MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

Worst-Case Plastic Limit Analysis of Trusses under Uncertain Loads via Mixed 0-1 Programming

Yoshihiro Kanno and Izuru Takewaki

METR 2006–39 July 2006

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.i.u-tokyo.ac.jp/mi/mi-e.htm

Worst-Case Plastic Limit Analysis of Trusses under Uncertain Loads via Mixed 0-1 Programming

Yoshihiro Kanno [a], Izuru Takewaki [b]

[a] *Department of Mathematical Informatics, The University of Tokyo, Tokyo 113-8656, Japan*
E-mail: `kanno@mist.i.u-tokyo.ac.jp`

[b] *Department of Urban and Environmental Engineering, Kyoto University, Kyoto 606-8501, Japan*
E-mail: `takewaki@archi.kyoto-u.ac.jp`

Abstract

The paper presents a global optimization method to compute the minimum limit load factor of trusses under the unknown-but-bounded load uncertainty. We assume that the external forces consist of a part proportional to a load factor and a part that is uncertain around its nominal value. The worst-case limit load factor is introduced as the smallest limit load factor realized with some uncertain parameters. By reformulating the worst-case determination problem as a mixed 0-1 programming problem, we propose a global optimization algorithm as a combination of a branch-and-boud method based on the linear programming relaxations and a cutting plane method based on the disjunctive or lift-and-project cuts. The worst-case limit loads, as well as the corresponding critical loading patterns, are computed to demonstrate that our method converges to the global optimal solutions successfully.

Keywords

Data uncertainty, limit analysis, integer programming, cutting plane, branch-and-bound, global optimization

1 Introduction

In designing civil, mechanical and aerospace structures, plastic limit analysis has been used widely for decades as a means of estimating the ultimate strength of structures. It is the case that, while the dead and live loads are uncertain around their nominal values, the disturbance load is applied proportionally with a load factor. This paper discusses a global optimization technique for computing the smallest limit load factor of truss structures, where the applied dead and live loads are imprecisely known.

The limit analysis still receives much attention by numerous researchers from the view points of algorithms [2, 14, 23, 29] and issues relevant to the finite element method [25, 33]. Based on the probabilistic uncertainty models of structural systems, various approaches to stochastic limit analysis have also been proposed [24, 27, 30, 31]. In the framework of probabilistic uncertainties, reliability-based structural design methods have been investigated extensively [22, 34].

Besides these probabilistic uncertainty models, non-probabilistic uncertainty models have also been developed, where a mechanical system is assumed to contain *unknown-but-bounded* uncertain parameters. Ben-Haim and Elishakoff [6] developed the well-known *convex model* approach, with which Ganzerli and Pantelides [18] proposed a robust truss optimization method. The interval linear algebra has been well developed for uncertain linear equations [1], and has been employed in structural analyses with uncertainties [10]. In contrast to probabilistic modelings, these non-probabilistic uncertainty modelings require only upper bounds on the magnitudes of uncertain parameters, and engineers need not to estimate the probabilistic density distributions of uncertain parameters.

Elishakoff *et al.* [17] proposed a structural optimization scheme under the unknown-but-bounded uncertainty by using the anti-optimization. The bi-level optimization problems were formulated and solved numerically for the robust structural design against the worst case [11, 16]. Gu *et al.* [19] proposed an estimation method for the worst case of the propagated uncertainty in a multidisciplinary system. A unified methodology of *robust counterpart* of various convex optimization problems was developed by Ben-Tal and Nemirovski [8], which was applied to robust compliance minimization of trusses [7]. The authors proposed the methods for robustness analysis and robust optimization of structures [20, 21, 32] based on the info-gap uncertainty model [5].

A serious difficulty in the worst-case detection arises when the uncertain parameters in the worst-case are defined as an optimal solution of a nonconvex optimization problem. The conventional methods for linear worst-case analysis, e.g., the convex model [6], can be applied only to the cases in which sufficiently small variation of the uncertain parameters is allowed, or in which the structural response considered is represented as a linear function of the uncertain parameters. In these cases, the worst case can be detected by solving a convex optimization problem.

Unfortunately, in many practical situations, the variation of uncertain parameters is not small and we are interested in nonlinear responses of structures. Then the worst case is defined via a nonconvex optimization problem. In general, the conventional nonlinear programming approach converges to a local optimal solution of that problem. However, a local minimum solution, that is not globally optimal, does not correspond to the worst case. Obviously, the worst case corresponds to a global optimal solution. Thus, we have to find a global optimal solution of the nonconvex problem and guarantee that the solution obtained is globally optimal, which prevent us to use the conventional nonlinear programming algorithms.

In this paper, we aim at developing a *global optimization* method for the worst-case detection. We consider the limit load factor of a truss under the load uncertainty. The external forces applied to a truss are supposed to consist of a constant part and a part proportional to a load factor, where the former part cannot be known precisely and is assumed to be unknown but bounded. The worst-case limit load factor is defined as the minimum value among all the possible limit load factors realized by some uncertain parameters belonging to the given closed set.

We define the worst-case limit load factor by using a nonconvex optimization problem, which can be rewritten as a *mixed 0-1 programming* problem. Based on a *linear programming* (LP) relaxation, a simple *branch-and-bound* algorithm is proposed to obtain a global optimal solution of the presented mixed 0-1 programming problem. To strengthen the LP relaxation, we generate some *cutting planes* at the root node of the branch-and-bound tree. This approach is called the *cut-and-branch method* [15]. We formulate an LP problem to generate the deepest *disjunctive cut*. By adding the generated cuts to LP relaxation problems, it is possible to reduce drastically the number of LP problems that should be solved in the branch-and-bound method. It is guaranteed that the solution obtained by using the cut-and-branch method is a global optimal solution of the worst-case determination problem, i.e., it is assured that there exists no uncertain parameter with which the limit load factor becomes smaller than the obtained optimal value.

Recently, there have been renewed interests in cutting planes, or cuts, that are valid linear inequalities of a mixed integer programming problem; see, e.g., the review paper [26]. Especially, the branch-and-cut method [3, 15], that is an LP based branch-and-bound method with cuts added, is considered as one of the most successful approaches to solving the mixed integer program. Among various cuts, a disjunctive cut (or lift-and-project cut) is defined as a linear inequality selected among inequalities valid for a disjunctive programming relaxation of the mixed 0-1 program [3, 4, 9]. We utilize disjunctive cuts to strengthen the LP relaxation problems that are solved at nodes of the branch-and-bound tree.

This paper is organized as follows. In section 2, in order to make this paper self-contained, we prepare the LP problems for the conventional limit analysis as well as the notation used in this paper. Section 3 introduces the notion of uncertain limit analysis by defining the uncertainty model of external load and the worst-case limit load factor. In section 4, we present the mixed 0-1 programming formulation for the uncertain limit analysis, and for the solution we propose a branch-and-bound method. In order to strengthen the LP relaxation problems solved in the branch-and-bound tree, an LP problem that generates the disjunctive cutting plane is proposed in section 5. Numerical experiments are presented in section 6 for various trusses by using the cut-and-branch method presented, while conclusions are drawn in section 7.

2 Notation and preliminary results

2.1 Notation

All vectors are assumed to be column vectors in this paper. For an n-tuple $p_{m+1}, \dots, p_{m+n}$, we let denote $(p_i \mid i = m+1, \dots, m+n)$ and $\{p_i \mid i = m+1, \dots, m+n\}$, respectively, the n-dimensional vector $(p_{m+1}, \dots, p_{m+n})^\top$ and the set that consists of $p_{m+1}, \dots, p_{m+n}$. The vector $(p_i \mid i = 1, \dots, n) \in \mathbb{R}^n$ is often simplified as $(p_i) \in \mathbb{R}^n$. The ℓ^1, ℓ^2 (or standard Euclidean), and ℓ^∞ norms of the vector $\boldsymbol{p} = (p_i) \in \mathbb{R}^n$, denoted by $\|\boldsymbol{p}\|_1$, $\|\boldsymbol{p}\|_2$, and $\|\boldsymbol{p}\|_\infty$, respectively, are defined as

$$\|\boldsymbol{p}\|_1 = \sum_{i=1}^{n} |p_i|,$$
$$\|\boldsymbol{p}\|_2 = (\boldsymbol{p}^\top \boldsymbol{p})^{1/2},$$
$$\|\boldsymbol{p}\|_\infty = \max_{i \in \{1,\dots,n\}} |p_i|.$$

For vectors $\boldsymbol{p} = (p_i) \in \mathbb{R}^n$ and $\boldsymbol{q} = (q_i) \in \mathbb{R}^n$, we write $\boldsymbol{p} \geq \boldsymbol{0}$ and $\boldsymbol{p} \geq \boldsymbol{q}$, respectively, if $p_i \geq 0$, $i = 1, \dots, n$ and $\boldsymbol{p} - \boldsymbol{q} \geq \boldsymbol{0}$. The $(m+n)$-dimensional column vector $(\boldsymbol{p}^\top, \boldsymbol{q}^\top)^\top$ is often written simply as $(\boldsymbol{p}, \boldsymbol{q})$. Moreover, $(\boldsymbol{p}, \boldsymbol{q})_i$ denotes the ith component of the vector $(\boldsymbol{p}^\top, \boldsymbol{q}^\top)^\top$. Define the vectors $\mathbf{1} \in \mathbb{R}^n$ and $\boldsymbol{e}^j \in \mathbb{R}^n$, $j = 1, \dots, n$, as

$$\mathbf{1} = (1, \dots, 1)^\top,$$

$$\boldsymbol{e}^j = (e^j_i \,|\, i = 1, \dots, n), \quad e^j_i = \begin{cases} e^j_i = 0, & \text{for } i \neq j, \\ e^j_j = 1, \end{cases}$$

i.e., $\boldsymbol{e}^j$ is the jth column vector of the identity matrix. Define $\mathbb{R}^n_+ \subset \mathbb{R}^n$ by

$$\mathbb{R}^n_+ = \{\boldsymbol{p} \in \mathbb{R}^n \,|\, \boldsymbol{p} \geq \mathbf{0}\}.$$

For two sets $\mathcal{A} \subseteq \mathbb{R}^m$ and $\mathcal{B} \subseteq \mathbb{R}^n$, their Cartesian product is defined by $\mathcal{A} \times \mathcal{B} = \{(\boldsymbol{a}^\top, \boldsymbol{b}^\top)^\top \in \mathbb{R}^{m+n} \,|\, \boldsymbol{a} \in \mathcal{A},\ \boldsymbol{b} \in \mathcal{B}\}$. Particularly, we write $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. Let '$\mathrm{conv}\, \mathcal{A}$' denote the *convex hull* of $\mathcal{A}$, that is the smallest convex set that contains $\mathcal{A}$. The *closure* of $\mathcal{A}$, that is the smallest closed set that contains $\mathcal{A}$, is denoted by '$\mathrm{cl}\, \mathcal{A}$'. The cardinality of the set $\mathcal{A}$ is denoted by $|\mathcal{A}|$. The empty set is denoted by $\emptyset$.

2.2 Basic problem for plastic limit analysis

Consider an elastic/perfectly-plastic truss in the two- or three-dimensional space. Small rotations and small strains are assumed. Let $\boldsymbol{f} \in \mathbb{R}^{n^{\mathrm{d}}}$ denote the vector of the external forces, where n^{d} denotes the number of degrees of freedom of displacements. Letting n^{m} denote the number of members, the vector of member axial forces is denoted by $\boldsymbol{q} = (q_i) \in \mathbb{R}^{n^{\mathrm{m}}}$. The system of equilibrium equations in terms of $\boldsymbol{f}$ and $\boldsymbol{q}$ can be written in the form of

$$\boldsymbol{B}\boldsymbol{q} = \boldsymbol{f}, \tag{1}$$

where $\boldsymbol{B} \in \mathbb{R}^{n^{\mathrm{d}} \times n^{\mathrm{m}}}$ is a constant matrix.

Let $\boldsymbol{u} \in \mathbb{R}^{n^{\mathrm{d}}}$ and c_i denote the vector of nodal displacements and the corresponding elongation of the ith member, respectively. We often write $\boldsymbol{c} = (c_i) \in \mathbb{R}^{n^{\mathrm{m}}}$. The ith column vector of $\boldsymbol{B}$ is denoted by $\boldsymbol{b}_i \in \mathbb{R}^{n^{\mathrm{d}}}$, $i = 1, \dots, n^{\mathrm{m}}$. The compatibility relation between $\boldsymbol{u}$ and c_i can be written as

$$c_i = \boldsymbol{b}_i^\top \boldsymbol{u}, \quad i = 1, \dots, n^{\mathrm{m}}. \tag{2}$$

Suppose that the external load $\boldsymbol{f}$ consists of the constant part $\boldsymbol{f}_{\mathrm{D}}$ and proportionally increasing part $\lambda \boldsymbol{f}_{\mathrm{R}}$ as

$$\boldsymbol{f} = \boldsymbol{f}_{\mathrm{D}} + \lambda \boldsymbol{f}_{\mathrm{R}}. \tag{3}$$

Notice here that $\lambda \boldsymbol{f}_{\mathrm{R}}$ is defined by the monotonically increasing load parameter $\lambda \in \mathbb{R}_+$ and the constant reference load $\mathbb{R}^{n^{\mathrm{d}}} \ni \boldsymbol{f}_{\mathrm{R}} \neq \mathbf{0}$. In civil engineering, $\boldsymbol{f}_{\mathrm{D}}$ consists of the dead load, live load, etc, while $\lambda \boldsymbol{f}_{\mathrm{R}}$ is referred to as the live or disturbance load caused by earthquakes, winds, etc. In this paper, $\boldsymbol{f}_{\mathrm{D}}$ is simply called *dead load* and $\boldsymbol{f}_{\mathrm{R}}$ is called *reference disturbance load* for simplicity of presentation.

Let $\sigma^{\mathrm{y}}_i > 0$ and $-\sigma^{\mathrm{y}}_i$ denote the yield stresses of the ith member in tension and in compression, respectively, where we assume for simplicity that the yield stresses in tension and compression share the common absolute value. The member cross-sectional area is denoted by $a_i > 0$. Define q^{y}_i by

$$q^{\mathrm{y}}_i = a_i \sigma^{\mathrm{y}}_i, \quad i = 1, \dots, n^{\mathrm{m}},$$

which is the absolute value of the admissible axial force. Then, the yield functions can be written as

$$|q_i| - q_i^{\rm y} \le 0, \quad i = 1, \dots, n^{\rm m}. \tag{4}$$

From the static or lower-bound principle, and by using (1), (3), and (4), the limit load factor is obtained by solving the following *linear programming* (LP) problem:

$$\left.\begin{array}{ll} \max & \lambda \\ \text{s.t.} & \boldsymbol{B}\boldsymbol{q} = \boldsymbol{f}_{\rm D} + \lambda \boldsymbol{f}_{\rm R}, \\ & |q_i| - q_i^{\rm y} \le 0, \quad i = 1, \dots, n^{\rm m}, \end{array}\right\} \tag{5}$$

where the variables are λ and $\boldsymbol{q}$.

3 Uncertain limit analysis

In this section, we introduce the uncertainty model of external load, and define the worst-case limit load factor rigorously.

3.1 Uncertainty model

In this paper, we suppose that only $\boldsymbol{f}_{\rm D}$ in (3) possesses the uncertainty, i.e., $\boldsymbol{f}_{\rm D}$ cannot be known precisely. The model of uncertainty of $\boldsymbol{f}_{\rm D}$ is motivated by the non-probabilistic information-gap model [5].

Let $\widetilde{\boldsymbol{f}}_{\rm D} \in \mathbb{R}^{n^{\rm d}}$ denote the nominal value (or the best estimate) of $\boldsymbol{f}_{\rm D}$. We describe the uncertainty of $\boldsymbol{f}_{\rm D}$ in terms of the m-dimensional vector $\boldsymbol{\zeta} \in \mathbb{R}^m$, that is considered to be unknown but bounded. Suppose that $\boldsymbol{f}_{\rm D}$ depends on $\boldsymbol{\zeta}$ affinely as

$$\boldsymbol{f}_{\rm D} = \widetilde{\boldsymbol{f}}_{\rm D} + \boldsymbol{T}\boldsymbol{\zeta}, \tag{6}$$

where $\boldsymbol{T} \in \mathbb{R}^{n^{\rm d} \times m}$ is a constant matrix satisfying the following assumption:

Assumption 3.1. $\boldsymbol{T}$ *satisfies the following conditions:*

(i) $\{\boldsymbol{T}^\top \boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{R}^{n^{\rm d}}\} = \mathbb{R}^m$;

(ii) $\boldsymbol{f}_{\rm L}^\top \boldsymbol{T}\boldsymbol{\zeta} = 0$ *for any* $\boldsymbol{\zeta} \in \mathbb{R}^m$.

Assumption 3.1 (i) implies that the reference disturbance load $\boldsymbol{f}_{\rm R}$ does not have the uncertainty.

For a given parameter $\alpha \in \mathbb{R}_+$, define a set $\mathcal{Z}(\alpha) \subset \mathbb{R}^m$ by

$$\mathcal{Z}(\alpha) = \{ \boldsymbol{\zeta} \in \mathbb{R}^m \mid \alpha \ge \|\boldsymbol{\zeta}\|_\infty \}. \tag{7}$$

The uncertain parameters vector $\boldsymbol{\zeta}$ is assumed to be running through the *uncertain set* $\mathcal{Z}(\alpha)$ defined by (7), i.e.,

$$\boldsymbol{\zeta} \in \mathcal{Z}(\alpha). \tag{8}$$

From (6), (7), and (8) it follows that the uncertain $\boldsymbol{f}_{\rm D}$ satisfies

$$\boldsymbol{f}_{\rm D} \in \mathcal{F}_{\rm D}(\alpha) := \left\{ \boldsymbol{f} \in \mathbb{R}^{n^{\rm d}} \,\middle|\, \boldsymbol{f} = \widetilde{\boldsymbol{f}}_{\rm D} + \boldsymbol{T}\boldsymbol{\zeta},\ \alpha \geq \|\boldsymbol{\zeta}\|_{\infty} \right\}. \tag{9}$$

Roughly speaking, $\boldsymbol{f}_{\rm D}$ moves around the center-point $\widetilde{\boldsymbol{f}}_{\rm D}$. The greater the value of α, the greater the range of possible variation of $\boldsymbol{f}_{\rm D}$. In the context of the info-gap uncertainty model [5], α is called the *uncertainty parameter*.

Note that the uncertain set $\mathcal{F}_{\rm D}(\alpha)$ is bounded for any $\alpha \in \mathbb{R}_{+}$. Moreover, $\mathcal{F}_{\rm D}(\alpha)$ satisfies the two basic axioms for the info-gap model, i.e., (i) nesting: $0 \leq \alpha_1 < \alpha_2$ implies $\mathcal{F}_{\rm D}(\alpha_1) \subset \mathcal{F}_{\rm D}(\alpha_2)$; (ii) contraction: $\mathcal{F}_{\rm D}(0)$ is the singleton set $\widetilde{\boldsymbol{f}}_{\rm D}$.

3.2 Worst-case limit load factor

For a given (but uncertain) dead load $\boldsymbol{f}_{\rm D} \in \mathbb{R}^{n^{\rm d}}$, define a set $\mathcal{Q}(\boldsymbol{f}_{\rm D}) \subseteq \mathbb{R}^{n^{\rm m}+1}$ by

$$\mathcal{Q}(\boldsymbol{f}_{\rm D}) := \left\{ (\lambda, \boldsymbol{q}) \in \mathbb{R} \times \mathbb{R}^{n^{\rm m}} \,\middle|\, \boldsymbol{B}\boldsymbol{q} = \boldsymbol{f}_{\rm D} + \lambda \boldsymbol{f}_{\rm R},\ |q_i| - q_i^{\rm y} \leq 0,\ i = 1, \dots, n^{\rm m} \right\}, \tag{10}$$

which is the set of all statically admissible vector $(\lambda, \boldsymbol{q}) \in \mathbb{R} \times \mathbb{R}^{n^{\rm m}}$ associated with the fixed $\boldsymbol{f}_{\rm D}$. Define $\lambda^* : \mathbb{R}^{n^{\rm d}} \to \mathbb{R}$ by

$$\lambda^*(\boldsymbol{f}_{\rm D}) = \max_{\lambda, \boldsymbol{q}} \{ \lambda : (\lambda, \boldsymbol{q}) \in \mathcal{Q}(\boldsymbol{f}_{\rm D}) \}. \tag{11}$$

According to the static principle (5), $\lambda^*(\boldsymbol{f}_{\rm D})$ corresponds to the limit load factor under the dead load $\boldsymbol{f}_{\rm D}$.

We next introduce a concept of the *worst-case limit load factor*. This is essentially motivated by the fact that the limit load factor can be regarded as a function of the dead load vector as seen in (11), while $\boldsymbol{f}_{\rm D}$ is uncertain and running through $\mathcal{F}_{\rm D}(\alpha)$. Obviously, to evaluate the robustness of trusses quantitatively, we are interested in the most severe situation, if any, in which the limit load factor happens to decrease unexpectedly from the nominal limit load factor corresponding to $\widetilde{\boldsymbol{f}}_{\rm D}$ because of the uncertainty of $\boldsymbol{f}_{\rm D}$. To this end, we attempt to compute the minimum value of the limit load factor that can be attained at some dead load satisfying $\boldsymbol{f}_{\rm D} \in \mathcal{F}_{\rm D}(\alpha)$. This is naturally realized by introducing $\lambda_{\rm min} : \mathbb{R}_{+} \to \mathbb{R}$ by

$$\lambda_{\rm min}(\alpha) = \min_{\boldsymbol{f}_{\rm D}} \{ \lambda^*(\boldsymbol{f}_{\rm D}) : \boldsymbol{f}_{\rm D} \in \mathcal{F}_{\rm D}(\alpha) \}. \tag{12}$$

Substitution of (9) into (12) reads

$$\lambda_{\rm min}(\alpha) = \min_{\boldsymbol{\zeta}} \{ \lambda^*(\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta})) : \alpha \geq \|\boldsymbol{\zeta}\|_{\infty} \}. \tag{13}$$

Let $\boldsymbol{\zeta}^{\rm cr}$ denote an optimal solution of Problem (13). Given the uncertainty parameter α, we refer to $\lambda_{\rm min}(\alpha)$ defined by (13) as the *worst-case limit load factor*, that is the minimum value among limit load factors $\lambda^*(\boldsymbol{f}_{\rm D})$ corresponding to some dead loads satisfying $\boldsymbol{f}_{\rm D} \in \mathcal{F}_{\rm D}(\alpha)$. The corresponding dead load $\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta}^{\rm cr})$ is referred to as the *critical dead load*. We call $\boldsymbol{\zeta}^{\rm cr}$ the *critical uncertain parameters vector*. In the case without uncertainty, we easily see that the relationship

$$\lambda_{\rm min}(0) = \lambda^*(\widetilde{\boldsymbol{f}}_{\rm D}) = \lambda^*(\boldsymbol{f}_{\rm D}(\boldsymbol{0}))$$

holds. We refer to $\lambda^*(\widetilde{\boldsymbol{f}}_{\rm D})$ as the *nominal limit load factor*, that indicates the limit load factor corresponding to the nominal dead load $\widetilde{\boldsymbol{f}}_{\rm D}$. The objective of this paper is to propose a solution technique to compute $\lambda_{\rm min}(\alpha)$ as well as $\boldsymbol{\zeta}^{\rm cr}$.

Remark 3.2. In this paper we suppose that only the dead load has the uncertainty, and that the stiffness of each member is certain. This is because, the worst-case associated with the uncertainty of member stiffness can be found easily for the limit analysis. Indeed, the limit load factor monotonically decreases if a member stiffness decreases. Hence, the set of critical member stiffness corresponds to the trivial case in which the stiffness of each member coincides with its lower bound. On the contrary, the loading pattern of the dead load is not trivial, which motivates us to confine attention to the uncertainty of the dead load. □

Remark 3.3. It should be noted that the limit load factor can be computed easily if the loading pattern of the additional dead load is fixed a priori. Suppose that $\boldsymbol{\zeta}$ in (9) is defined as $\boldsymbol{\zeta} = \beta\boldsymbol{\zeta}^0$ with a given constant $\boldsymbol{\zeta}^0$ and a parameter β. After finding the nominal limit load factor $\lambda^*(\boldsymbol{f}_{\rm D}(\boldsymbol{0}))$ by employing the conventional limit analysis, the variation of $\lambda^*(\boldsymbol{f}_{\rm D}(\beta\boldsymbol{\zeta}^0))$ with respect to β can be computed by simply using the *parametric linear programming* [12] approach. In our problem, the direction of $\boldsymbol{\zeta}$ is unknown and should be determined so as to minimize $\lambda^*(\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta}))$. It should be noted again that we attempt to find a global optimal solution of Problem (13) which is essentially nonconvex. □

3.3 Some relevant problems

The remainder of this section prepares the reformulation of Problem (13) into the mixed 0-1 programming problem which will be presented in section 4.

Defining

$$\mathcal{U} = \left\{ (\boldsymbol{u}, \boldsymbol{z}) \in \mathbb{R}^{n^{\rm m}} \times \mathbb{R}^{n^{\rm d}} \;\middle|\; \boldsymbol{f}_{\rm R}^\top \boldsymbol{u} = 1,\ z_i \ge |\boldsymbol{b}_i^\top \boldsymbol{u}|,\ i = 1, \dots, n^{\rm m} \right\}, \tag{14}$$

consider the following problem in the variables $(\boldsymbol{u}, \boldsymbol{z}) \in \mathbb{R}^{n^{\rm d}} \times \mathbb{R}^{n^{\rm m}}$:

$$v^*(\boldsymbol{f}_{\rm D}) := \min_{\boldsymbol{u}, \boldsymbol{z}} \left\{ -\boldsymbol{f}_{\rm D}^\top \boldsymbol{u} + \boldsymbol{q}^{{\rm y}\top} \boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U} \right\}. \tag{15}$$

We first show that Problem (15) is equivalent to Problem (5) in the following sense:

Proposition 3.4 (Relation between Problems (5) **and** (15)**).** *Let* $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}})$ *denote an optimal solution of Problem* (15)*. Then,*

(i) $v^*(\boldsymbol{f}_{\rm D}) = \lambda^*(\boldsymbol{f}_{\rm D})$*;*

(ii) $\overline{\boldsymbol{u}}$ *corresponds to a collapse mode associated with* $\boldsymbol{f}_{\rm D}$*;*

(iii) $\overline{z}_i$ *corresponds to the member elongation compatible to* $\overline{\boldsymbol{u}}$*.*

Proof. We prove this proposition by showing that Problem (15) corresponds to the dual problem of the static principle problem (5). In association with the constraint

$$q_i^{\rm y} \ge |q_i| \tag{16}$$

in Problem (5), observe that q_i satisfies (16) if and only if the inequality

$$(q_i^{\rm y}, q_i) \cdot (z_i, w_i) \ge 0$$

holds for any (z_i, w_i) satisfying

$$z_i \geq |w_i|. \tag{17}$$

This observation justifies that the function

$$L(\lambda, \boldsymbol{q}, \boldsymbol{u}, \boldsymbol{z}, \boldsymbol{w}) = \begin{cases} \lambda + \boldsymbol{u}^{\top}(\boldsymbol{B}\boldsymbol{q} - \boldsymbol{f}_{\mathrm{D}} - \lambda \boldsymbol{f}_{\mathrm{R}}) + (\boldsymbol{z}^{\top}\boldsymbol{q}^{\mathrm{y}} + \boldsymbol{w}^{\top}\boldsymbol{q}), & \text{if } z_i \geq |w_i|, \ i = 1, \dots, n^{\mathrm{m}}, \\ -\infty, & \text{otherwise} \end{cases}$$

corresponds to the Lagrangian of Problem (5), where $\boldsymbol{u} \in \mathbb{R}^{n^{\mathrm{d}}}$, $\boldsymbol{z} \in \mathbb{R}^{n^{\mathrm{m}}}$, and $\boldsymbol{w} \in \mathbb{R}^{n^{\mathrm{m}}}$ are the Lagrangian multipliers. Then the Lagrangian dual of Problem (5) is defined by

$$\min_{(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{w})} \sup \left\{ L(\lambda, \boldsymbol{q}, \boldsymbol{u}, \boldsymbol{z}, \boldsymbol{w}) : (\lambda, \boldsymbol{q}) \in \mathbb{R} \times \mathbb{R}^{n^{\mathrm{m}}} \right\},$$

the explicit form of which is easily obtained as

$$\left. \begin{array}{ll} \min & -\boldsymbol{f}_{\mathrm{D}}^{\top}\boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top}\boldsymbol{z} \\ \mathrm{s.t.} & w_i = -\boldsymbol{b}_i^{\top}\boldsymbol{u}, \quad \boldsymbol{f}_{\mathrm{R}}^{\top}\boldsymbol{u} = 1, \quad z_i \geq |w_i| \quad i = 1, \dots, n^{\mathrm{m}}. \end{array} \right\} \tag{18}$$

Elimination of $\boldsymbol{w}$ from Problem (18) yields Problem (15). Hence, the LP problem (15) is dual to the LP problem (5), from which and the strong duality of LP [12] we obtain the assertions (i) and (ii). Optimal solutions of Problems (5) and (18) satisfy the complementarity condition

$$z_i q_i^{\mathrm{y}} + w_i q_i = 0 \tag{19}$$

over the constraints (16) and (17). Since (16), (17), and (19) imply $z_i = -w_i$, we see that $\overline{z}_i = \boldsymbol{b}_i^{\top}\overline{\boldsymbol{u}}$ is satisfied at an optimal solution of Problem (15), which concludes the assertion (iii). □

Remark 3.5. Note that the upper-bound principle (15) is different from the well-known formulation for trusses; see, e.g., [29]. Observe that the yield condition (4) in Problem (5) can be rewritten as

$$q_i^{\mathrm{y}} - q_i \geq 0, \quad q_i^{\mathrm{y}} + q_i \geq 0, \quad i = 1, \dots, n^{\mathrm{m}}. \tag{20}$$

The elongation c_i defined in (2) is divided into the two parts as

$$c_i = c_i^{+} - c_i^{-}, \quad c_i^{+} \geq 0, \quad c_i^{-} \geq 0, \quad i = 1, \dots, n^{\mathrm{m}}. \tag{21}$$

By using (20) and (21), it is known that the set of relations governing the elastic/plastic behavior is written as

$$\boldsymbol{B}\boldsymbol{q} = \boldsymbol{f}_{\mathrm{D}} + \lambda \boldsymbol{f}_{\mathrm{R}}, \qquad \text{(equilibrium)} \tag{22a}$$

$$\boldsymbol{q}^{\mathrm{y}} - \boldsymbol{q} \geq \boldsymbol{0}, \quad \boldsymbol{q}^{\mathrm{y}} + \boldsymbol{q} \geq \boldsymbol{0}, \qquad \text{(yield conditions)} \tag{22b}$$

$$\boldsymbol{f}_{\mathrm{R}}^{\top}\boldsymbol{u} = 1, \qquad \text{(normalization)} \tag{22c}$$

$$\boldsymbol{c}^{+} - \boldsymbol{c}^{-} = \boldsymbol{B}^{\top}\boldsymbol{u}, \qquad \text{(compatibility)} \tag{22d}$$

$$\boldsymbol{c}^{+} \geq \boldsymbol{0}, \quad \boldsymbol{c}^{-} \geq \boldsymbol{0}, \qquad \text{(plastic elongation)} \tag{22e}$$

$$(\boldsymbol{q}^{\mathrm{y}} - \boldsymbol{q})^{\top}\boldsymbol{c}^{+} = 0, \quad (\boldsymbol{q}^{\mathrm{y}} + \boldsymbol{q})^{\top}\boldsymbol{c}^{-} = 0. \qquad \text{(complementarity)} \tag{22f}$$

From (21) and (22) it follows that the dual to Problem (5) can be formulated in the variables $\boldsymbol{u} \in \mathbb{R}^{n^{\rm d}}$, $\boldsymbol{c}^{+} \in \mathbb{R}^{n^{\rm m}}$, and $\boldsymbol{c}^{-} \in \mathbb{R}^{n^{\rm m}}$ as

$$\left.\begin{array}{ll} \min & -\boldsymbol{f}_{\rm D}^{\top}\boldsymbol{u} + \boldsymbol{q}^{{\rm y}\top}(\boldsymbol{c}^{+} + \boldsymbol{c}^{-}) \\ \mathrm{s.t.} & \boldsymbol{f}_{\rm R}^{\top}\boldsymbol{u} = 1, \\ & \boldsymbol{c}^{+} - \boldsymbol{c}^{-} = \boldsymbol{B}^{\top}\boldsymbol{u}, \\ & \boldsymbol{c}^{+} \ge \boldsymbol{0}, \quad \boldsymbol{c}^{-} \ge \boldsymbol{0}, \end{array}\right\} \tag{23}$$

which coincides with the conventional formulation of upper-bound principle [29]. However, the number of variables in Problem (23) is larger than that of Problem (15), which may imply an advantage of Problem (15) over Problem (23). □

For $\alpha \in \mathbb{R}_{+}$, consider the following nonconvex problem in the variables $(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\zeta}) \in \mathbb{R}^{n^{\rm d}} \times \mathbb{R}^{n^{\rm m}} \times \mathbb{R}^{m}$:

$$v_{\min}(\alpha) := \min_{\boldsymbol{u},\boldsymbol{z},\boldsymbol{\zeta}} \left\{ -(\widetilde{\boldsymbol{f}}_{\rm D} + \boldsymbol{T}\boldsymbol{\zeta})^{\top}\boldsymbol{u} + \boldsymbol{q}^{{\rm y}\top}\boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U},\ \alpha \ge \|\boldsymbol{\zeta}\|_{\infty} \right\}. \tag{24}$$

The following proposition shows that Problem (24) corresponds to the kinematic version of the uncertain limit analysis (13):

Proposition 3.6 (Relation between Problems (13) and (24)). *Let $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}, \overline{\boldsymbol{\zeta}})$ denote an optimal solution of Problem (24). Then,*

(i) $v_{\min}(\alpha) = \lambda_{\min}(\alpha)$*;*

(ii) $\overline{\boldsymbol{\zeta}}$ *is an optimal solution of Problem (13).*

(iii) $\overline{\boldsymbol{u}}$ *corresponds to a collapse mode associated with the external dead load* $\boldsymbol{f}_{\rm D}(\overline{\boldsymbol{\zeta}})$*;*

(iv) $\overline{z}_i$ *corresponds to the member elongation compatible with* $\overline{\boldsymbol{u}}$*.*

Proof. By using the definition (11) of λ^{*}, Problem (13) can be rewritten equivalently as

$$\min_{\boldsymbol{\zeta}} \left\{ \max_{\lambda,\boldsymbol{q}} \{ \lambda : (\lambda, \boldsymbol{q}) \in \mathcal{Q}(\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta})) \} : \alpha \ge \|\boldsymbol{\zeta}\|_{\infty} \right\} \tag{25}$$

without changing the optimal value. Let $(\widehat{\boldsymbol{\zeta}}, \widehat{\lambda}, \widehat{\boldsymbol{q}})$ denote an optimal solution of Problem (25). It is obvious that $\widehat{\boldsymbol{\zeta}}$ is an optimal solution of Problem (13), and that $\widehat{\lambda} = \lambda_{\min}(\alpha)$. Since the inner problem of Problem (25) coincides with the static principle (13), $\widehat{\boldsymbol{q}}$ corresponds to the vector of axial forces at the collapse mode. By using Proposition 3.4, we can rewrite the inner problem of Problem (25) as

$$\min_{\boldsymbol{\zeta}} \left\{ \min_{\boldsymbol{u},\boldsymbol{z}} \left\{ -\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta})^{\top}\boldsymbol{u} + \boldsymbol{q}^{{\rm y}\top}\boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U} \right\} : \alpha \ge \|\boldsymbol{\zeta}\|_{\infty} \right\} \tag{26}$$

without changing the optimal value. Obviously, $\widehat{\boldsymbol{\zeta}}$ is an optimal solution of Problem (25) if and only if it is an optimal solution of Problem (26). Moreover, Proposition 3.4 guarantees that, at an optimal solution of Problem (26), $\boldsymbol{u}$ and z_i, respectively, coincide with the collapse mode and the member elongation corresponding to $\widehat{\boldsymbol{q}}$. From (6), we see that Problems (24) and (26) share the same optimal value and same optimal solutions, which concludes the proof. □

Proposition 3.6 justifies to solve Problem (24) instead of the bi-level optimization problem (13), i.e., the worst-case limit load factor is obtained as the optimal value of Problem (24), and the critical dead load and the corresponding collapse mode can be obtained simultaneously as the optimal variables. Note that Problem (24) is a nonconvex (but single-level) problem, since the objective function includes the nonconvex quadratic term $\boldsymbol{\zeta}^\top \boldsymbol{T}^\top \boldsymbol{u}$. Hence, the conventional nonlinear programming approach converges to local optimal or stationary solutions in general. It should be emphasized that, for the purpose of the robustness analysis, the proof of global optimum of Problem (24) is strongly desired, since it guarantees that the limit load factor cannot be smaller than the obtained optimal objective value. This is the major difficulty of the uncertain limit analysis. To overcome this difficulty, in the following section we propose an algorithm that converges to a global optimal solution of Problem (24).

4 Global optimization for uncertain limit analysis

In this section, we propose an algorithm to find a global optimal solution of Problem (13) based on the enumeration. We start with reformulating Problem (13) as a mixed 0-1 programming problem.

4.1 Mixed 0-1 programming formulation

Letting

$$\mathcal{C}^0 := \mathbb{R}^{n^{\mathrm{m}}} \times \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \times \mathbb{R}^m,$$

define a set $\mathcal{K} \subseteq \mathcal{C}^0$ by

$$\mathcal{K} = \left\{ (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{C}^0 \;\middle|\; \begin{array}{l} \boldsymbol{f}_{\mathrm{R}}^\top \boldsymbol{u} = 1 \\ \boldsymbol{z} - \boldsymbol{B}^\top \boldsymbol{u} \geq \boldsymbol{0}, \quad \boldsymbol{z} + \boldsymbol{B}^\top \boldsymbol{u} \geq \boldsymbol{0} \\ \boldsymbol{\gamma} - \boldsymbol{T}^\top \boldsymbol{u} \leq M(\boldsymbol{1} - \boldsymbol{\tau}), \quad \boldsymbol{\gamma} + \boldsymbol{T}^\top \boldsymbol{u} \leq M\boldsymbol{\tau} \\ \boldsymbol{0} \leq \boldsymbol{\tau} \leq \boldsymbol{1} \end{array} \right\}, \tag{27}$$

where $M \in \mathbb{R}_+$ is a sufficiently large constant. Let

$$\mathcal{K}^{\mathbb{Z}} = \{ (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{K} \,|\, \boldsymbol{\tau} \in \{0,1\}^m \}. \tag{28}$$

Consider the following optimization problem in the variables $(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{C}^0$:

$$\min \left\{ -\alpha \boldsymbol{1}^\top \boldsymbol{\gamma} - \tilde{\boldsymbol{f}}_{\mathrm{D}}^\top \boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} : (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{K}^{\mathbb{Z}} \right\}. \tag{29}$$

Note that Problem (29) is referred to as the *mixed 0-1 programming problem*, that has binary constraints on $\boldsymbol{\tau}$, linear inequality constraints, and a linear objective function.

Let $\boldsymbol{t}_j \in \mathbb{R}^{n^{\mathrm{d}}}$, $j = 1, \dots, m$, denote the jth column vector of $\boldsymbol{T}$. The following proposition, together with Proposition 3.6, implies that the optimal solution of Problem (13) can be obtained from an optimal solution of Problem (29):

Proposition 4.1 (Relation between Problems (24) **and** (29)**).** *A feasible solution* $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}, \overline{\boldsymbol{\zeta}})$ *of Problem* (24) *satisfying*

$$\overline{\boldsymbol{\zeta}}^\top \boldsymbol{T}^\top \overline{\boldsymbol{u}} = \alpha \| \boldsymbol{T}^\top \overline{\boldsymbol{u}} \|_1 \tag{30}$$

is optimal if and only if a feasible solution $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}, \overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\tau}})$ *of Problem* (29) *satisfying*

$$\overline{\gamma}_j = |\boldsymbol{t}_j^\top \overline{\boldsymbol{u}}|, \quad j = 1, \dots, m, \tag{31}$$

$$\begin{cases} \overline{\tau}_j = 1, & \text{if } \boldsymbol{t}_j^\top \overline{\boldsymbol{u}} > 0, \\ \overline{\tau}_j = 0, & \text{if } \boldsymbol{t}_j^\top \overline{\boldsymbol{u}} < 0, \\ \overline{\tau}_j \in \{0, 1\}, & \text{if } \boldsymbol{t}_j^\top \overline{\boldsymbol{u}} = 0 \end{cases} \tag{32}$$

is optimal. Moreover, Problems (24) *and* (29) *share the same optimal value, that is equal to* $\lambda_{\min}(\alpha)$*.*

Proof. Observe that, in Problem (24), only $\alpha \geq \|\boldsymbol{\zeta}\|_\infty$ is the constraint on $\boldsymbol{\zeta}$, which is independent of the remaining variables $\boldsymbol{z}$ and $\boldsymbol{u}$. Hence, Problem (24) is equivalently rewritten as

$$\left. \begin{array}{ll} \min\limits_{\boldsymbol{u}, \boldsymbol{z}} & \min\limits_{\boldsymbol{\zeta}} \left\{ -(\widetilde{\boldsymbol{f}}_{\mathrm{D}} + \boldsymbol{T}\boldsymbol{\zeta})^\top \boldsymbol{u} : \alpha \geq \|\boldsymbol{\zeta}\|_\infty \right\} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} \\ \text{s.t.} & (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U} \end{array} \right\} \tag{33}$$

without changing the optimal value and optimal solution. From the Hölder inequality [28, Chap. 9], we see that

$$(\boldsymbol{T}\boldsymbol{\zeta})^\top \boldsymbol{u} \geq \|\boldsymbol{\zeta}\|_\infty \|\boldsymbol{T}^\top \boldsymbol{u}\|_1 \tag{34}$$

holds for any fixed $\boldsymbol{u}$. Moreover, Assumption 3.1 (i) guarantees that there exists a $\boldsymbol{\zeta}$ satisfying

$$(\boldsymbol{T}\boldsymbol{\zeta})^\top \boldsymbol{u} = \|\boldsymbol{\zeta}\|_\infty \|\boldsymbol{T}^\top \boldsymbol{u}\|_1. \tag{35}$$

From (34) and (35), we obtain

$$\min_{\boldsymbol{\zeta}} \left\{ -(\boldsymbol{T}\boldsymbol{\zeta})^\top \boldsymbol{u} : \alpha \geq \|\boldsymbol{\zeta}\|_\infty \right\} = \min_{\boldsymbol{\zeta}} \left\{ -\|\boldsymbol{\zeta}\|_\infty \|\boldsymbol{T}^\top \boldsymbol{u}\|_1 : \alpha \geq \|\boldsymbol{\zeta}\|_\infty \right\} = -\alpha \|\boldsymbol{T}^\top \boldsymbol{u}\|_1,$$

where an optimal $\boldsymbol{\zeta}$ satisfies (35). Consequently, the variable $\boldsymbol{\zeta}$ can be eliminated from Problem (33) as

$$\left. \begin{array}{ll} \min\limits_{\boldsymbol{u}, \boldsymbol{z}} & -\alpha \|\boldsymbol{T}^\top \boldsymbol{u}\|_1 - \widetilde{\boldsymbol{f}}_{\mathrm{D}}^\top \boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} \\ \text{s.t.} & (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U}. \end{array} \right\} \tag{36}$$

Note that an optimal solution $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}})$ of (36) can be converted to an optimal solution $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}, \overline{\boldsymbol{\zeta}})$ of (33) by defining $\overline{\boldsymbol{\zeta}}$ by (30), and these two problems share the same objective value. By introducing new variables $\boldsymbol{\gamma} \in \mathbb{R}^m$, Problem (36) is equivalently rewritten as

$$\left. \begin{array}{ll} \min & -\alpha \mathbf{1}^\top \boldsymbol{\gamma} - \widetilde{\boldsymbol{f}}_{\mathrm{D}}^\top \boldsymbol{u} + \boldsymbol{q}^{\mathrm{y}\top} \boldsymbol{z} \\ \text{s.t.} & (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{U}, \\ & \left(\gamma_j = \boldsymbol{t}_j^\top \boldsymbol{u} \right) \vee \left(\gamma_j = -\boldsymbol{t}_j^\top \boldsymbol{u} \right), \quad j = 1, \dots, m. \end{array} \right\} \tag{37}$$

where $\vee$ denotes logical 'or'. Note that (31) holds at an optimal solution of Problem (37). By using a sufficiently large constant M, we see that the disjunction

$$\left(\gamma_j \leq \boldsymbol{t}_j^\top \boldsymbol{u} \right) \vee \left(\gamma_j \leq -\boldsymbol{t}_j^\top \boldsymbol{u} \right)$$

is equivalently rewritten to

$$\gamma_j \leq \boldsymbol{t}_j^\top \boldsymbol{u} + M(1 - \tau_j), \quad \gamma_j \leq -\boldsymbol{t}_j^\top \boldsymbol{u} + M\tau_j, \quad \tau_j \in \{0, 1\}$$

with the relation (32), which completes the proof. □

4.2 Branch-and-bound method for Problem (29)

The LP relaxation of the mixed 0-1 programming problem (29) is obtained by ignoring the binary constraints on $\boldsymbol{\tau}$ as

$$\min\left\{-\alpha\mathbf{1}^\top\boldsymbol{\gamma}-\widetilde{\boldsymbol{f}}_{\mathrm{D}}^\top\boldsymbol{u}+\boldsymbol{q}^{\mathrm{y}\top}\boldsymbol{z} : (\boldsymbol{u},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\tau})\in\mathcal{K}\right\}. \tag{38}$$

Define a set $\mathcal{C}$ as

$$\mathcal{C}=\left\{(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\tau})\in\mathcal{C}^0 \;\middle|\; \boldsymbol{A}_u^\top\boldsymbol{u}+\boldsymbol{A}_z^\top\boldsymbol{z}+\boldsymbol{A}_\gamma^\top\boldsymbol{\gamma}+\boldsymbol{A}_\tau^\top\boldsymbol{\tau}\geq\boldsymbol{b}\right\}, \tag{39}$$

where $\boldsymbol{A}_u$, $\boldsymbol{A}_z$, $\boldsymbol{A}_\gamma$, $\boldsymbol{A}_\tau$, and $\boldsymbol{b}$ are constant matrices and a constant vector with appropriate sizes. Assume that $\mathcal{C}$ satisfies

$$\mathrm{cl}\ \mathrm{conv}\,\mathcal{K}\subseteq\mathcal{C}\subseteq\mathcal{C}^0. \tag{40}$$

Throughout this section we set $\mathcal{C}:=\mathcal{C}^0$, while in section 5 we discuss how to generate a proper subset $\mathcal{C}$ of $\mathcal{C}^0$.

Let $\mathcal{J}_0^k$ and $\mathcal{J}_1^k$ denote the subsets of indices satisfying

$$\mathcal{J}_0^k\subseteq\{1,\dots,m\},\quad \mathcal{J}_1^k\subseteq\{1,\dots,m\},\quad \mathcal{J}_0^k\cap\mathcal{J}_1^k=\emptyset.$$

Define a set $\mathcal{K}(\mathcal{C},\mathcal{J}_0^k,\mathcal{J}_1^k)$ by

$$\mathcal{K}(\mathcal{C},\mathcal{J}_0^k,\mathcal{J}_1^k)=\left\{(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\tau})\in\mathcal{K}\cap\mathcal{C}\;\middle|\;\tau_j=0\text{ for }j\in\mathcal{J}_0^k,\ \tau_j=1\text{ for }j\in\mathcal{J}_1^k\right\},$$

where $\mathcal{K}$ and $\mathcal{C}$ have been defined in (27) and (39). Consider the following LP problem in the variables $(\boldsymbol{z},\boldsymbol{u},\boldsymbol{\gamma},\boldsymbol{\tau})\in\mathcal{C}^0$:

$$\mathrm{LP}(\mathcal{C},\mathcal{J}_0^k,\mathcal{J}_1^k):\quad v^k:=\min\left\{-\alpha\mathbf{1}^\top\boldsymbol{\gamma}-\widetilde{\boldsymbol{f}}_{\mathrm{D}}^\top\boldsymbol{u}+\boldsymbol{q}^{\mathrm{y}\top}\boldsymbol{z} : (\boldsymbol{u},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\tau})\in\mathcal{K}(\mathcal{C},\mathcal{J}_0^k,\mathcal{J}_1^k)\right\}. \tag{41}$$

Explicitly, Problem (41) is written as

$$\begin{aligned}
\min\quad & -\alpha\mathbf{1}^\top\boldsymbol{\gamma}-\widetilde{\boldsymbol{f}}_{\mathrm{D}}^\top\boldsymbol{u}+\boldsymbol{q}^{\mathrm{y}\top}\boldsymbol{z} && \text{(42a)}\\
\mathrm{s.t.}\quad & \boldsymbol{f}_{\mathrm{R}}^\top\boldsymbol{u}=1, && \text{(42b)}\\
& \boldsymbol{z}-\boldsymbol{B}^\top\boldsymbol{u}\geq\mathbf{0}, && \text{(42c)}\\
& \boldsymbol{z}+\boldsymbol{B}^\top\boldsymbol{u}\geq\mathbf{0}, && \text{(42d)}\\
& \boldsymbol{\gamma}-\boldsymbol{T}^\top\boldsymbol{u}\leq M(\mathbf{1}-\boldsymbol{\tau}), && \text{(42e)}\\
& \boldsymbol{\gamma}+\boldsymbol{T}^\top\boldsymbol{u}\leq M\boldsymbol{\tau}, && \text{(42f)}\\
& \mathbf{0}\leq\boldsymbol{\tau}\leq\mathbf{1}, && \text{(42g)}\\
& \boldsymbol{A}_u^\top\boldsymbol{u}+\boldsymbol{A}_z^\top\boldsymbol{z}+\boldsymbol{A}_\gamma^\top\boldsymbol{\gamma}+\boldsymbol{A}_\tau^\top\boldsymbol{\tau}\geq\boldsymbol{b}, && \text{(42h)}\\
& \tau_j=0\text{ for }j\in\mathcal{J}_0^k,\quad \tau_j=1\text{ for }j\in\mathcal{J}_1^k. && \text{(42i)}
\end{aligned}$$

We solve $\mathrm{LP}(\mathcal{C},\mathcal{J}_0^k,\mathcal{J}_1^k)$ at the nodes of enumeration tree. Note that $\mathrm{LP}(\mathcal{C}^0,\emptyset,\emptyset)$ coincides with the LP relaxation (38).

The following is a branch-and-bound method for solving the mixed 0-1 programming problem (29) based on the LP relaxation:

Algorithm 4.2 (Branch-and-Bound Algorithm for Problem (29)).

Step 0: (Initialization) Set $k = 0$, $\mathcal{J}_1^0 = \emptyset$, $\mathcal{J}_0^0 = \emptyset$, and $v^{\mathrm{U}} = \infty$. Choose the small tolerance $\epsilon > 0$ and $\mathcal{C}$ satisfying (40) (set $\mathcal{C} := \mathcal{C}^0$ throughout this section).

Step 1: (Solving subproblem) Solve the linear program $\mathrm{LP}(\mathcal{C}, \mathcal{J}_0^k, \mathcal{J}_1^k)$ defined in (41). If the problem is infeasible, go to Step 5; otherwise, let $(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$ and v^k denote its optimal solution and optimal objective value, respectively.

Step 2: (Fathoming) If $v^k \geq v^{\mathrm{U}}$, go to Step 5.

Step 3: (Branching) If

$$
{\boldsymbol{\tau}^k}^\top (\mathbf{1} - \boldsymbol{\tau}^k) \leq \epsilon,
$$

then go to Step 4; otherwise, select an index j_1 such that

$$
j_1 = \arg \max_{j \in \{1,\ldots,m\}} \left\{ \tau_j^k (1 - \tau_j^k) \right\}.
$$

Set

$$
\begin{aligned}
\mathcal{J}_1^{k+1} &:= \mathcal{J}_1^k \cup \{j_1\}, \\
\boldsymbol{p}^{k+1} &:= (\boldsymbol{p}^k, j_1)^\top,
\end{aligned}
$$

update $k \leftarrow k + 1$, and go to Step 1.

Step 4: (Updating) Put $v^{\mathrm{U}} := v^k$ and $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}, \overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\tau}}) := (\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$. Go to Step 5.

Step 5: (Backtracking) If $\boldsymbol{p}^k < \mathbf{0}$, then go to Step 6. Otherwise branch to a new live node as follows. Letting L denote the size of the vector $\boldsymbol{p}^k$, define l_1 by

$$
l_1 = \max \left\{ l \in \{1, \ldots, L\} \,\middle|\, p_l^k > 0 \right\}.
$$

Divide $\boldsymbol{p}^k$ into the three parts as

$$
\boldsymbol{p}_1 = \left(p_l^k \,\middle|\, l = 1, \ldots, l_1 - 1 \right), \quad p_2 = p_{l_1}^k, \quad \mathcal{P}_3 = \left\{ -p_l^k \,\middle|\, l = l_1 + 1, \ldots, L \right\}.
$$

Set

$$
\begin{aligned}
\boldsymbol{p}^{k+1} &:= \left(\boldsymbol{p}_1^\top, -p_2 \right)^\top, \\
\mathcal{J}_0^{k+1} &:= \left\{ \mathcal{J}_0^k \cup \{p_2\} \right\} \setminus \mathcal{P}_3, \\
\mathcal{J}_1^{k+1} &:= \mathcal{J}_1^k \setminus \{p_2\},
\end{aligned}
$$

update $k \leftarrow k + 1$, and go to Step 1.

Step 6: (Termination) Declare $(\overline{\boldsymbol{z}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\tau}})$ as the optimal solution, and stop.

Remark 4.3. Essentially, Algorithm 4.2 is designed by employing the *depth first search* (see, e.g., [13]) as a strategy for selecting the next live subproblem at Step 5. The condition $\boldsymbol{p}^k < \boldsymbol{0}$ implies that there exists no live node. Among the live subproblems, we always select the subproblem with the largest level in the branch-and-bound tree. The vector $\boldsymbol{p}^k$ plays the role of bookkeeping the path from the root node to the current node in the branch-and-bound tree. The size L of $\boldsymbol{p}^k = (p_j^k)$ coincides with the current depth of the tree, and we see that the relations

$$\mathcal{J}_0^k = \left\{ p_j^k \,\middle|\, p_j^k \le 0,\ j = 1, \dots, L \right\},$$
$$\mathcal{J}_1^k = \left\{ -p_j^k \,\middle|\, p_j^k \ge 0,\ j = 1, \dots, L \right\}$$

hold, i.e., the components of $\boldsymbol{p}^k$ correspond to the indices of τ_j, possibly with opposite signs, that are fixed in the current subproblem $\mathrm{LP}(\mathcal{C}, \mathcal{J}_0^k, \mathcal{J}_1^k)$; the remaining τ_j are not fixed in $\mathrm{LP}(\mathcal{C}, \mathcal{J}_0^k, \mathcal{J}_1^k)$. The order of an element p_j^k of $\boldsymbol{p}^k$ is determined by its level in the tree. □

Remark 4.4. Observe that the binary constraints $\boldsymbol{\tau} \in \{0, 1\}^m$ are equivalent to the following complementarity conditions:

$$\boldsymbol{\tau} \ge \boldsymbol{0}, \quad \boldsymbol{1} - \boldsymbol{\tau} \ge \boldsymbol{0}, \tag{43}$$
$$\tau_j(1 - \tau_j) = 0, \quad j = 1, \dots, m. \tag{44}$$

Notice here that any feasible solution of $\mathrm{LP}(\mathcal{C}, \mathcal{J}_0^k, \mathcal{J}_1^k)$ satisfies (43). At Step 3, we make a check if the current solution $(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$ satisfies the complementarity conditions (44) or not. Satisfaction (possibly with small tolerance in practice) implies that $(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$ is a feasible solution of Problem (29). Alternatively, if (44) is not satisfied, then the variable τ_j with the largest residual of the complementarity (44) is used as the branching variable in Step 3. □

Remark 4.5. Note that it is not difficult to randomly generate $\boldsymbol{f}'_{\mathrm{D}}$ satisfying $\boldsymbol{f}'_{\mathrm{D}} \in \mathcal{F}_{\mathrm{D}}(\alpha)$. Then the corresponding limit load factor $\lambda^*(\boldsymbol{f}'_{\mathrm{D}})$ provides an upper bound of the mixed 0-1 problem (29). At Step 0, we can obtain an upper bound v^{U} by solving Problem (11) several times for randomly sampled $\boldsymbol{f}'_{\mathrm{D}}$. We simply set $v^{\mathrm{U}} = \infty$ if this process is skipped. □

4.3 Duality and simplification

The remainder of this section is devoted to some practical issues on implementation of Algorithm 4.2. In fact, to obtain $(\boldsymbol{u}^k, \boldsymbol{z}^k, \boldsymbol{\gamma}^k, \boldsymbol{\tau}^k)$ at Step 1, we do not solve Problem (41) directly but solve its Lagrangian dual problem, that is denoted by $\mathrm{LP}^*(\mathcal{C}, \mathcal{J}_0^k, \mathcal{J}_1^k)$, by using the simplex method. Then the solution of Problem (41) is obtained as the optimal Lagrange multipliers. It is observed from our preliminary numerical experiments that the CPU time required to solve the dual problem is much smaller than that required to solve the original problem (41). Indeed, after the branching process of Step 3, at the new node it is easy to obtain a feasible solution of the dual problem from an optimal solution of the dual problem solved at the previous node.

Let

$$\mathcal{C}^{0*} = \mathbb{R} \times \mathbb{R}^{n^{\mathrm{m}}} \times \mathbb{R}^{n^{\mathrm{m}}} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m.$$

From the LP duality [12] it follows that the dual problem of the LP relaxation $\mathrm{LP}(\mathcal{C},\emptyset,\emptyset)$ is formulated in the variables $(\rho^{\lambda},\boldsymbol{q}^{+},\boldsymbol{q}^{-},\boldsymbol{\zeta}^{+},\boldsymbol{\zeta}^{-},\boldsymbol{\rho}^{+},\boldsymbol{\rho}^{-},\boldsymbol{\mu})\in\mathcal{C}^{0*}\times\mathbb{R}^{n^{\mathrm{c}}}$ as

$$
\begin{aligned}
\mathrm{LP}^{*}(\mathcal{C},\emptyset,\emptyset):\quad \max\quad & \rho^{\lambda}-\mathbf{1}^{\top}(M\boldsymbol{\rho}^{+}+\boldsymbol{\rho}^{-})+\boldsymbol{b}^{\top}\boldsymbol{\mu} && \text{(45a)}\\
\text{s.t.}\quad & \boldsymbol{B}(\boldsymbol{q}^{+}-\boldsymbol{q}^{-})=\widetilde{\boldsymbol{f}}_{\mathrm{D}}+\boldsymbol{T}(\boldsymbol{\zeta}^{+}-\boldsymbol{\zeta}^{-})+\boldsymbol{f}_{\mathrm{R}}\rho^{\lambda}+\boldsymbol{A}_{u}\boldsymbol{\mu}, && \text{(45b)}\\
& \boldsymbol{q}^{+}+\boldsymbol{q}^{-}+\boldsymbol{A}_{z}\boldsymbol{\mu}=\boldsymbol{q}^{\mathrm{y}}, && \text{(45c)}\\
& \boldsymbol{\zeta}^{+}+\boldsymbol{\zeta}^{-}=\alpha\mathbf{1}+\boldsymbol{A}_{\gamma}\boldsymbol{\mu}, && \text{(45d)}\\
& M(\boldsymbol{\zeta}^{+}-\boldsymbol{\zeta}^{-})=\boldsymbol{\rho}^{+}-\boldsymbol{\rho}^{-}+\boldsymbol{A}_{\tau}\boldsymbol{\mu}, && \text{(45e)}\\
& \boldsymbol{q}^{+},\boldsymbol{q}^{-},\boldsymbol{\zeta}^{+},\boldsymbol{\zeta}^{-},\boldsymbol{\rho}^{+},\boldsymbol{\rho}^{-},\boldsymbol{\mu}\geq\mathbf{0}. && \text{(45f)}
\end{aligned}
$$

Let $\mathrm{LP}^{*}(\mathcal{C},\mathcal{J}_0^k,\mathcal{J}_1^k)$ denote the dual of $\mathrm{LP}(\mathcal{C},\mathcal{J}_0^k,\mathcal{J}_1^k)$ in which some variables are fixed. To obtain $\mathrm{LP}^{*}(\mathcal{C},\mathcal{J}_0^k,\mathcal{J}_1^k)$, we make the following modification and simplification on Problem (45):

(a) For $j\in\mathcal{J}_0^k$, the variable τ_j in the primal problem (42) is set as $\tau_j=0$. This is realized in the dual problem as follows: (i) The jth row of (42f) should be rewritten as $-\boldsymbol{t}_j^{\top}\boldsymbol{u}-\gamma_j\geq 0$. Consequently, the variable ζ_j^{-} should be eliminated from the jth row of (45e). (ii) The jth row of (42e) becomes redundant. Hence, the variable ζ_j^{+} can be eliminated from Problem (45). (iii) The jth row of (42g) becomes redundant. Hence, the variables ρ_j^{+} and ρ_j^{-} can be eliminated from Problem (45). (iv) If the jth row vector of $\boldsymbol{A}_{\tau}$ is a zero vector, then the constraint (45e) itself can be eliminated.

(b) For $j\in\mathcal{J}_1^k$, the variable τ_j in the primal problem (42) is set as $\tau_j=1$. This is realized in the dual problem as follows: (i) The jth row of (42e) should be rewritten as $\boldsymbol{t}_j^{\top}\boldsymbol{u}-\gamma_j\geq 0$. Consequently, the variable ζ_j^{+} should be eliminated from the jth row of (45e). (ii) The jth row of (42f) becomes redundant. Hence, the variable ζ_j^{-} can be eliminated from Problem (45). (iii) The jth row of (42g) becomes redundant. Hence, the variables ρ_j^{+} and ρ_j^{-} can be eliminated from Problem (45). (iv) If the jth row vector of $\boldsymbol{A}_{\tau}$ is a zero vector, then the constraint (45e) itself can be eliminated.

5 Cutting plane algorithm

It is guaranteed that Algorithm 4.2 converges to a global optimal solution of the mixed 0-1 programming problem (29). However, it is possible that the algorithm is no better than the enumeration of all binary variables $\boldsymbol{\tau}$. The efficiency of the algorithm depends partially on the tightness of the LP relaxation problem solved at each node of the branch-and-bound tree. In order to strengthen the LP relaxation problems, we propose an LP problem that generates the disjunctive cutting planes.

5.1 Disjunctive cut generation

Recall that $\mathcal{K}^{\mathbb{Z}}$ and $\mathcal{K}$, defined in (28) and (27), correspond to the feasible sets of the mixed 0-1 program (29) and its LP relaxation (38), respectively. Let $(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\gamma}},\widehat{\boldsymbol{\tau}})$ denote the optimal solution of the LP relaxation (38). Suppose that $\widehat{\boldsymbol{\tau}}$ does not satisfy the binary constraints in $\mathcal{K}^{\mathbb{Z}}$, i.e., $(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\gamma}},\widehat{\boldsymbol{\tau}})\notin\mathcal{K}^{\mathbb{Z}}$. The *cutting plane* is an additional linear inequality that the point $(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\gamma}},\widehat{\boldsymbol{\tau}})$ does not satisfy but is valid for $\mathcal{K}^{\mathbb{Z}}$. If a cutting plane is generated successfully, then we can add it to the LP relaxation as the constraint without extracting any feasible solution in $\mathcal{K}^{\mathbb{Z}}$. If the new optimal solution of the obtained LP problem is feasible for $\mathcal{K}^{\mathbb{Z}}$, then it is a global optimal solution of the original mixed 0-1 program problem (29); otherwise, we may continue to generate cutting planes.

In the following, the cutting plane generation is performed over the so-called *disjunctive programming* relaxation of $\mathcal{K}^{\mathbb{Z}}$ instead of $\mathcal{K}^{\mathbb{Z}}$. Then an obtained valid inequality is called the *disjunctive cut* [9]. Define sets $P_j(\mathcal{K})$ by

$$P_j(\mathcal{K}) = \mathrm{cl\ conv}\,\{(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in \mathcal{K} \mid \tau_j \in \{0, 1\}\}\,, \quad j = 1, \dots, m,$$

which is a disjunctive programming relaxation of the closure of $\mathrm{conv}\,\mathcal{K}^{\mathbb{Z}}$. We attempt to find an linear inequality that cuts off $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})$ but is valid for $P_j(\mathcal{K})$. Although the characterization of $P_j(\mathcal{K})$ is essentially nonlinear, a polyhedral representation can be obtained easily [3, 4], i.e., we have

$$(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \in P_j(\mathcal{K})$$

if and only if there exist $(\boldsymbol{w}^u, \boldsymbol{w}^z, \boldsymbol{w}^\gamma, \boldsymbol{w}^\tau) \in \mathcal{C}^0$, $w_0 \in \mathbb{R}$, $(\boldsymbol{y}^u, \boldsymbol{y}^z, \boldsymbol{y}^\gamma, \boldsymbol{y}^\tau) \in \mathcal{C}^0$, and $y_0 \in \mathbb{R}$ satisfying

$$(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) = (\boldsymbol{w}^u, \boldsymbol{w}^z, \boldsymbol{w}^\gamma, \boldsymbol{w}^\tau) + (\boldsymbol{y}^u, \boldsymbol{y}^z, \boldsymbol{y}^\gamma, \boldsymbol{y}^\tau), \tag{46a}$$
$$\boldsymbol{f}_{\mathrm{R}}^\top \boldsymbol{w}^u = w_0, \tag{46b}$$
$$\boldsymbol{w}^z - \boldsymbol{B}^\top \boldsymbol{w}^u \geq \boldsymbol{0}, \quad \boldsymbol{w}^z + \boldsymbol{B}^\top \boldsymbol{w}^u \geq \boldsymbol{0}, \tag{46c}$$
$$\boldsymbol{T}^\top \boldsymbol{w}^u - \boldsymbol{w}^\gamma - M\boldsymbol{w}^\tau \geq -Mw_0\boldsymbol{1}, \quad -\boldsymbol{T}^\top \boldsymbol{w}^u - \boldsymbol{w}^\gamma + M\boldsymbol{w}^\tau \geq \boldsymbol{0}, \tag{46d}$$
$$\boldsymbol{0} \leq \boldsymbol{w}^\tau \leq w_0\boldsymbol{1}, \tag{46e}$$
$$w_j^\tau \leq 0, \tag{46f}$$
$$\boldsymbol{f}_{\mathrm{R}}^\top \boldsymbol{y}^u = y_0, \tag{46g}$$
$$\boldsymbol{y}^z - \boldsymbol{B}^\top \boldsymbol{y}^u \geq \boldsymbol{0}, \quad \boldsymbol{y}^z + \boldsymbol{B}^\top \boldsymbol{y}^u \geq \boldsymbol{0}, \tag{46h}$$
$$\boldsymbol{T}^\top \boldsymbol{y}^u - \boldsymbol{y}^\gamma - M\boldsymbol{y}^\tau \geq -My_0\boldsymbol{1}, \quad -\boldsymbol{T}^\top \boldsymbol{y}^u - \boldsymbol{y}^\gamma + M\boldsymbol{y}^\tau \geq \boldsymbol{0}, \tag{46i}$$
$$\boldsymbol{0} \leq \boldsymbol{y}^\tau \leq y_0\boldsymbol{1}, \tag{46j}$$
$$y_j^\tau \leq y_0, \tag{46k}$$
$$w_0 + y_0 = 1. \tag{46l}$$

Define a set $P_j^*(\mathcal{K}) \subseteq \mathcal{C}^0 \times \mathbb{R}$ so that

$$(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau, \beta) \in P_j^*(\mathcal{K}) \tag{47}$$

holds if and only if there exist $(\xi^\lambda, \boldsymbol{\xi}^{q+}, \boldsymbol{\xi}^{q-}, \boldsymbol{\xi}^{\zeta+}, \boldsymbol{\xi}^{\zeta-}, \boldsymbol{\xi}^{\rho+}, \boldsymbol{\xi}^{\rho-}) \in \mathcal{C}^{0*}$, $\xi^0 \in \mathbb{R}$, $(\eta^\lambda, \boldsymbol{\eta}^{q+}, \boldsymbol{\eta}^{q-}, \boldsymbol{\eta}^{\zeta+}, \boldsymbol{\eta}^{\zeta-}$,

$\boldsymbol{\eta}^{\rho+}, \boldsymbol{\eta}^{\rho-}) \in \mathcal{C}^{0*}$, and $\eta^0 \in \mathbb{R}$ satisfying

$$\boldsymbol{\alpha}_u = \boldsymbol{f}_{\mathrm{R}}\xi^{\lambda} - \boldsymbol{B}(\boldsymbol{\xi}^{q+} - \boldsymbol{\xi}^{q-}) + \boldsymbol{T}(\boldsymbol{\xi}^{\zeta+} - \boldsymbol{\xi}^{\zeta-}), \tag{48a}$$

$$\boldsymbol{\alpha}_z = \boldsymbol{\xi}^{q+} + \boldsymbol{\xi}^{q-}, \tag{48b}$$

$$\boldsymbol{\alpha}_\gamma = -\boldsymbol{\xi}^{\zeta+} - \boldsymbol{\xi}^{\zeta-}, \tag{48c}$$

$$\boldsymbol{\alpha}_\tau = -M(\boldsymbol{\xi}^{\zeta+} - \boldsymbol{\xi}^{\zeta-}) + \boldsymbol{\xi}^{\rho+} - \boldsymbol{\xi}^{\rho-} - \xi^0 \boldsymbol{e}^j, \tag{48d}$$

$$\beta = \xi^{\lambda} - \mathbf{1}^{\top}(M\boldsymbol{\xi}^{\zeta+} + \boldsymbol{\xi}^{\rho-}), \tag{48e}$$

$$\boldsymbol{\xi}^{q+}, \boldsymbol{\xi}^{q-}, \boldsymbol{\xi}^{\zeta+}, \boldsymbol{\xi}^{\zeta-}, \boldsymbol{\xi}^{\rho+}, \boldsymbol{\xi}^{\rho-}, \xi^0 \geq \mathbf{0}, \tag{48f}$$

$$\boldsymbol{\alpha}_u = \boldsymbol{f}_{\mathrm{R}}\eta^{\lambda} - \boldsymbol{B}(\boldsymbol{\eta}^{q+} - \boldsymbol{\eta}^{q-}) + \boldsymbol{T}(\boldsymbol{\eta}^{\zeta+} - \boldsymbol{\eta}^{\zeta-}), \tag{48g}$$

$$\boldsymbol{\alpha}_z = \boldsymbol{\eta}^{q+} + \boldsymbol{\eta}^{q-}, \tag{48h}$$

$$\boldsymbol{\alpha}_\gamma = -\boldsymbol{\eta}^{\zeta+} - \boldsymbol{\eta}^{\zeta-}, \tag{48i}$$

$$\boldsymbol{\alpha}_\tau = -M(\boldsymbol{\eta}^{\zeta+} - \boldsymbol{\eta}^{\zeta-}) + \boldsymbol{\eta}^{\rho+} - \boldsymbol{\eta}^{\rho-} + \eta^0 \boldsymbol{e}^j, \tag{48j}$$

$$\beta = \eta^{\lambda} - \mathbf{1}^{\top}(M\boldsymbol{\eta}^{\zeta+} + \boldsymbol{\eta}^{\rho-}) + \eta^0, \tag{48k}$$

$$\boldsymbol{\eta}^{q+}, \boldsymbol{\eta}^{q-}, \boldsymbol{\eta}^{\zeta+}, \boldsymbol{\eta}^{\zeta-}, \boldsymbol{\eta}^{\rho+}, \boldsymbol{\eta}^{\rho-}, \eta^0 \geq \mathbf{0}. \tag{48l}$$

Then, an inequality

$$(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau) \cdot (\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\gamma}, \boldsymbol{\tau}) \geq \beta \tag{49}$$

is valid for $P_j(\mathcal{K})$ if it satisfies (47).

Thus, for a point $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}}) \notin P_j(\mathcal{K})$, we are interested in the following problem in the variables $(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau, \beta) \in \mathcal{C} \times \mathbb{R}$:

$$\left.\begin{array}{ll} \max & \beta - (\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau) \cdot (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}}) \\ \text{s.t.} & (\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau, \beta) \in P_j^*(\mathcal{K}), \end{array}\right\} \tag{50}$$

because a feasible solution of Problem (50) defines a valid inequality (in the form of (49)) for $P_j(\mathcal{K})$, that is violated at $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})$. However, some normalization constraints should be appended to Problem (50), since Problem (50) itself is unbounded.

As a normalization scheme, we add the constraints restricting the magnitude of the vector $(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau)$ [9]. Define the index sets $\mathcal{I}$ and $\overline{\mathcal{I}}$ of a partition of $\{1, \dots, n^{\mathrm{d}} + n^{\mathrm{m}} + 2m\}$ by

$$\overline{\mathcal{I}} = \left\{ i \in \{1, \dots, n^{\mathrm{d}} + n^{\mathrm{m}} + 2m\} \,\middle|\, (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})_i = 0 \right\},$$
$$\mathcal{I} = \{1, \dots, n^{\mathrm{d}} + n^{\mathrm{m}} + 2m\} \setminus \overline{\mathcal{I}}.$$

We assume $\mathcal{I} \neq \emptyset$. Consider the following cut generation problem:

$$\left.\begin{array}{lll} (\mathrm{CGLP})_j: & \max & \beta - (\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau) \cdot (\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}}) \\ & \text{s.t.} & (\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau, \beta) \in P_j^*(\mathcal{K}), \\ & & \|(\boldsymbol{\alpha}_u, \boldsymbol{\alpha}_z, \boldsymbol{\alpha}_\gamma, \boldsymbol{\alpha}_\tau)_{\mathcal{I}}\|_\infty \leq 1, \end{array}\right\} \tag{51}$$

where $P_j^*(\mathcal{K})$ has been defined in (48). Note that (51) is an LP problem. In Problem (51), we attempt to find the *deepest* cut in the sense that a distance from $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{z}}, \widehat{\boldsymbol{\gamma}}, \widehat{\boldsymbol{\tau}})$ to a separating hyperplane is maximized. The dual to Problem (51) is formulated as [9]

$$\left.\begin{array}{ll} \min & \|(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\tau})-(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\gamma}},\widehat{\boldsymbol{\tau}})\|_1 \\ \text{s.t.} & (\boldsymbol{u},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\tau}) \in P_j(\mathcal{K}), \\ & (\boldsymbol{u},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\tau})_{\overline{\mathcal{I}}} = (\widehat{\boldsymbol{u}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\gamma}},\widehat{\boldsymbol{\tau}})_{\overline{\mathcal{I}}}, \end{array}\right\} \tag{52}$$

where $P_j(\mathcal{K})$ has been defined in (46).

At the root node of the enumeration tree of Algorithm 4.2, we employ the following procedure for generating some disjunctive cuts:

Algorithm 5.1 (Cut Generation for Problem (29)**).**

Step 0: Set $\mathcal{C}^0 = \mathbb{R}^{n^{\mathrm{m}}} \times \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \times \mathbb{R}^m$, $\mathcal{J}^0_{\mathrm{res}} = \{1,\dots,m\}$, $\mathcal{J} = \{1,\dots,m\}$, and $k=1$. Let $(\boldsymbol{u}^0,\boldsymbol{z}^0,\boldsymbol{\gamma}^0,\boldsymbol{\tau}^0)$ an optimal solution of $\mathrm{LP}(\mathcal{C}^0,\emptyset,\emptyset)$.

Step 1: Select $j_2 \in \{1,\dots,m\}$ by

$$j_2 = \arg\max_{j\in\mathcal{J}} \left\{\tau_j^{k-1}(1-\tau_j^{k-1})\right\}.$$

Step 2: Solve $(\mathrm{CGLP})_{j_2}$, with the definition (48) of $P_j^*(\mathcal{K})$, at $(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{z}},\widehat{\boldsymbol{\gamma}},\widehat{\boldsymbol{\tau}}) = (\boldsymbol{u}^{k-1},\boldsymbol{z}^{k-1},\boldsymbol{\gamma}^{k-1},\boldsymbol{\tau}^{k-1})$ to find an optimal solution $(\boldsymbol{\alpha}_u^k,\boldsymbol{\alpha}_z^k,\boldsymbol{\alpha}_\gamma^k,\boldsymbol{\alpha}_\tau^k,\beta^k)$.

Step 3: Letting

$$\mathcal{C}_{\mathrm{cur}} := \left\{(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\tau}) \in \mathcal{C}^{k-1} \,\middle|\, (\boldsymbol{\alpha}_u^k,\boldsymbol{\alpha}_z^k,\boldsymbol{\alpha}_\gamma^k,\boldsymbol{\alpha}_\tau^k)\cdot(\boldsymbol{u},\boldsymbol{z},\boldsymbol{\gamma},\boldsymbol{\tau}) \geq \beta^k\right\},$$

solve $\mathrm{LP}(\mathcal{C}_{\mathrm{cur}},\emptyset,\emptyset)$ to find an optimal solution $(\boldsymbol{u}^k,\boldsymbol{z}^k,\boldsymbol{\gamma}^k,\boldsymbol{\tau}^k)$.

Step 4: Let

$$\mathcal{J}^k_{\mathrm{res}} = \left\{j \in \{1,\dots,m\} \,\middle|\, \tau_j^k(1-\tau_j^k) > \epsilon\right\}.$$

If $|\mathcal{J}^k_{\mathrm{res}}| \leq |\mathcal{J}^{k-1}_{\mathrm{res}}|$, then let $\mathcal{C}^k := \mathcal{C}_{\mathrm{cur}}$ and $\mathcal{J} := \{1,\dots,m\}$; otherwise, $\mathcal{C}^k := \mathcal{C}^{k-1}$ and $\mathcal{J} := \{1,\dots,m\}\setminus j_2$.

Step 5: If the termination condition is satisfied, then stop; otherwise, update $k \leftarrow k+1$, and go to Step 1.

Remark 5.2. If $\mathcal{J}^k_{\mathrm{res}} = \emptyset$ at Step 4, then stop, because the current solution $(\boldsymbol{u}^k,\boldsymbol{z}^k,\boldsymbol{\gamma}^k,\boldsymbol{\tau}^k)$ is a global optimal solution of the original problem (29). However, it often requires large computational time to solve Problem(29) only by Algorithm 5.1. In practice, we restrict the maximum number of iterations as $k \leq 1.8m$ and then employ Algorithm 4.2. The set of disjunctive cuts $\mathcal{C}^k$ generated by Algorithm 5.1 plays a role to strengthen the LP relaxation problems solved in Algorithm 4.2. □

Remark 5.3. At Step 1, as is done in Step 3 of Algorithm 4.2, we select the variable τ_j with the largest residual of the complementarity condition (44). Then the variable τ_{j_2} is used at Step 2 to define the disjunctive constraint. □

5.2 Simplification of cut generating LP

We further analyze the simplifications that take place for the cut generating LP problem (51). This is motivated by the fact that, in the dual problem (52), we can eliminate some of variables corresponding to the index set $\overline{\mathcal{I}}$. Recall that $P_j(\mathcal{K})$ in the constraints of Problem (52) has been defined in (46).

(a) For i such that $\widehat{z}_i = 0$:
Observe that $(\boldsymbol{\alpha}_z)_i$ does not contribute to the objective function of Problem (51). By replacing the ith rows of (48b) and (48h) with

$$\xi_i^{q+} + \xi_i^{q-} = \eta_i^{q+} + \eta_i^{q-}, \tag{53}$$

we can remove the variable $(\boldsymbol{\alpha}_z)_i$ from (48) without changing the optimal solution. At an optimal solution, we can complete $(\overline{\boldsymbol{\alpha}}_z)_i$ by letting

$$(\overline{\boldsymbol{\alpha}}_z)_i := \overline{\xi}_i^{q+} + \overline{\xi}_i^{q-}. \tag{54}$$

However, instead of (53), we append more restrictive constraints

$$\xi_i^{q+} = \eta_i^{q-}, \quad \xi_i^{q-} = \eta_i^{q+}$$

to (48), which enables us to remove the variables η_i^{q+} and η_i^{q-}. Then an optimal solution of the simplified problem can be completed to an optimal solution of Problem (51) by using (54).

(b) For i such that $\widehat{\tau}_i = 0$:
In the system of (46), observe that (46a), (46e), (46j), and $\widehat{\tau}_i = 0$ imply

$$w_i^\tau = y_i^\tau = 0 \tag{55}$$

(b1) Assume that there exists an $l \in \{1, \dots, m\}$ such that $\widehat{\tau}_l \neq 0$. Then, in the system (46), (55) and the lth row of (46e) make the constraint

$$w_i^\tau \leq w_0 \tag{56}$$

redundant. Similarly, it follows from the lth row of (46j) and (55) that the constraint

$$y_i^\tau \leq y_0 \tag{57}$$

is redundant. Then, we see that eliminating (56) and (57) from (46) is equivalent to eliminating the variables $\xi_i^{\rho-}$ and $\eta_i^{\rho-}$ from (48).

(b2) In (46), it follows from (55) that the jth rows of (46d) can be replaced with

$$\boldsymbol{t}_i^\top \boldsymbol{w}^u - w_i^\gamma \geq -M w_0, \quad -\boldsymbol{t}_i^\top \boldsymbol{w}^u - w_i^\gamma \geq 0$$

without changing $P_j(\mathcal{K})$. Then, in (48), the ith row of (48d) is replaced with

$$(\boldsymbol{\alpha}^\tau)_i = \xi_i^{\rho+} - \xi_0 e_i^j. \tag{58}$$

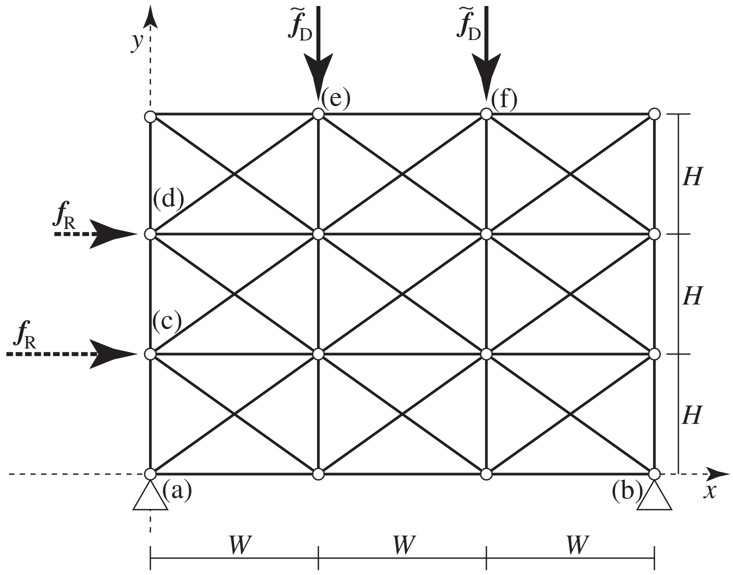

Figure 1: 3×3 plane grid truss.

Similarly, the ith row of (48j) is replaced with

$$(\boldsymbol{\alpha}^{\tau})_i = \eta_i^{\rho+} + \eta_0 e_i^j. \tag{59}$$

Note that, in Problem (51), the variables $\xi_i^{\rho+}$ and $\eta_i^{\rho+}$ appear only in the constraints (58) and (59), respectively. Moreover, $(\boldsymbol{\alpha}_{\tau})_i$ does not contribute to the objective function. Consequently, from Problem (51), we can eliminate the variables $(\boldsymbol{\alpha}_{\tau})_i$, $\xi_i^{\rho+}$, and $\eta_i^{\rho+}$ and the constraints (58) and (59). From the nonnegativity of $\xi_i^{\rho+}$ and $\eta_i^{\rho+}$ it follows that an optimal solution of the simplified problem can be completed to an optimal solution of Problem (51) by defining the eliminated variables as

$$(\overline{\boldsymbol{\alpha}}_{\tau})_i := \max\left\{-\overline{\xi}_0 e_i^j,\ \overline{\eta}_0 e_i^j\right\}.$$

6 Numerical experiments

The worst-case limit load factors are computed for trusses by using Algorithm 4.2 and Algorithm 5.1. Computation has been carried out on a Pentium M (1.5 GHz with 1 GB memory) with MATLAB Ver. 6.5.1 [36]. The LP problems are solved by using the simplex method at Step 1 of Algorithm 4.2 and at Steps 2 and 3 of Algorithm 5.1. As an implementation of the simplex method, we use MATLAB built-in function `linprog` of Optimization Toolbox Ver. 2.1 [35], where the options '`LargeScale`' and '`Simplex`' are set to '`off`' and '`on`', respectively.

In the following examples, the yield stress is $\sigma_i^{\rm y} = 400$ MPa and cross-sectional area is $a_i = 20.0\ {\rm cm}^2$ for each member. We set $M = 5.0$ in Algorithm 4.2 and Algorithm 5.1.

6.1 3×3 truss

Consider a plane truss illustrated in Fig.1, where $W = 70.0$ cm, $H = 50.0$ cm, $n^{\rm d} = 28$, and $n^{\rm m} = 42$. The nodes (a) and (b) are pin-supported.

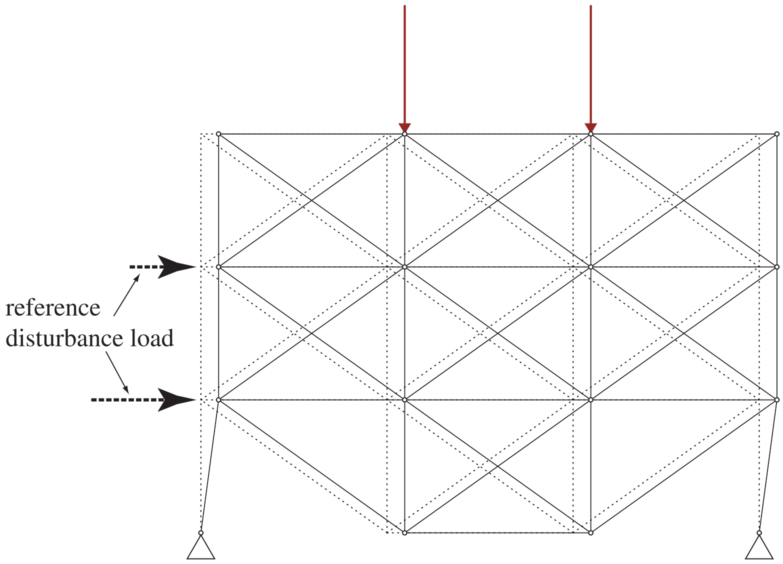

Figure 2: Collapse mode and the dead load of the 42-bar truss without the uncertainty in dead load ($\lambda^*(\widetilde{\boldsymbol{f}}_{\rm D}) = 48.3662$).

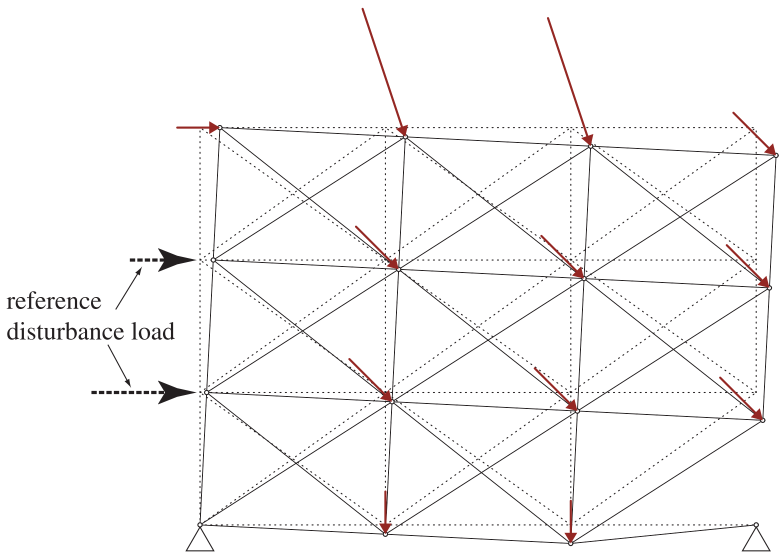

Figure 3: Collapse mode and the critical dead load of the 42-bar truss in the worst case for $\alpha_1 =$ 40.0 kN ($\lambda_{\rm min}(\alpha_1) = 37.0120$).

As the nominal dead load $\widetilde{\boldsymbol{f}}_{\rm D}$, we apply the external forces $(0, -120.0)$ kN at the nodes (e) and (f) as shown in Fig.1. The reference disturbance load $\boldsymbol{f}_{\rm R}$ is defined such that $(40.0, 0)$ kN and $(20.0, 0)$ kN, respectively, are applied at the nodes (c) and (d). The limit load factor under the nominal dead loads is computed as $\lambda^*(\widetilde{\boldsymbol{f}}_{\rm D}) = 48.3662$ by employing the usual limit analysis, i.e., by solving the LP (5). The collapse mode corresponds to the sway-type with horizontal displacements of the joints shown in Fig.2, where the vanishing members experience plastic deformations.

In accordance with Assumption 3.1, the uncertain load $\boldsymbol{T\zeta}$ are assumed to exist possibly at all free nodes except for the components that $\widetilde{\boldsymbol{f}}_{\rm D}$ and $\boldsymbol{f}_{\rm R}$ are nonzero, i.e. $m = 24$. For $\alpha_1 = 40.0$ kN, the worst-case limit load factor is computed as $\lambda_{\rm min}(\alpha_1) = 37.0120$ by using Algorithm 4.2 and

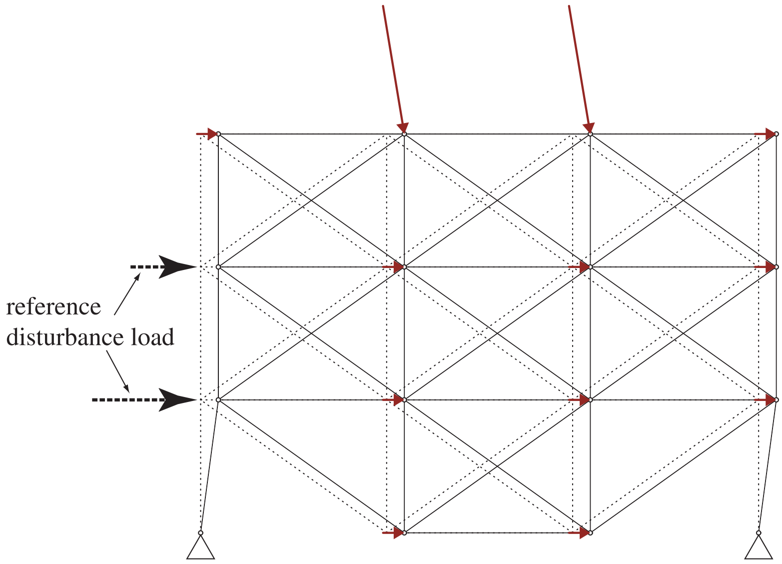

Figure 4: Collapse mode and the critical dead load of the 42-bar truss in the worst case for $\alpha_2 =$ 20.0 kN ($\lambda_{\min}(\alpha_2) = 44.3662$).

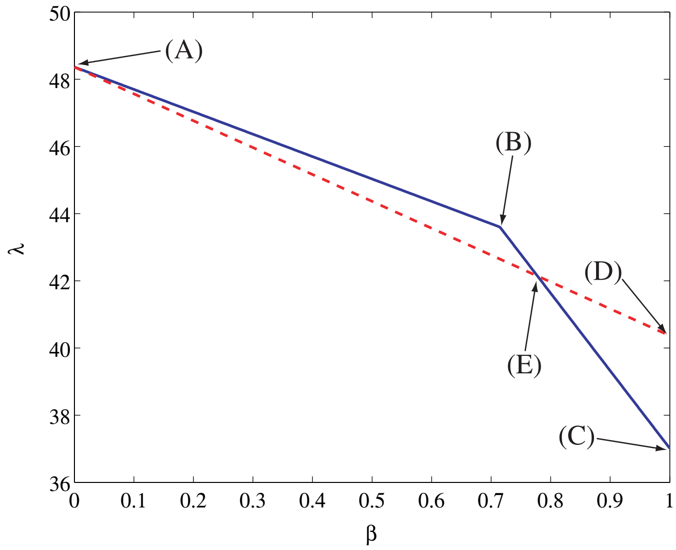

Figure 5: Limit load factor of the 3×3 truss; $\lambda^*(\boldsymbol{f}_{\rm D}(\beta\boldsymbol{\zeta}_1))$: solid line; $\lambda^*(\boldsymbol{f}_{\rm D}(2\beta\boldsymbol{\zeta}_2))$: dashed line.

Algorithm 5.1. Let $\boldsymbol{\zeta}_1^{\rm cr}$ denote the optimal solution of Problem (13). The corresponding critical dead load $\widetilde{\boldsymbol{f}}_{\rm D} + \boldsymbol{T}\boldsymbol{\zeta}_1^{\rm cr}$ and collapse mode are shown in Fig.3.

It is observed from Fig.3 that the collapse mode in the worst case is different from the sway-type mode observed in the nominal case of Fig.2. On the contrary, for $\alpha_2 = 20.0$ kN, the collapse mode in the worst case coincides with the sway-type as illustrated in Fig.4. The corresponding worst-case limit load factor is $\lambda_{\min}(\alpha_2) = 44.3662$. The distribution of critical dead load $\widetilde{\boldsymbol{f}}_{\rm D} + \boldsymbol{T}\boldsymbol{\zeta}_2^{\rm cr}$ is shown in Fig.4, which is different from the critical dead load in the case of Fig.3.

We next investigate the variation of the limit load factor by proportionally increasing the uncertain dead load, i.e., we employ the usual limit analyses repeatedly by putting $\boldsymbol{\zeta} = \beta\boldsymbol{\zeta}_1^{\rm cr}$ with increas-

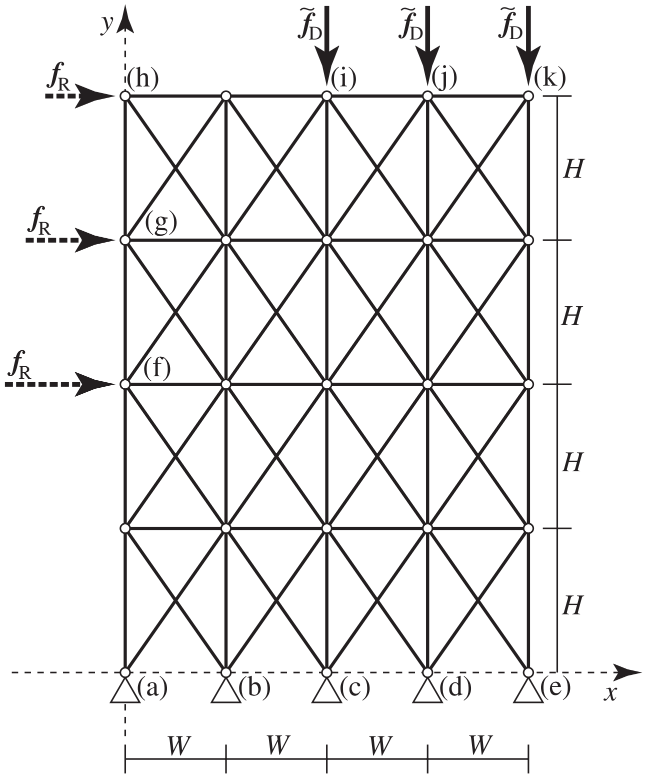

Figure 6: 4×4 plane truss.

ing β gradually. In Fig.5, the solid curve (A)→(B)→(C) depicts the the variations of $\lambda^*(\boldsymbol{f}_{\rm D}(\beta\boldsymbol{\zeta}_1))$ with respect to β. The collapse mode coincides with the sway-type shown in Fig.2 between the points (A) and (B), while the mode of Fig.3 is observed between (B) and (C). The variation of $\lambda^*(\boldsymbol{f}_{\rm D}(2\beta\boldsymbol{\zeta}_2^{\rm cr}))$ (note that $\alpha_1 = 2\alpha_2$ implies $\|\boldsymbol{\zeta}_1^{\rm cr}\| = 2\|\boldsymbol{\zeta}_2^{\rm cr}\|$) with respect to β is indicated by the dashed line (A)→(D) in Fig.5. The collapse mode coincides with the sway-type shown in Fig.4 between (A) and (D). The curve (A)→(E)→(C) corresponds to the variation of the worst-case limit load factor $\lambda_{\min}(\beta\alpha_1)$ with respect to β. This illustrates that the critical dead loads as well as the corresponding collapse modes depend on the level of uncertainty α.

6.2 4×4 truss

Consider a 68-bar plane truss illustrated in Fig.6, where $n^{\rm m} = 68$, $n^{\rm d} = 40$, $W = 35.0$ cm, and $H = 50.0$ cm. The nodes (a)–(e) are pin-supported. As the nominal dead load $\widetilde{\boldsymbol{f}}_{\rm D}$, we apply the external forces $(0, -800.0)$ kN at the nodes (i)–(k). The reference disturbance load $\boldsymbol{f}_{\rm R}$ are defined such that $(52.0, 0)$ kN, $(40.0, 0)$ kN, and $(28.0, 0)$ kN are applied at the nodes (f), (g), and (h), respectively. The nominal limit load factor is computed as $\lambda^*(\widetilde{\boldsymbol{f}}_{\rm D}) = 14.2650$ by employing the usual limit analysis. The corresponding collapse mode is shown in Fig.7, where the vanishing members experience plastic deformations.

The uncertain dead load $\boldsymbol{T\zeta}$ are supposed to possibly exist at all free nodes except for the components that $\widetilde{\boldsymbol{f}}_{\rm D}$ and $\boldsymbol{f}_{\rm R}$ are nonzero, i.e. $m = 34$. We set $\alpha = 40.0$ kN. By using Algorithm 4.2 and Algorithm 5.1, the worst-case limit load factor is computed as $\lambda_{\min}(\alpha) = 7.7296$. The CPU

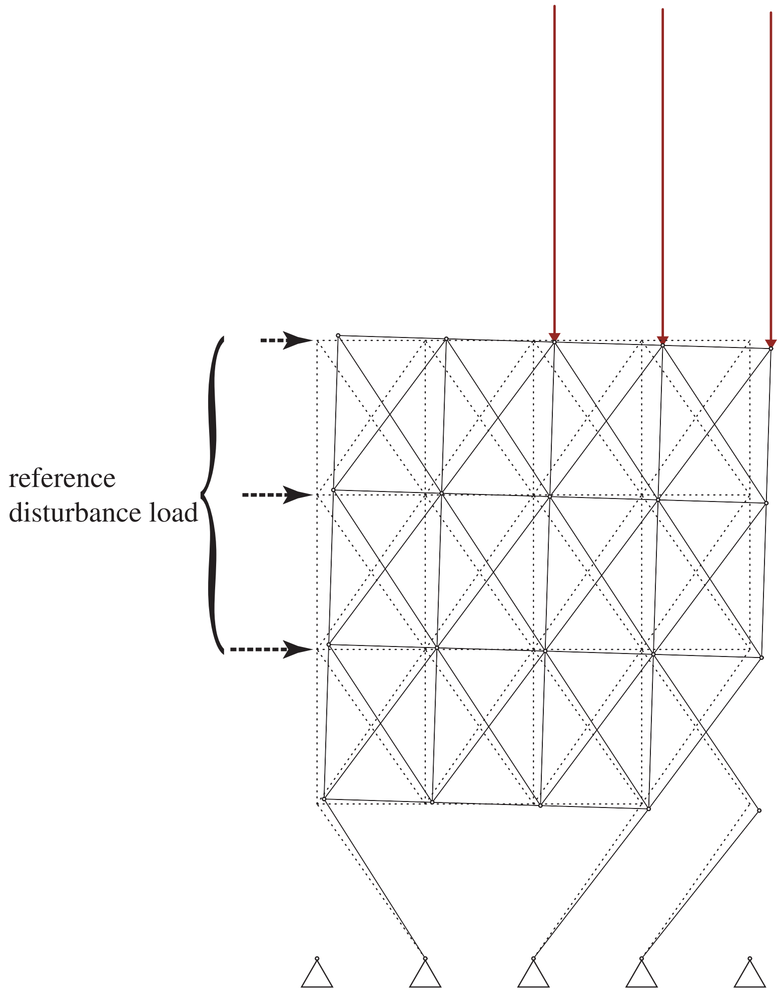

Figure 7: Collapse mode and the dead load of the 4×4 truss without the uncertainty in dead load ($\lambda^*(\widetilde{\boldsymbol{f}}_{\rm D}) = 14.2650$).

time required by Algorithm 5.1 is 229.5 sec, and the 34 cutting planes are generated within 61 iterations. Afterward, Algorithm 4.2 terminates by solving only 9 LP problems, where the CPU time required is 28.5 sec. This result demonstrates that the generated cutting planes at the root node of the branch-and-bound tree can reduce drastically the number of nodes that have to be visited in Algorithm 4.2.

Note that the worst-case limit load factor is almost half of the nominal one, in spite of the fact that the level of uncertainty α is relatively small compared with the norm of the nominal dead loads vector $\widetilde{\boldsymbol{f}}_{\rm D}$. The critical dead load $\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta}^{\rm cr})$ and the corresponding collapse mode are shown in Fig.8. It is observed from Fig.8 that the collapse mode in the worst case is the same as that in the nominal case illustrated in Fig.7.

For comparison, we select a sample of uncertain parameters vector $\boldsymbol{\zeta}'$ satisfying $\|\boldsymbol{\zeta}'\|_\infty = \alpha$ as the nodal forces shown in Fig.9. The corresponding limit load factor is $\lambda^*(\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta}')) = 8.4268$, which is larger than the worst case. The corresponding collapse mode is shown in Fig.9, which is different from the mode shown in Fig.8. Thus, it is not easy to find the critical dead loads vector in a heuristic way.

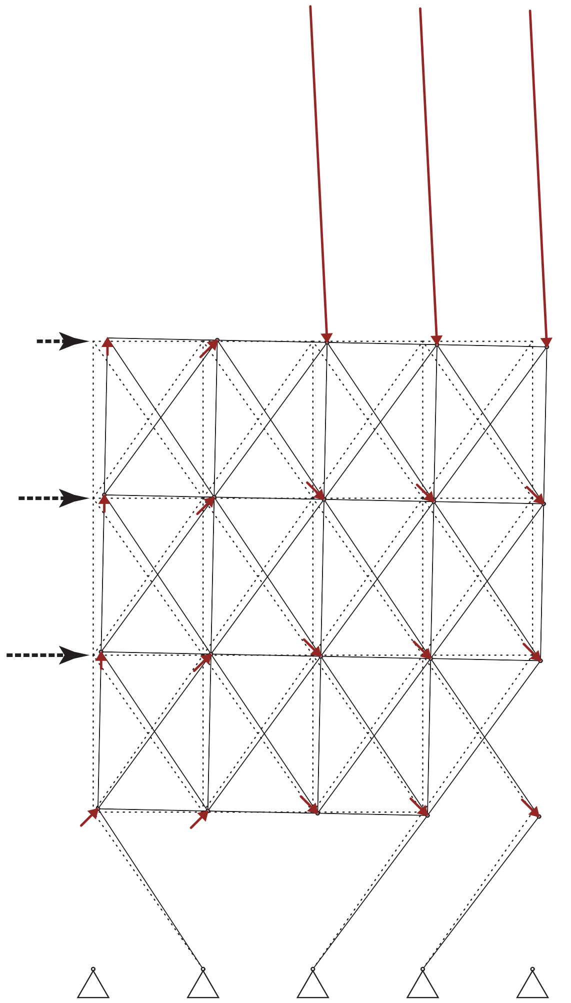

Figure 8: Collapse mode and the critical dead load of the 4×4 truss in the worst case for $\alpha = 40.0$ kN ($\lambda_{\min}(\alpha) = 7.7296$).

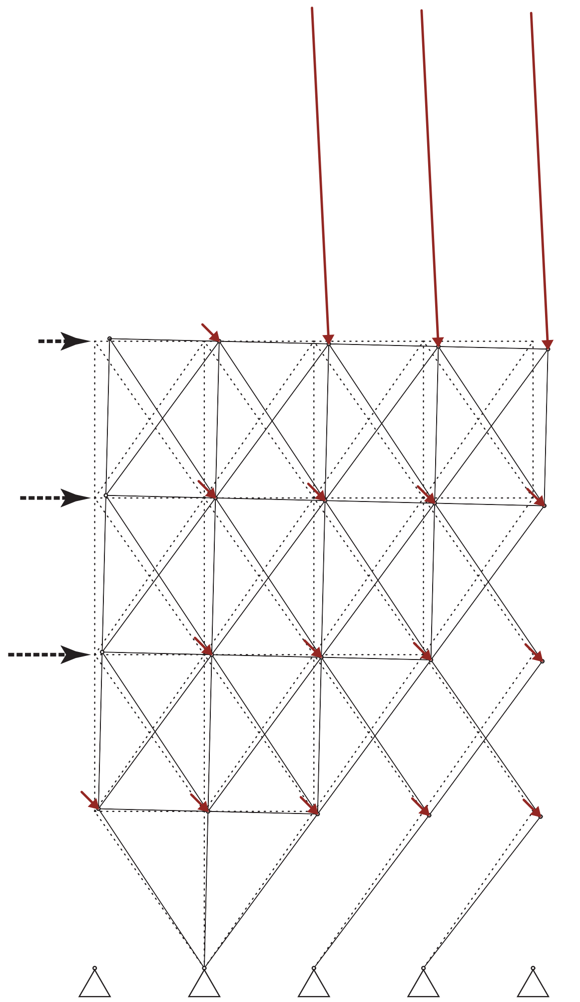

Figure 9: Definition of the dead load with $\boldsymbol{\zeta}'$ and the corresponding collapse mode of the 4×4 truss ($\lambda^*(\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta}')) = 8.4268$).

We randomly generate a number of $\boldsymbol{\zeta}$ satisfying $\|\boldsymbol{\zeta}\|_\infty = \alpha$, and perform the (conventional) limit analyses. The limit load factors $\lambda^*(\boldsymbol{f}_{\rm D}(\boldsymbol{\zeta}))$ obtained are shown in Fig.10 as number of points.

It is observed from Fig.10 that all generated $\lambda^*(\boldsymbol{\zeta})$ are larger than the worst-case limit load factor $\lambda^*(\boldsymbol{\zeta}^{\rm cr}) = \lambda_{\min}(\alpha)$, which supports that the obtained solution by using the proposed algorithms is a global optimal solution of the nonconvex problem (13).

7 Conclusions

In this paper, we have investigated the worst-case detection in the plastic limit analysis of trusses affected by unknown-but-bounded dead and live loads. While the imprecisely-known dead and live loads are constrained into a bounded set, the live or disturbance load are amplified with the load factor. A global optimization technique has been presented to compute the worst-case limit load factor as well as the critical dead load.

We suppose that the dead and live loads applied to a truss has the unknown-but-bounded

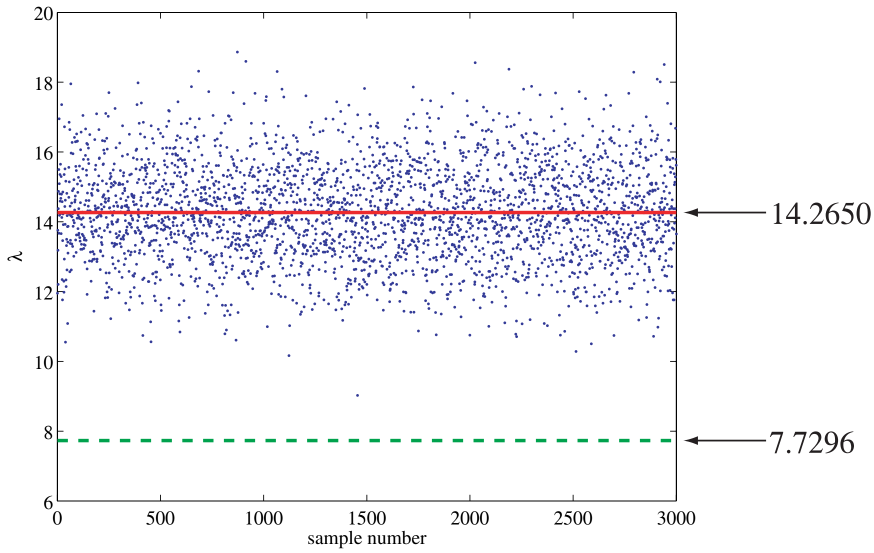

Figure 10: Limit load factor for randomly generated $\boldsymbol{\zeta}$; solid line: $\lambda^*(\widetilde{\boldsymbol{f}}_{\rm D})$; dashed line: $\lambda_{\min}(\alpha)$.

uncertainty. The worst-case limit load factor has been defined as the minimum value among limit load factors attained with some loading patterns belonging to a given closed set. Then the worst-case detection problem has been formulated as a mixed 0-1 programming problem. To obtain a global optimal solution of the present problem, we have proposed a cut-and-branch method based on the LP relaxation and the disjunctive cut, where a cutting plane is generated by solving another LP problem. Since the proposed method converges to a global optimal solution, it is theoretically guaranteed that there exits no uncertain parameter with which the limit load factor becomes smaller than the obtained optimal value.

It has been shown in the numerical examples that the proposed cut-and-branch method can find the worst-case limit load factors. The comparison with the limit load factors for randomly generated dead and live loads demonstrates that the obtained limit load factors correspond to the global optimal solutions of the mixed 0-1 programming problem presented. We have also illustrated through numerical examples that the process of cutting plane generation at the root node of the enumeration tree can reduce the number of LP relaxation problems that should be solved in the successive branch-and-bound procedure, though no theoretical result is to date available that suggests how many cutting planes should be generated.

References

[1] Alefeld, G. and Mayer, G., Interval analysis: theory and applications, *Journal of Computational and Applied Mathematics*, **121** (2000) 421–464.

[2] Andersen, K., Christiansen, E. and Overton, M.L., Computing limit loads by minimizing a sum of norms, *SIAM Journal on Scientific Computing*, **19** (1998) 1046–1062.

[3] Balas, E., Ceria, S. and Cornuéjols, G., Mixed 0-1 programming by lift-and-project in a branch-and-cut framework, *Management Science*, **42** (1996) 1229–1246.

[4] Balas, E. and Perregaard, M., Lift-and-project for mixed 0-1 programming: recent progress, *Discrete Applied Mathematics*, **123** (2003) 129–154.

[5] Ben-Haim, Y., *Information-gap Decision Theory*, Academic Press, London, UK, 2001.

[6] Ben-Haim, Y. and Elishakoff, I., *Convex Models of Uncertainty in Applied Mechanics*, Elsevier, Amsterdam, The Netherlands, 1990.

[7] Ben-Tal, A. and Nemirovski, A., Robust truss topology optimization via semidefinite programming, *SIAM Journal on Optimization*, **7** (1997) 991–1016.

[8] Ben-Tal, A. and Nemirovski, A., Robust optimization — methodology and applications, *Mathematical Programming*, **B92** (2002) 453–480.

[9] Ceria, S. and Soares, J., Disjunctive cut generation for mixed 0-1 programs: duality and lifting, Technical Report, Computational Operations Research Center, Columbia University, New York, NY, March 1997.

[10] Chen, S. Lian, H. and Yang, X., Interval static displacement analysis for structures with interval parameters, *International Journal for Numerical Methods in Engineering*, **53** (2002) 393–407.

[11] Cheng, Y.S., Au, F.T.K., Tham, L.G. and Zeng, G.W., Optimal and robust design of docking blocks with uncertainty, *Engineering Structures*, **26** (2004) 499–510.

[12] Chvátal, V., *Linear Programming*, W.H. Freeman and Company, New York, 1983.

[13] Clausen, J. and Perregaard, M., On the best search strategy in parallel branch-and-bound: best-first search versus lazy depth-first search, *Annals of Operations Research*, **90** (1999) 1–17.

[14] Cocchetti, G. and Maier, G., Elastic-plastic and limit-state analyses of frames with softening plastic-hinge models by mathematical programming, *International Journal of Solids and Structures*, **40** (2003) 7219–7244.

[15] Cordier, C., Marchand, H., Laundy, R. and Wolsey, L.A., *bc-opt*: a branch-and-cut code for mixed integer programs, *Mathematical Programming*, **86** (1999) 335–353.

[16] Craig, K.J., Stander, N. and Balasubramanyam, S., Worst-case design in head impact crashworthiness optimization, *International Journal for Numerical Methods in Engineering*, **57** (2003) 795–817.

[17] Elishakoff, I., Haftka, R.T. and Fang, J., Structural design under bounded uncertainty—optimization with anti-optimization, *Conputers & Structures*, **53** (1994) 1401–1405.

[18] Ganzerli, S. and Pantelides, C.P., Load and resistance convex models for optimum design, *Structural Optimization*, **17** (1999) 259–268.

[19] Gu, X.,Renaud, J.E., Batill, S.M., Brach, R.M. and Budhiraja, A.S., Worst-case propagated uncertainty of multidisciplinary systems in robust design optimizaion, *Structural and Multidisciplinary Optimization*, **20** (2000) 90–213.

[20] Kanno, Y. and Takewaki, I., Robustness analysis of trusses with separable load and structural uncertainties, *International Journal of Solids and Structures*, **43** (2006) 2646–2669.

[21] Kanno, Y. and Takewaki, I., Sequential semidefinite program for robust truss optimization based on robustness functions associated with stress constraints, *Journal of Optimization Theory and Applications*, to appear.

[22] Kharmanda, G., Olhoff, N., Mohamed, A. and Lemaire, M., Reliability-Based Topology Optimization, *Structural and Multidisciplinary Optimization*, **26** (2004) 295–307.

[23] Krabbenhoft, K. and Damkilde, L., A general non-linear optimization algorithm for lower bound limit analysis, *International Journal for Numerical Methods in Engineering*, **56** (2003) 165–184.

[24] Llyoyd Smith, D., Chuang, P.-H. and Munro, J., Fuzzy linear programming in plastic limit design. D. Lloyd Smith, ed., *Mathematical Programming Methods in Structural Plasticity*, Springer–Verlag, Wien, Austria, 1990, 425–435.

[25] Lyamin, A.V., Sloan, S.W., Krabbenhøft, K. and Hjiaj, M., Lower bound limit analysis with adaptive remeshing, *International Journal for Numerical Methods in Engineering*, **63** (2005) 1961–1974.

[26] Marchand, H., Martin, A., Weismantel, R. and Wolsey, L., Cutting planes in integer and mixed integer programming, *Discrete Applied Mathematics*, **123** (2002) 397–446.

[27] Marti, K. and Stoeckel, G., Stochastic linear programming methods in limit load analysis and optimal plastic design under stochastic uncertainty, *ZAMM*, **10** (2004) 666–677.

[28] Michael Sttele, J., *The Cauchy–Schwarz Master Class*, Cambridge University Press, New York, NY, 2004.

[29] Muralidhar, R. and Jagannatha Rao, J.R., New models for optimal truss topology in limit design based on unified elastic/plastic analysis, *Computer Methods in Applied Mechanics and Engineering*, **140** (1997) 109–138.

[30] Rocho, P. and Sonnenberg, S., Limit analysis of frames — Application to structural reliability. M. Staat and M. Heitzer, eds., *Numerical Methods for Limit and Shakedown Analysis*, John von Neumann Institute for Computing, Jülich, Germany, 2003, 269–282.

[31] Staat, M. and Heitzer, M., Probabilistic limit and shakedown problems. M. Staat and M. Heitzer, eds., *Numerical Methods for Limit and Shakedown Analysis*, John von Neumann Institute for Computing, Jülich, Germany, 2003, 217–268.

[32] Takewaki, I. and Ben-Haim, Y., Info-gap robust design with load and model uncertainties, *Journal of Sound and Vibration*, **288** (2005) 551–570.

[33] Tin-Loi, F. and Ngo, N.S., Performance of the p-version finite element method for limit analysis, *International Journal of Mechanical Sciences*, **45** (2003) 1149–1166.

[34] Zang, C., Friswell, M.I. and Mottershead, J.E., A review of robust optimal design and its application in dynamics, *Conputers & Structures*, **83** (2005) 315–326.

[35] *Optimization Toolbox User's Guide*, The MathWorks, Natick, MA, 2000.

[36] *Using MATLAB*, The MathWorks, Natick, MA, 2002.