MATHEMATICAL ENGINEERING TECHNICAL REPORTS

Hierarchical orbital decompositions and extended decomposable distributions

Hidehiko KAMIYA and Akimichi TAKEMURA

METR 2006–38 June 2006

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.i.u-tokyo.ac.jp/mi/mi-e.htm

Hierarchical orbital decompositions and extended decomposable distributions

Hidehiko Kamiya
Okayama University
and
Akimichi Takemura
University of Tokyo

June 2006

Abstract

Elliptically contoured distributions can be considered to be the distributions for which the contours of the density functions are proportional ellipsoids. Kamiya, Takemura and Kuriki (2006) generalized the elliptically contoured distributions to star-shaped distributions, for which the contours are allowed to be arbitrary proportional star-shaped sets. This was achieved by considering the so-called orbital decomposition of the sample space in the general framework of group invariance. In the present paper, we extend their results by conducting the orbital decompositions in steps and obtaining a further, hierarchical decomposition of the sample space. This allows us to construct probability models and distributions with further independence structures. The general results are applied to the star-shaped distributions with a certain symmetric structure, the distributions related to the two-sample Wishart problem and the distributions of preference rankings.

Key words: action, decomposable distribution, elliptically contoured distribution, global cross section, Haar measure, isotropy subgroup, orbital decomposition, ranking, star-shaped distribution.

1 Introduction

Elliptically contoured distributions are defined to be the distributions for which the contours of the density functions are proportional ellipsoids. As a natural generalization of the multivariate normal distributions, they are widely used as a distributional assumption (Kelker (1970), Cambanis, Huang and Simons (1981), Fang and Anderson (1990)). Fang and Zhang (1990) discuss "generalized multivariate analysis" based on elliptically contoured distribuitons. In the meantime, from another perspective, elliptically contoured distributions can be obtained from spherical distributions by affine transformation. Extending the l_2-norm in spherical distributions to the l_q-norm, $q > 0$, Osiewalski and Steel (1993) introduced l_q-spherical distributions.

Generalizing elliptically contoured distributions and l_q-spherical distributions, Kamiya, Takemura and Kuriki (2006) defined the so-called star-shaped distributions, for which the contours of the density functions are proportional to (the boundaries of) arbitrary star-shaped sets $\tilde{\mathcal{Z}}$, called cross sections (see also Fernández, Osiewalski and Steel (1995) and Ferreira and Steel (2005)). They showed that basic facts about the independence of the "length" and "direction" continue to hold for star-shaped distributions. However, when the star-shaped set has a symmetric structure, we can make a more detailed investigation into this distribution.

In the star-shaped distribution, the cross section $\tilde{\mathcal{Z}} \subset \mathbb{R}^p - \{\mathbf{0}\}$ is allowed to be an arbitrary star-shaped set—a set which intersects each ray emanating from the origin exactly once, and the density is assumed to be constant on each proportional star-shaped set $g\tilde{\mathcal{Z}} = \{g\tilde{\boldsymbol{z}} : \tilde{\boldsymbol{z}} \in \tilde{\mathcal{Z}}\}$, $g > 0$. However, there are some cases where we have some symmetry; as in the case of elliptically contoured distributions, we might be able to assume that $\tilde{\mathcal{Z}}$ is symmetric about the origin: $\tilde{\mathcal{Z}} = -\tilde{\mathcal{Z}} = \{-\tilde{\boldsymbol{z}} : \tilde{\boldsymbol{z}} \in \tilde{\mathcal{Z}}\}$. In those cases, $\tilde{\mathcal{Z}}$ may be obtained as $\tilde{\mathcal{Z}} = \{\pm 1\}\mathcal{Z} = \{\pm \boldsymbol{z} : \boldsymbol{z} \in \mathcal{Z}\} = \mathcal{Z} \cup (-\mathcal{Z})$ in terms of a set $\mathcal{Z} \subset \mathbb{R}^p - \{\mathbf{0}\}$ which intersects each line through the origin exactly once. As long as this condition is satisfied, $\mathcal{Z}$ is allowed to be an arbitrary set. Now, suppose $\boldsymbol{x} \in \mathbb{R}^p$ is distributed according to a star-shaped distribution with respect to such a symmetric $\tilde{\mathcal{Z}} = \mathcal{Z} \cup (-\mathcal{Z})$. Then the density of this distribution is constant on each $g\tilde{\mathcal{Z}}$, $g > 0$, and the distribution of $-\boldsymbol{x}$ is the same as that of $\boldsymbol{x}$.

In the above situation, we cannot deal with the skewness of the distributions. However, we can go further and consider those distributions whose densities are constant on each $g\mathcal{Z}$, $g \neq 0$, but not necessarily constant on each $g\tilde{\mathcal{Z}} = g\mathcal{Z} \cup (-g\mathcal{Z})$, $g > 0$, where $\tilde{\mathcal{Z}} = \mathcal{Z} \cup (-\mathcal{Z})$. That is, the value of the density on $g\mathcal{Z}$ can differ from the value on $-g\mathcal{Z}$ for $g > 0$. In such a case, distributions of $\boldsymbol{x}$ and $-\boldsymbol{x}$ are not the same.

These types of distributions can be studied by decomposing $\boldsymbol{x}$ $(\neq \mathbf{0})$ uniquely as $\boldsymbol{x} = \epsilon h \boldsymbol{z}$, $\epsilon = \pm 1$, $h > 0$, $\boldsymbol{z} \in \mathcal{Z}$, with respect to $\mathcal{Z}$. As with Kamiya, Takemura and Kuriki (2006), these problems can be treated as special cases of a general discussion in terms of abstract group invariance, and we choose to do so in this paper. This approach enables us to apply the obtained general results to the distributions of random matrices and moreover discrete distributions.

In the general framework of group invariance, the present problem corresponds to the decomposition of the sample space under the action of an invariance group, called the orbital decomposition (Wijsman (1990)). Conducting this decomposition twice, we obtain a hierarchical decomposition into three parts. When a group $\mathcal{G}$ acts on a sample space $\mathcal{X}$, any subgroup $\mathcal{H}$ of $\mathcal{G}$ acts on each $\mathcal{G}$-orbit. By choosing an appropriate $\mathcal{H}$ and performing a hierarchical orbital decomposition, we can construct probability models with the corresponding hierarchical independence structures. This allows us to propose in our general framework new probability models for various statistical problems, as illustrated with modeling of preference rankings in Section 4.3.

The organization of this paper is as follows. In Section 2, we summarize some fundamental facts about group actions and orbital decompositions, and review the results about the decomposable distributions studied in Kamiya, Takemura and Kuriki (2006). In Section 3 we introduce a further, hierarchical decomposition by means of a subgroup action and define extended decomposable distributions. We establish various facts on hierarchical orbital decompositions and derive distributional properties of extended decomposable distributions. In the final section, we apply the general results to the star-shaped distributions (Section 4.1), the two-sample Wishart problem (Section 4.2) and the distributions of preference rankings (Section 4.3).

2 Orbital decomposition and decomposable distribution

In this section we summarize some fundamental facts about group actions and orbital decompositions, and review the results about the decomposable distributions. For group invariance in statistics, the reader is referred to Eaton (1989), Barndorff-Nielsen, Blæsild and Eriksen (1989) and Wijsman (1990). For global cross sections and orbital decompositions in particular, see Wijsman (1967, 1986), Koehn (1970), Bondar (1976) and Kamiya (1996).

2.1 Orbital decomposition

Let a group $\mathcal{G}$ act on a space $\mathcal{X}$ (typically the sample space) from the left $(g,x) \mapsto gx : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$. We write the action of $\mathcal{G}$ on $\mathcal{X}$ as $(\mathcal{G}, \mathcal{X})$.

Let $\mathcal{G}x = \{gx : g \in \mathcal{G}\}$ be the *orbit* containing $x \in \mathcal{X}$, and let $\mathcal{X}/\mathcal{G} = \{\mathcal{G}x : x \in \mathcal{X}\}$ be the *orbit space*, i.e., the set of all orbits. When $\mathcal{X}$ consists of a single orbit $\mathcal{X} = \mathcal{G}x$, the action is said to be *transitive*.

Indicate by $\mathcal{G}_x = \{g \in \mathcal{G} : gx = x\}$ the *isotropy subgroup* at $x \in \mathcal{X}$. When $\mathcal{G}_x = \{e\}$ for all $x \in \mathcal{X}$, the action is said to be *free,* where e denotes the identity element of $\mathcal{G}$. In general, the isotropy subgroups at two points on a common orbit are conjugate to each other: $\mathcal{G}_{gx} = g\mathcal{G}_x g^{-1}$, $g \in \mathcal{G}$, $x \in \mathcal{X}$.

The set of *left cosets* $g\mathcal{G}_x = \{gg' : g' \in \mathcal{G}_x\}$, $g \in \mathcal{G}$, is called the *left coset space* of $\mathcal{G}$ modulo $\mathcal{G}_x$, and is denoted by $\mathcal{G}/\mathcal{G}_x = \{g\mathcal{G}_x : g \in \mathcal{G}\}$. The *canonical map* $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{G}_x$ is defined by $\pi(g) = g\mathcal{G}_x$, $g \in \mathcal{G}$. The group $\mathcal{G}$ or more generally its subgroup $\mathcal{H} < \mathcal{G}$ acts on $\mathcal{G}/\mathcal{G}_x$ by

(1) $$(h, g\mathcal{G}_x) \mapsto (hg)\mathcal{G}_x, \quad h \in \mathcal{H},\ g \in \mathcal{G}.$$

This action is not transitive unless $\mathcal{H}$ includes a complete set of representatives of $g\mathcal{G}_x$, $g \in \mathcal{G}$, i.e., $\mathcal{G} = \bigcup_{h\in\mathcal{H}} h\mathcal{G}_x$.

A subset $\mathcal{Z} \subset \mathcal{X}$ is said to be a *cross section* if $\mathcal{Z}$ intersects each orbit $\mathcal{G}x$, $x \in \mathcal{X}$, exactly once. So any cross section $\mathcal{Z}$ is in one-to-one correspondence with the orbit space $\mathcal{X}/\mathcal{G}$ by $z \leftrightarrow \mathcal{G}z$, $z \in \mathcal{Z}$. A cross section $\mathcal{Z}$ having a common isotropy subgroup is called a *global cross section*: $\mathcal{G}_z = \mathcal{G}_0$, say, for all $z \in \mathcal{Z}$. Unlike a mere cross section, a global cross section does not always exist. A global cross section exists if and only if the isotropy subgroups $\mathcal{G}_x$, $x \in \mathcal{X}$, are all conjugate to one another.

Unless otherwise stated, however, we assume from now on that there does exist a global cross section $\mathcal{Z}$. Then, we have the following one-to-one correspondence:

(2) $$\begin{aligned} \mathcal{X} &\leftrightarrow \mathcal{Y} \times \mathcal{Z}, \\ x &\leftrightarrow (y,z), \quad x = gz, \quad y = \pi(g), \quad g \in \mathcal{G}, \end{aligned}$$

where $\mathcal{Y} = \mathcal{G}/\mathcal{G}_0$ with $\mathcal{G}_0 = \mathcal{G}_z,\ z \in \mathcal{Z}$. Decomposition (2) is called the *orbital decomposition* of $\mathcal{X}$ (or x) with respect to $\mathcal{Z}$. In (2) we can think of y and z as functions $y = y(x)$ and $z = z(x)$ of x. Under the action of $\mathcal{G}$ on $\mathcal{X}$, $y(x)$ is equivariant and $z(x)$ is invariant: $y(gx) = gy(x),\ z(gx) = z(x),\quad g \in \mathcal{G},\ x \in \mathcal{X}$.

We move on to reviewing some properties of global cross sections obtained in Kamiya, Takemura and Kuriki (2006).

Let $\mathcal{Z}$ be a global cross section. Then $g\mathcal{Z} = \{gz : z \in \mathcal{Z}\}$ for each $g \in \mathcal{G}$ is again a global cross section. We say $g\mathcal{Z},\ g \in \mathcal{G}$, are *proportional to* $\mathcal{Z}$, and call $\{g\mathcal{Z} : g \in \mathcal{G}\}$ the *family of proportional global cross sections.* In $\mathcal{X} = \bigcup_{g\in\mathcal{G}} g\mathcal{Z}$, it holds that $g_1\mathcal{Z} \cap g_2\mathcal{Z} \neq \emptyset$ for $g_1, g_2 \in \mathcal{G}$ implies $g_1\mathcal{Z} = g_2\mathcal{Z}$, so the family of proportional global cross sections gives a partition of $\mathcal{X}$.

From a given global cross section $\mathcal{Z}$, we can construct a general cross section $\mathcal{Z}'$ by changing the points of $\mathcal{Z}$ within their orbits. For $\mathcal{Z}'$ to be global, i.e., for the isotropy subgroups to be the same on the whole of $\mathcal{Z}'$, these changes of the points have to be made subject to some restriction as follows. Let $\mathcal{N} = \{g \in \mathcal{G} : g\mathcal{G}_0 g^{-1} = \mathcal{G}_0\}$ be the *normalizer* of the common isotropy subgroup $\mathcal{G}_0$ on $\mathcal{Z}$. Then a subset $\mathcal{Z}' \subset \mathcal{X}$ is a global cross section if and only if it can be written as

$$\mathcal{Z}' = \{g_0 n_z z : z \in \mathcal{Z}\} \tag{3}$$

for some $g_0 \in \mathcal{G}$ and $n_z \in \mathcal{N},\ z \in \mathcal{Z}$.

Under the change from $\mathcal{Z}$ to $\mathcal{Z}'$ in (3), the equivariant part transforms as follows. Let $x \leftrightarrow (y, z)$ be the orbital decomposition with respect to $\mathcal{Z}$, and let $x \leftrightarrow (y', z')$ be the orbital decomposition with respect to the $\mathcal{Z}'$ in (3). Then we have

$$y' = y n_z^{-1} g_0^{-1}. \tag{4}$$

2.2 Decomposable distribution

In this subsection, we review the decomposable distributions defined in Kamiya, Takemura and Kuriki (2006).

Throughout the rest of the paper, we make the following assumptions: (a) $\mathcal{X}$ is a locally compact Hausdorff space; (b) $\mathcal{G}$ is a second countable, locally compact Hausdorff topological group acting continuously on $\mathcal{X}$; (c) $\mathcal{G}_0$ is compact; and (d) $\mathcal{Z}$ is locally compact and the bijection $x \leftrightarrow (y, z)$ with respect to $\mathcal{Z}$ is bimeasurable.

We consider distributions on $\mathcal{X}$ which have densities $f(x)$ with respect to a dominating measure λ. Measure λ is assumed to be relatively invariant with multiplier χ : $\lambda(d(gx)) = \chi(g)\lambda(dx),\ g \in \mathcal{G}$. Then, we say a distribution $f(x)\lambda(dx)$ is a *decomposable distribution* with respect to $\mathcal{Z}$ if it is of the form $f(x)\lambda(dx) = f_{\mathcal{Y}}(y(x))f_{\mathcal{Z}}(z(x))\lambda(dx)$. In particular, we say it is *cross-sectionally contoured* if $f_{\mathcal{Z}}(z)$ is constant, and *orbitally contoured* if $f_{\mathcal{Y}}(y)$ is constant. We mainly study cross-sectionally contoured distributions because a decomposable distribution $f_{\mathcal{Y}}(y(x))f_{\mathcal{Z}}(z(x))\lambda(dx)$ can always be thought of as a cross-sectionally contoured distribution with density $f_{\mathcal{Y}}(y(x))$ with respect to $\tilde{\lambda}(dx) := f_{\mathcal{Z}}(z(x))\lambda(dx)$ Obviously, a distribution $f(x)\lambda(dx)$ is cross-sectionally contoured with respect to $\mathcal{Z}$ if and only if $f(x)$ is constant on each proportional global cross section $g\mathcal{Z},\ g \in \mathcal{G}$.

Topological assumption (b) about $\mathcal{G}$ implies that there exists a left Haar measure $\mu_{\mathcal{G}}$ on $\mathcal{G}$, which is unique up to a multiplicative constant. By the compactness of $\mathcal{G}_0$ assumed in (c), we have the induced measure $\mu_{\mathcal{Y}} = \pi(\mu_{\mathcal{G}}) = \mu_{\mathcal{G}}\pi^{-1}$ on $\mathcal{Y}$ (Proposition 2.3.5 and Corollary 7.4.4 of Wijsman (1990)). Again by the same assumption (c), we can define $\bar{\chi}(y)$, $y \in \mathcal{Y}$, by $\bar{\chi}(y) = \chi(g)$ with $g \in \pi^{-1}(\{y\})$. By abuse of notation, we will write $\chi(y)$ for $\bar{\chi}(y)$.

In terms of these, $\lambda(dx)$ is factored as

$$\lambda(dx) = \chi(y)\mu_{\mathcal{Y}}(dy)\nu_{\mathcal{Z}}(dz) \tag{5}$$

(Theorem 7.5.1 of Wijsman (1990), Theorem 10.1.2 of Farrell (1985)). Here, we are identifying $\mathcal{X}$ with $\mathcal{Y} \times \mathcal{Z}$. Existence of a density $f(x)$ with respect to λ implies $\nu_{\mathcal{Z}}$ is a finite measure, so from now on we assume that $\nu_{\mathcal{Z}}(dz)$ is standardized to be a probability measure on $\mathcal{Z}$. From (5) we immediately obtain the following result.

Proposition 2.1. (Kamiya, Takemura and Kuriki (2006)) *Suppose that x is distributed according to a cross-sectionally contoured distribution $f_{\mathcal{Y}}(y(x))\lambda(dx)$. Then we have:*

1. *$y = y(x)$ and $z = z(x)$ are independently distributed.*
2. *The distribution of y is $f_{\mathcal{Y}}(y)\chi(y)\mu_{\mathcal{Y}}(dy)$.*
3. *The distribution of z does not depend on $f_{\mathcal{Y}}$.*

Note that since $\nu_{\mathcal{Z}}(dz)$ is taken to be a probability measure, we do not need a normalizing constant in $f_{\mathcal{Y}}(y)\chi(y)\mu_{\mathcal{Y}}(dy)$. (To put it another way, the version of $\mu_{\mathcal{Y}}$ is taken in this way.)

3 Hierarchical orbital decomposition and extended decomposable distribution

In this section we introduce a further, hierarchical decomposition and define extended decomposable distributions.

3.1 Hierarchical orbital decomposition

In this subsection, we give a further factorization of the $\mathcal{G}$-orbital decomposition. This is obtained by decomposing the equivariant part $\mathcal{G}/\mathcal{G}_0$ by means of the action of a subgroup of $\mathcal{G}$.

We continue to assume that there exists a global cross section $\mathcal{Z}$ with the common isotropy subgroup $\mathcal{G}_0$. Furthermore, let $\mathcal{H}$ be a subgroup of $\mathcal{G}$.

As in Section 2.1, we have the decomposition

$$\mathcal{X} \leftrightarrow \mathcal{G}/\mathcal{G}_0 \times \mathcal{Z}. \tag{6}$$

Now, $\mathcal{H}$ acts on $\mathcal{G}/\mathcal{G}_0$ by (1) with $\mathcal{G}_x = \mathcal{G}_0$. Note that instead of this action we may equivalently consider the action of $\mathcal{H}$ on $\mathcal{G}z_0 : (h, gz_0) \mapsto (hg)z_0$, $z_0 \in \mathcal{Z}$. In particular, we have $\mathcal{H}_{g\mathcal{G}_0} = \mathcal{H}_{gz_0}$, $g \in \mathcal{G}$, from which we obtain the following lemma:

Lemma 3.1. *Action* $(\mathcal{H},\mathcal{X})$ *is free if and only if action* $(\mathcal{H},\mathcal{G}/\mathcal{G}_0)$ *is free.*

Now suppose a global cross section $\mathcal{V}\subset\mathcal{G}/\mathcal{G}_0$ exists for action $(\mathcal{H},\mathcal{G}/\mathcal{G}_0)$. The existence of a global cross section $\mathcal{V}$ leads to a further decomposition of (6) as follows.

Denote the common isotropy subgroup at the points of $\mathcal{V}$ by $\mathcal{H}_0$. Then $\mathcal{G}/\mathcal{G}_0$ is decomposed as

$$\mathcal{G}/\mathcal{G}_0 \leftrightarrow \mathcal{H}/\mathcal{H}_0\times\mathcal{V}. \tag{7}$$

We can take $\mathcal{V}$ in such a way that $\mathcal{G}_0\in\mathcal{V}$; in that case, we can write $\mathcal{H}_0$ as

$$\mathcal{H}_0=\mathcal{H}_{\mathcal{G}_0}=\{h\in\mathcal{H}: h\mathcal{G}_0=\mathcal{G}_0\}=\mathcal{H}\cap\mathcal{G}_0.$$

From now on, we always take $\mathcal{V}$ in this way.

Combining (6) and (7), we have the decomposition

$$\mathcal{X}\leftrightarrow\mathcal{H}/\mathcal{H}_0\times\mathcal{V}\times\mathcal{Z}. \tag{8}$$

Our questions are:

(i) specifying the condition for $\mathcal{V}$ to exist, and

(ii) expressing $\mathcal{V}$ in a concrete form.

Note that the orbits under $(\mathcal{H},\mathcal{G}/\mathcal{G}_0)$ are of the form

$$\{hg\mathcal{G}_0: h\in\mathcal{H}\}\subset\mathcal{G}/\mathcal{G}_0,\quad g\in\mathcal{G}.$$

This suggests that the above questions are closely related to the properties of the double cosets

$$\begin{aligned}\mathcal{H}g\mathcal{G}_0 &= \{hgg_0: h\in\mathcal{H},\ g_0\in\mathcal{G}_0\}\\ &= \pi^{-1}\left(\{hg\mathcal{G}_0: h\in\mathcal{H}\}\right)\subset\mathcal{G},\quad g\in\mathcal{G},\end{aligned}$$

in $\mathcal{G}$. The following lemma indicates this fact.

Lemma 3.2. *Let* $\mathcal{G}'\subset\mathcal{G}$. *Then* $\mathcal{V}=\{g'\mathcal{G}_0: g'\in\mathcal{G}'\}\subset\mathcal{G}/\mathcal{G}_0$ *is a cross section for the action of* $\mathcal{H}$ *on* $\mathcal{G}/\mathcal{G}_0$ *such that* $g'\mathcal{G}_0\neq g''\mathcal{G}_0$ *for* $g'\neq g''$, $g',g''\in\mathcal{G}'$, *if and only if* $\mathcal{G}'$ *is a complete set of representatives of the double cosets* $\mathcal{H}g\mathcal{G}_0$, $g\in\mathcal{G}$, *in* $\mathcal{G}$:

$$\mathcal{G}=\bigsqcup_{g'\in\mathcal{G}'}\mathcal{H}g'\mathcal{G}_0\quad\text{(disjoint union)}.$$

Proof. *Necessity*: Suppose $\mathcal{V}=\{g'\mathcal{G}_0: g'\in\mathcal{G}'\}$ is a cross section for action $(\mathcal{H},\mathcal{G}/\mathcal{G}_0)$ and that $g'\mathcal{G}_0=g''\mathcal{G}_0$ for $g',g''\in\mathcal{G}'$ implies $g'=g''$. We want to prove $\mathcal{G}=\bigsqcup_{g'\in\mathcal{G}'}\mathcal{H}g'\mathcal{G}_0$. It suffices to verify (a) $\mathcal{G}\subset\bigcup_{g'\in\mathcal{G}'}\mathcal{H}g'\mathcal{G}_0$; and (b) $\mathcal{H}g'\mathcal{G}_0=\mathcal{H}g''\mathcal{G}_0$ for $g',g''\in\mathcal{G}'$ implies $g'=g''$. First, (a) is shown as follows. Let g be an arbitrary element of $\mathcal{G}$. Then, since $\mathcal{V}$ intersects the orbit containing $g\mathcal{G}_0\in\mathcal{G}/\mathcal{G}_0$ at least once, there exist $h\in\mathcal{H}$ and $g'\in\mathcal{G}'$ such that $g\mathcal{G}_0=hg'\mathcal{G}_0$. Thus g can be written as $g=hg'g_0$ with some $g_0\in\mathcal{G}_0$. Therefore, $g\in\mathcal{H}g'\mathcal{G}_0\subset\bigcup_{g''\in\mathcal{G}'}\mathcal{H}g''\mathcal{G}_0$. Next, (b) is proved as follows. Suppose $\mathcal{H}g'\mathcal{G}_0 = \mathcal{H}g''\mathcal{G}_0$ for $g', g'' \in \mathcal{G}'$. Then $g' = hg''g_0$ for some $h \in \mathcal{H}$ and $g_0 \in \mathcal{G}_0$, and thus we have $g'\mathcal{G}_0 = hg''\mathcal{G}_0$. Now, since $\mathcal{V}$ intersects the orbit containing $g''\mathcal{G}_0$ at most once, we obtain $g'\mathcal{G}_0 = g''\mathcal{G}_0$. Therefore, we get $g' = g''$ by our assumption.

Sufficiency: Suppose $\mathcal{G} = \bigsqcup_{g' \in \mathcal{G}'} \mathcal{H}g'\mathcal{G}_0$. We want to show (a) $\mathcal{V} = \{g'\mathcal{G}_0 : g' \in \mathcal{G}'\}$ intersects each orbit under $(\mathcal{H}, \mathcal{G}/\mathcal{G}_0)$ at least once; (b) $\mathcal{V}$ intersects each orbit at most once; and (c) $g'\mathcal{G}_0 = g''\mathcal{G}_0$ for $g', g'' \in \mathcal{G}'$ implies $g' = g''$. We begin by showing (a). Let $g\mathcal{G}_0 \in \mathcal{G}/\mathcal{G}_0$, $g \in \mathcal{G}$, be arbitrarily given. Pick any $g_1 \in g\mathcal{G}_0 \subset \mathcal{G} = \bigcup_{g' \in \mathcal{G}'} \mathcal{H}g'\mathcal{G}_0$. Then g_1 can be written as $g_1 = hg'g_0$ for some $h \in \mathcal{H}$, $g' \in \mathcal{G}'$ and $g_0 \in \mathcal{G}_0$. Hence $g\mathcal{G}_0 = g_1\mathcal{G}_0 = hg'\mathcal{G}_0$. This shows that the orbit containing $g\mathcal{G}_0$ intersects $\mathcal{V}$ at least once. Next we verify (b). Suppose $g'\mathcal{G}_0 = hg''\mathcal{G}_0$ for $g', g'' \in \mathcal{G}'$ and $h \in \mathcal{H}$. Then $\mathcal{H}g'\mathcal{G}_0 = \mathcal{H}g''\mathcal{G}_0$, which implies $g' = g''$ since $\mathcal{G} = \bigsqcup_{g' \in \mathcal{G}'} \mathcal{H}g'\mathcal{G}_0$ is a disjoint union. Hence we have $g'\mathcal{G}_0 = g''\mathcal{G}_0$. This observation shows (b). Finally, (c) can be verified similarly as (b). ∎

Now, concerning the existence of $\mathcal{V}$, we state the following theorem.

Theorem 3.1. *Suppose that there exists a global cross section $\mathcal{Z}$ for the action of $\mathcal{G}$ on $\mathcal{X}$, with the common isotropy subgroup denoted by $\mathcal{G}_0$. Let $\mathcal{G}'$ be a complete set of representatives of the double cosets $\mathcal{H}g\mathcal{G}_0$, $g \in \mathcal{G}$, in $\mathcal{G}$. Then a global cross section $\mathcal{V}$ exists for the action of $\mathcal{H}$ on $\mathcal{G}/\mathcal{G}_0$ if and only if*

$$\mathcal{H} \cap g'\mathcal{G}_0 g'^{-1}, \quad g' \in \mathcal{G}',$$

are all conjugate to one another in $\mathcal{H}$.

Proof. First note that a global cross section $\mathcal{V}$ exists for action $(\mathcal{H}, \mathcal{G}/\mathcal{G}_0)$ if and only if the isotropy subgroups $\mathcal{H}_{g\mathcal{G}_0}$, $g \in \mathcal{G}$, are all conjugate in $\mathcal{H}$.

Every $g \in \mathcal{G} = \bigsqcup_{g' \in \mathcal{G}'} \mathcal{H}g'\mathcal{G}_0$ can be written in the form $g = hg'g_0$, $h \in \mathcal{H}$, $g' \in \mathcal{G}'$, $g_0 \in \mathcal{G}_0$, and thus we have $\mathcal{H}_{g\mathcal{G}_0} = \mathcal{H}_{hg'g_0\mathcal{G}_0} = h\mathcal{H}_{g'\mathcal{G}_0}h^{-1}$. Therefore, $\mathcal{V}$ exists if and only if $\mathcal{H}_{g'\mathcal{G}_0}$, $g' \in \mathcal{G}'$, are all conjugate in $\mathcal{H}$. Here we can write $\mathcal{H}_{g'\mathcal{G}_0}$ as $\mathcal{H}_{g'\mathcal{G}_0} = \{h \in \mathcal{H} : hg'\mathcal{G}_0 = g'\mathcal{G}_0\} = \{h \in \mathcal{H} : g'^{-1}hg' \in \mathcal{G}_0\} = \mathcal{H} \cap g'\mathcal{G}_0 g'^{-1}$. ∎

Remark 3.1. *The condition that all $\mathcal{H} \cap g'\mathcal{G}_0 g'^{-1}$, $g' \in \mathcal{G}'$, be conjugate does not depend on the choice of a complete set $\mathcal{G}'$.*

Let us now move on to the second problem—expressing $\mathcal{V}$ in a concrete form. The following theorem gives a useful explicit expression of $\mathcal{V}$. We omit the proof because it is a simple consequence of Lemma 3.2.

Theorem 3.2. *Suppose that there exists a global cross section $\mathcal{Z}$ for the action of $\mathcal{G}$ on $\mathcal{X}$, with the common isotropy subgroup $\mathcal{G}_0$. Suppose furthermore that there exists a complete set $\mathcal{G}' = \{g_i : i \in I\} \subset \mathcal{G}$ of representatives of the double cosets $\mathcal{H}g\mathcal{G}_0$, $g \in \mathcal{G}$, in $\mathcal{G}$ such that $\mathcal{H}_{g_i\mathcal{G}_0}$ does not depend on $i \in I$. Then*

$$\mathcal{V} = \{g_i\mathcal{G}_0 : i \in I\}$$

is a global cross section for the action of $\mathcal{H}$ on $\mathcal{G}/\mathcal{G}_0$.

Remark 3.2. *When the action of* $\mathcal{H}$ *on* $\mathcal{G}/\mathcal{G}_0$ *is free, any complete set of representatives of* $\mathcal{H}g\mathcal{G}_0$*,* $g \in \mathcal{G}$*, in* $\mathcal{G}$ *satisfies the condition of* $\mathcal{G}'$ *in Theorem 3.2:* $\mathcal{H}_{g_i\mathcal{G}_0} = \{e\}$ *for all* $i \in I$*.*

Under the assumption of Theorem 3.2, $\mathcal{X}$ is decomposed as (8). We now prove that $\mathcal{V} \times \mathcal{Z}$ in (8) is a global cross section for action $(\mathcal{H}, \mathcal{X})$.

Theorem 3.3. *Suppose that there exists a global cross section* $\mathcal{Z}$ *for the action of* $\mathcal{G}$ *on* $\mathcal{X}$*, with the common isotropy subgroup* $\mathcal{G}_0$*. Suppose moreover that there exists a complete set* $\mathcal{G}' = \{g_i : i \in I\}$ *of representatives of the double cosets* $\mathcal{H}g\mathcal{G}_0$*,* $g \in \mathcal{G}$*, in* $\mathcal{G}$ *satisfying the condition of Theorem 3.2, and let* $\mathcal{V} = \{g_i\mathcal{G}_0 : i \in I\}$*. Then* $\mathcal{V} \times \mathcal{Z}$ *is in one-to-one correspondence with*

$$\tilde{\mathcal{Z}} := \mathcal{G}'\mathcal{Z} = \{g_i z : \ i \in I, \ z \in \mathcal{Z}\}, \tag{9}$$

and $\tilde{\mathcal{Z}}$ *is a global cross section for the action of* $\mathcal{H}$ *on* $\mathcal{X}$*.*

Proof. It is easy to see that the correspondence $(g_i\mathcal{G}_0, z) \leftrightarrow g_i z$ between $\mathcal{V} \times \mathcal{Z}$ and $\mathcal{G}'\mathcal{Z}$ is a bijection. We show below that $\mathcal{G}'\mathcal{Z}$ is a global cross section for action $(\mathcal{H}, \mathcal{X})$.

Let x be an arbitrary element of $\mathcal{X}$. Then x can be written as $x = gz$, $g \in \mathcal{G}$, $z \in \mathcal{Z}$. Furthermore, this g can be written as $g = hg_ig_0$ for some $h \in \mathcal{H}$, $i \in I$ and $g_0 \in \mathcal{G}_0$. Hence, $x = hg_ig_0z = hg_iz$ and so $\mathcal{H}x = \mathcal{H}g_iz \ni g_iz$. This implies that $\mathcal{G}'\mathcal{Z}$ intersects each orbit $\mathcal{H}x$, $x \in \mathcal{X}$, under $(\mathcal{H}, \mathcal{X})$ at least once.

Next we show that $\mathcal{G}'\mathcal{Z}$ intersects each $\mathcal{H}x$, $x \in \mathcal{X}$, at most once. Suppose that $hg_iz \in \mathcal{G}'\mathcal{Z}$ for $h \in \mathcal{H}$, $i \in I$ and $z \in \mathcal{Z}$. Then there exist $i' \in I$ and $z' \in \mathcal{Z}$ such that $hg_iz = g_{i'}z'$. Since $\mathcal{Z}$ is a cross section for action $(\mathcal{G}, \mathcal{X})$, we have $z = z'$ and $hg_i\mathcal{G}_0 = g_{i'}\mathcal{G}_0$, which implies $\mathcal{H}g_i\mathcal{G}_0 = \mathcal{H}g_{i'}\mathcal{G}_0$ and thus $i = i'$. Therefore, we obtain $hg_iz = g_iz$. This observation shows that $\mathcal{G}'\mathcal{Z}$ intersects each $\mathcal{H}x$, $x \in \mathcal{X}$, at most once.

It remains to be proved that the isotropy subgroups $\mathcal{H}_{g_iz}$ at the points $g_iz \in \mathcal{G}'\mathcal{Z}$ are all common. But this is obvious because $\mathcal{H}_{g_iz} = \mathcal{H}_{g_i\mathcal{G}_0}$ does not depend on $i \in I$ by the assumption of Theorem 3.2. ∎

We call a global cross section $\tilde{\mathcal{Z}}$ for action $(\mathcal{H}, \mathcal{X})$ of the form (9) a *decomposable global cross section.* Of course, a general global cross section for $(\mathcal{H}, \mathcal{X})$ is not necessarily decomposable.

Let us delve into Theorems 3.2 and 3.3 in two specific cases.

First, consider the case where the action of $\mathcal{G}$ on $\mathcal{X}$ is free. In that case, we want to decompose $\mathcal{G}$ by considering the action of $\mathcal{H}$ on $\mathcal{G} : (h, g) \mapsto hg$, $h \in \mathcal{H}$, $g \in \mathcal{G}$.

Corollary 3.1. *The action of* $\mathcal{H}$ *on* $\mathcal{G}$ *is free, and any complete set* $\{g_i : i \in I\}$ *of representatives of the right cosets* $\mathcal{H}g$*,* $g \in \mathcal{G}$*, in* $\mathcal{G}$ *is a cross section for this action.*

Proof. It is trivial to see that action $(\mathcal{H}, \mathcal{G})$ is free. The rest is obvious from Theorem 3.2 with $\mathcal{G}_0 = \{e\}$. ∎

The latter statement of the corollary is also apparent from the fact that $\mathcal{H}g$, $g \in \mathcal{G}$, are the orbits under $(\mathcal{H}, \mathcal{G})$ and $\mathcal{G} = \bigsqcup_{i \in I} \mathcal{H}g_i$.

In the case of Corollary 3.1, $\mathcal{G}$ is decomposed as

$$\begin{aligned}\mathcal{G} &\leftrightarrow \mathcal{H} \times \{g_i : i \in I\}\\ &\leftrightarrow \mathcal{H} \times \mathcal{H}\backslash\mathcal{G},\end{aligned}$$

where $\mathcal{H}\backslash\mathcal{G}$ is the right coset space

$$\mathcal{H}\backslash\mathcal{G} := \{\mathcal{H}g : g \in \mathcal{G}\} = \{\mathcal{H}g_i : i \in I\}.$$

Corollary 3.2. *Suppose that the action of $\mathcal{G}$ on $\mathcal{X}$ is free, and let $\mathcal{Z}$ be a cross section for this action. Then the action of $\mathcal{H}$ on $\mathcal{X}$ is free, and for any complete set $\mathcal{G}' = \{g_i : i \in I\}$ of representatives of the right cosets $\mathcal{H}g,\ g \in \mathcal{G},$ in $\mathcal{G},$ the set $\tilde{\mathcal{Z}} = \mathcal{G}'\mathcal{Z} = \{g_i z : i \in I,\ z \in \mathcal{Z}\}$ is a cross section for this action.*

Proof. It is evident that the action of $\mathcal{H}$ on $\mathcal{X}$ is free. The rest follows immediately from Theorem 3.3 with Lemma 3.1 and Remark 3.2. ∎

As an example, consider the actions related to the star-shaped distributions—the actions of $\mathcal{G} = \mathbb{R}^*_\times$ (the multiplicative group of nonzero real numbers) and $\mathcal{H} = \mathbb{R}^*_+$ (the multiplicative group of positive real numbers) on $\mathcal{X} = \mathbb{R}^p - \{\mathbf{0}\}$ by scalar multiplication.

In that case, $\mathcal{G}$ acts on $\mathcal{X}$ freely. Moreover, $\{\pm 1\}$ is a complete set of representatives of $\mathcal{H}g,\ g \in \mathcal{G},$ and is thus a cross section for action $(\mathcal{H}, \mathcal{G})$ by Corollary 3.1. Accordingly, we have a one-to-one correspondence

$$\mathbb{R}^*_\times \leftrightarrow \mathbb{R}^*_+ \times \{\pm 1\}.$$

Furthermore, we have by Corollary 3.2 that $\tilde{\mathcal{Z}} = \mathcal{G}'\mathcal{Z} = \mathcal{Z} \cup (-\mathcal{Z})$ with $\mathcal{G}' = \{\pm 1\}$ is a cross section for the action of $\mathcal{H} = \mathbb{R}^*_+$ on $\mathcal{X} = \mathbb{R}^p - \{\mathbf{0}\}$. Let us take $\mathcal{Z}$ as

$$\mathcal{Z} = \left\{(x_1, \ldots, x_p)^t \in \mathbb{S}^{p-1} : x_p > 0\right\} \cup \left\{(x_1, \ldots, x_{p-1}, 0)^t \in \mathbb{S}^{p-1} : (x_1, \ldots, x_{p-1})^t \in \mathcal{Z}_{-1}\right\},$$

where $\mathbb{S}^{p-1}$ denotes the $(p-1)$-dimensional unit sphere and $\mathcal{Z}_{-1}$ is a cross section for the action of $\mathcal{G} = \mathbb{R}^*_\times$ on $\mathbb{R}^{p-1} - \{\mathbf{0}\},\ \mathbf{0} \in \mathbb{R}^{p-1}$. Since it is clear that $\mathcal{G}'\mathcal{Z} = \mathbb{S}^{p-1}$ is true for $p = 2,$ we see by induction on p that for all $p,$

$$\begin{aligned}\tilde{\mathcal{Z}} &= \mathcal{G}'\mathcal{Z}\\ &= \left\{(x_1, \ldots, x_p)^t \in \mathbb{S}^{p-1} : x_p \neq 0\right\} \cup \left\{(x_1, \ldots, x_{p-1}, 0)^t \in \mathbb{S}^{p-1}\right\}\\ &= \mathbb{S}^{p-1},\end{aligned}$$

which is clearly a cross section for the action of $\mathcal{H} = \mathbb{R}^*_+$ on $\mathcal{X} = \mathbb{R}^p - \{\mathbf{0}\}$. In fact, we can take any $\mathcal{Z} \subset \mathcal{X} = \mathbb{R}^p - \{\mathbf{0}\}$ which intersects each line through the origin in exactly one point.

Next we treat the case which covers the two-sample Wishart problem.

Corollary 3.3. *Let $\mathcal{G}_0$ be a subgroup of $\mathcal{G}$. Suppose that there exists a subgroup $\mathcal{K}$ of $\mathcal{G}$ satisfying the following conditions:*

(i) *Every* $g \in \mathcal{G}$ *can be written uniquely in the form* $g = hk,\ h \in \mathcal{H},\ k \in \mathcal{K}$.

(ii) $\mathcal{G}_0$ *is a subgroup of* $\mathcal{K}$.

Then, the action of $\mathcal{H}$ *on* $\mathcal{G}/\mathcal{G}_0$ *is free, and*

$$\mathcal{V} = \mathcal{K}/\mathcal{G}_0 = \{k\mathcal{G}_0 : k \in \mathcal{K}\}$$

is a cross section for this action.

Proof. Noting that $\mathcal{H} \cap \mathcal{K} = \{e\}$ by assumption (i) and that $k\mathcal{G}_0 k^{-1} \subset \mathcal{K},\ k \in \mathcal{K}$, by assumption (ii), we have $\mathcal{H} \cap k\mathcal{G}_0 k^{-1} = \{e\}$ for any $k \in \mathcal{K}$. Therefore, $\mathcal{H}_{g\mathcal{G}_0} = \mathcal{H} \cap g\mathcal{G}_0 g^{-1}$ is trivial for any g in $\mathcal{K}$ and thus for any g in $\mathcal{G}$:

$$\mathcal{H}_{g\mathcal{G}_0} = \mathcal{H}_{hk\mathcal{G}_0} = h\mathcal{H}_{k\mathcal{G}_0}h^{-1} = h\{e\}h^{-1} = \{e\}, \quad g = hk,\ h \in \mathcal{H},\ k \in \mathcal{K}.$$

Hence, action $(\mathcal{H}, \mathcal{G}/\mathcal{G}_0)$ is free.

Let $\mathcal{G}' \subset \mathcal{K}$ be a complete set of representatives of the left cosets $k\mathcal{G}_0,\ k \in \mathcal{K}$, in $\mathcal{K}$: $\mathcal{K} = \bigsqcup_{g' \in \mathcal{G}'} g'\mathcal{G}_0$. Then $\mathcal{K}/\mathcal{G}_0 = \{g'\mathcal{G}_0 : g' \in \mathcal{G}'\}$, and by Theorem 3.2 it suffices to show that $\mathcal{G}'$ is a complete set of representatives of the double cosets $\mathcal{H}g\mathcal{G}_0,\ g \in \mathcal{G}$, in $\mathcal{G}$: $\mathcal{G} = \bigsqcup_{g' \in \mathcal{G}'} \mathcal{H}g'\mathcal{G}_0$. Since $\bigcup_{g' \in \mathcal{G}'} \mathcal{H}g'\mathcal{G}_0 = \mathcal{H}(\bigcup_{g' \in \mathcal{G}'} g'\mathcal{G}_0) = \mathcal{H}\mathcal{K} = \mathcal{G}$, it remains to show that

$$\mathcal{H}g'\mathcal{G}_0 = \mathcal{H}g''\mathcal{G}_0, \quad g', g'' \in \mathcal{G}', \tag{10}$$

implies $g' = g''$. Suppose that (10) holds. Then, there exist $h \in \mathcal{H}$ and $g_0 \in \mathcal{G}_0$ such that $g' = hg''g_0$. By assumptions (i) and (ii), we have $g' = g''g_0$ and thus $g'\mathcal{G}_0 = g''\mathcal{G}_0$. The definition of $\mathcal{G}'$ implies $g' = g''$. ∎

Remark 3.3. *As was seen in the proof of Corollary 3.3, if* $\mathcal{G}' \subset \mathcal{K}$ *is a complete set of representatives of the left cosets* $k\mathcal{G}_0,\ k \in \mathcal{K}$, *in* $\mathcal{K}$, *then* $\mathcal{G}'$ *is also a complete set of representatives of the double cosets* $\mathcal{H}g\mathcal{G}_0,\ g \in \mathcal{G}$, *in* $\mathcal{G}$. *As can be shown in a similar manner, if we assume* $\mathcal{G}' \subset \mathcal{K}$, *the converse is true as well and the two conditions are in fact equivalent.*

In the case of Corollary 3.3, $\mathcal{G}/\mathcal{G}_0$ is decomposed as

$$\mathcal{G}/\mathcal{G}_0 \leftrightarrow \mathcal{H} \times \mathcal{K}/\mathcal{G}_0.$$

Corollary 3.4. *Suppose that there exists a global cross section* $\mathcal{Z}$ *for the action of* $\mathcal{G}$ *on* $\mathcal{X}$, *with the common isotropy subgroup denoted by* $\mathcal{G}_0$. *Suppose moreover that there exists a subgroup* $\mathcal{K}$ *satisfying conditions* (i) *and* (ii) *in Corollary 3.3. Then, the action of* $\mathcal{H}$ *on* $\mathcal{X}$ *is free, and* $\tilde{\mathcal{Z}} = \mathcal{K}\mathcal{Z}$ *is a cross section for this action.*

Proof. By Lemma 3.1 and Corollary 3.3, action $(\mathcal{H}, \mathcal{X})$ is free. We show below that $\mathcal{K}\mathcal{Z}$ is a cross section for $(\mathcal{H}, \mathcal{X})$.

Let $\mathcal{G}' \subset \mathcal{K}$ be some complete set of representatives of the left cosets $k\mathcal{G}_0,\ k \in \mathcal{K}$, in $\mathcal{K}$. Then by Remark 3.3, $\mathcal{G}'$ is a complete set of representatives of the double cosets

$\mathcal{H}g\mathcal{G}_0,\ g \in \mathcal{G},$ in $\mathcal{G}$ as well. Theorem 3.3 implies that $\tilde{\mathcal{Z}} = \mathcal{G}'\mathcal{Z}$ is a cross section for $(\mathcal{H}, \mathcal{X})$. Thus, the proof will be finished if we verify that $\mathcal{G}'\mathcal{Z} = \mathcal{K}\mathcal{Z}$. But this follows from $\mathcal{K} = \bigsqcup_{g' \in \mathcal{G}'} g'\mathcal{G}_0 = \mathcal{G}'\mathcal{G}_0$. ∎

As an example, consider the situation related to the two-sample Wishart problem.

Let $\mathcal{G} = GL(p)$ (the general linear group) and

$$\begin{aligned}(11) \qquad \mathcal{X} = \{(W_1, W_2) \in PD(p) \times PD(p) :& \\ \text{the } p \text{ roots of } \det(W_1 - \lambda(W_1 + W_2)) = 0 \text{ are all distinct}\}.&\end{aligned}$$

The action is $(B,\ (W_1, W_2)) \mapsto (BW_1B^t,\ BW_2B^t),\ B \in GL(p)$. Let us take

$$\mathcal{Z} = \{(\Lambda, I_p - \Lambda) : \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_p),\ 1 > \lambda_1 > \cdots > \lambda_p > 0\}, \tag{12}$$

where I_p denotes the $p \times p$ identity matrix. Then we have

$$\mathcal{G}_0 = \{\operatorname{diag}(\epsilon_1, \ldots, \epsilon_p) : \epsilon_1 = \pm 1, \ldots, \epsilon_p = \pm 1\}. \tag{13}$$

Now, as a subgroup of $\mathcal{G}$ consider $\mathcal{H} = LT(p)$, the group of $p \times p$ lower triangular matrices with positive diagonal elements. Then the orthogonal group $O(p)$ can serve as the $\mathcal{K}$ in Corollary 3.3. With this $\mathcal{K}$,

$$\mathcal{V} = \mathcal{K}/\mathcal{G}_0 = \{C\mathcal{G}_0 : C \in O(p)\}$$

is the set of $p \times p$ orthogonal matrices with the sign of each column ignored.

Corollary 3.4 implies that the action of $\mathcal{H} = LT(p)$ on $\mathcal{X}$ is free and that $\tilde{\mathcal{Z}} = \mathcal{K}\mathcal{Z}$ with $\mathcal{K} = O(p)$ is a cross section for $(\mathcal{H}, \mathcal{X})$. We can write $\tilde{\mathcal{Z}}$ as

$$\begin{aligned}\tilde{\mathcal{Z}} &= \mathcal{K}\mathcal{Z} \\ &= \left\{(C\Lambda C^t,\ I_p - C\Lambda C^t) : C \in O(p),\ \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_p),\ 1 > \lambda_1 > \cdots > \lambda_p > 0\right\} \\ &= \{(U,\ I_p - U) : O < U < I_p, \text{ and the eigenvalues of } U \text{ are all distinct}\},\end{aligned}$$

where O denotes the null matrix and $A < B$ means that $B - A$ is positive definite for symmetric matrices A and B.

3.2 Extended decomposable distribution

In this subsection, we discuss the distributional aspect of the hierarchical decompositions in the preceding subsection.

Suppose that there exists a subgroup $\mathcal{L}$ of $\mathcal{G}$ of the form

$$\mathcal{L} = \mathcal{G}'\mathcal{G}_0, \tag{14}$$

where $\mathcal{G}' = \{g_i : i \in I\}$ is a complete set of representatives of the double cosets $\mathcal{H}g\mathcal{G}_0,\ g \in \mathcal{G},$ in $\mathcal{G}$ such that $\mathcal{H}_{g_i\mathcal{G}_0}$ does not depend on $i \in I$.

Then we have

$$\mathcal{G} = \mathcal{H}\mathcal{G}'\mathcal{G}_0 = \mathcal{H}\mathcal{L}.$$

Therefore, every $g \in \mathcal{G}$ can be written in the form $g = hl$ (or $g = hl^{-1}$), $h \in \mathcal{H}$, $l \in \mathcal{L}$. Moreover, by considering the transitive action $((h,l),g) \mapsto hgl^{-1}$, $h \in \mathcal{H}$, $l \in \mathcal{L}$, $g \in \mathcal{G}$, of the product group $\mathcal{H} \times \mathcal{L}$ on $\mathcal{G}$, we have a bijection

$$\mathcal{G} \leftrightarrow (\mathcal{H} \times \mathcal{L})/\mathcal{F}^*,$$

where $\mathcal{F}^* = \{(g,g) : g \in \mathcal{F} = \mathcal{H} \cap \mathcal{L}\}$ is the isotropy subgroup at $e \in \mathcal{G}$.

Before proceeding further, let us see the two specific cases considered in the preceding subsection.

First, consider the situation in Corollaries 3.1 and 3.2. Suppose that the action of $\mathcal{G}$ on $\mathcal{X}$ is free and that a complete set $\mathcal{G}'$ of representatives of the right cosets $\mathcal{H}g$, $g \in \mathcal{G}$, forms a subgroup of $\mathcal{G}$. Then $\mathcal{G}'$ can serve as $\mathcal{L}$. For instance, consider the example immediately after Corollary 3.2—the actions related to the star-shaped distributions. Then $\mathcal{G}' = \{\pm 1\}$ forms a subgroup of $\mathcal{G} = \mathbb{R}^*_\times$ and thus can serve as $\mathcal{L}$.

Next, consider the situation in Corollaries 3.3 and 3.4. Suppose that a subgroup $\mathcal{K}$ of $\mathcal{G}$ satisfies conditions (i) and (ii) of Corollary 3.3, and let $\mathcal{G}' \subset \mathcal{K}$ be a complete set of representatives of the left cosets $k\mathcal{G}_0$, $k \in \mathcal{K}$, in $\mathcal{K}$: $\mathcal{K} = \bigsqcup_{g' \in \mathcal{G}'} g'\mathcal{G}_0$. Then we have $\mathcal{K} = \mathcal{G}'\mathcal{G}_0$, and $\mathcal{G}'$ is a complete set of representatives of the double cosets $\mathcal{H}g\mathcal{G}_0$, $g \in \mathcal{G}$, in $\mathcal{G}$ as well (Remark 3.3). Thus $\mathcal{K}$ can serve as $\mathcal{L}$. For instance, consider the example immediately after Corollary 3.4—the actions related to the two-sample Wishart problem. Then $O(p)$ can serve as $\mathcal{L}$.

Now we have by Theorem 3.2 and (14) that

$$\mathcal{V} = \{g_i\mathcal{G}_0 : i \in I\} = \{l\mathcal{G}_0 : l \in \mathcal{L}\}$$

is a global cross section for action $(\mathcal{H}, \mathcal{G}/\mathcal{G}_0)$, and thus we obtain the decomposition

$$\begin{aligned} \mathcal{X} &\leftrightarrow \mathcal{U} \times \mathcal{V} \times \mathcal{Z}, \qquad \mathcal{U} = \mathcal{H}/\mathcal{H}_0, \\ x &\leftrightarrow (u,v,z), \quad x = hlz, \quad u = h\mathcal{H}_0, \ v = l\mathcal{G}_0, \end{aligned} \tag{15}$$

where $\mathcal{H}_0 = \mathcal{H} \cap \mathcal{G}_0$, since $\mathcal{L}$ contains $e \in \mathcal{G}$ and thus $\mathcal{G}_0 \in \mathcal{V}$. When $\mathcal{G}'$ is taken in such a way that $e \in \mathcal{G}'$, we have $\mathcal{G}_0 < \mathcal{G}'\mathcal{G}_0 = \mathcal{L}$ and thus $\mathcal{V} = \mathcal{L}/\mathcal{G}_0$ is the left coset space. We can always take $\mathcal{G}'$ in this way, and we decide to do so.

Concerning topological questions, we make the following assumptions in addition to (a) through (d) at the beginning of Section 2.2.

Assumption 3.1.

1. *$\mathcal{H}$ and $\mathcal{L}$ are closed subgroups of $\mathcal{G}$.*
2. *$\mathcal{F}$ is compact.*

Note that under our assumptions, $\mathcal{H}_0 = \mathcal{H} \cap \mathcal{G}_0$ is compact, since $\mathcal{G}_0$ is compact and $\mathcal{H} \cap \mathcal{G}_0$ is closed in the relative topology of $\mathcal{G}_0$. Note also that the one-to-one correspondence $\mathcal{G} \leftrightarrow (\mathcal{H} \times \mathcal{L})/\mathcal{F}^*$ is a homeomorphism since $\mathcal{G}$ is second countable (p.92 of Wijsman (1990)).

As before, let λ be a relatively invariant measure on $\mathcal{X}$ under the action of $\mathcal{G}$ with multiplier χ. Now we define the *extended decomposable distributions* as follows.

Definition 3.1. *A distribution on* $\mathcal{X}$ *is said to be an extended decomposable distribution with respect to a pair of global cross sections* $(\mathcal{Z}, \mathcal{V})$ *if it is of the form*

$$f(x)\lambda(dx) = f_{\mathcal{U}}(u(x))f_{\mathcal{V}}(v(x))f_{\mathcal{Z}}(z(x))\lambda(dx).$$

The following theorem gives the distributions of u, v and z when x is distributed according to an extended decomposable distribution.

Theorem 3.4. *Suppose that* x *is distributed according to an extended decomposable distribution* $f_{\mathcal{U}}(u(x))f_{\mathcal{V}}(v(x))f_{\mathcal{Z}}(z(x))\lambda(dx)$*. Then* $u = u(x) = h\mathcal{H}_0$, $v = v(x) = l\mathcal{G}_0$ *and* $z = z(x)$ $(x = hlz)$ *are independently distributed with the joint distribution*

$$\begin{aligned}&f_{\mathcal{U}}(u)\chi(u)\mu_{\mathcal{U}}(du)\\&\quad\times f_{\mathcal{V}}(v)\chi(v)\Delta^{\mathcal{G}}(v)\Delta^{\mathcal{L}}(v)^{-1}\mu_{\mathcal{V}}(dv)\\&\quad\times f_{\mathcal{Z}}(z)\nu_{\mathcal{Z}}(dz),\end{aligned}$$

where $\Delta^{\mathcal{G}}$ *(resp.* $\Delta^{\mathcal{L}}$*) is the right-hand modulus of* $\mathcal{G}$ *(resp.* $\mathcal{L}$*), measure* $\mu_{\mathcal{U}}$ *(resp.* $\mu_{\mathcal{V}}$*) is a version of the invariant measures on* $\mathcal{U} = \mathcal{H}/\mathcal{H}_0$ *(resp.* $\mathcal{V} = \mathcal{L}/\mathcal{G}_0$*), and* $\nu_{\mathcal{Z}}$ *is the probability measure in* (5)*.*

This theorem can be proved by Proposition 7.6.1 and (7.6.5) of Wijsman (1990).

4 Examples

In this section, we apply Theorem 3.4 to the star-shaped distributions, the two-sample Wishart problem and the distributions of rankings.

4.1 Star-shaped distributions with symmetry

Consider the actions related to the star-shaped distributions—the actions of $\mathcal{G} = \mathbb{R}^*_\times$ and $\mathcal{H} = \mathbb{R}^*_+$ on $\mathcal{X} = \mathbb{R}^p - \{\mathbf{0}\}$.

Then we have $\mathcal{G}_0 = \mathcal{H}_0 = \{1\}$. If we take $\mathcal{L} = \mathcal{G}' = \{\pm 1\}$, we obtain the bijection

$$\begin{aligned}\mathcal{X} &\leftrightarrow \mathcal{H} \times \mathcal{L} \times \mathcal{Z},\\ \boldsymbol{x} &\leftrightarrow (h,\ \epsilon,\ \boldsymbol{z}), \quad \boldsymbol{x} = \epsilon h \boldsymbol{z},\end{aligned}$$

where $\mathcal{Z}$ is a cross section for action $(\mathcal{G}, \mathcal{X})$. Furthermore, $\mathcal{G} = \mathbb{R}^*_\times$ and $\mathcal{L} = \{\pm 1\}$ are unimodular: $\Delta^{\mathcal{G}} = 1$, $\Delta^{\mathcal{L}} = 1$.

First, suppose that $\boldsymbol{x}$ is distributed according to a star-shaped distribution with respect to decomposable cross section $\tilde{\mathcal{Z}} = \mathcal{G}'\mathcal{Z} = \mathcal{L}\mathcal{Z} = \{\epsilon \boldsymbol{z} : \epsilon = \pm 1,\ \boldsymbol{z} \in \mathcal{Z}\} = \mathcal{Z} \cup (-\mathcal{Z})$:

$$f(h(\boldsymbol{x}))d\boldsymbol{x}.$$

This distribution can be regarded as the extended decomposable distribution with

$$\begin{aligned}f_{\mathcal{U}}(h) &= f_{\mathcal{H}}(h) = f(h),\\ f_{\mathcal{V}}(\epsilon) &= f_{\mathcal{L}}(\epsilon) \equiv 1,\\ f_{\mathcal{Z}}(\boldsymbol{z}) &\equiv 1,\end{aligned}$$

and

$$\lambda(d\boldsymbol{x}) = d\boldsymbol{x}.$$

Dominating measure λ is relatively invariant under the action of $\mathcal{G}$ with multiplier

$$\chi(g) = |g|^p, \quad g \in \mathcal{G},$$

where $|\cdot|$ denotes the absolute value. Therefore, we have by Theorem 3.4 that h, ϵ and $\boldsymbol{z}$ are independently distributed according to

$$\frac{1}{c_0} f(h) h^p h^{-1} dh = \frac{1}{c_0} f(h) h^{p-1} dh, \quad \mu_{\mathcal{V}} = \mu_{\mathcal{L}} \quad \text{and} \quad \nu_{\mathcal{Z}}, \tag{16}$$

respectively, where $c_0 = \int_0^\infty f(h) h^{p-1} dh$ and $\mu_{\mathcal{L}}(\{1\}) = \mu_{\mathcal{L}}(\{-1\}) = 1/2$. We can see that the distributions of $\boldsymbol{x}$ and $-\boldsymbol{x}$ are the same by $-\boldsymbol{x} \leftrightarrow (h, -\epsilon, \boldsymbol{z})$ for $\boldsymbol{x} \leftrightarrow (h, \epsilon, \boldsymbol{z})$. Under the additional assumption that $h(\boldsymbol{x})$ is piecewise of class C^1, we have

$$\nu_{\mathcal{Z}}(d\boldsymbol{z}) = 2c_0 \langle \boldsymbol{z}, \boldsymbol{n}_{\boldsymbol{z}} \rangle d\boldsymbol{z},$$

where $\boldsymbol{n}_{\boldsymbol{z}}$ is the outward unit normal vector of $\mathcal{Z}$, $d\boldsymbol{z}$ on the right-hand side is the volume element of $\mathcal{Z}$ and $\langle \cdot , \cdot \rangle$ denotes the standard inner product (Section 4 of Kamiya, Takemura and Kuriki (2006)).

Next, by taking nonconstant $f_{\mathcal{L}}(\epsilon)$ instead, we can make an asymmetric distribution of $\boldsymbol{x}$ as follows. Suppose

$$\boldsymbol{x} \sim c(\epsilon(\boldsymbol{x})) f(h(\boldsymbol{x})) d\boldsymbol{x}, \tag{17}$$

where

$$c(\epsilon) := \begin{cases} c & \text{if } \epsilon = 1, \\ 2 - c & \text{if } \epsilon = -1 \end{cases} \tag{18}$$

with $0 \le c \le 2$. Then again Theorem 3.4 implies that h, ϵ and $\boldsymbol{z}$ are independently distributed as (16), but this time with $\mu_{\mathcal{V}} = \mu_{\mathcal{L}}$ replaced by $\tilde{\mu}_{\mathcal{L}}(\{1\}) = c/2$, $\tilde{\mu}_{\mathcal{L}}(\{-1\}) = 1 - (c/2)$. In this case, the distribution of $-\boldsymbol{x}$ is different from that of $\boldsymbol{x}$ unless $c = 1$.

Note that we can make a more general skewed distribution by considering a cross-sectionally contoured distribution $\tilde{f}(\epsilon(\boldsymbol{x}) h(\boldsymbol{x})) d\boldsymbol{x}$ with respect to $\mathcal{Z}$. But in that case, we have to specify the function $\tilde{f}$ defined on the whole of $\mathbb{R}_\times = (-\infty, 0) \cup (0, \infty)$. By contrast, in the case of (17) we have only to specify f defined on $\mathbb{R}_+ = (0, \infty)$ and one value c in (18).

4.2 Two-sample Wishart problem

Consider the action of $\mathcal{G} = GL(p)$ on the sample space $\mathcal{X}$ in (11). The cross section $\mathcal{Z}$ is taken as (12) with the isotropy subgroup $\mathcal{G}_0$ in (13).

We continue to take $\mathcal{H} = LT(p)$ and $\mathcal{L} = \mathcal{K} = O(p)$. Then we obtain the bijection

$$\begin{aligned} \mathcal{X} \quad &\leftrightarrow \quad \mathcal{H} \times \mathcal{L}/\mathcal{G}_0 \times \mathcal{Z}, \\ (W_1, W_2) \quad &\leftrightarrow \quad (T,\ C\mathcal{G}_0,\ (\Lambda,\ I_p - \Lambda)), \quad (W_1, W_2) = \left(TC\Lambda C^t T^t,\ TC(I_p - \Lambda) C^t T^t \right). \end{aligned}$$

Furthermore, $\mathcal{G} = GL(p)$ and $\mathcal{L} = O(p)$ are unimodular: $\Delta^{\mathcal{G}} = 1$, $\Delta^{\mathcal{L}} = 1$.

Suppose that the random matrices W_1 and W_2 are independently distributed according to $W_p(n_1, \Sigma)$ and $W_p(n_2, \Sigma)$, respectively. Then the distribution of (W_1, W_2) can be regarded as the extended decomposable distribution with

$$f_{\mathcal{U}}(T) = f_{\mathcal{H}}(T) \propto \operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}TT^{t}\right), \quad f_{\mathcal{V}}(C\mathcal{G}_0) = f_{\mathcal{L}/\mathcal{G}_0}(C\mathcal{G}_0) \equiv 1, \quad f_{\mathcal{Z}}\left((\Lambda,\ I_p - \Lambda)\right) \equiv 1$$

and

$$\lambda(d(W_1, W_2)) = (\det W_1)^{a - \frac{p+1}{2}} (\det W_2)^{b - \frac{p+1}{2}} dW_1 dW_2, \tag{19}$$

where $a = n_1/2$, $b = n_2/2$, $W_1 = (w_{1,ij})$, $W_2 = (w_{2,ij})$, $dW_1 = \prod_{i \geq j} dw_{1,ij}$, $dW_2 = \prod_{i \geq j} dw_{2,ij}$. The dominating measure $\lambda(d(W_1, W_2))$ is relatively invariant with multiplier

$$\chi(B) = (\det B)^{2(a+b)} = (\det B)^{n_1 + n_2}, \quad B \in GL(p),$$

(Wijsman (1990), (9.1.4)). It follows from Theorem 3.4 that T, $C\mathcal{G}_0$ and Λ are independently distributed. The distributions of these parts are given in standard textbooks of multivariate statistical theory (see Anderson (2003) or Muirhead (1982), for example). In particular, $C\mathcal{G}_0$ is distributed according to the invariant probability measure on $\mathcal{V} = \mathcal{L}/\mathcal{G}_0 = O(p)/\mathcal{G}_0$ induced by the Haar measure on $O(p)$.

A nonstandard distribution is given as follows. Since the normalizer of $\mathcal{G}_0$ is

$$\mathcal{N} = \{P \in GL(p) : P \text{ has exactly one nonzero element in each row and in each column}\},$$

we know from (3) that a general global cross section $\mathcal{Z}'$ is of the form

$$\mathcal{Z}' = \left\{ \left(BP(\Lambda)\Lambda P(\Lambda)^{t} B^{t},\ BP(\Lambda)(I_p - \Lambda)P(\Lambda)^{t} B^{t}\right) : \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_p),\ 1 > \lambda_1 > \cdots > \lambda_p > 0 \right\} \tag{20}$$

with $B \in GL(p)$ and $P(\Lambda) \in \mathcal{N}$. Without loss of generality, we assume $B = I_p$ in (20). Let $B(W)$ denote the equivariant part of $W = (W_1, W_2)$ with respect to $\mathcal{Z}$. Then the equivariant part with respect to $\mathcal{Z}'$ is $B(W)P(\Lambda(W))^{-1}\mathcal{G}_0$ by (4). Writing the latter as

$$B(W)P(\Lambda(W))^{-1}\mathcal{G}_0 = T'(W)C'(W)\mathcal{G}_0, \qquad T'(W) \in LT(p),\ C'(W) \in O(p),$$

we obtain the decomposition

$$W \leftrightarrow \left(T'(W),\ C'(W)\mathcal{G}_0,\ z'(W)\right),$$

where $z'(W) = (P(\Lambda(W))\Lambda(W)P(\Lambda(W))^{t},\ P(\Lambda(W))(I_p - \Lambda(W))P(\Lambda(W))^{t}) \in \mathcal{Z}'$ is the invariant part of W with respect to $\mathcal{Z}'$. Suppose that the density $f(W)$ with respect to λ in (19) with general $a, b > (p+1)/2$ is factored as

$$f(W) = f_{\mathcal{H}}(T'(W)) f_{\mathcal{L}/\mathcal{G}_0}(C'(W)\mathcal{G}_0) f_{\mathcal{Z}'}(z'(W)).$$

Then we can get from Theorem 3.4 the distributions of the three parts $T' = T'(W)$, $C'\mathcal{G}_0 = C'(W)\mathcal{G}_0$ and $z' = z'(W)$ using $\Delta^{\mathcal{G}} = \Delta^{\mathcal{L}} = 1$, $\chi(B) = (\det B)^{2(a+b)}$ and $\mu_{\mathcal{H}}(dT) = \mu_{LT(p)}(dT) = \prod_{i=1}^{p} t_{ii}^{-i} dT$, $dT = \prod_{i \geq j} dt_{ij}$, $T = (t_{ij})$ (Wijsman (1990), (7.7.2)). Specifically,

$$
\begin{aligned}
T' &\sim \frac{1}{c_{\mathcal{H}}} f_{\mathcal{H}}(T') \prod_{i=1}^{p} t_{ii}'^{2a+2b-i} dT', \\
C'\mathcal{G}_0 &\sim f_{\mathcal{L}/\mathcal{G}_0}(C'\mathcal{G}_0) d\mu_{\mathcal{L}/\mathcal{G}_0}(C'\mathcal{G}_0), \\
z' &\sim \frac{1}{c_{\mathcal{Z}'}} f_{\mathcal{Z}'}(z') d\nu_{\mathcal{Z}'}(z'),
\end{aligned}
$$

where $c_{\mathcal{H}} = \int_{LT(p)} f_{\mathcal{H}}(T) \prod_{i=1}^{p} t_{ii}^{2a+2b-i} dT$, $c_{\mathcal{Z}'} = \int_{\mathcal{Z}'} f_{\mathcal{Z}'}(z) d\nu_{\mathcal{Z}'}(z)$ and $\mu_{\mathcal{L}/\mathcal{G}_0}$ is an appropriate version of the invariant measures on $\mathcal{L}/\mathcal{G}_0 = O(p)/\mathcal{G}_0$.

4.3 Decompositions of rankings

Our general discussion can be applied to discrete distributions as well. In this subsection, let us look at one such example—distributions of preference rankings. For the analysis of ranking data in general, the reader is referred to the excellent books by Critchlow (1985), Diaconis (1988) and Marden (1995). Other interesting problems about preference rankings can be found in Kamiya, Orlik, Takemura and Terao (2006).

Let us consider rankings of m objects $1, 2, \ldots, m$. We denote rankings as $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(m))$, where $\sigma(i)$ stands for the rank given to object i. Then we can regard

$$
\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(m)) = \begin{pmatrix} 1 & 2 & \cdots & m \\ \sigma(1) & \sigma(2) & \cdots & \sigma(m) \end{pmatrix}
$$

as an element of the symmetric group S_m on $\{1, 2, \ldots, m\}$. We can deal with distributions of rankings $\sigma \in S_m$ by considering probability functions on $\mathcal{X} = S_m$.

Here we define an action of $\mathcal{G} = S_{m-1} < S_m$ on $\mathcal{X} = S_m$ as follows. Thinking of $\tau \in S_{m-1}$ as permutations of $\{2, \ldots, m\}$, we express τ as

$$
\tau = (\tau(2), \ldots, \tau(m)) = \begin{pmatrix} 2 & \cdots & m \\ \tau(2) & \cdots & \tau(m) \end{pmatrix}.
$$

Permutation $\tau \in S_{m-1}$ changes rank $k \in \{2, \ldots, m\}$ to rank $\tau(k) \in \{2, \ldots, m\}$. Now we consider the action $(\tau, \sigma) \mapsto \tau\sigma$. Here $\tau\sigma \in S_m$ means $(\tau\sigma)(i) = \tau(\sigma(i))$, $i \in \{1, 2, \ldots, m\}$, where we agree that $\tau(1) = 1$. Note that this action is free: $\mathcal{G}_0 = \{e\}$, $e = (2, \ldots, m) = (1, 2, \ldots, m)$.

Under the above action, the orbit containing $\sigma \in S_m$ is

$$
\mathcal{G}\sigma = S_{m-1}\sigma = \{\tilde{\sigma} \in S_m : \tilde{\sigma}^{-1}(1) = \sigma^{-1}(1)\}
$$

and $\mathcal{X} = S_m$ consists of the m orbits

$$
\{\tilde{\sigma} \in S_m : \tilde{\sigma}(i) = 1\}, \quad i = 1, 2, \ldots, m.
$$

Orbit $\{\tilde{\sigma} \in S_m : \tilde{\sigma}(i) = 1\}$ is seen to be the set of all rankings that rank object i first, and we will call this the i-orbit.

One way of selecting a representative σ_i of the i-orbit is choosing σ_i as

$$\sigma_i^{-1}(1) = i, \quad \sigma_i^{-1}(2) < \cdots < \sigma_i^{-1}(m). \tag{21}$$

Let us take the cross section $\mathcal{Z}$ consisting of these representatives σ_i, $i = 1, 2, \ldots, m$, i.e., $\mathcal{Z} = \{\sigma_i : i = 1, 2, \ldots, m\}$. Then, with respect to this cross section, we can write an arbitrary $\sigma \in \mathcal{X} = S_m$ uniquely as

$$\sigma = \tau s = \tau(\sigma)s(\sigma), \qquad \tau \in \mathcal{G} = S_{m-1}, \quad s \in \mathcal{Z}.$$

Note that $s(\sigma)$ can be expressed explicitly as $s(\sigma) = \sigma_{\sigma^{-1}(1)}$. For example, when $m = 4$, ranking

$$\sigma = (4, 2, 1, 3) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

belongs to the 3-orbit, and the corresponding $\tau(\sigma)$ and $s(\sigma)$ are given as

$$\tau(\sigma) = (4, 2, 3) = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix}, \quad s(\sigma) = \sigma_3 = (2, 3, 1, 4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}.$$

We are now in a position to introduce a family of distributions on $\mathcal{X} = S_m$. Consider the following type of probability functions:

$$p(\sigma) = \tilde{p}_{\mathcal{G}}(d(\sigma, s(\sigma)))p_{\mathcal{Z}}(s(\sigma)), \quad \sigma \in S_m, \tag{22}$$

where $d(\,\cdot\,,\,\cdot\,)$ is a right-invariant metric on S_{m-1} (note that $\sigma^{-1}(1) = s(\sigma)^{-1}(1)$). We usually take $\tilde{p}_{\mathcal{G}}$ to be a decreasing function. In that case, $\tilde{p}_{\mathcal{G}}(d(\sigma, s(\sigma)))$ expresses a unimodal distribution in the orbit with the representative $s(\sigma)$ as the modal ranking; the farther away from the representative $s(\sigma)$, the smaller probability of that ranking σ. An example of a specification of $\tilde{p}_{\mathcal{G}}$ is provided by Mallows' model $\tilde{p}_{\mathcal{G}}(d(\sigma, s(\sigma))) = c\exp(\theta d(\sigma, s(\sigma)))$ with $\theta \le 0$, where c is the normalizing constant (Mallows (1957), Feigin and Cohen (1978), Fligner and Verducci (1986)). Now, writing $p_{\mathcal{G}}(\tau) := \tilde{p}_{\mathcal{G}}(d(\tau, e))$, we can express (22) as

$$p(\sigma) = p_{\mathcal{G}}(\tau(\sigma))p_{\mathcal{Z}}(s(\sigma)), \quad \sigma \in S_m.$$

This can be regarded as a decomposable distribution under our action of $\mathcal{G} = S_{m-1}$ on $\mathcal{X} = S_m$. We are writing $p(\sigma)$ instead of $f(\sigma)$ in order to emphasize that we are dealing with discrete probability functions. Of course, the density is with respect to the counting measure, so $\chi = 1$.

In the discussions so far, the representatives $\sigma_1, \ldots, \sigma_m$ of the orbits have been chosen as (21), but this is just one way of choosing them and other choices are also possible. The choice is arbitrary as long as they form a complete set of representatives. In fact, the selection rule can differ from orbit to orbit. For example, we may consider taking the representative $\sigma_{i_0} \in \{\tilde{\sigma} \in S_m : \tilde{\sigma}(i_0) = 1\}$ for some object $i_0 \in \{1, 2, \ldots, m\}$ as follows: for some object j_0,

$$\sigma_{i_0}^{-1}(1) = i_0, \quad \sigma_{i_0}^{-1}(2) < \cdots < \sigma_{i_0}^{-1}(m-1), \quad \sigma_{i_0}^{-1}(m) = j_0. \tag{23}$$

For the other orbits, we continue to select representatives $\sigma_i,\ i \neq i_0,$ as in (21).

We now discuss motivations of the above modeling. Imagine we are considering people's preference rankings of a league of m sports teams $1, 2, \ldots, m$. Suppose that we are faced with the following situation: while people are interested in their favorite team, they do not care much about the differences among the rest and tend to simply rank the second to last preferred teams according to the ranks in the standings (based on winning percentages). In that case, we can describe the situation by modeling the distribution of the rankings as follows.

We label the m teams with the ranks in the standings. Then, we choose the representatives of all the orbits as in (21). With these choices, the distribution (22) of people's preference rankings implies that their top rank is distributed according to $p_{\mathcal{Z}}(s(\sigma))$, while the rest of the ranks are distributed based on the distances to the modal ranking, which in this case is the ranking (having the same relative ranks of the non-top objects as the ranking) in the standings; the closer to the ranking in the standings, the larger percentage of people with that ranking. So we have succeeded in describing the case in question.

Alternatively, there may be some cases where the fans of a certain team i_0 have a strong sense of rivalry with some other team j_0. In those cases, the choice (23) will be more appropriate.

Now, we can go further and consider decompositions into three parts. Suppose that people are very interested in the top rank, interested to some degree in ranks $2, \ldots, m'$ $(2 \leq m' \leq m-1)$ and totally indifferent about the rest of the ranks $m'+1, \ldots, m$.

Let us consider the action of $\mathcal{H} = S_{m-m'}$ on $\mathcal{G} = S_{m-1}$, with $S_{m-m'}$ being regarded as the set of permutations of $\{m'+1, \ldots, m\}$. Denote a cross section of this action by $\mathcal{V}$. Then any $\tau \in \mathcal{G}$ can be written uniquely as $\tau = ht,\ h \in \mathcal{H},\ t \in \mathcal{V}$. Hereafter we will write $S_m = S_{\{1,\ldots,m\}},\ S_{m-1} = S_{\{2,\ldots,m\}}$ and $S_{m-m'} = S_{\{m'+1,\ldots,m\}}$.

By writing $\sigma \in \mathcal{X} = S_{\{1,\ldots,m\}}$ as

$$\begin{aligned} \sigma &= \tau s \quad (\tau \in \mathcal{G} = S_{\{2,\ldots,m\}},\ s \in \mathcal{Z}) \\ &= hts \quad (h \in \mathcal{H} = S_{\{m'+1,\ldots,m\}},\ t \in \mathcal{V}), \end{aligned}$$

we obtain bijections

$$\begin{aligned} \sigma &\leftrightarrow \big(\tau,\ S_{\{2,\ldots,m\}}\sigma\big)\ (\in S_{\{2,\ldots,m\}} \times S_{\{2,\ldots,m\}}\backslash S_{\{1,\ldots,m\}}) \\ &\leftrightarrow \big(h,\ S_{\{m'+1,\ldots,m\}}\tau,\ S_{\{2,\ldots,m\}}\sigma\big) \\ &\qquad\qquad (\in S_{\{m'+1,\ldots,m\}} \times S_{\{m'+1,\ldots,m\}}\backslash S_{\{2,\ldots,m\}} \times S_{\{2,\ldots,m\}}\backslash S_{\{1,\ldots,m\}}) \\ &\leftrightarrow (h,\ t,\ s) \in S_{\{m'+1,\ldots,m\}} \times \mathcal{V} \times \mathcal{Z}. \end{aligned} \tag{24}$$

Remember that $\mathcal{H}\backslash\mathcal{G}$ for $\mathcal{H} < \mathcal{G}$ denotes the right coset space: $\mathcal{H}\backslash\mathcal{G} = \{\mathcal{H}g : g \in \mathcal{G}\}$.

Example 4.1. *Let $m = 6,\ m' = 3,$ and take $\mathcal{Z} = \{\sigma_i : i = 1, 2, \ldots, 6\}$ with $\sigma_1, \ldots, \sigma_6$ as in (21) and $\mathcal{V} = \{\sigma_{(i,j)} \in S_{\{2,\ldots,6\}} : \{i, j\} \subset \{2, \ldots, 6\},\ i \neq j\}$ with $\sigma_{(i,j)}$ such that $\sigma^{-1}_{(i,j)}(2) = i,\ \sigma^{-1}_{(i,j)}(3) = j,\ \sigma^{-1}_{(i,j)}(4) < \sigma^{-1}_{(i,j)}(5) < \sigma^{-1}_{(i,j)}(6)$. Then,*

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 1 & 4 & 2 \end{pmatrix} &= \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 5 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 5 & 6 \\ 6 & 5 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 5 & 6 \end{pmatrix}, \end{aligned}$$

so

$$
\begin{aligned}
& S_{\{1,2,3,4,5,6\}} \ni \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 1 & 4 & 2 \end{pmatrix} \\
\leftrightarrow \quad & \left(\begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 4 & 2 \end{pmatrix}, \; S_{\{2,3,4,5,6\}} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 1 & 4 & 2 \end{pmatrix} \right) \\
& \qquad\qquad (\in S_{\{2,3,4,5,6\}} \times S_{\{2,3,4,5,6\}} \backslash S_{\{1,2,3,4,5,6\}}) \\
\leftrightarrow \quad & \left(\begin{pmatrix} 4 & 5 & 6 \\ 6 & 5 & 4 \end{pmatrix}, \; S_{\{4,5,6\}} \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 4 & 2 \end{pmatrix}, \; S_{\{2,3,4,5,6\}} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 1 & 4 & 2 \end{pmatrix} \right) \\
& \qquad\qquad (\in S_{\{4,5,6\}} \times S_{\{4,5,6\}} \backslash S_{\{2,3,4,5,6\}} \times S_{\{2,3,4,5,6\}} \backslash S_{\{1,2,3,4,5,6\}}) \\
\leftrightarrow \quad & \left(\begin{pmatrix} 4 & 5 & 6 \\ 6 & 5 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 5 & 6 \end{pmatrix} \right) \in S_{\{4,5,6\}} \times \mathcal{V} \times \mathcal{Z}.
\end{aligned}
$$

Now consider probability functions $p(\sigma)$ which can be factored with respect to (24) as

$$
p(\sigma) = p_{\mathcal{H}\backslash\mathcal{G}}(S_{\{m'+1,\ldots,m\}}\tau) p_{\mathcal{Z}}(s) \tag{25}
$$

with $p_{\mathcal{H}\backslash\mathcal{G}}(S_{\{m'+1,\ldots,m\}}\tau) := \tilde{p}_{\mathcal{H}\backslash\mathcal{G}}(d'(S_{\{m'+1,\ldots,m\}}\tau, S_{\{m'+1,\ldots,m\}}))$, where d' is the Hausdorff metric on $\mathcal{H}\backslash\mathcal{G} = S_{\{m'+1,\ldots,m\}}\backslash S_{\{2,\ldots,m\}}$ induced by the metric d on $\mathcal{G} = S_{\{2,\ldots,m\}}$ (Critchlow (1985)). Note that distribution (25) can be seen as an extended decomposable distribution with respect to (24). Now, label the teams with the ranks in the standings and take $\mathcal{Z} = \{\sigma_i : i = 1, 2, \ldots, m\}$ with σ_i, $i = 1, 2, \ldots, m$, in (21) as before. Then (25) describes the situation stated earlier. Note that in (25) the choice of $\mathcal{V}$ is irrelevant, because the rest of the ranks are uniform.

Remark 4.1. *Here we have studied the permutations of ranks. By considering the orderings*

$$(\textit{top ranked object, second ranked object, } \ldots)$$

instead of rankings

$$(\textit{object 1's rank, object 2's rank, } \ldots),$$

we can also deal with the permutations of objects in a similar manner.

References

[1] Anderson, T. W. (2003). *An Introduction to Multivariate Statistical Analysis*, Third Edition. John Wiley & Sons, New Jersey.

[2] Barndorff-Nielsen, O. E., Blæsild, P. and Eriksen, P. S. (1989). *Decomposition and Invariance of Measures, and Statistical Transformation Models*. Lecture Notes in Statistics, Vol. 58, Springer-Verlag, Berlin.

[3] Bondar, J. V. (1976). Borel cross-sections and maximal invariants. *Ann. Statist.*, **4**, 866–877.

[4] Cambanis, S., Huang, S. and Simons, G. (1981). On the theory of elliptically contoured distributions. *J. Multivariate Anal.*, **11**, 368–385.

[5] Critchlow, D. E. (1985). *Metric Methods for Analyzing Partially Ranked Data*. Lecture Notes in Statistics, Vol. 34, Springer-Verlag, Berlin.

[6] Diaconis, P. (1988). *Group Representations in Probability and Statistics*. Lecture Notes–Monograph Series, Vol. 11, Institute of Mathematical Statistics, Hayward, California.

[7] Eaton, M. L. (1989). *Group Invariance Applications in Statistics*. Regional Conference Series in Probability and Statistics, Vol. 1, Institute of Mathematical Statistics, Hayward, California.

[8] Fang, K. T. and Anderson, T. W. (1990). *Statistical Inference in Elliptically Contoured and Related Distributions*. Allerton Press, New York.

[9] Fang, K. T. and Zhang, Y. T. (1990). *Generalized Multivariate Analysis*. Springer-Verlag, New York.

[10] Farrell, R. H. (1985). *Multivariate Calculation: Use of the Continuous Groups*. Springer-Verlag, New York.

[11] Feigin, P. D. and Cohen, A. (1978). On a model for concordance between judges. *J. Roy. Statist. Soc.* B, **40**, 203–213.

[12] Fernández, C., Osiewalski, J. and Steel, M. F. J. (1995). Modeling and inference with v-spherical distributions. *J. Amer. Statist. Assoc.*, **90**, 1331–1340.

[13] Ferreira, J. T. A. S. and Steel, M. F. J. (2005). Modelling directional dispersion through hyperspherical log-splines. *J. Roy. Statist. Soc.* B, **67**, 599–616.

[14] Fligner, M. A. and Verducci, J. S. (1986). Distance based ranking models. *J. Roy. Statist. Soc.* B, **48**, 359–369.

[15] Kamiya, H. (1996). Borel isomorphism between the sample space and a product space. *Math. Methods Statist.*, **5**, 237–243.

[16] Kamiya, H., Orlik, P., Takemura, A. and Terao, H. (2006). Arrangements and ranking patterns. *Ann. Comb.*, **10**, 267-284.

[17] Kamiya, H., Takemura, A. and Kuriki, S. (2006). Star-shaped distributions and their generalizations. *J. Statist. Plann. Inference*, to appear. Available at **http://arxiv.org/abs/math.ST/0605600**.

[18] Kelker, D. (1970). Distribution theory of spherical distributions and a location-scale parameter generalization. *Sankhyā* A, **32**, 419–430.

[19] Koehn, U. (1970). Global cross sections and the densities of maximal invariants. *Ann. Math. Statist.*, **41**, 2045–2056.

[20] Mallows, C. L. (1957). Non-null ranking models. I. *Biometrika*, **44**, 114–130.

[21] Marden, J. I. (1995). *Analyzing and Modeling Rank Data.* Chapman & Hall, London.

[22] Muirhead, R. W. (1982). *Aspects of Multivariate Statistical Theory.* John Wiley & Sons, New York.

[23] Osiewalski, J. and Steel, M. F. J. (1993). Robust Bayesian inference in l_q-spherical models. *Biometrika*, **80**, 456–460.

[24] Wijsman, R. A. (1967). Cross-sections of orbits and their application to densities of maximal invariants. *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability*, Vol. I: Statistics 389–400. Univ. California Press, Berkeley, California.

[25] Wijsman, R. A. (1986). Global cross sections as a tool for factorization of measures and distribution of maximal invariants. *Sankhyā* A, **48**, 1–42.

[26] Wijsman, R. A. (1990). *Invariant Measures on Groups and Their Use in Statistics.* Lecture Notes–Monograph Series, Vol. 14, Institute of Mathematical Statistics, Hayward, California.