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Approximation Algorithms for the Single Allocation Problem in Hub-and-Spoke Networks and Related Metric Labeling Problems

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Abstract

This paper deals with a single allocation problem in hub-and-spoke networks. We present a simple deterministic 3-approximation algorithm and randomized 2-approximation algorithm based on a linear relaxation problem and a randomized rounding procedure. We handle the case where the number of hubs is three, which is known to be NP-hard, and present a (5/4)-approximation algorithm.

The single allocation problem includes a special class of the metric labeling problem, defined by introducing an assumption that both objects and labels are embedded in a common metric space. Under this assumption, we can apply our algorithms to the metric labeling problem without loosing theoretical approximation ratios. As a byproduct, we also obtain a (4/3)-approximation algorithm for an ordinary metric labeling problem with three labels.

Keywords: hub location, metric labeling, approximation algorithm, dependent rounding

1 Introduction

In this paper, we consider a single allocation problem in hub-and-spoke networks. Given a set of hub nodes and a set of non-hub nodes, the problem allocates each non-hub node to exactly one of hub nodes so that a total transportation cost is minimized where required amount of flow and a transportation cost per unit flow are given for each pair of nodes. In hub-and-spoke networks, it is assumed that flows between any pair of nodes are sent via hub nodes.

First, we describe a simple 3-approximation algorithm. Next, we propose a 2-approximation algorithm based on a linear programming relaxation and a randomized rounding procedure. Lastly, we handle the case where the number of hubs is three, which is known to be NP-hard, and present a (5/4)-approximation algorithm.

By substituting objects and labels for non-hubs and hubs, respectively, the single allocation problem becomes a special class of the metric labeling problem, which is investigated by Kleinberg and Tardos in [10], defined by introducing an assumption that both objects and labels are embedded in a common metric space. Under this assumption, we can apply our algorithms to the metric labeling problem without loosing theoretical approximation ratios. As a byproduct, we also obtain a (4/3)-approximation algorithm for an ordinary metric labeling problem with three labels.

The hub-and-spoke structure is based on the situation where some nodes, called non-hub nodes, can interact only via a set of completely interconnected nodes, called hub nodes. The structure arises in the airline industry, telecommunications and postal delivery systems. In 1987, O'Kelly [13] considered a hub location problem, which chooses hub nodes from given nodes and allocates remained nodes to exactly one of hub nodes so that a total transportation cost is minimized. After his work, a wide variety of studies have been done on this topic (e.g., [4, 6]). Due to the hardness of the problem, most of the researches centered on the development of heuristics to solve this problem. Many of those heuristics are surveyed by Bryan and O'Kelly [4]. Exact algorithms are found, for example, in [9, 11, 12].

The single allocation problem is a subproblem of the hub location problem obtained by fixing hub locations. In many practical situations, the hub locations are fixed for some time interval because of costs of moving equipment on hubs. In this case, the decision of optimally allocating non-hub nodes to one of given hub nodes is important for efficient operation of the network.

The single allocation problem is first considered by Sohn and Park [17]. They showed the polynomial time solvability of the problem when the number of hub nodes is equal to two. When the number of hub nodes is greater than or equal to three, this problem is proved to be NP-hard [18]. To the best of our knowledge, polynomial time approximation algorithms for the problem have not been studied in the literature.

As we will see in a later section, the single allocation problem is a special class of metric labeling problem. The metric labeling problem was introduced by Kleinberg and Tardos in [10], which has connections to Markov random field and classification problems that arise in computer vision and related areas. They proposed a 2-approximation algorithm for the uniform metric case, which is defined by assuming that all distances between labels (hubs) are the same. For general case, they proposed an $\mathrm{O}(\log h \log\log h)$-approximation algorithm where h is the number of labels (hubs). Chuzhoy and Naor [8] showed that there is no polynomial time approximation algorithm with a constant ratio for the problem unless P=NP. Thus, our results give a practically important class of the metric labeling problem, which has polynomial time approximation algorithms with constant approximation ratios.

This paper is organized as follows: Section 2 formulates the problem as a quadratic 0-1 integer programming problem and derives an LP relaxation of the problem through a mixed integer linear programming reformulation. We also present a simple 3-approximation algorithm. In Section 3, we propose a 2-approximation algorithm. Section 4 deals with the case where the number of hubs is equal to three and proposes a (5/4)-approximation algorithm for the single allocation problem and a (4/3)-approximation algorithm for an ordinary metric labeling problem with three labels. The last section states conclusions.

2 Problem formulations

In this section, we show a formulation of the single allocation problem. Let H and N be sets of hub nodes with $|H| = h$ and non-hub nodes with $|N| = n$, respectively. We define $\widetilde{N^2} \stackrel{\text{def.}}{=} \{(p, q) \in N^2 \mid p \neq q\}$. For any pair of nodes $(p, q) \in \widetilde{N^2} \cup (N \times H) \cup (H \times N)$, a given non-negative amount of flow from p to q is denoted by $w_{pq} (\geq 0)$. For any pair of nodes $(i, j) \in (H \times H) \cup (H \times N) \cup (N \times H)$, a given non-negative transportation cost per unit flow is denoted by $c_{ij} (\geq 0)$. Throughout this paper, we assume the following.

Assumption 1. *A given cost* c_{ij} *satisfies*
(i) $c_{ii} = 0$ *for any* $i \in H$,
(ii) *triangle inequalities among hubs, i.e.,* $c_{ij} \le c_{ik} + c_{kj}$ *for any* $(i, j, k) \in H^3$,
(iii) *symmetry, i.e.,* $c_{ij} = c_{ji}$ $(\forall (i,j) \in (H \times H) \cup (H \times N) \cup (N \times H))$.

In some sections, we also assume the following.

Assumption 2. *A given cost* c_{ij} *satisfies* $c_{ij} \le c_{pi} + c_{pj}$ $(\forall (p, i, j) \in N \times H^2)$.

This assumption stems from an ordinary triangle inequality and the fact that there is an economy of scale with respect to the transportation among hubs in practical situations.

We introduce variables $x_{pi} \in \{0, 1\}$ $(\forall (p, i) \in N \times H)$ where $x_{pi} = 1$ when non-hub node p is connected to hub node i and $x_{pi} = 0$ otherwise. Then the single allocation problem is formulated as follows:

$$\begin{array}{lll} \text{QIP}: & \text{min.} & \displaystyle\sum_{(p,q)\in \widetilde{N^2}} w_{pq} \left(\sum_{i\in H} c_{pi} x_{pi} + \sum_{i\in H}\sum_{j\in H} c_{ij} x_{pi} x_{qj} + \sum_{j \in H} c_{jq} x_{qj} \right) \\ & & \displaystyle + \sum_{(p,j)\in N\times H} w_{pj} \sum_{i\in H} (c_{pi} + c_{ij}) x_{pi} + \sum_{(i,q)\in H\times N} w_{iq} \sum_{j\in H} (c_{ij} + c_{jq}) x_{qj} \\ & \text{s. t.} & \sum_{i\in H} x_{pi} = 1 \qquad\qquad\qquad\qquad\qquad\qquad (\forall p \in N), \\ & & x_{pi} \in \{0, 1\} \qquad\qquad\qquad\qquad\qquad (\forall (p, i) \in N \times H). \end{array}$$

Note that we omit the transportation cost between hub nodes in the objective function because it is a constant term.

Here we briefly mention a relation between the single allocation problem and the metric labeling problem. Roughly speaking, by replacing hub nodes and non-hub nodes with labels and objects, respectively, the problem QIP becomes the metric labeling, which was first considered by Kleinberg and Tardos in [10]. The problem is called “metric labeling” since distances among labels (hub nodes) satisfy the axioms of metric spaces, which correspond to Assumption 1. When objects and labels are embedded in a common metric space, we can assume Assumption 2. Thus, under Assumption 1, QIP is essentially equivalent to the metric labeling problem and by adding Assumption 2, QIP includes a special class of the metric labeling problem defined by introducing the assumption that objects and labels are embedded in a common metric space.

An immediate relaxation problem of QIP is obtained by substituting non-negativity of variables for integrality. It is easy to show that the obtained continuous optimization problem has a 0-1 valued optimal solution (This is discussed for a similar quadratic 0-1 programming problem in [16]). Thus, the difficulty of QIP stems from quadratic terms of the objective function. Adams and Sherali [1] proposed a tight linearization for general zero-one quadratic programming problems. By simply applying their technique, we can transform QIP to the following mixed integer programming (MIP) problem:

$$
\begin{aligned}
\text{MIP}: \quad \text{min.} \quad & \sum_{(p,q)\in\widetilde{N^2}} w_{pq}\left(\sum_{i\in H} c_{pi}x_{pi} + \sum_{i\in H}\sum_{j\in H} c_{ij}y_{piqj} + \sum_{j\in H} c_{jq}x_{qj}\right) \\
& + \sum_{(p,j)\in N\times H} w_{pj}\sum_{i\in H}(c_{pi}+c_{ij})x_{pi} + \sum_{(i,q)\in H\times N} w_{iq}\sum_{j\in H}(c_{ij}+c_{jq})x_{qj} \\
\text{s. t.} \quad & \textstyle\sum_{i\in H} x_{pi} = 1 && (\forall p\in N), \\
& \textstyle\sum_{j\in H} y_{piqj} = x_{pi} && (\forall (p,q)\in\widetilde{N^2},\ \forall i\in H), \\
& \textstyle\sum_{i\in H} y_{piqj} = x_{qj} && (\forall (p,q)\in\widetilde{N^2},\ \forall j\in H), \\
& x_{pi}\in\{0,1\} && (\forall (p,i)\in N\times H), \\
& y_{piqj}\geq 0 && (\forall (p,q)\in\widetilde{N^2},\ \forall (i,j)\in H^2).
\end{aligned}
$$

The formulation MIP can be obtained by replacing $x_{pi}x_{qj}$ with a new variable y_{piqj}, multiplying $\sum_{i\in H} x_{pi} = 1$ by x_{qj} to derive $\sum_{i\in H} y_{piqj} = x_{qj}$ ($\sum_{j\in H} y_{piqj} = x_{pi}$ is derived in the same manner). Throughout this paper, the objective function of MIP is denoted by $\widehat{\boldsymbol{w}}^\top \boldsymbol{x} + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}$ for simplicity. We remark that these QIP and MIP formulations are also studied under the name of the quadratic semi-assignment problem (for details, see a polyhedral study [14] and references therein).

We consider the linear programming relaxation of MIP, called LPR, obtained by substituting non-negativity constraints $x_{pi}\geq 0$ for 0-1 constraints $x_{pi}\in\{0,1\}$. In [15], two of authors performed computational experiments with widely used data set called CAB data [13]. Their results indicate the tightness of LPR. In succeeding sections, we propose rounding procedures and bound the objective value produced by applying them to an optimal solution of LPR.

We close this section by presenting a simple 3-approximation algorithm, called "`Nearest Neighbor Algorithm`," that only connects each non-hub node to the nearest hub node.

Theorem 1. *Under Assumptions 1 and 2,* `Nearest Neighbor Algorithm` *yields a 3-approximation algorithm.*

Proof. We consider a pair of non-hub nodes $(p,q) \in \widetilde{N^2}$. Let i' and j' be the nearest hub nodes from the non-hub nodes p and q, respectively. Suppose that p and q are connected to i^* and j^* in an optimal allocation, respectively. It is clear that $c_{pi'} \leq c_{pi^*}$ and $c_{qj'} \leq c_{qj^*}$ hold. From Assumptions 1 and 2, a transportation cost per unit associated with $(p,q) \in \widetilde{N^2}$ is bounded by

$$\begin{aligned} c_{pi'} + c_{i'j'} + c_{qj'} &\leq c_{pi'} + (c_{i'i^*} + c_{i^*j^*} + c_{j^*j'}) + c_{qj'} \\ &\leq c_{pi'} + (c_{i'p} + c_{pi^*}) + c_{i^*j^*} + (c_{j^*q} + c_{qj'}) + c_{qj'} \\ &\leq 3c_{pi^*} + c_{i^*j^*} + 3c_{qj^*} \leq 3(c_{pi^*} + c_{i^*j^*} + c_{qj^*}). \end{aligned}$$

This property also holds even if $i' = j'$ or $i^* = j^*$, since $c_{i'i'} = c_{i^*i^*} = 0$. For any pair in $(N \cup H) \cup (H \cup N)$, a transportation cost per unit is bounded in a similar way. Thus we have a desired result. □

A solution obtained by `Nearest Neighbor Algorithm` only depends on the transportation cost per unit and thus robust with respect to changes and/or uncertainties in flow values (w_{pq}).

3 2-approximation algorithm

We propose a 2-approximation algorithm for the single allocation problem. Our algorithm, called "`Independent Rounding Algorithm`," independently connects each non-hub node $p \in N$ to a hub node $i \in H$ with probability x^*_{pi} where $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an optimal solution of LPR. In the rest of this section, we show that our algorithm gives a 2-approximation algorithm under Assumptions 1 and 2.

First, we present a key lemma of this section. Recall that the objective function of MIP is denoted by $\widehat{\boldsymbol{w}}^\top \boldsymbol{x} + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}$.

Lemma 1. *Let $(\boldsymbol{x}, \boldsymbol{y})$ and $(\boldsymbol{x'}, \boldsymbol{y'})$ be feasible solutions of LPR with $\boldsymbol{x} = \boldsymbol{x'}$. Under Assumptions 1 and 2, the inequality $\widetilde{\boldsymbol{w}}^\top \boldsymbol{y'} \leq \widehat{\boldsymbol{w}}^\top \boldsymbol{x} + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}$ holds.*

The above lemma, which we will prove later, directly implies a main result of this section.

Theorem 2. *Under Assumptions 1 and 2,* `Independent Rounding Algorithm` *gives a 2-approximation algorithm.*

Proof. We denote an optimal solution of LPR by $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ and $\boldsymbol{X}$ be a vector of random variables (indexed by $N \times H$) obtained by applying `Independent Rounding Algorithm` to $(\boldsymbol{x}^*, \boldsymbol{y}^*)$. The objective function value with respect to $\boldsymbol{X}$ is $\widehat{\boldsymbol{w}}^\top \boldsymbol{X} + \widetilde{\boldsymbol{w}}^\top \boldsymbol{Y}$ where $Y_{piqj} = X_{pi}X_{qj}$. Since X_{pi} and X_{qj} are independent if $p \neq q$, the equality $\mathrm{E}[Y_{piqj}] = \mathrm{E}[X_{pi}]\mathrm{E}[X_{qj}] = x^*_{pi}x^*_{qj}$ holds. Thus the expectation of the objective value with respect to $\boldsymbol{X}$ is

$$\mathrm{E}[\widehat{\boldsymbol{w}}^\top \boldsymbol{X} + \widetilde{\boldsymbol{w}}^\top \boldsymbol{Y}] = \widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}'$$

where $y'_{piqj} = x^*_{pi}x^*_{qj}$. It is clear that the pair $(\boldsymbol{x}^*, \boldsymbol{y}')$ is feasible to LPR and thus Lemma 1 directly implies that

$$\widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}' \leq \widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + (\widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}^*) \leq 2(\widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}^*).$$

Since $\widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}^*$ gives a lower bound of the optimal value of QIP, we obtained a desired result. □

In the rest of this section, we prove Lemma 1. First, we describe a property obtained from Assumptions 1 and 2. We denote the set of hub nodes by $H = \{1, 2, \ldots, h\}$. For any pair of non-hub nodes $(p, q) \in \widetilde{N^2}$, we introduce a complete directed bipartite graph $G^{pq} = (V_p, V_q, E_1 \cup E_2)$ where $V_p \stackrel{\text{def.}}{=} \{p_1, \ldots, p_h\}$, $V_q \stackrel{\text{def.}}{=} \{q_1, \ldots, q_h\}$, $E_1 \stackrel{\text{def.}}{=} V_p \times V_q$, and $E_2 \stackrel{\text{def.}}{=} V_q \times V_p$. For each arc $(p_i, q_j) \in E_1$ and $(q_j, p_i) \in E_2$, we associate an arc cost c_{ij} and c_{ji}, respectively. For each vertex $p_i \in V_p$ and $q_j \in V_q$, we associate a vertex cost c_{pi} and c_{qj}, respectively. Given an arc subset $E' \subseteq E_1 \cup E_2$ and a vertex subset $V' \subseteq V_p \cup V_q$, we denote the sum of costs of arcs in E' and vertices in V' by $\boldsymbol{c}_E(E')$ and $\boldsymbol{c}_V(V')$, respectively. For any elementary dicycle C in G^{pq}, ∂C denotes a set of vertices covered by C. We denote a set of arcs in a dicycle C by C, if there is no ambiguity.

Lemma 2. *Under Assumptions 1 and 2, every elementary dicycle C in G^{pq} satisfies that*

$$\boldsymbol{c}_E(C \cap E_1) \leq \boldsymbol{c}_V(\partial C) + \boldsymbol{c}_E(C \cap E_2).$$

Proof. Since C is an elementary dicycle, we can express C by a sequence of vertices $(p_{i_1}, q_{j_1}, p_{i_2}, q_{j_2}, \dots, p_{i_k}, q_{j_k})$ where $2k$ denotes the length of C. In the following, we identify (p_{i_0}, q_{j_0}) with (p_{i_k}, q_{j_k}), and $(p_{i_{k+1}}, q_{j_{k+1}})$ with (p_{i_1}, q_{j_1}). From Assumption 1, the cost $c_{i_\ell j_\ell}$ of arc $(p_{i_\ell}, q_{j_\ell}) \in C \cap E_1$ satisfies that

$$\begin{aligned} c_{i_\ell j_\ell} &\leq c_{i_\ell i_{\ell+1}} + c_{i_{\ell+1} j_\ell} = c_{i_\ell i_{\ell+1}} + c_{j_\ell i_{\ell+1}}, \\ c_{i_\ell j_\ell} &\leq c_{i_\ell j_{\ell-1}} + c_{j_{\ell-1} j_\ell} = c_{j_{\ell-1} i_\ell} + c_{j_{\ell-1} j_\ell}. \end{aligned}$$

Assumption 2 implies that

$$c_{i_\ell i_{\ell+1}} \leq c_{p i_\ell} + c_{p i_{\ell+1}} \quad \text{and} \quad c_{j_{\ell-1} j_\ell} \leq c_{q j_{\ell-1}} + c_{q j_\ell}.$$

The above inequalities yield that

$$\begin{aligned} \boldsymbol{c}_E(C \cap E_1) &= \textstyle\sum_{\ell=1}^k c_{i_\ell j_\ell} \leq (1/2) \sum_{\ell=1}^k (c_{i_\ell i_{\ell+1}} + c_{j_\ell i_{\ell+1}} + c_{j_{\ell-1} i_\ell} + c_{j_{\ell-1} j_\ell}) \\ &\leq (1/2) \textstyle\sum_{\ell=1}^k (c_{p i_\ell} + c_{p i_{\ell+1}} + c_{j_\ell i_{\ell+1}} + c_{j_{\ell-1} i_\ell} + c_{q j_{\ell-1}} + c_{q j_\ell}) \\ &= (1/2) \textstyle\sum_{\ell=1}^k (c_{p i_\ell} + c_{p i_{\ell+1}} + c_{q j_{\ell-1}} + c_{q j_\ell}) + (1/2) \sum_{\ell=1}^k (c_{j_\ell i_{\ell+1}} + c_{j_{\ell-1} i_\ell}) \\ &= \textstyle\sum_{\ell=1}^k (c_{p i_\ell} + c_{q j_\ell}) + \sum_{\ell=1}^k c_{j_\ell i_{\ell+1}} = \boldsymbol{c}_V(\partial C) + \boldsymbol{c}_E(C \cap E_2). \end{aligned}$$

□

Given a feasible solution $(\boldsymbol{x}, \boldsymbol{y})$ of LPR and a pair of non-hub nodes $(p, q) \in \widetilde{N^2}$, $\boldsymbol{x}|_{pq}$ denotes a subvector of $\boldsymbol{x}$ consists of elements $\{x_{pi} \mid i \in H\} \cup \{x_{qj} \mid j \in H\}$. A subvector of $\boldsymbol{y}$ consists of elements $\{y_{piqj} \mid (i,j) \in H^2\}$ is denoted by $\boldsymbol{y}|_{pq}$. When $(\boldsymbol{x}, \boldsymbol{y})$ is feasible to LPR, the pair $\boldsymbol{x}|_{pq}$ and $\boldsymbol{y}|_{pq}$ satisfies that

$$\textstyle\sum_{j \in H} y_{piqj} = x_{pi} \ (i \in H) \quad \text{and} \quad \sum_{i \in H} y_{piqj} = x_{qj} \ (j \in H).$$

We denote the above equality system by $M^{pq} \boldsymbol{y}|_{pq} = \boldsymbol{x}|_{pq}$.

Now we give a proof of Lemma 1.

Proof of Lemma 1. The outline of the proof is as follows. For any $(p, q) \in \widetilde{N^2}$, we introduce a flow $f : E_1 \cup E_2 \to \mathbf{R}$, on the digraph G^{pq}, defined by

$$f(e) \stackrel{\text{def.}}{=} \begin{cases} y'_{piqj} & (e = (p_i, q_j) \in E_1), \\ y_{piqj} & (e = (q_j, p_i) \in E_2). \end{cases}$$

First, we show that flow f is a circulation flow on G^{pq}. Next, we decompose f into cycles. Lastly, we apply Lemma 2 to each cycle and show the inequality.

We show that f is a circulation flow. Since both $(\boldsymbol{x}, \boldsymbol{y})$ and $(\boldsymbol{x'}, \boldsymbol{y'})$ are feasible to LPR, the equalities $M^{pq} \boldsymbol{y}|_{pq} = \boldsymbol{x}|_{pq} = \boldsymbol{x'}|_{pq} = M^{pq} \boldsymbol{y'}|_{pq}$ hold and thus flow f satisfies the conservation law for each vertex in G^{pq}. It implies that f is a non-negative circulation flow on G^{pq}.

A well-known "flow decomposition theorem" says that a circulation flow is represented by a non-negative combination of cycle flows (see, e.g., [2]). Let $\mathcal{C}^{pq}$ be a set of all elementary dicycles in G^{pq}. A *cycle flow* with respect to dicycle $C \in \mathcal{C}^{pq}$ is defined by introducing a unit flow for each arc in C. Let λ be a vector of non-negative coefficients indexed by $\mathcal{C}^{pq}$ which represents flow f by a non-negative combination of cycle flows. We denote an element of λ indexed by a dicycle C by λ_C.

For any cycle $C \in \mathcal{C}^{pq}$, we denote characteristic vectors of $C \cap E_1$ and $C \cap E_2$ by ψ^C and χ^C, respectively, i.e.,

$$\psi^C_{piqj} = \begin{cases} 1 & ((p_i, q_j) \in C \cap E_1), \\ 0 & ((p_i, q_j) \in E_1 \setminus C), \end{cases} \quad \text{and} \quad \chi^C_{piqj} = \begin{cases} 1 & ((q_j, p_i) \in C \cap E_2), \\ 0 & ((q_j, p_i) \in E_2 \setminus C). \end{cases}$$

Similarly, we define the characteristic vector $\chi^{\partial C}$ of $\partial(C)$ by

$$\chi^{\partial C}_v = \begin{cases} 1 & (v \in \partial C), \\ 0 & (v \in (V_p \cup V_q) \setminus \partial C). \end{cases}$$

Every cycle $C \in \mathcal{C}^{pq}$ satisfies the equality $M^{pq}\chi^C = \chi^{\partial C}$, since $C \cap E_2$ is a matching.

We express a transportation cost per unit associated with (p, q) as follows

$$\sum_{i \in H} c_{pi} x_{pi} + \sum_{i \in H} \sum_{j \in H} c_{ij} y_{piqj} + \sum_{j \in H} c_{jq} x_{qj} = \widehat{\boldsymbol{c}^{pq}}^\top \boldsymbol{x}|_{pq} + \widetilde{\boldsymbol{c}^{pq}}^\top \boldsymbol{y}|_{pq}$$

by introducing appropriate vectors $\widehat{\boldsymbol{c}^{pq}}$ and $\widetilde{\boldsymbol{c}^{pq}}$. The above definitions yield that

$$\begin{aligned} \widetilde{\boldsymbol{w}}^\top \boldsymbol{y'} &= \sum_{(p,q) \in \widetilde{N^2}} w_{pq} \sum_{i \in H} \sum_{j \in H} c_{ij} y'_{piqj} = \sum_{(p,q) \in \widetilde{N^2}} w_{pq} \widetilde{\boldsymbol{c}^{pq}}^\top \boldsymbol{y'}|_{pq} \\ &= \sum_{(p,q) \in \widetilde{N^2}} w_{pq} \widetilde{\boldsymbol{c}^{pq}}^\top \left(\sum_{C \in \mathcal{C}^{pq}} \lambda_C \psi^C \right) = \sum_{(p,q) \in \widetilde{N^2}} w_{pq} \sum_{C \in \mathcal{C}^{pq}} \lambda_C \left(\widetilde{\boldsymbol{c}^{pq}}^\top \psi^C \right) \\ &= \sum_{(p,q) \in \widetilde{N^2}} w_{pq} \sum_{C \in \mathcal{C}^{pq}} \lambda_C \ \boldsymbol{c}_E(C \cap E_1). \end{aligned}$$

Lemma 2 implies that

$$
\begin{aligned}
\widetilde{\boldsymbol{w}}^\top \boldsymbol{y'} &\le \sum_{(p,q)\in\widetilde{N^2}} w_{pq} \sum_{C\in\mathcal{C}^{pq}} \lambda_C \left(\boldsymbol{c}_E(C\cap E_2) + \boldsymbol{c}_V(\partial C)\right) \\
&= \sum_{(p,q)\in\widetilde{N^2}} w_{pq} \sum_{C\in\mathcal{C}^{pq}} \lambda_C (\widetilde{\boldsymbol{c}^{pq}}^\top \chi^C + \widehat{\boldsymbol{c}^{pq}}^\top \chi^{\partial C}) \\
&= \sum_{(p,q)\in\widetilde{N^2}} w_{pq} \left(\widetilde{\boldsymbol{c}^{pq}}^\top \left(\sum_{C\in\mathcal{C}^{pq}} \lambda_C \chi^C \right) + \widehat{\boldsymbol{c}^{pq}}^\top \left(\sum_{C\in\mathcal{C}^{pq}} \lambda_C M^{pq} \chi^C \right)\right) \\
&= \sum_{(p,q)\in\widetilde{N^2}} w_{pq} \left(\widetilde{\boldsymbol{c}^{pq}}^\top \boldsymbol{y}|_{pq} + \widehat{\boldsymbol{c}^{pq}}^\top M^{pq} \left(\sum_{C\in\mathcal{C}^{pq}} \lambda_C \chi^C \right)\right) \\
&= \sum_{(p,q)\in\widetilde{N^2}} w_{pq} \left(\widetilde{\boldsymbol{c}^{pq}}^\top \boldsymbol{y}|_{pq} \right) + \sum_{(p,q)\in\widetilde{N^2}} w_{pq} \left(\widehat{\boldsymbol{c}^{pq}}^\top M^{pq} \boldsymbol{y}|_{pq} \right) \\
&= \widetilde{\boldsymbol{w}}^\top \boldsymbol{y} + \sum_{(p,q)\in\widetilde{N^2}} w_{pq} \left(\widehat{\boldsymbol{c}^{pq}}^\top \boldsymbol{x}|_{pq} \right) \\
&\le \widetilde{\boldsymbol{w}}^\top \boldsymbol{y} + \sum_{(p,q)\in\widetilde{N^2}} w_{pq} \left(\widehat{\boldsymbol{c}^{pq}}^\top \boldsymbol{x}|_{pq} \right) + \sum_{(p,j)\in N\times H} w_{pj} \sum_{i\in H} (c_{pi} + c_{ij}) x_{pi} \\
&\quad + \sum_{(i,q)\in H\times N} w_{iq} \sum_{j\in H} (c_{ij} + c_{jq}) x_{qj} = \widetilde{\boldsymbol{w}}^\top \boldsymbol{y} + \widehat{\boldsymbol{w}}^\top \boldsymbol{x}.
\end{aligned}
$$

□

4 Approximation algorithms for three hubs

In this section, we introduce a new rounding technique. Let Π be a set of all the total orders of hubs, i.e., $\{\pi(1), \pi(2), \ldots, \pi(h)\} = H$ $(\forall \pi \in \Pi)$. For any $\pi \in \Pi$, we define a procedure "`Dependent Rounding` π" as follows. Given a feasible solution $(\boldsymbol{x}, \boldsymbol{y})$ of LPR, we generate a random variable U following a uniform distribution defined on $[0, 1)$. For each non-hub node $p \in N$, we connect p to a hub $\pi(i)$ where $i \in \{1, 2, \ldots, h\}$ is the minimum index satisfying $U < x_{p\pi(1)} + \cdots + x_{p\pi(i)}$. A vector of random variables X^π, indexed by $N \times H$, denotes a solution obtained by `Dependent Rounding` π. Randomized rounding method with such a dependency among 0-1 variables was named "dependent rounding" by Bertsimas, Teo, and Vohra in [3]. They also devised (approximation) algorithms for several combinatorial optimization problems.

In the following, we discuss the probability $\Pr[X^{\pi}_{pi}X^{\pi}_{qj} = 1]$. If $(\boldsymbol{x}, \boldsymbol{y})$ is feasible to LPR, $\boldsymbol{y}$ satisfies that for any pair $(p, q) \in \widetilde{N^2}$, the subvector $\boldsymbol{y}|_{pq}$ is feasible to the following Hitchcock transportation problem

$$\begin{array}{rll} \mathrm{HTP}_{pq}(\boldsymbol{x}): & \text{min.} & \sum_{i\in H}\sum_{j\in H} c_{ij}\widetilde{y}_{piqj} \\ & \text{s. t.} & \sum_{j\in H}\widetilde{y}_{piqj} = x_{pi} \qquad (\forall i \in H), \\ & & \sum_{i\in H}\widetilde{y}_{piqj} = x_{qj} \qquad (\forall j \in H), \\ & & \widetilde{y}_{piqj} \geq 0 \qquad (\forall (i,j) \in H^2), \end{array}$$

where $\{\widetilde{y}_{piqj} \mid (i,j) \in H^2\}$ is a set of variables. Given a feasible solution $(\boldsymbol{x}, \boldsymbol{y})$ of LPR and a total order $\pi \in \Pi$, we define $(\boldsymbol{x}, \boldsymbol{y}^{\pi})$ be a solution obtained by applying a classical "North-West Corner Rule" with respect to π to transportation problems $\mathrm{HTP}_{pq}(\boldsymbol{x})$ (for each $(p,q) \in \widetilde{N^2}$). More precisely, $\boldsymbol{y}^{\pi}$ is the unique vector satisfying the equalities

$$\sum_{i=1}^{i'}\sum_{j=1}^{j'} y^{\pi}_{p\pi(i)q\pi(j)} = \min\left\{\sum_{i=1}^{i'} x_{p\pi(i)}\ ,\ \sum_{j=1}^{j'} x_{q\pi(j)}\right\}\left(\begin{array}{l}\forall (p,q) \in \widetilde{N^2}, \\ \forall (i',j') \in \{1,2,\ldots,h\}^2\end{array}\right).$$

Lemma 3. *Let $(\boldsymbol{x}, \boldsymbol{y})$ be a feasible solution of LPR and $\pi \in \Pi$ a total order of H. A vector of random variables $\boldsymbol{X}^{\pi}$ obtained by applying procedure* `Dependent Rounding` π *to $(\boldsymbol{x}, \boldsymbol{y})$ satisfies that $\Pr[X^{\pi}_{pi}X^{\pi}_{qj} = 1] = y^{\pi}_{piqj}$ $(\forall (p,q) \in \widetilde{N^2}, \forall (i,j) \in H^2)$ where $(\boldsymbol{x}, \boldsymbol{y}^{\pi})$ is North-West Corner Rule solution with respect to π.*

Proof. We denote $\Pr[X^{\pi}_{pi}X^{\pi}_{qj} = 1]$ by y'_{piqj} for simplicity. Then the vector $\boldsymbol{y}'$ satisfies that for any pairs $(p,q) \in \widetilde{N^2}$ and $(i',j') \in H^2$,

$$\begin{aligned} \textstyle\sum_{i=1}^{i'}\sum_{j=1}^{j'} y'_{p\pi(i)q\pi(j)} &= \Pr\left[\left[\textstyle\sum_{i=1}^{i'} X_{p\pi(i)} = 1\right] \wedge \left[\textstyle\sum_{j=1}^{j'} X_{q\pi(j)} = 1\right]\right] \\ &= \Pr\left[\left[U < \textstyle\sum_{i=1}^{i'} x_{p\pi(i)}\right] \wedge \left[U < \textstyle\sum_{j=1}^{j'} x_{q\pi(j)}\right]\right] \\ &= \Pr\left[U < \min\left\{\textstyle\sum_{i=1}^{i'} x_{p\pi(i)}, \textstyle\sum_{j=1}^{j'} x_{q\pi(j)}\right\}\right] \\ &= \min\left\{\textstyle\sum_{i=1}^{i'} x_{p\pi(i)}, \textstyle\sum_{j=1}^{j'} x_{q\pi(j)}\right\}. \end{aligned}$$

From the above, $(\boldsymbol{x}, \boldsymbol{y}')$ is North-West Corner Rule solution with respect to π and the uniqueness of North-West Corner Rule solution implies $\boldsymbol{y}' = \boldsymbol{y}^{\pi}$. □

Chekuri et al. [7] also discussed procedure `Dependent Rounding` π in the context of the metric labeling problem. They dealt with a line metric case and pointed out a relation to Monge property. In the above lemma, we explicitly showed a relation between procedure `Dependent Rounding` π and North-West Corner Rule solution, which is independent of Monge property.

Now we consider the case where the number of hubs is equal to three, i.e., $h = 3$. We propose a "`Dependent Rounding Algorithm`" which executes `Dependent Rounding` π for every $\pi \in \Pi$ to an optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ of LPR and outputs a best solution. For discussing an approximation ratio of `Dependent Rounding Algorithm`, we introduce an artificial rounding procedure described below. In the rest of this section, we denote $H = \{1, 2, 3\}$ and $a = c_{12}$, $b = c_{23}$, and $c = c_{31}$ for simplicity. Next, we introduce three non-negative parameters $(\theta_1, \theta_2, \theta_3)$ defined by

$$\begin{aligned}
&\theta_1 \stackrel{\text{def.}}{=} b(b+c-a)(a+b-c)/K, \\
&\theta_2 \stackrel{\text{def.}}{=} c(c+a-b)(b+c-a)/K, \\
&\theta_3 \stackrel{\text{def.}}{=} a(a+b-c)(c+a-b)/K, \\
&K \stackrel{\text{def.}}{=} b(b+c-a)(a+b-c) + c(c+a-b)(b+c-a) + a(a+b-c)(c+a-b).
\end{aligned}$$

Assumption 1 (ii) implies that $K \geq 0$. We can assume that $K > 0$, since $K = 0$ implies that $a = b = c = 0$ which gives a trivial case. Obviously, $\theta_1 + \theta_2 + \theta_3 = 1$ holds. We also denote three total orders $(2, 1, 3), (3, 2, 1), (1, 3, 2)$ of H by π^1, π^2, π^3, respectively. An artificial rounding procedure "`Dependent Rounding` θ" executes one of three procedures `Dependent Rounding` π^1, π^2, or π^3 with probability θ_1, θ_2, or θ_3, respectively. First we show the following.

Lemma 4. *Let $(\boldsymbol{x}, \boldsymbol{y})$ be a feasible solution of LPR. Under Assumption 1, a vector of random variables $\boldsymbol{X}^\theta$ obtained by applying* `Dependent Rounding` θ *to $(\boldsymbol{x}, \boldsymbol{y})$ satisfies that*

$$\begin{array}{rr}
\mathrm{E}[X^\theta_{pi}] = x_{pi} & (\forall (p,i) \in N \times H), \\
\mathrm{E}\left[\sum_{i \in H} \sum_{j \in H} c_{ij} X^\theta_{pi} X^\theta_{qj}\right] \leq (4/3) \sum_{i \in H} \sum_{j \in H} c_{ij} y_{piqj} & (\forall (p,q) \in \widetilde{N^2}).
\end{array}$$

Proof. We discuss the second inequalities because the first equalities are trivial. It is well-known that North-West Corner Rule solution is optimal to a Hitchcock transportation problem if a given cost matrix has Monge Property (e.g., see a survey [5]). For any total order $\pi \in \Pi$, a matrix defined by

$$\begin{bmatrix}
c_{\pi(1)\pi(1)}(=0) & c_{\pi(1)\pi(2)} & c_{\pi(1)\pi(2)} + c_{\pi(2)\pi(3)} \\
c_{\pi(2)\pi(1)} & c_{\pi(2)\pi(2)}(=0) & c_{\pi(2)\pi(3)} \\
c_{\pi(3)\pi(2)} + c_{\pi(2)+\pi(1)} & c_{\pi(3)\pi(2)} & c_{\pi(3)\pi(3)}(=0)
\end{bmatrix}$$

has Monge property. Hence, North-West Corner Rule solution $(\boldsymbol{x}, \boldsymbol{y}^{\pi})$ satisfies that subvector $\boldsymbol{y}^{\pi}|_{pq}$ of $\boldsymbol{y}^{\pi}$ is optimal to a Hitchcock transportation problem obtained from $\mathrm{HTP}_{pq}(\boldsymbol{x})$ by substituting $c_{\pi(1)\pi(2)} + c_{\pi(2)\pi(3)}$ for $c_{\pi(1)\pi(3)}$. We express $\sum_{i\in H}\sum_{j\in H} c_{ij} y_{piqj} = \widetilde{\boldsymbol{c}^{pq}}^{\top} \boldsymbol{y}|_{pq}$ by introducing an appropriate vector $\widetilde{\boldsymbol{c}^{pq}}$. We define a modified cost vector $\widetilde{\boldsymbol{c}^{pq}_{\pi}}$ which is obtained from $\widetilde{\boldsymbol{c}^{pq}}$ by substituting $c_{\pi(1)\pi(2)} + c_{\pi(2)\pi(3)}$ for $c_{\pi(1)\pi(3)}$. The optimality of North-West Corner Rule solution implies that $\widetilde{\boldsymbol{c}^{pq}_{\pi}}^{\top} \boldsymbol{y}|_{pq} \geq \widetilde{\boldsymbol{c}^{pq}_{\pi}}^{\top} \boldsymbol{y}^{\pi}|_{pq}$. Triangle inequalities and symmetry (Assumption 1 (ii)(iii)) imply that $\widetilde{\boldsymbol{c}^{pq}_{\pi}}^{\top} \boldsymbol{y}^{\pi}|_{pq} \geq \widetilde{\boldsymbol{c}^{pq}}^{\top} \boldsymbol{y}^{\pi}|_{pq}$. From the above, we have that for any $\pi \in \Pi$,

$$
\begin{aligned}
\widetilde{\boldsymbol{c}^{pq}_{\pi}}^{\top} \boldsymbol{y}|_{pq} &\geq \widetilde{\boldsymbol{c}^{pq}_{\pi}}^{\top} \boldsymbol{y}^{\pi}|_{pq} \geq \widetilde{\boldsymbol{c}^{pq}}^{\top} \boldsymbol{y}^{\pi}|_{pq} = \sum_{i\in H}\sum_{j\in H} c_{ij} y^{\pi}_{piqj} \\
&= \sum_{i\in H}\sum_{j\in H} c_{ij} \Pr[X^{\pi}_{pi} X^{\pi}_{qj} = 1] = \mathrm{E}\left[\sum_{i\in H}\sum_{j\in H} c_{ij} X^{\pi}_{pi} X^{\pi}_{qj}\right],
\end{aligned}
$$

where $\boldsymbol{X}^{\pi}$ is a vector of random variables obtained by applying `Dependent Rounding` π to $(\boldsymbol{x}, \boldsymbol{y})$.

From Assumption 1 (i) and the above, we obtain the following;

$$
\begin{aligned}
&\mathrm{E}\left[\textstyle\sum_{i\in H}\sum_{j\in H} c_{ij} X^{\theta}_{pi} X^{\theta}_{qj}\right] = \mathrm{E}\left[\textstyle\sum_{i\in H}\sum_{j\in H} c_{ij} \sum_{\ell=1}^{3} \theta_{\ell} X^{\pi^{\ell}}_{pi} X^{\pi^{\ell}}_{qj}\right] \\
&= \mathrm{E}\left[\theta_1 \sum_{i\in H}\sum_{j\in H} c_{ij} X^{\pi^1}_{pi} X^{\pi^1}_{qj} + \theta_2 \sum_{i\in H}\sum_{j\in H} c_{ij} X^{\pi^2}_{pi} X^{\pi^2}_{qj} + \theta_3 \sum_{i\in H}\sum_{j\in H} c_{ij} X^{\pi^3}_{pi} X^{\pi^3}_{qj}\right] \\
&\leq \theta_1 \widetilde{\boldsymbol{c}^{pq}_{\pi^1}}^{\top} \boldsymbol{y}|_{pq} + \theta_2 \widetilde{\boldsymbol{c}^{pq}_{\pi^2}}^{\top} \boldsymbol{y}|_{pq} + \theta_3 \widetilde{\boldsymbol{c}^{pq}_{\pi^3}}^{\top} \boldsymbol{y}|_{pq} \\
&= \theta_1(c_{12} r + (c_{21} + c_{13}) s + c_{31} t) \\
&\quad + \theta_2(c_{12} r + c_{23} s + (c_{32} + c_{21}) t) \\
&\quad + \theta_3((c_{13} + c_{32}) r + c_{23} s + c_{31} t)
\end{aligned}
$$

where $r \overset{\text{def.}}{=} (y_{p1q2} + y_{p2q1})$, $s \overset{\text{def.}}{=} (y_{p2q3} + y_{p3q2})$, $t \overset{\text{def.}}{=} (y_{p1q3} + y_{p3q1})$. Then

$\theta_1 + \theta_2 + \theta_3 = 1$ implies that

$$\begin{array}{rl}
& \theta_1 \widetilde{\boldsymbol{c}^{pq}_{\pi^1}}^\top \boldsymbol{y}|_{pq} + \theta_2 \widetilde{\boldsymbol{c}^{pq}_{\pi^2}}^\top \boldsymbol{y}|_{pq} + \theta_3 \widetilde{\boldsymbol{c}^{pq}_{\pi^3}}^\top \boldsymbol{y}|_{pq} \\
= & \theta_1(\quad ar + (a+c)s + \quad ct) \\
& +\theta_2(\quad ar + \quad bs + (b+a)t) \\
& +\theta_3((c+b)r + \quad bs + \quad ct) \\
= & (ar + bs + ct) + \theta_3(b+c-a)r + \theta_1(a+c-b)s + \theta_2(a+b-c)t \\
= & (ar + bs + ct)(1 + (a+b-c)(c+a-b)(b+c-a)/K) \\
= & \sum_{i \in H} \sum_{j \in H} c_{ij} y_{piqj} (1 + (a+b-c)(c+a-b)(b+c-a)/K).
\end{array}$$

Lastly, we show that

$$(a+b-c)(c+a-b)(b+c-a)/K \leq 1/3.$$

If either $b+c-a=0$, $a+c-b=0$, or $a+b-c=0$ holds, then the above inequality is trivial. Thus we need to consider the problem

$$\min \left\{ \frac{K}{(a+b-c)(c+a-b)(b+c-a)} \;\middle|\; \begin{array}{l} a+b>c,\ c+a>b, \\ b+c>a,\ (a,b,c) \geq \mathbf{0} \end{array} \right\}$$

in order to bound the left-hand-side of the above inequality for every possible a, b, c (≥ 0) satisfying triangle inequalities (Assumption 1 (ii)). Because of the equality

$$\frac{K}{(a+b-c)(c+a-b)(b+c-a)} = \frac{c}{a+b-c} + \frac{b}{c+a-b} + \frac{a}{b+c-a},$$

we can assume $a+b+c=1$ without loss of generality. Therefore the function

$$\frac{c}{a+b-c} + \frac{b}{c+a-b} + \frac{a}{b+c-a} = \frac{c}{1-2c} + \frac{b}{1-2b} + \frac{a}{1-2a}$$

is a convex function of variables a, b, and c. From the symmetry of variables, the minimum is attained at $a=b=c=1/3$ and thus

$$\frac{c}{1-2c} + \frac{b}{1-2b} + \frac{a}{1-2a} \geq 3.$$

From the above, we obtain a desired result. □

Theorem 3. *Under Assumption 1,* `Dependent Rounding Algorithm` *yields a (4/3)-approximation algorithm.*

Proof. Let $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ be an optimal solution of LPR and $\boldsymbol{X}$ be a solution obtained by `Dependent Rounding Algorithm`. Then the expectation of the objective value with respect to $\boldsymbol{X}$ satisfies that

$$\begin{aligned}
&\mathrm{E}\left[\widehat{\boldsymbol{w}}^\top \boldsymbol{X} + \textstyle\sum_{(p,q)\in\widetilde{N^2}} w_{pq} \sum_{i\in H}\sum_{j\in H} c_{ij} X_{pi} X_{qj}\right] \\
&\quad\leq\ \mathrm{E}\left[\widehat{\boldsymbol{w}}^\top \boldsymbol{X}^\theta + \textstyle\sum_{(p,q)\in\widetilde{N^2}} w_{pq} \sum_{i\in H}\sum_{j\in H} c_{ij} X^\theta_{pi} X^\theta_{qj}\right]
\end{aligned}$$

where $\boldsymbol{X}^\theta$ is a vector of random variables obtained by applying `Dependent Rounding` θ to $(\boldsymbol{x}^*, \boldsymbol{y}^*)$. Lemma 4 implies that

$$\begin{aligned}
&\mathrm{E}\left[\widehat{\boldsymbol{w}}^\top \boldsymbol{X}^\theta + \textstyle\sum_{(p,q)\in\widetilde{N^2}} w_{pq} \sum_{i\in H}\sum_{j\in H} c_{ij} X^\theta_{pi} X^\theta_{qj}\right] \\
&\quad\leq\ \widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + \textstyle\sum_{(p,q)\in\widetilde{N^2}} w_{pq} (4/3) \sum_{i\in H}\sum_{j\in H} c_{ij} y^*_{piqj} \\
&\quad=\ \widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + (4/3)\widetilde{\boldsymbol{w}}^\top \boldsymbol{y}^* \leq (4/3)(\widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}^*).
\end{aligned}$$

The optimality of $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ to LPR implies a desired result. □

In the above proof, we do not need Assumption 2. Consequently, our (4/3)-approximation algorithm is also applicable to the metric labeling problem with three labels maintaining theoretical approximation ratio, whereas our 2-approximation algorithm presented in the previous section is not.

Lastly, we propose a (5/4)-approximation algorithm.

Theorem 4. *Under Assumptions 1, 2 and that the number of hubs is equal to three, a better of two solutions given by* `Independent Rounding Algorithm` *and* `Dependent Rounding Algorithm` *satisfies that the expectation of the corresponding objective value is less than or equal to (5/4) times the optimal value of the original problem QIP.*

Proof. Let $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ be an optimal solution of LPR. Let Z_1 and Z_2 denote the objective function value obtained by `Independent Rounding Algorithm` and `Dependent Rounding Algorithm`, respectively. The proof of Theorem 2 shows that $\mathrm{E}[Z_1] \leq 2\widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}^*$. The proof of Theorem 3 implies that $\mathrm{E}[Z_2] \leq \widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + (4/3)\widetilde{\boldsymbol{w}}^\top \boldsymbol{y}^*$. Combining the above results, we obtain that

$$\mathrm{E}[\min\{Z_1, Z_2\}] \leq (1/4)\mathrm{E}[Z_1] + (3/4)\mathrm{E}[Z_2] \leq (5/4)(\widehat{\boldsymbol{w}}^\top \boldsymbol{x}^* + \widetilde{\boldsymbol{w}}^\top \boldsymbol{y}^*). \quad \square$$

5 Conclusion

We proposed a formulation of the single allocation problem in hub-and-spoke networks and presented a simple 3-approximation algorithm and randomized approximation algorithms based on LP relaxation and randomized rounding techniques. Our algorithms can be derandomized using the method of conditional probabilities.

We remark that it is nontrivial to extend our algorithms in Section 4 from $h = 3$ to the general case, because the analysis depends on a modification of a given cost matrix to convex combination of Monge matrices (Lemma 4).

Obtaining approximation algorithms for the hub location problem is a challenging open problem.

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