp \left[-\frac{(x-z)^2}{2r^2h^2} \right] \right| \frac{|f(z)|}{|\sin(\pi z/h)|} |dz| \\ & \leq 4d \cdot \frac{A+B[(n+1/2)h+L+d]^\alpha}{[(n+1/2)h-L]^{k-l+1}} \left[\sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} \frac{2^{j-2i} j!}{i!(j-2i)!} \left(\frac{(n+1/2)h+L+d}{\sqrt{2rh}} \right)^{j-2i} \right] \\ & \cdot \exp \left[-\frac{[(n+1/2)h-L]^2 - d^2}{2r^2h^2} \right]. \end{aligned}$$

This implies (5.4).

Estimate of the integral on U_n . By (5.2), we have

$$\begin{aligned} & \left| \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{U_n} \frac{d^m}{dx^m} \left(\frac{\sin(\pi x/h)}{z-x} \exp \left[-\frac{(x-z)^2}{2r^2h^2} \right] \right) \frac{f(z)}{\sin(\pi z/h)} dz \right| \\ & \leq \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{U_n} \sum_{k=0}^m \sum_{l=0}^k \frac{m!}{l!(m-k)!} \left(\frac{1}{\sqrt{2rh}} \right)^{m-k} \frac{1}{|z-x|^{k-l+1}} \left| H_{m-k} \left(\frac{x-z}{\sqrt{2rh}} \right) \right| \\ & \cdot \left| \exp \left[-\frac{(x-z)^2}{2r^2h^2} \right] \right| \frac{|f(z)|}{|\sin(\pi z/h)|} |dz|. \end{aligned} \quad (5.5)$$

For $z = t \pm di$ ($t \in \mathbb{R}$) we have

$$\begin{aligned} & \frac{1}{|z-x|^{k-l+1}} = \frac{1}{|t-x \pm di|^{k-l+1}} \leq \frac{1}{d^{k-l+1}} \leq \max \left\{ \frac{1}{d}, \frac{1}{d^{m+1}} \right\} \leq \frac{1}{d} + \frac{1}{d^{m+1}}, \\ & \left| \exp \left[-\frac{(x-z)^2}{2r^2h^2} \right] \right| = \left| \exp \left[-\frac{(x-t)^2 \mp 2(x-t)di - d^2}{2r^2h^2} \right] \right| \\ & = \exp \left[-\frac{(x-t)^2}{2r^2h^2} \right] \exp \left(\frac{d^2}{2r^2h^2} \right), \\ & |f(z)| \leq A + B|t \pm di|^\alpha \leq A + B(|t| + d)^\alpha, \\ & \frac{1}{|\sin(\pi z/h)|} \leq \frac{1}{\sinh(\pi d/h)} \leq 4 \exp \left(-\frac{\pi d}{h} \right), \end{aligned}$$

where the assumption $h \leq 2\pi d/\log 2$ is used in the last inequality. These estimates and (5.5) imply that

$$\begin{aligned}
& \left| \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{U_n} \frac{d^m}{dx^m} \left(\frac{\sin(\pi x/h)}{z-x} \exp \left[-\frac{(x-z)^2}{2r^2 h^2} \right] \right) \frac{f(z)}{\sin(\pi z/h)} dz \right| \\
& \leq \frac{2m!}{\pi} \left(\frac{1}{d} + \frac{1}{d^{m+1}} \right) \exp \left(-\frac{\pi d}{h} + \frac{d^2}{2r^2 h^2} \right) \\
& \quad \cdot \int_{-\infty}^{\infty} \exp \left[-\frac{(x-t)^2}{2r^2 h^2} \right] [A + B(|t| + d)^\alpha] \sum_{k=0}^m \sum_{l=0}^k \frac{1}{l!(m-k)!} \left(\frac{1}{\sqrt{2rh}} \right)^{m-k} \\
& \quad \cdot \left[\left| H_{m-k} \left(\frac{x-t-di}{\sqrt{2rh}} \right) \right| + \left| H_{m-k} \left(\frac{x-t+di}{\sqrt{2rh}} \right) \right| \right] dt \\
& \leq \frac{2(m+1)!}{\pi} \left(\frac{1}{d} + \frac{1}{d^{m+1}} \right) \exp \left(-\frac{\pi d}{h} + \frac{d^2}{2r^2 h^2} \right) \\
& \quad \cdot \int_{-\infty}^{\infty} \exp \left[-\frac{(x-t)^2}{2r^2 h^2} \right] [A + B(|t| + d)^\alpha] \sum_{k=0}^m \frac{1}{(m-k)!} \left(\frac{1}{\sqrt{2rh}} \right)^{m-k} \\
& \quad \cdot \left[\left| H_{m-k} \left(\frac{x-t-di}{\sqrt{2rh}} \right) \right| + \left| H_{m-k} \left(\frac{x-t+di}{\sqrt{2rh}} \right) \right| \right] dt. \tag{5.6}
\end{aligned}$$

With $\xi = (x-t)/(rh)$, the integral in (5.6) becomes

$$\begin{aligned}
& \int_{-\infty}^{\infty} \exp \left(-\frac{\xi^2}{2} \right) [A + B(|x - rh\xi| + d)^\alpha] \\
& \quad \cdot \sum_{j=0}^m \frac{1}{j!} \left(\frac{1}{\sqrt{2rh}} \right)^j \left[\left| H_j \left(\frac{rh\xi - di}{\sqrt{2rh}} \right) \right| + \left| H_j \left(\frac{rh\xi + di}{\sqrt{2rh}} \right) \right| \right] rh d\xi. \tag{5.7}
\end{aligned}$$

In (5.7), it follows from $-L \leq x \leq L$ that

$$\begin{aligned}
& (|x - rh\xi| + d)^\alpha \leq (|x| + rh|\xi| + d)^\alpha \\
& \leq [(L + d) + rh|\xi|]^\alpha \leq 2^\alpha [(L + d)^\alpha + (rh)^\alpha |\xi|^\alpha], \tag{5.8}
\end{aligned}$$

where the last inequality is due to Lemma 5.1. The Hermite polynomial in (5.7) is evaluated as follows. By (5.3), for $\zeta \in \mathbb{C}$ we have

$$|H_j(\zeta)| \leq 2^j j! \sum_{i=0}^{\lfloor \frac{j}{2} \rfloor} |\zeta|^{j-2i} \leq 2^j j! \frac{j+2}{2} \max\{1, |\zeta|^j\} \leq 2^{j-1} j!(j+2)(1 + |\zeta|^j),$$

and therefore

$$\begin{aligned}
\sum_{j=0}^m \frac{1}{j!} \left(\frac{1}{\sqrt{2}rh} \right)^j |H_j(\zeta)| &\leq \sum_{j=0}^m \left(\frac{\sqrt{2}}{rh} \right)^j \frac{(j+2)}{2} (1 + |\zeta|^j) \\
&\leq \frac{(m+2)}{2} \max \left\{ 1, \left(\frac{\sqrt{2}}{rh} \right)^m \right\} \sum_{j=0}^m (1 + |\zeta|^j) \\
&\leq \frac{(m+2)}{2} \left[1 + \left(\frac{\sqrt{2}}{rh} \right)^m \right] \sum_{j=0}^m (1 + |\zeta|^j) \\
&\leq \frac{(m+2)}{2} \left[1 + \left(\frac{\sqrt{2}}{rh} \right)^m \right] (m+1)(1 + \max\{1, |\zeta|^m\}) \\
&\leq \frac{(m+1)(m+2)}{2} \left[1 + \left(\frac{\sqrt{2}}{rh} \right)^m \right] (2 + |\zeta|^m).
\end{aligned}$$

Substituting $\zeta = (rh\xi \pm di)/\sqrt{2}rh$, we obtain

$$\begin{aligned}
&\sum_{j=0}^m \frac{1}{j!} \left(\frac{1}{\sqrt{2}rh} \right)^j \left| H_j \left(\frac{rh\xi \pm di}{\sqrt{2}rh} \right) \right| \\
&\leq \frac{(m+1)(m+2)}{2} \left[1 + \left(\frac{\sqrt{2}}{rh} \right)^m \right] \left(2 + \left| \frac{\xi}{\sqrt{2}} \pm \frac{di}{\sqrt{2}rh} \right|^m \right) \\
&\leq \frac{(m+1)(m+2)}{2} \left[1 + \left(\frac{\sqrt{2}}{rh} \right)^m \right] \left[2 + \left(\left| \frac{\xi}{\sqrt{2}} \right| + \left| \frac{d}{\sqrt{2}rh} \right| \right)^m \right] \\
&\leq \frac{(m+1)(m+2)}{2} \left[1 + \left(\frac{\sqrt{2}}{rh} \right)^m \right] \left[2 + 2^m \left(\left| \frac{\xi}{\sqrt{2}} \right|^m + \left| \frac{d}{\sqrt{2}rh} \right|^m \right) \right], \tag{5.9}
\end{aligned}$$

where the last inequality is due to Lemma 5.1. Note that this inequality holds good even when $m = 0$.

It follows from (5.6), (5.7), (5.8), and (5.9) that

$$\begin{aligned}
&\left| \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{U_n} \frac{d^m}{dx^m} \left(\frac{\sin(\pi x/h)}{z-x} \exp \left[-\frac{(x-z)^2}{2r^2h^2} \right] \right) \frac{f(z)}{\sin(\pi z/h)} dz \right| \\
&\leq \frac{2(m+1)!}{\pi} \left(\frac{1}{d} + \frac{1}{d^{m+1}} \right) \exp \left(-\frac{\pi d}{h} + \frac{d^2}{2r^2h^2} \right) rh(m+1)(m+2) \\
&\quad \cdot \left(1 + \left(\frac{\sqrt{2}}{rh} \right)^m \right) \int_{-\infty}^{\infty} \exp \left(-\frac{\xi^2}{2} \right) [A + 2^\alpha B[(L+d)^\alpha + (rh)^\alpha |\xi|^\alpha]] \\
&\quad \cdot \left[2 + 2^m \left(\left| \frac{\xi}{\sqrt{2}} \right|^m + \left| \frac{d}{\sqrt{2}rh} \right|^m \right) \right] d\xi \\
&\leq \exp \left(-\frac{\pi d}{h} + \frac{d^2}{2r^2h^2} \right) \cdot C_0 \left[C_1 C_3 \sqrt{2\pi} + C_2 C_3 2^{\frac{\alpha+1}{2}} \Gamma \left(\frac{\alpha+1}{2} \right) \right. \\
&\quad \left. + C_1 2^{\frac{2m+1}{2}} \Gamma \left(\frac{m+1}{2} \right) + C_2 2^{\frac{\alpha+2m+1}{2}} \Gamma \left(\frac{\alpha+m+1}{2} \right) \right],
\end{aligned}$$

where C_0, \dots, C_3 are defined in (2.3)–(2.6).

5.2 Proof of Lemma 2.2

We prepare the following lemma.

Lemma 5.2. *Let p be a nonnegative integer, N a positive integer, and r a positive number. If $N \geq \sqrt{pr}$, we have*

$$\sum_{n=N+1}^{\infty} n^p \exp\left(-\frac{n^2}{2r^2}\right) \leq \frac{(p+1)!! \max\{N^p, r^p\} r^2}{N} \exp\left(-\frac{N^2}{2r^2}\right).$$

Proof. Put $\eta(x) := x^p \exp[-x^2/(2r^2)]$. Since

$$\eta'(x) = \left(px^{p-1} - \frac{x^{p+1}}{r^2}\right) \exp\left(-\frac{x^2}{2r^2}\right),$$

$\eta(x)$ decreases at $x \geq \sqrt{pr}$. Since $N \geq \sqrt{pr}$ by the assumption, we have an estimate by an integral:

$$\sum_{n=N+1}^{\infty} n^p \exp\left(-\frac{n^2}{2r^2}\right) \leq \int_N^{\infty} x^p \exp\left(-\frac{x^2}{2r^2}\right) dx. \quad (5.10)$$

Denote by I_p the integral of the right hand side in (5.10). By integration by parts, we have

$$\begin{aligned} I_p &= -r^2 \int_N^{\infty} x^{p-1} \left(-\frac{x}{r^2}\right) \exp\left(-\frac{x^2}{2r^2}\right) dx = -r^2 \int_N^{\infty} x^{p-1} \frac{d}{dx} \exp\left(-\frac{x^2}{2r^2}\right) dx \\ &= -r^2 \left[-N^{p-1} \exp\left(-\frac{N^2}{2r^2}\right) - (p-1) \int_N^{\infty} x^{p-2} \exp\left(-\frac{x^2}{2r^2}\right) dx \right] \\ &= N^{p-1} r^2 \exp\left(-\frac{N^2}{2r^2}\right) + (p-1)r^2 I_{p-2}. \end{aligned}$$

Iterating the similar operation, we obtain

$$\begin{aligned} I_p &= \\ &= (p-1)!! \left[\sum_{i=0}^{k-1} \frac{N^{p-2i-1} r^{2i}}{(p-2i-1)!!} \right] r^2 \exp\left(-\frac{N^2}{2r^2}\right) + \frac{r^{2k}}{(p-2k-1)!!} I_{p-2k} \end{aligned} \quad (5.11)$$

for $k = 0, 1, \dots, \lfloor p/2 \rfloor$, where we set $(-1)!! = 0!! = 1$. In the case of $k = \lfloor p/2 \rfloor$, it holds that $I_{p-2k} = I_1$ or I_0 . These integrals are evaluated as follows:

$$I_1 = r^2 \exp\left(-\frac{N^2}{2r^2}\right), \quad (5.12)$$

$$\begin{aligned} I_0 &= \sqrt{2r} \int_{N/(\sqrt{2r})}^{\infty} \exp(-u^2) du \\ &\leq \sqrt{2r} \cdot \frac{\sqrt{2r}}{2N} \exp\left(-\frac{N^2}{2r^2}\right) = \frac{r^2}{N} \exp\left(-\frac{N^2}{2r^2}\right), \end{aligned} \quad (5.13)$$

where the latter is due to the general fact that

$$\int_t^{\infty} \exp(-u^2) du \leq \frac{\exp(-t^2)}{2t}$$

holds for $t > 0$. It follows from (5.11), (5.12), and (5.13) that

$$\begin{aligned}
I_p &\leq (p-1)!! \left[\sum_{i=0}^{\lfloor p/2 \rfloor} \frac{N^{p-2i-1} r^{2i}}{(p-2i-1)!!} \right] r^2 \exp\left(-\frac{N^2}{2r^2}\right) \\
&\leq N^{p-1} (p-1)!! \left(\sum_{i=0}^{\lfloor p/2 \rfloor} \frac{r^{2i}}{N^{2i}} \right) r^2 \exp\left(-\frac{N^2}{2r^2}\right) \\
&\leq N^{p-1} (p-1)!! \left(\sum_{i=0}^{\lfloor p/2 \rfloor} \max\left\{1, \frac{r^p}{N^p}\right\} \right) r^2 \exp\left(-\frac{N^2}{2r^2}\right) \\
&\leq N^{p-1} (p+1)!! \max\left\{1, \frac{r^p}{N^p}\right\} r^2 \exp\left(-\frac{N^2}{2r^2}\right) \\
&= \frac{(p+1)!! \max\{N^p, r^p\} r^2}{N} \exp\left(-\frac{N^2}{2r^2}\right).
\end{aligned}$$

With this expression and (5.10), we obtain the conclusion. \square

We now prove Lemma 2.2. In a similar manner to (5.2), we see

$$\begin{aligned}
&\frac{d^m}{dx^m} \left(\frac{\sin(\pi(x-kh)/h)}{\pi(x-kh)/h} \exp\left[-\frac{(x-kh)^2}{2r^2 h^2}\right] \right) \\
&= \sum_{k=0}^m \sum_{l=0}^k \frac{m!}{l!(m-k)!} \frac{(-1)^{m-l}}{(\sqrt{2}rh)^{m-k}} \left(\frac{\pi}{h}\right)^{l-1} \\
&\quad \cdot \frac{\sin(\pi(x-kh)/h + \pi l/2)}{(x-kh)^{k-l+1}} H_{m-k} \left(\frac{x-kh}{\sqrt{2}rh} \right) \exp\left[-\frac{(x-kh)^2}{2r^2 h^2}\right],
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left| \left(\mathcal{G}_h^{(m)} f \right) (x) - \left(\mathcal{T}_{N,h}^{(m)} f \right) (x) \right| \\
&\leq \sum_{\substack{k-\lceil x/h \rceil > N \text{ or} \\ k-\lfloor x/h \rfloor < -N}} |f(kh)| \sum_{k=0}^m \sum_{l=0}^k \frac{m!}{l!(m-k)!} \frac{1}{(\sqrt{2}rh)^{m-k}} \left(\frac{\pi}{h}\right)^{l-1} \\
&\quad \cdot \frac{1}{h^{k-l+1} |x/h - k|^{k-l+1}} \left| H_{m-k} \left(\frac{x/h - k}{\sqrt{2}r} \right) \right| \exp\left[-\frac{(x/h - k)^2}{2r^2}\right]. \tag{5.14}
\end{aligned}$$

We assume that $-L \leq x \leq L$.

We estimate the sum in (5.14) in the case of $k - \lceil x/h \rceil > N$. Note that

$$N < k - \left\lceil \frac{x}{h} \right\rceil \leq k - \frac{x}{h} \leq k - \left\lfloor \frac{x}{h} \right\rfloor + 1. \tag{5.15}$$

By (5.3), setting

$$a_i(t) := \frac{j!(2t)^{j-2i}}{i!(j-2i)!}$$

for $t \in \mathbb{R}$, we have that

$$H_j(t) = \sum_{i=0}^{\lfloor j/2 \rfloor} (-1)^i a_i(t),$$

and that $a_i(t)$ ($i = 0, 1, \dots, \lfloor j/2 \rfloor$) have the same sign. Furthermore, if $|t| \geq j/2$, we have $|a_0(t)| \geq \dots \geq |a_{\lfloor j/2 \rfloor}(t)|$, and therefore $|H_j(t)| \leq |a_0(t)|$.

Since $k - x/h > N \geq mr/\sqrt{2}$ holds by the assumption, setting $t = (k - x/h)/(\sqrt{2}r)$ in the above discussion, we have

$$\left| H_j \left(\frac{x/h - k}{\sqrt{2}r} \right) \right| \leq \left(\frac{\sqrt{2}|x/h - k|}{r} \right)^j = \left[\frac{\sqrt{2}(k - x/h)}{r} \right]^j \quad (5.16)$$

for $j = 0, 1, \dots, m$.

By (5.15) and (5.16), we have

$$\begin{aligned} & \sum_{k - \lfloor x/h \rfloor > N} |f(kh)| \sum_{k=0}^m \sum_{l=0}^k \frac{m!}{l!(m-k)!} \frac{1}{(\sqrt{2}rh)^{m-k}} \left(\frac{\pi}{h} \right)^{l-1} \\ & \cdot \frac{1}{h^{k-l+1} |x/h - k|^{k-l+1}} \left| H_{m-k} \left(\frac{x/h - k}{\sqrt{2}r} \right) \right| \exp \left[-\frac{(x/h - k)^2}{2r^2} \right] \\ & \leq \frac{1}{Nh^m} \sum_{k - \lfloor x/h \rfloor > N} \left[A + B \left[\left(k - \left\lfloor \frac{x}{h} \right\rfloor \right) h + (L + h) \right]^\alpha \right] \\ & \cdot \sum_{k=0}^m \sum_{l=0}^k \frac{m! \pi^{l-1}}{l!(m-k)!} \left(\frac{k - \lfloor x/h \rfloor + 1}{r^2} \right)^{m-k} \exp \left[-\frac{(k - \lfloor x/h \rfloor)^2}{2r^2} \right] \\ & \leq \frac{1}{Nh^m} \sum_{n=N+1}^{\infty} [A + B [nh + (L + h)]^\alpha] \\ & \cdot \sum_{k=0}^m \sum_{l=0}^k \frac{m! \pi^{l-1}}{l!(m-k)!} \left(\frac{n+1}{r^2} \right)^{m-k} \exp \left(-\frac{n^2}{2r^2} \right). \end{aligned} \quad (5.17)$$

Furthermore, noting that

$$\begin{aligned} [nh + (L + h)]^\alpha & \leq 2^\alpha [(nh)^\alpha + (L + h)^\alpha] \\ & \leq 2^\alpha [(L + h)^\alpha + 2^\alpha h^\alpha] + 2^{2\alpha} h^\alpha (n - 1)^\alpha, \\ \frac{\pi^{l-1}}{l!} & = \frac{\pi^l}{\pi l!} \leq \frac{e^\pi}{\pi}, \\ \sum_{k=0}^m \frac{1}{(m-k)!} \left(\frac{n+1}{r^2} \right)^{m-k} & \leq \exp \left(\frac{n+1}{r^2} \right), \end{aligned}$$

where Lemma 5.1 is used, we have

$$\begin{aligned}
& \frac{1}{Nh^m} \sum_{n=N+1}^{\infty} [A + B[nh + (L + h)]^\alpha] \sum_{k=0}^m \sum_{l=0}^k \frac{m! \pi^{l-1}}{l!(m-k)!} \left(\frac{n+1}{r^2}\right)^{m-k} \exp\left(-\frac{n^2}{2r^2}\right) \\
& \leq \frac{(m+1)! e^\pi}{Nh^m \pi} \sum_{n=N+1}^{\infty} [A + 2^\alpha B[(L+h)^\alpha + 2^\alpha h^\alpha] + 2^{2\alpha} B h^\alpha (n-1)^\alpha] \\
& \quad \cdot \exp\left(\frac{n+1}{r^2}\right) \exp\left(-\frac{n^2}{2r^2}\right) \\
& \leq \frac{(m+1)! e^\pi e^{\frac{3}{2r^2}}}{Nh^m \pi} \sum_{n=N+1}^{\infty} [A + 2^\alpha B[(L+h)^\alpha + 2^\alpha h^\alpha] + 2^{2\alpha} B h^\alpha (n-1)^\alpha] \\
& \quad \cdot \exp\left[-\frac{(n-1)^2}{2r^2}\right]. \tag{5.18}
\end{aligned}$$

Applying Lemma 5.2 (by the assumption $N \geq \sqrt{\lceil \alpha \rceil} r + 1$, we have only to replace N with $N - 1$ and set $p = 0, \lceil \alpha \rceil$), we have

$$\begin{aligned}
& \frac{(m+1)! e^\pi e^{\frac{3}{2r^2}}}{Nh^m \pi} \sum_{n=N+1}^{\infty} [A + 2^\alpha B[(L+h)^\alpha + 2^\alpha h^\alpha] + 2^{2\alpha} B h^\alpha (n-1)^\alpha] \\
& \quad \cdot \exp\left[-\frac{(n-1)^2}{2r^2}\right] \\
& \leq \frac{(m+1)! e^\pi e^{\frac{3}{2r^2}} r^2}{N(N-1)h^m \pi} (C'_1 + C'_2) \exp\left[-\frac{(N-1)^2}{2r^2}\right], \tag{5.19}
\end{aligned}$$

where C'_1 and C'_2 are defined in (2.8) and (2.9), respectively.

Noting that the sum in (5.14) in the case of $k - \lfloor x/h \rfloor < -N$ can be estimated in a similar manner, we see from (5.14), (5.17), (5.18), and (5.19) that

$$\left| \left(\mathcal{G}_h^{(m)} f\right)(x) - \left(\mathcal{T}_{N,h}^{(m)} f\right)(x) \right| \leq C'_0 (C'_1 + C'_2) \exp\left[-\frac{(N-1)^2}{2r^2}\right],$$

where C'_0 is defined in (2.7). Thus we have proven Lemma 2.2.

Acknowledgement

This work is supported by the 21st Century COE Program on Information Science and Technology Strategic Core and a Grant-in-Aid of the Ministry of Education, Culture, Sports, Science and Technology of Japan. The first author is supported by the Research Fellowship of the Japan Society for the Promotion of Science for Young Scientists.

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