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Abstract

Induction (or transformation) by bipartite graphs is one of the most important operations on matroids, and it is well known that the induction of a matroid by a bipartite graph is again a matroid. As an abstract form of this fact, the induction of a matroid by a linking system is known to be a matroid.

M-convex functions are quantitative extensions of matroidal structures, and they are known as discrete convex functions. As with matroids, it is known that the induction of an M-convex function by networks generates an M-convex function. As an abstract form of this fact, this paper shows that the induction of an M-convex function by linking systems generates an M-convex function. Furthermore, we show that this result also holds for M-convex functions on constant-parity jump systems. Previously known operations such as aggregation, splitting, and induction by networks can be understood as special cases of this construction.

1 Introduction

As a generalization of matroids, many matroidal structures and its quantitative extensions are studied. The concept of M-convex functions on base polyhedra introduced in [13] is a generalization of valuated matroids [4], [6], which are quantitative extensions of matroids. M-convex functions are known as discrete convex functions, and play a central role in discrete convex analysis [15]. Recently, the concept of M-convex functions on constant-parity jump systems is introduced in [16] as a generalization of M-convex functions on base polyhedra. The definitions of M-convex functions and jump systems are to be described later. To distinguish between M-convex functions on base polyhedra and those on constant-parity jump systems, we sometimes refer to the former as M^{B}-convex functions and the latter as M^{J}-convex functions. For M-convex functions global minimality is equivalent to local minimality defined in an appropriate manner. In addition, discrete duality theorems such as discrete separation and min-max formula hold for M^{B}-convex functions, whereas they fail for M^{J}-convex functions.

In the study of matroidal structures and their extensions, a number of natural operations and transformations are considered. To be specific, induction by bipartite graphs is one of the most important operations on matroids. Given a bipartite graph $G = (S, T; E)$ and a matroid $\mathcal{M} = (S, \mathcal{I})$ with the collection of independent sets $\mathcal{I} \subseteq 2^S$, the induction of $\mathcal{M}$ by G is defined as $\mathcal{M}' = (T, \mathcal{I}')$ with

$$\mathcal{I}' = \{\partial M \cap T \mid M : \text{matching}, (\partial M \cap S) \in \mathcal{I}\},$$

where ∂M is the set of vertices covered by M. It is known that $\mathcal{M}'$ forms a matroid [17]. As an abstract form of this fact, the induction of a matroid by a linking system, which is introduced as a generalization of bipartite matchings, is known to be a matroid [18].

Operations on M^{B}-convex functions and M^{J}-convex functions are studied in [13] and [10], respectively. It is known that the induction of an M^{B}-convex function (respectively, an M^{J}-convex function) by a network is an M^{B}-convex function (respectively, an M^{J}-convex function). As a natural generalization of this operation, we introduce a new operation called induction of M-convex functions by integral poly-linking systems, where integral poly-linking systems are generalizations of integral flows in networks. Our main results (Theorems 5 and 6) show that the induction of an M-convex function by an integral poly-linking system is M-convex.

2 Preliminaries

Let V be a finite set. For $x = (x(v)), y = (y(v)) \in \mathbf{Z}^V$ define

$$\begin{aligned} x(U) &= \sum_{v \in U} x(v) \quad (U \subseteq V), \\ \mathrm{supp}^+(x) &= \{v \in V \mid x(v) > 0\}, \\ \mathrm{supp}^-(x) &= \{v \in V \mid x(v) < 0\}, \end{aligned}$$

$$[x, y] = \{z \in \mathbf{Z}^V \mid \min(x(v), y(v)) \leq z(v) \leq \max(x(v), y(v)), \forall v \in V\}.$$

We denote by $\mathbf{0}$ the zero vector of an appropriate dimension. For $U \subseteq V$ we denote by χ_U the *characteristic vector* of U, with $\chi_U(v) = 1$ for $v \in U$ and $\chi_U(v) = 0$ for $v \in V \setminus U$. For $u \in V$ we denote $\chi_{\{u\}}$ simply by χ_u. A vector $s \in \mathbf{Z}^V$ is called an *(x, y)-increment* if $s = \chi_u$ or $s = -\chi_u$ for some $u \in V$ and $x + s \in [x, y]$.

2.1 Base polyhedra and jump systems

The set of integral points in a base polyhedra of integral polymatroids has the following exchange property [15].

(B-EXC) For any $x, y \in B$ and for any $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

A nonempty set of integer points $B \subseteq \mathbf{Z}^V$ is defined to be an *M^B-convex set* if it satisfies (B-EXC) above. It is known that M^{B}-convex sets are, up to translations, the same as base polyhedra of integral submodular systems.

Next we consider a generalized concept of M^{B}-convex sets called *jump systems* [3] (see also [9], [12]). A nonempty set $J \subseteq \mathbf{Z}^V$ is said to be a jump system if it satisfies an exchange axiom, called the *2-step axiom*: for any $x, y \in J$ and for any (x, y)-increment s with $x + s \notin J$, there exists an $(x+s, y)$-increment t such that $x+s+t \in J$. A set $J \subseteq \mathbf{Z}^V$ is a *constant-sum system* if $x(V) = y(V)$ for any $x, y \in J$, and a *constant-parity system* if $x(V) - y(V)$ is even for any $x, y \in J$.

For constant-parity jump systems, a stronger exchange axiom is relevant:

(J-EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$ and $y - s - t \in J$.

This property characterizes a constant-parity jump system, a fact communicated to one of the authors by J. Geelen (see [16] for a proof).

Theorem 1 ([8])**.** *A nonempty set J is a constant-parity jump system if and only if it satisfies* (J-EXC).

2.2 M-convex functions

M^{B}-convex functions and M^{J}-convex functions are defined as quantitative extensions of M^{B}-convex sets and constant-parity jump systems, respectively.

M-convex functions on base polyhedra, to be denoted M^{B}-convex functions in this paper, are introduced by Murota [13], and they play a central role in discrete convex analysis [15]. We call $f : B \to \mathbf{R}$ an *M^B-convex function* if it satisfies the following exchange axiom:

(M^{B}-EXC) For any $x, y \in B$ and for any $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that $x - \chi_u + \chi_v \in B$, $y + \chi_u - \chi_v \in B$, and

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

It follows from (M^{B}-EXC) that B satisfies (B-EXC) and hence is an M^{B}-convex set. We adopt the convention that $f(x) = +\infty$ for $x \notin B$.

As a common generalization of valuated delta-matroids [5] and M^{B}-convex functions, M-convex functions on constant-parity jump systems, to be denoted M^{J}-convex functions in this paper, are introduced in [16]. We call $f : J \to \mathbf{R}$ an *M^J-convex function* if it satisfies the following exchange axiom:

(M^{J}-EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$, $y - s - t \in J$, and

$$f(x) + f(y) \geq f(x + s + t) + f(y - s - t).$$

It follows from (M^{J}-EXC) that J satisfies (J-EXC) and hence is a constant-parity jump system. We adopt the convention that $f(x) = +\infty$ for $x \notin J$.

A separable convex function on the degree sequences of a graph is a typical example of M^{J}-convex functions [1], [2]. The definition of an M^{J}-convex function is consistent with the previously considered special cases where (i) J is a constant-sum jump system, and (ii) J is a constant-parity jump system contained in $\{0, 1\}^V$. Case (i) is equivalent to J being the set of integer points in the base polyhedron of an integral submodular system [7] (or the $\mathrm{M^B}$-convex set), and then $\mathrm{M^J}$-convex functions are the same as the $\mathrm{M^B}$-convex functions investigated in [13], [15]. Case (ii) is equivalent to J being an even delta-matroid [21], [22], and then f is $\mathrm{M^J}$-convex if and only if $-f$ is a valuated delta-matroid in the sense of [5].

In [10], a number of natural operations on $\mathrm{M^J}$-convex functions, which preserve $\mathrm{M^J}$-convexity, are defined. Here we refer to two of them called convolution and composition.

For two functions $f_1 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, their (infimum) *convolution* is the function $f_1 \Box f_2 : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty, -\infty\}$ given by

$$(f_1 \Box f_2)(x) = \inf\{f_1(x_1) + f_2(x_2) \mid x_1 + x_2 = x,\ x_1 \in \mathbf{Z}^V,\ x_2 \in \mathbf{Z}^V\}.$$

Convolution is a quantitative extension of (Minkowski) sum, and it preserves $\mathrm{M^J}$-convexity.

Theorem 2 ([10])**.** *If f_1 and f_2 are M^J-convex functions, then their convolution $f_1 \Box f_2$ is M^J-convex, provided $f_1 \Box f_2 > -\infty$.*

Let $f_1 : \mathbf{Z}^{S_1} \to \mathbf{R} \cup \{+\infty\}$ and $f_2 : \mathbf{Z}^{S_2} \to \mathbf{R} \cup \{+\infty\}$ be $\mathrm{M^J}$-convex functions. Put $V_0 = S_1 \cap S_2$, $V_1 = S_1 \setminus V_0$, and $V_2 = S_2 \setminus V_0$. The *composition* of f_1 and f_2 is the function $f : \mathbf{Z}^{V_1 \cup V_2} \to \mathbf{R} \cup \{+\infty, -\infty\}$ given by

$$f(x_1, x_2) = \inf\{f_1(x_1, y_1) + f_2(x_2, y_2) \mid y_1 = y_2 \in \mathbf{Z}^{V_0}\} \quad (x_1 \in \mathbf{Z}^{V_1}, x_2 \in \mathbf{Z}^{V_2}).$$

Composition also preserves $\mathrm{M^J}$-convexity.

Theorem 3 ([10])**.** *The composition of two M^J-convex functions is M^J-convex, provided it does not take the value $-\infty$.*

2.3 Poly-linking systems

The concept of poly-linking systems is originally introduced by A. Schrijver [18], [19]. In [18], poly-linking systems are defined to be sets of vectors whose components are nonnegative real numbers, and it is shown that they are associated with base polyhedra of polymatroids.

We consider poly-linking systems which are sets of integral vectors whose components are allowed to be negative. We use the following definition which differs from the original one. Our definition, however, is justified by the fact that poly-linking systems defined below are associated with base polyhedra of integral polymatroids (Proposition 4).

An *integral poly-linking system* is a triple (S, T, L) where S and T are finite sets and L is a nonempty subset of $\mathbf{Z}^S \times \mathbf{Z}^T$ satisfying the following conditions.

1. $(\mathbf{0}, \mathbf{0}) \in L$.
2. For any $(x_1, y_1), (x_2, y_2) \in L$ and $u \in \mathrm{supp}^+(x_1 - x_2)$, at least one of the following conditions holds.
    - There exists $v \in \mathrm{supp}^-(x_1 - x_2)$ such that $(x_1 - \chi_u + \chi_v, y_1) \in L$ and $(x_2 + \chi_u - \chi_v, y_2) \in L$.

- There exists $v \in \mathrm{supp}^+(y_1-y_2)$ such that $(x_1-\chi_u, y_1-\chi_v) \in L$ and $(x_2+\chi_u, y_2+\chi_v) \in L$.

3. For any $(x_1, y_1), (x_2, y_2) \in L$ and $u \in \mathrm{supp}^+(y_1 - y_2)$, at least one of the following conditions holds.
   - There exists $v \in \mathrm{supp}^-(y_1-y_2)$ such that $(x_1, y_1-\chi_u+\chi_v) \in L$ and $(x_2, y_2+\chi_u-\chi_v) \in L$.
   - There exists $v \in \mathrm{supp}^+(x_1-x_2)$ such that $(x_1-\chi_v, y_1-\chi_u) \in L$ and $(x_2+\chi_v, y_2+\chi_u) \in L$.

Integral poly-linking systems are intimately related with M^{B}-convex sets. The following proposition can be easily derived from the definitions of M^{B}-convex sets and integral poly-linking systems. In what follows, we identify $\mathbf{Z}^S \times \mathbf{Z}^T$ with $\mathbf{Z}^{S \cup T}$.

Proposition 4. *Suppose that S and T are finite sets and L is a nonempty subset of $\mathbf{Z}^S \times \mathbf{Z}^T$. Then (S, T, L) is an integral poly-linking system if and only if the set $B \subseteq \mathbf{Z}^{S\cup T}$ defined by*

$$B = \{(x, y) \mid (-x, y) \in L\}$$

is an M^B-convex set with $(\mathbf{0}, \mathbf{0}) \in B$.

Proof. First, $(\mathbf{0}, \mathbf{0}) \in B$ corresponds to the first condition in the definition of integral poly-linking systems. The exchange axiom of M^{B}-convex sets corresponds to the second and the third conditions in the definition of integral poly-linking systems. □

A typical example of integral poly-linking systems is integral flows in directed graphs.

Example. Let $G = (V, A; S, T)$ be a directed graph with vertex set V, arc set A, entrance set S, and exit set T, where S and T are disjoint subsets of V. Let $c \in \mathbf{Z}_+^A$ represent the capacities of the arcs. Define

$$L = \big\{(x, y) \in \mathbf{Z}^S \times \mathbf{Z}^T \mid \exists \xi \in \mathbf{Z}^A,\ \partial\xi = (x, -y, \mathbf{0}) \in \mathbf{Z}^S \times \mathbf{Z}^T \times \mathbf{Z}^{V\setminus(S\cup T)},\ \mathbf{0} \le \xi \le c\big\},$$

where $\partial\xi \in \mathbf{Z}^V$ is the vector given by

$$\partial\xi(v) = \sum_{a:\ a \text{ leaves } v} \xi(a) - \sum_{a:\ a \text{ enters } v} \xi(a) \quad (v \in V).$$

Then (S, T, L) is an integral poly-linking system. The first condition, $(\mathbf{0}, \mathbf{0}) \in L$, in the definition of integral poly-linking systems is obvious. The second and the third conditions can be shown by an alternating path argument. This fact can be derived from [15] (Note 2.19), and the non-integral version is considered in [18].

An integral poly-linking system that satisfies $L \subseteq \{0, 1\}^S \times \{0, 1\}^T$ is called a *linking system* [18] (or *bimatroid* [11]), and it corresponds to the collection of bases of a matroid.

3 Induction by integral poly-linking systems

3.1 Main theorems

Given a function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{+\infty\}$ and an integral poly-linking system (S, T, L), the function $\tilde{f} : \mathbf{Z}^T \to \mathbf{R} \cup \{+\infty, -\infty\}$ defined by

$$\tilde{f}(y) = \inf\{f(x) \mid x \in \mathbf{Z}^S,\ (x, y) \in L\} \quad (y \in \mathbf{Z}^T)$$

is called the *induction of f by (S, T, L)*. If such x does not exist, we define $\tilde{f}(y) = +\infty$.

Theorem 5. *Assume that f is an M^B-convex function defined on $\mathbf{Z}^S$. Then the function $\tilde{f}$ induced by an integral poly-linking system (S, T, L) is M^B-convex, provided $\tilde{f} > -\infty$.*

A proof of Theorem 5 will be given after the proof of Theorem 6.

Theorem 6. *Assume that f is an M^J-convex function defined on $\mathbf{Z}^S$. Then the function $\tilde{f}$ induced by an integral poly-linking system (S, T, L) is M^J-convex, provided $\tilde{f} > -\infty$.*

Proof. By Proposition 4, there is an $\mathrm{M^B}$-convex set B which satisfies

$$(x, y) \in L \iff (-x, y) \in B$$

for any vector $(x, y) \in \mathbf{Z}^S \times \mathbf{Z}^T$. We define a function $g : \mathbf{Z}^{S \cup T} \to \mathbf{R} \cup \{+\infty\}$ that represents the structure of L as follows:

$$g(x, y) = \begin{cases} 0 & (\text{if } (x, y) \in B) \\ +\infty & (\text{otherwise}). \end{cases}$$

Since B is an $\mathrm{M^B}$-convex set, g is an $\mathrm{M^B}$-convex function and hence an $\mathrm{M^J}$-convex function.

Let $f' : \mathbf{Z}^{S \cup T} \to \mathbf{R} \cup \{+\infty\}$ be the function defined by

$$f'(x, y) = \begin{cases} f(x) & (\text{if } y = \mathbf{0}) \\ +\infty & (\text{otherwise}). \end{cases}$$

Then f' is also an $\mathrm{M^J}$-convex function. By the definition of $\tilde{f}$, we have

$$\begin{aligned} \tilde{f}(y) &= \inf\{f(x) \mid x \in \mathbf{Z}^S,\ (x, y) \in L\} \\ &= \inf\{f(x) \mid x \in \mathbf{Z}^S,\ (-x, y) \in B\} \\ &= \inf_{x \in \mathbf{Z}^S} \{f(x) + g(-x, y)\} \\ &= \inf_{x \in \mathbf{Z}^S} \{f'(x, \mathbf{0}) + g(-x, y)\} \end{aligned} \tag{1}$$

for any $y \in \mathbf{Z}^T$.

On the other hand, the convolution of two $\mathrm{M^J}$-convex functions f' and g defined by

$$(f' \Box g)(z) = \inf_{z' \in \mathbf{Z}^{S \cup T}} \{f'(z') + g(z - z')\} \quad (z \in \mathbf{Z}^{S \cup T})$$

is an $\mathrm{M^J}$-convex function by Theorem 2. If $z = (\mathbf{0}, y)$ for some $y \in \mathbf{Z}^T$, we have

$$\begin{aligned}(f'\Box g)(\mathbf{0}, y) &= \inf_{z' \in \mathbf{Z}^{S \cup T}} \{f'(z') + g((\mathbf{0}, y) - z')\} \\ &= \inf_{x \in \mathbf{Z}^S} \{f'(x, \mathbf{0}) + g(-x, y)\}. \end{aligned} \tag{2}$$

By (1) and (2), we have $\tilde{f}(y) = (f'\Box g)(\mathbf{0}, y)$ for each $y \in \mathbf{Z}^T$. Thus $\tilde{f}$ is $\mathrm{M^J}$-convex since $f'\Box g$ is $\mathrm{M^J}$-convex and the restriction of an $\mathrm{M^J}$-convex function is $\mathrm{M^J}$-convex. □

We can show Theorem 5 in a similar way.

Proof of Theorem 5. In the proof of Theorem 6, the function f' is $\mathrm{M^B}$-convex if f is $\mathrm{M^B}$-convex. Then $f'\Box g$ is $\mathrm{M^B}$-convex, since g is also an $\mathrm{M^B}$-convex function and the convolution of two $\mathrm{M^B}$-convex functions is known to be $\mathrm{M^B}$-convex [13], [15] (Theorem 6.13). Thus $\tilde{f}$, which is a restriction of $f'\Box g$, is $\mathrm{M^B}$-convex. □

3.2 Special cases

3.2.1 Network

Induction by networks is a fundamental operation investigated in [10], [13]. We explain here that this can be seen as a special case of the induction by integral poly-linking systems.

Just as in Example, let $G = (V, A; S, T)$ be a directed graph with vertex set V, arc set A, entrance set S, and exit set T, where S and T are disjoint subsets of V. Let $c \in \mathbf{Z}^A_+$ represent the capacities of the arcs.

Given a function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{+\infty\}$ associated with the entrance set S of the network, we define a function $\tilde{f} : \mathbf{Z}^T \to \mathbf{R} \cup \{+\infty, -\infty\}$ on the exit set T by

$$\begin{aligned}\tilde{f}(y) = \inf_{\xi, x} \big\{ f(x) \mid \partial\xi = (x, -y, \mathbf{0}),\ \mathbf{0} \le \xi \le c, \\ \xi \in \mathbf{Z}^A,\ (x, -y, \mathbf{0}) \in \mathbf{Z}^S \times \mathbf{Z}^T \times \mathbf{Z}^{V \setminus (S \cup T)} \big\} \quad (y \in \mathbf{Z}^T).\end{aligned}$$

If such ξ and x do not exist, we define $\tilde{f}(y) = +\infty$. We regard $\tilde{f}$ as a result of the *induction of f by the network*.

Corollary 7 ([10], [13] (see also [15], [20]))**.** *Assume that f is an M^J-convex function (respectively, M^B-convex function). Then the function $\tilde{f}$ induced by a network $G = (V, A; S, T)$ is M^J-convex (respectively, M^B-convex), provided $\tilde{f} > -\infty$.*

Proof. Let $L \subseteq \mathbf{Z}^S \times \mathbf{Z}^T$ be the set given by

$$L = \big\{ (x, y) \mid \exists \xi \in \mathbf{Z}^A,\ \partial\xi = (x, -y, \mathbf{0}) \in \mathbf{Z}^S \times \mathbf{Z}^T \times \mathbf{Z}^{V \setminus (S \cup T)},\ \mathbf{0} \le \xi \le c \big\}.$$

Then (S, T, L) is an integral poly-linking system. Since $\tilde{f}$ is the induction of f by (S, T, L), by Theorem 6 (respectively, Theorem 5), $\tilde{f}$ is $\mathrm{M^J}$-convex (respectively, $\mathrm{M^B}$-convex). □

3.2.2 Set system

We can also consider the induction of set systems by integral poly-linking systems. Given a set $B \subseteq \mathbf{Z}^S$ and an integral poly-linking system (S, T, L), the set $\tilde{B} \subseteq \mathbf{Z}^T$ defined by

$$\tilde{B} = \{y \in \mathbf{Z}^T \mid \exists x \in B,\ (x, y) \in L\} \tag{3}$$

is called the *induction of* B *by* (S, T, L).

Corollary 8. *Let $B \subseteq \mathbf{Z}^S$ be an M^B-convex set, and $\tilde{B} \subseteq \mathbf{Z}^T$ be the induction of B by an integral poly-linking system (S, T, L). Then $\tilde{B}$ is an M^B-convex set, provided $\tilde{B} \neq \emptyset$.*

Proof. Since B is an M^B-convex set, the function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} 0 & (x \in B) \\ +\infty & (\text{otherwise}) \end{cases}$$

is an M^B-convex function. Let $\tilde{f}$ be the induction of f by (S, T, L). Then $\tilde{f}$ is M^B-convex by Theorem 5. Hence $\tilde{B} = \{x \in \mathbf{Z}^T \mid \tilde{f}(x) < +\infty\}$ is an M^B-convex set. □

Corollary 9. *Let $J \subseteq \mathbf{Z}^S$ be a constant-parity jump system, and $\tilde{J} \subseteq \mathbf{Z}^T$ be the induction of J by an integral poly-linking system (S, T, L). Then $\tilde{J}$ is a constant-parity jump system, provided $\tilde{J} \neq \emptyset$.*

Proof. Since J is a constant-parity jump system, the function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} 0 & (x \in J) \\ +\infty & (\text{otherwise}) \end{cases}$$

is an M^J-convex function. Let $\tilde{f}$ be the induction of f by (S, T, L). Then $\tilde{f}$ is M^J-convex by Theorem 6. Hence $\tilde{J} = \{x \in \mathbf{Z}^T \mid \tilde{f}(x) < +\infty\}$ is a constant-parity jump system. □

The fact that the induction of a matroid by a linking system is a matroid can be seen as a special case of Theorem 5. For a set family $\mathcal{B} \subseteq 2^V$ in general, we define $\chi(\mathcal{B}) = \{\chi_B \in \{0, 1\}^V \mid B \in \mathcal{B}\}$. Suppose that $\mathcal{B} \subseteq 2^S$ is the collection of bases of a matroid, that is $\chi(\mathcal{B})$ is an M^B-convex set, and (S, T, L) is a linking system. A set family $\mathcal{B}' \subseteq 2^T$ is called the *induction of* $\mathcal{B}$ *by* (S, T, L), if $\chi(\mathcal{B}')$ is the induction of $\chi(\mathcal{B})$ by (S, T, L) in the sense of (3). Since $\chi(\mathcal{B})$ is an M^B-convex set, $\chi(\mathcal{B}')$ is an M^B-convex set by Corollary 8. Hence $\mathcal{B}'$ is the collection of bases of a matroid.

Corollary 10 ([18])**.** *Let $\mathcal{B}$ be the collection of bases of a matroid, and $\mathcal{B}'$ be the induction of $\mathcal{B}$ by a linking system. Then $\mathcal{B}'$ is the collection of bases of a matroid, provided $\mathcal{B}' \neq \emptyset$.*

3.2.3 Valuated delta-matroid

In [14] (Theorem 5.2.22), it is shown that the induction of a valuated matroid is again a valuated matroid. Here we refer to the induction of a valuated delta-matroid, which is equivalent to a special case of M^J-convex functions.

It is easy to see that $\delta : \{0,1\}^V \to \mathbf{R} \cup \{-\infty\}$ is a *valuated delta-matroid* [5], if and only if the function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \begin{cases} -\delta(x) & (x \in \{0,1\}^V) \\ +\infty & (\text{otherwise}) \end{cases} \tag{4}$$

is M^{J}-convex. Given a valuated delta-matroid $\delta : \{0,1\}^S \to \mathbf{R} \cup \{-\infty\}$ and a linking system (S,T,L), the function $\tilde{\delta} : \{0,1\}^T \to \mathbf{R} \cup \{-\infty\}$ defined by

$$\tilde{\delta}(y) = \sup\{\delta(x) \mid x \in \{0,1\}^S,\ (x,y) \in L\} \quad (y \in \{0,1\}^T) \tag{5}$$

is called the *supremum induction of* δ *by* (S,T,L). If such x does not exist, we define $\tilde{\delta}(y) = -\infty$.

Corollary 11. *Assume that δ is a valuated delta-matroid defined on $\{0,1\}^S$. Then the supremum induction of δ by a linking system (S,T,L), the function $\tilde{\delta}$ in (5), is a valuated delta-matroid.*

Proof. Since δ is a valuated delta-matroid, the function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{+\infty\}$ as defined in (4) with $V = S$ is M^{J}-convex. Thus the function $\tilde{f} : \mathbf{Z}^T \to \mathbf{R} \cup \{+\infty\}$ induced of f by (S,T,L) is M^{J}-convex by Theorem 6. Since

$$\tilde{f}(x) = \begin{cases} -\tilde{\delta}(x) & (x \in \{0,1\}^T) \\ +\infty & (\text{otherwise}) \end{cases}$$

holds, $\tilde{\delta}$ is a valuated delta-matroid. □

4 Discussion

We mention the relation between composition and induction by integral poly-linking systems. By the equation (1) in the proof of Theorem 6, $\tilde{f}$ can be represented as

$$\tilde{f}(y) = \inf_{-x \in \mathbf{Z}^S} \{\bar{f}(-x) + g(-x,y)\}$$

with the M^{J}-convex function $\bar{f} : \mathbf{Z}^S \to \mathbf{R} \cup \{+\infty\}$ defined by $\bar{f}(x) = f(-x)$. This equation means that $\tilde{f}$ is the composition of two M^{J}-convex functions $\bar{f}$ and g. Thus induction by integral poly-linking systems can be seen as a special case of composition.

On the basis of this observation, we could define a jump system version of the linking system. Suppose that S and T are finite sets and L is a nonempty subset of $\mathbf{Z}^S \times \mathbf{Z}^T$. Let us call a triple (S,T,L) a *jump linking system* if the set $J \subseteq \mathbf{Z}^{S \cup T}$ defined by

$$J = \{(x,y) \mid (-x,y) \in L\}$$

is a constant-parity jump system. Note that the jump linking system is a generalization of the integral poly-linking system.

We could also define the induction of a function by a jump linking system. Given a function $f : \mathbf{Z}^S \to \mathbf{R} \cup \{+\infty\}$ and a jump linking system (S,T,L), we call the function $\tilde{f} : \mathbf{Z}^T \to \mathbf{R} \cup \{+\infty, -\infty\}$ defined by

$$\tilde{f}(y) = \inf\{f(x) \mid x \in \mathbf{Z}^S,\ (x,y) \in L\} \quad (y \in \mathbf{Z}^T)$$

the *induction of* f *by* (S, T, L). If such x does not exist, we define $\tilde{f}(y) = +\infty$. Then the induction $\tilde{f}$ of the $\mathrm{M^J}$-convex function f is $\mathrm{M^J}$-convex, because $\tilde{f}$ is the composition of two $\mathrm{M^J}$-convex functions $\bar{f}$ and g defined as in the proof of Theorem 6.

Theorem 12. *Assume that* f *is an* M^J*-convex function defined on* $\mathbf{Z}^S$*. Then the function* $\tilde{f}$ *induced by a jump linking system* (S, T, L) *is* M^J*-convex, provided* $\tilde{f} > -\infty$*.*

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