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Global Stabilization of N-dimensional Quantum Spin Systems via Continuous Feedback *

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Abstract

In this paper, a stabilization problem of quantum spin systems in general dimension under continuous measurement is considered and it is shown that the global stabilization at an eigenstate by continuous inputs is possible. Quantum states of spin systems under continuous measurement by mutual interference with laser beams can be continuously estimated by quantum filtering and with its information, the intension of magnetic field, which is applied to atoms, can be controlled. Because of the structural symmetry of the nonlinear dynamics, this stabilization problem by continuous feedback has been considered to be hard and recently a switching rule was introduced which attains the global stability. Our proposing control rule is the sum of two terms: a term which attracts the quantum states to targets and the other term which draw apart from the other equilibrium points. The proof is done by the strict analysis on the sample paths of solutions of a stochastic differential equation. We also show numerical examples to demonstrate the efficiency of the proposing control rule.

1 Introduction

Quantum information technologies such as quantum computing or code, have been expected to attain technological breakthroughs and investigated recently in the broad fields of physics, information theory and other science and technologies[12]. Enhancement of quantum theory in many phases is necessary for their realizations and in particular, quantum control theory, which attains the realization and reservation of quantum bits, is indispensable.

*This paper is the technical report version of the conference paper [16]

Feedforward control of quantum systems has been investigated in 1980's at first. In general terms, however, feedforward control is not robust in noisy realistic systems with unmodeled dynamics and it is not effective for quantum control.

Such situation drastically changed with quantum control systems under "continuous measurement." Belavkin [2] and others [20] showed that the time evolution of estimated quantum states under continuous measurement can be described by a classical stochastic differential equation in the early 1990's. After that, research on feedback control by using estimated quantum states has been actively investigated [19, 4, 21] and its effectiveness has been also demonstrated by real experiments [6]. Feedback control is robust for noise or unmodeled dynamics and it is more realistic.

A recent notable result [17] is on feedback control of single spin 1/2 systems by using a continuous control rule. This result is important for showing a possibility of feedback control, however it is also limited with respect to its applications from the point of the generalization of the dimension.

This limitation has been solved recently by Mirrahimi & van Handel [11]. They proposed a switching control for a group of atoms to globally stabilize the angular moments at any eigenstate. The switching control operates a control input around a targe state, which attracts the quantum states to the targets, and switches to another constant control input when the quantum states are near the other equilibrium points of the former control input in order to draw away from them. The proof is done by the strict analysis on the sample paths of the quantum state. This is the first result to show the global stability for quantum spin systems in general dimensions.

With this result, our interest naturally moves to a question on the global stabilizability of the quantum system by continuous feedback. This problem is important from the viewpoint of realizability of apparatus or pure physics and mathematics. This problem has been open in this area and it is the main subject of this paper. The related work on an almost global stability was reported in [1] and global stability except for some special points was shown to be possible. The difficulty of showing the strict global stability is caused by the symmetry of the quantum dynamics. A stochastic version of Lyapunov method is a standard approach to show the stability, however, in this case, trivial control inputs cause plural equilibrium points and it is hard to find globally effective Lyapunov functions. The proposing continuous control rule in this paper is a sum of two terms: a control signal which attracts the quantum states to a target state and another signal which draws away from the other equilibrium point. This control scheme was considered in [9] and its effectiveness was demonstrated by numerical examples. This paper proves that the global stability at an eigenstate is possible for N-dimensional quantum spin systems.

This paper is organized as follows. In section 2, we introduce the problem setting and the previous result by Mirrahimi & van Handel [11]. In section 3 we give the main result of this paper and its proof. In the proof, we use several lemmas and propositions. The method for the proof is similar to that of the switching control by Mirrahimi & van Handel [11]. In section 4, we show some examples in order to demonstrate the efficiency of our proposing control rule and in section 5, we conclude this paper.

2 Formulation

In this section, at first we introduce several notations on quantum mechanics. Then, we show quantum dynamics dealt with in this paper and define its stabilization problem. We also introduce the previous result [11] which uses a switching control for the stabilization problem. We employ the approach of the proof for showing the proof of our proposing control scheme.

In this paper, we deal with quantum state ρ, an operator on a Hilbert space $\mathcal{C}^N$, which belongs to the set:

$$\mathcal{S} = \{\rho \in \mathcal{C}^{N\times N} : \rho = \rho^*,\ \mathrm{tr}\,(\rho) = 1,\ \rho \geq 0\} \tag{1}$$

where ρ^* denotes Hermitian conjugation of ρ.

In quantum mechanics, an *observable* is regarded as an operator on the Hilbert space which is associated with an observed physical quantity. For the finite dimensional systems, an observable is an Hermite matrix. When an observable C has different eigenvalues λ_i, we observe one of the eigenvalues as the physical quantity by using *orthogonal measurement* such as Stern-Gerlach's experiment. The realization is completely random and its expectation is given by $\mathrm{tr}\,(C\rho)$. Moreover, the probability of an observation λ_i is given by $\mathrm{tr}\,(\rho P_i)$ where P_i is the corresponding orthogonal projection.

When the orthogonal measurement is operated, the quantum state jumps to the corresponding eigenstate. Because of this phenomenon, feedback control of quantum mechanics can be considered difficult in the case of orthogonal measurement. Another type of measurement is called *continuous measurement*. Fig. 1 is a sketch of its typical realization [17, 18, 11] and it is dealt with in this paper.

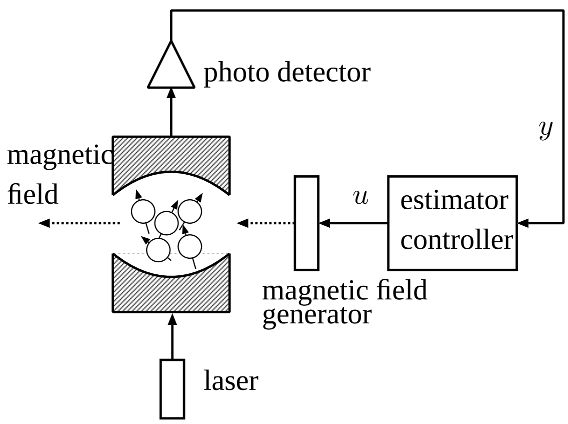

Fig.1: Quantum spin system under continuous measurement

A group of atoms is held in a cavity. When the number of atoms is n, the dimension of the quantum state on the angular moment is $N = 2J+1$ where J is the absolute value of the moment. The mutual interaction between the laser beam and the atoms is observed by a photo detector where the intensity of the interference laser beam has the information on the angular moment of the atoms. The observation of this indirect information causes an effect on the quantum state of the atoms. This process can be regarded as a continuous operation of a generalized observation called positive operator valued measurement (POVM) and the probability of an observation k, where k is the index corresponding to the observed intensity of the laser beam, can be given by

$$\mathrm{tr}\,(\rho\Omega(k)^*\Omega(k)), \tag{2}$$

where $\Omega(k)$ is an appropriate operator.

Magnetic field is also applied to the group of the atoms and its intension is controlled. By using the above probability with the history of the indirectly observed information, the conditional expectation of the observation can be calculated [2]. This is called *quantum filtering* and the time evolution of the estimated quantum state becomes a quantum version of a classical Kushner-Stratonovich equation [2, 3, 17].

When we observe the angular moment on z-axis and apply the magnetic field along y-axis, the corresponding nonlinear Itô stochastic differential equation is:

$$\begin{aligned} d\rho_t =& - iu_t[F_y, \rho_t]dt - \frac{1}{2}[F_z, [F_z, \rho_t]]dt \\ &+ \sqrt{\eta}(F_z\rho_t + \rho_t F_z - 2\mathrm{tr}\,(F_z\rho_t)\rho_t)dW_t, & (3) \\ dy =& 2\sqrt{\eta}\mathrm{tr}\,(F_z\rho)dt + dW_t & (4) \end{aligned}$$

where

ρ_t : a quantum state at time t,

dW_t : an infinitesimal Wiener increment satisfying $\mathrm{E}[(dW_t)^2] = dt,\ \mathrm{E}[dW_t] = 0$,

u_t : control input $(u_t \in \mathcal{R})$,

y_t : output $(y_t \in \mathcal{R})$,

η : the detector efficiency$(0 < \eta \leq 1)$.

F_y: the angular momentum along the axis y of the form [10]

$$F_y = \frac{1}{2i}\begin{bmatrix} 0 & -c_1 & & & \\ c_1 & 0 & -c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{2J-1} & 0 & -c_{2J} \\ & & & c_{2J} & 0 \end{bmatrix},$$

$$c_m = \sqrt{(2J+1-m)m}. \tag{5}$$

F_z: the angular momentum along the axis z of the form [10]

$$F_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix}. \tag{6}$$

This is called SME (*stochastic master equation*) and it has been mainly investigated in the research field of quantum control. It should be noted that the solution of (3) is continuous in time [13] if u_t is continuous. We also define some notations:

$$\psi_i := [0 \cdots 0 \underbrace{1}_{i-\mathrm{th}} 0 \cdots 0]^*, \tag{7}$$

$$\rho_{\psi_i} := \psi_i \psi_i^*, \tag{8}$$

$$V_{\rho_\mathrm{f}}^{\mathrm{I}}(\rho) := 1 - \mathrm{tr}\,(\rho\rho_\mathrm{f}), \tag{9}$$

$$V_{\rho_\mathrm{f}}^{\mathrm{II}}(\rho) := 1 - (\mathrm{tr}\,(\rho\rho_\mathrm{f}))^2, \tag{10}$$

where $\rho_\mathrm{f} \in \mathcal{S}$ is an eigenstate. The control objective is to globally converge the quantum state ρ_t to some desired state ρ_f by controlling the intensity u_t of the magnetic field which is decided by ρ_t or its record. Note that $0 \leq V_{\rho_\mathrm{f}}^{\bullet}(\rho) \leq 1$, and $V_{\rho_\mathrm{f}}^{\bullet}(\rho) = 0$ iff $\rho = \rho_\mathrm{f}$. Moreover define

$$\mathcal{S}_{\rho_\mathrm{f}}^{\epsilon} := \left\{\rho \,|\, V_{\rho_\mathrm{f}}^{\mathrm{I}}(\rho) = \epsilon\right\}, \tag{11}$$

$$\mathcal{S}_{\rho_\mathrm{f}}^{<\epsilon} := \left\{\rho \,|\, V_{\rho_\mathrm{f}}^{\mathrm{I}}(\rho) < \epsilon\right\}, \tag{12}$$

$$\mathcal{S}_{\rho_\mathrm{f}}^{>\epsilon} := \left\{\rho \,|\, V_{\rho_\mathrm{f}}^{\mathrm{I}}(\rho) > \epsilon\right\}. \tag{13}$$

We define the stochastic stability of (3) as follows.

Definition 2.1 *[8] Let ρ_e be an equilibrium point of* (3)*, i.e.* $d\rho_t|_{\rho_t=\rho_\mathrm{e}} = 0$*. Then*

1. *the equilibrium ρ_e is said to be stable in probability if*

$$\forall \epsilon > 0 \quad \lim_{\rho_0 \to \rho_\mathrm{e}} \Pr\left(\sup_{0<t<\infty} \|\rho_t - \rho_\mathrm{e}\| \geq \epsilon\right) = 0. \tag{14}$$

   where $\|\cdot\|$ is an arbitrary norm of a matrix in $\mathcal{C}^{N\times N}$.

2. *The equilibrium ρ_e is globally stable if it is stable in probability and additionally*

$$\forall \rho_0 \in \mathcal{S} \quad \Pr\left(\lim_{t\to\infty} \rho_t = \rho_\mathrm{e}\right) = 1. \tag{15}$$

For showing the stochastic stabilities, a stochastic version of the Lyapunov theorem is available. At first define a nonnegative real-valued continuous function $V(\cdot)$ on $\mathcal{S}$. Also define $\rho_t^z := \rho_t$ such that $\rho_0 = z$, a level set Q_ϵ such that $Q_\epsilon := \{\rho \in \mathcal{S} : V(\rho) < \epsilon\}$, $\tau_\epsilon := \inf\{t : \rho_t^z \notin Q_\epsilon\}$ and $\tilde{\rho}_t^z = \rho_{t\wedge\tau_\epsilon}^z$, $t \wedge \tau_\epsilon = \min(t, \tau_\epsilon)$, $\mathcal{L}_\epsilon$: infinitesimal operator, $\mathcal{L}_\epsilon$: restriction of $\mathcal{L}$ on $\tilde{\rho}_t$. Then, we get the following propositions.

Proposition 2.1 *[8] Let $\mathcal{L}_\epsilon V \le 0$ in Q_ϵ. Then, the followings hold:*

1. *$\lim_{t\to\infty} V(\tilde{\rho}_t^z)$ exists a.s., so $V(\rho_t^z)$ converges for a.e. path remaining in Q_ϵ.*

2. *$\mathrm{Pr}\text{–}\lim_{t\to\infty} \mathcal{L}_\epsilon V(\tilde{\rho}_t^z) = 0$, so $\mathcal{L}_\epsilon V(\rho_t^z) \to 0$ in probability as $t \to \infty$ for almost all paths which never leave Q_ϵ.*

3. *For $z \in Q_\epsilon$ and $\alpha \le \epsilon$ we have the uniform estimate*

$$\begin{aligned} \mathrm{Pr}(\sup_{0\le t<\infty} V(\rho_t^z) \ge \alpha) &= \mathrm{Pr}(\sup_{0\le t<\infty} V(\tilde{\rho}_t^z) \ge \alpha) \\ &\le \frac{V(z)}{\alpha}. \end{aligned} \tag{16}$$

4. *If $V(\tilde{z}) = 0$ and $V(\rho) \neq 0$ for $\rho \neq \tilde{z}$, then $\tilde{z}$ ($\tilde{z} \in Q_\epsilon$) is stable in probability.*

Definition 2.2 *An invariant set C is defined as a set with the property that if system's initial state is in C then its whole path (forward and backward) lies in C.*

Proposition 2.2 *[11] Assume the followings:*

1. *Q_ϵ is bounded and that $\mathcal{L}_\epsilon V(\rho) \le 0$ within Q_ϵ.*

2. *For any bounded scalar continuous function $g(\rho)$ and a fixed t, $\mathrm{E}[g(\rho_t^z)]$ is continuous on $z = \rho_0$.*

3. *For any positive real number κ and $z \in Q_\epsilon$, $\mathrm{Pr}(\|\rho - z\| > \kappa) \to 0$, $t \to 0$.*

Let R be the set of all points within Q_ϵ where $\mathcal{L}_\epsilon V(\rho) = 0$, and let M be the largest invariant set in R. Then, every solution ρ_t in Q_ϵ tends to M as $t \to \infty$

Here we consider the control problem:

Problem 2.1 *For the controlled spin system (3), find a globally stabilizing controller u_t on an eigenstate $\rho_\mathrm{f} = \rho_{\psi_1}$.*

This is not a trivial problem from the following reasons: 1) (3) is a nonlinear stochastic system, 2) there exist plural locally stable equilibrium points when $u = 0$ because of the nonlinearity, 3) because of a kind of symmetry of the dynamics, many of locally stabilizing control scheme on one of above equilibrium points also preserve the other equilibrium points.

Mirrahimi & van Handel found a globally stabilizing control scheme on any target eigenstates by introducing a switching rule in order to solve above difficulties[11].

Proposition 2.3 *[11] Consider the system* (3) *evolving in the set $\mathcal{S}$ and let $\gamma > 0$, $\rho_\mathrm{f} = \rho_{\psi_i}$ and*

$$u_1(\rho_t) := -\mathrm{tr}\,(i[F_y, \rho_t]\rho_\mathrm{f}). \tag{17}$$

Moreover, consider the following control scheme:

1. $u_t = u_1(\rho_t)$ *if* $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho_t) \leq 1 - \gamma$;

2. $u_t = 1$ *if* $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho_t) \geq 1 - \gamma/2$;

3. If $\rho_t \in \mathcal{B} = \{\rho : 1 - \gamma < V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho_t) < 1 - \gamma/2\}$*, then* $u_t = u_1(\rho_t)$ *if* ρ_t *last entered* $\mathcal{B}$ *through the boundary* $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) = 1 - \gamma$*, and* $u_t = 1$ *otherwise.*

Then $\exists \gamma > 0$ *s.t.* u_t *globally stabilizes* (3) *around* ρ_{f} *and* $\mathrm{E}[\rho_t] \to \rho_{\mathrm{f}}$ *as* $t \to \infty$*.*

Feedback stabilization of spin systems under continuous measurement was introduced in [17] for a special case of single spin 1/2 systems, however it is restricted for such special case. The above result is a generalization of the dimension of the spin systems and any eigenstates are possible to be the target.

The proof of Proposition 2.3 is composed of three parts and its approach is common for the proof of the main result of this paper, therefore, we refer it briefly:

1) ρ_{f} is stable in probability.

2) almost all sample paths which never leave the domain $\mathcal{S}^{<1-\gamma/2}_{\rho_{\mathrm{f}}}$ converge to ρ_{f}.

3) for almost all sample paths there exists a finite time T and after it, they never leave $\mathcal{S}^{<1-\gamma}_{\rho_{\mathrm{f}}}$.

Unfortunately the scheme in Proposition 2.3 is a switching control and it should be avoided from the view point of practical use. Moreover, the essential question whether the quantum spin systems can be globally stabilized by continuous feedback is interesting itself from the view point of physics or mathematics and it is one of main research subjects in this field.

3 Main Result

The stabilizability of (3) by a continuous feedback was demonstrated with numerical examples by Matsumoto [9]. This paper proves the global stabilizability at an eigenstate $\rho_{\mathrm{f}} = \rho_{\psi_1}$. We get the following theorem:

Theorem 3.1 *Consider the system* (3) *evolving in the set* $\mathcal{S}$*. Let* $\rho_{\mathrm{f}} = \rho_{\psi_1}$ *and* $\eta > 0$*. Then,*

$$\begin{aligned} u_t &= \alpha u_1(\rho_t) + \beta V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho_t) \\ & \quad \alpha,\ \beta > 0 \end{aligned} \tag{18}$$

globally stabilizes (3) *around* ρ_{f} *and* $\mathrm{E}[\rho_t] \to \rho_{\mathrm{f}}$ *as* $t \to \infty$ *when*

$$\frac{\beta^2}{8\alpha\eta} < 1. \tag{19}$$

Remark 3.1 This is the first result to show the global stabilizability of general finite dimensional quantum systems at an eigenstate (in this case, $\rho_{\rm f} = \rho_{\psi_1}$) by continuous feedback for the type of the master equation (3). Note that α and β are design parameters and we can always find them satisfying the condition (19) if $\eta > 0$.

We prove Theorem 3.1 in the followings. The procedure of the proof is similar to that of Proposition 2.3 and it is composed of the following three parts:

Step 1) $\rho_{\rm f} = \rho_{\psi_1}$ is stable in probability.

Step 2) there exists $0 < \gamma < 1$ and almost all sample paths which never leave the domain $\mathcal{S}_{\rho_{\rm f}}^{<1-\gamma}$ converge to $\rho_{\rm f}$.

Step 3) for almost all sample paths there exists a finite time T and after it, they never leave $\mathcal{S}_{\rho_{\rm f}}^{<1-\gamma}$.

Step 1)

In order to show the statement of the Step 1), we should find some Lyapunov function which satisfies the conditions of Proposition 2.1 around $\rho_{\rm f}$. We get a key lemma for it.

Lemma 3.1 *With the control input (18),*

$$\mathcal{L}_{1-\gamma_o} V_{\rho_{\rm f}}^{\rm II} \quad \leq \quad 0 \tag{20}$$

is satisfied in the subset $\mathcal{S}_{\rho_{\rm f}}^{<1-\gamma_o}$ where

$$\gamma_o = \frac{\beta^2}{8\alpha\eta} < 1. \tag{21}$$

Moreover, $\mathcal{L}_{1-\gamma_o} V_{\rho_{\rm f}}^{\rm II}(\rho) = 0$ in $\rho \in \mathcal{S}_{\rho_{\rm f}}^{<1-\gamma_o}$ iff $\rho = \rho_{\rm f}$.

Proof By the direct calculation of $\mathcal{L} V_{\rho_{\rm f}}^{\rm II}$, we get the following:

$$\begin{aligned}
\mathcal{L} V_{\rho_{\rm f}}^{\rm II} \quad &= \quad -2{\rm tr}\,(\rho_t\rho_{\rm f})\, u_t \,{\rm tr}\,(-i[F_y, \rho_t]\rho_{\rm f}) \\
&\qquad - 4\eta(J - {\rm tr}\,(F_z\rho_t))^2({\rm tr}\,(\rho_t\rho_{\rm f}))^2 \\
&= \quad -2{\rm tr}\,(\rho_t\rho_{\rm f})(\alpha u_1 + \beta V_{\rho_{\rm f}}^{\rm I}) u_1 \\
&\qquad - 4\eta(J - {\rm tr}\,(F_z\rho_t))^2({\rm tr}\,(\rho_t\rho_{\rm f}))^2 \\
&= \quad -2{\rm tr}\,(\rho_t\rho_{\rm f}) \left\{ (\alpha u_1 + \beta V_{\rho_{\rm f}}^{\rm I}) u_1 \right. \\
&\qquad \left. + 2\eta J^2({\rm tr}\,((I - J_z)\rho_t))^2 {\rm tr}\,(\rho_t\rho_{\rm f}) \right\},
\end{aligned} \tag{22}$$

where

$$J_z := \begin{bmatrix} 1 & & & & O \\ & \frac{J-1}{J} & & & \\ & & \ddots & & \\ & & & \frac{-J+1}{J} & \\ O & & & & -1 \end{bmatrix}. \tag{23}$$

The factor $\mathrm{tr}\,(\rho_t\rho_{\mathrm{f}})$ outside the curly brackets is always nonnegative, therefore, the factor of the curly brackets should be nonnegative for $\mathcal{L}V^{\mathrm{II}}_{\rho_{\mathrm{f}}}$ to be nonpositive. The factor of the brackets is

$$\begin{aligned} & \left(\alpha u_1 + \beta V^{\mathrm{I}}_{\rho_{\mathrm{f}}}\right) u_1 + 2\eta J^2(\mathrm{tr}\,((I-J_z)\rho))^2 \mathrm{tr}\,(\rho\rho_{\mathrm{f}}) \\ = \;& \alpha\left(u_1 + \frac{\beta}{\alpha}\frac{V^{\mathrm{I}}_{\rho_{\mathrm{f}}}}{2}\right)^2 + \mathrm{tr}\,(\rho\rho_{\mathrm{f}})2\eta J^2(\mathrm{tr}\,((I-J_z)\rho))^2 \\ & -\frac{\beta^2}{\alpha}\frac{(V^{\mathrm{I}}_{\rho_{\mathrm{f}}})^2}{4}. \end{aligned} \tag{24}$$

The first term is always nonnegative, therefore, when the sum of the second and the third terms is always nonnegative in some subset of ρ, then, we can derive $\mathcal{L}V^{\mathrm{II}}_{\rho_{\mathrm{f}}} \le 0$ in the subset.

In order to show that the sum of the second and third terms of (24) is nonnegative, here we consider an equivalence class $\mathcal{S}^{1-\gamma}_{\rho_{\mathrm{f}}}$ of ρ defined by (11) and find the worst case which minimize the second term in $\mathcal{S}^{1-\gamma}_{\rho_{\mathrm{f}}}$. Note that $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) = 1-\gamma$ for $\rho \in \mathcal{S}^{1-\gamma}_{\rho_{\mathrm{f}}}$. With regard of the form (23) of J_z and $\rho_{\mathrm{f}} = \rho_{\psi_1} = \mathrm{diag}[1\ 0\ \cdots\ 0]$, the worst ρ in $\mathcal{S}^{1-\gamma}_{\rho_{\mathrm{f}}}$ is:

$$\rho = \begin{bmatrix} \rho_{11} & & & & O \\ & 1-\rho_{11} & & & \\ & & 0 & & \\ & & & \ddots & \\ O & & & & 0 \end{bmatrix}, \tag{25}$$

where $\rho_{11} = \gamma$ and then we get:

$$\begin{aligned} (24) \ge\; & \alpha\left(u_1 + \frac{\beta}{\alpha}\frac{V^{\mathrm{I}}_{\rho_{\mathrm{f}}}}{2}\right)^2 \\ & + \mathrm{tr}\,(\rho\rho_{\mathrm{f}})2\eta J^2\left\{1 - \left(\rho_{11} + \frac{J-1}{J}(1-\rho_{11})\right)\right\}^2 \\ & -\frac{\beta^2}{\alpha}\frac{(V^{\mathrm{I}}_{\rho_{\mathrm{f}}})^2}{4}. \end{aligned} \tag{26}$$

The second and the third terms can be reformed as

$$\begin{aligned}
& \mathrm{tr}\,(\rho\rho_{\mathrm{f}})2\eta J^2 \left\{1 - \left(\rho_{11} + \frac{J-1}{J}(1-\rho_{11})\right)\right\}^2 \\
& - \frac{\beta^2}{\alpha}\frac{(V^{\mathrm{I}}_{\rho_{\mathrm{f}}})^2}{4} \\
= \;& \mathrm{tr}\,(\rho\rho_{\mathrm{f}})2\eta J^2 \left(\frac{1}{J} - \frac{1}{J}\rho_{11}\right)^2 - \frac{\beta^2}{\alpha}\frac{(V^{\mathrm{I}}_{\rho_{\mathrm{f}}})^2}{4} \\
= \;& 2\eta\rho_{11}(1-\rho_{11})^2 - \frac{\beta^2}{4\alpha}(1-\rho_{11})^2 \\
= \;& 2\eta(1-\rho_{11})^2\left(\rho_{11} - \frac{\beta^2}{8\alpha\eta}\right). \qquad (27)
\end{aligned}$$

Therefore, when

$$\frac{\beta^2}{8\alpha\eta} < 1 \qquad (28)$$

is satisfied, we can set

$$\gamma_0 := \frac{\beta^2}{8\alpha\eta} \qquad (29)$$

and for the case:

$$\gamma_0 < \rho_{11} \leq 1, \qquad (30)$$

we conclude $\mathcal{L}V^{\mathrm{II}}_{\rho_{\mathrm{f}}} \leq 0$. This means with the setting

$$\gamma = \gamma_0,\ \mathcal{S}^{<1-\gamma}_{\rho_{\mathrm{f}}} = \mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}, \qquad (31)$$

$\mathcal{L}_{1-\gamma_o}V^{\mathrm{II}}_{\rho_{\mathrm{f}}} \leq 0$ for any $\rho \in \mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$.

The latter half statement of the lemma can be also shown by direct calculation. □

With Lemma 3.1 and Proposition 2.1, $\rho = \rho_{\mathrm{f}}$ is stable in probability and in the subset $\mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$ around ρ_{f}, the statements in Proposition 2.1 are concluded.

Step 2)

From Lemma 3.1, the master equation (3) with the control input (18) satisfies the conditions in Proposition 2.2, therefore, the sample paths which never leave the subset $\mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$ converge to ρ_{f} in probability. Moreover, $V^{\mathrm{II}}_{\rho_{\mathrm{f}}}$ converges almost surely from Proposition 2.1. With this and Lebesgue's dominated convergence, we can show that almost all paths converge to ρ_{f} by employing the similar discussion in [11]:

Lemma 3.2 *ρ_t converges to ρ_{f} as $t \to \infty$ for almost all paths that never exit the set $\mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$.*

We omit the proof here.
Step 3)

We next show the behavior of the paths when they leave $\mathcal{S}_{\rho_{\rm f}}^{<1-\gamma_0}$ or the initial state is outside it. We get the following key lemma:

Lemma 3.3 *The solution ρ_t of (3) where $\rho(0) \in \mathcal{S}_{\rho_{\rm f}}^{>1-\gamma_0}$ satisfies*

$$\sup_{\rho_0 \in \mathcal{S}_{\rho_{\rm f}}^{>1-\gamma_0}} \mathrm{E}[\min\ t : \rho_t \notin \mathcal{S}_{\rho_{\rm f}}^{>1-\gamma_0}] < \infty. \tag{32}$$

At first introduce propositions which are used for the proof of Lemma 3.3.

Proposition 3.1 *[15, 7] Consider a Stratonovich's stochastic differential equation:*

$$d\varphi_t = f_0(\varphi_t, t)dt + \sum_{l=1}^{n} f_l(\varphi_t, t) \circ dW^l(t). \tag{33}$$

Assume that the coefficients $f_l(x,t)$, $l = 0, 1, 2, \dots, n$ are of the class $C_b^{k+1,\delta}$ for some $k \geq 2$ and $\delta > 0$ (see Appendix for the definition of $C_b^{k+1,\delta}$). Let φ_t be the Brownian flow determined by (33). Then the support of $\varphi(t) = \varphi_t$ as the C^{k-1}-flow is equal to the closure $\{\varphi_t : \xi \in \Xi\}$ of

$$\frac{d\varphi_t}{dt} = f_0(\varphi_t, t) + \sum_{l=1}^{n} f_l(\varphi_t, t)\xi^l(t) \tag{34}$$

in the space W_{k-1}, where Ξ is the set of all deterministic piecewise smooth function and $W_k = C([0,T] : C^k)$.

Proposition 3.2 *[5] Consider diffusion process $x_t \in E$ starting from x where E is the domain of x_t. Let Γ be a subset of E and $\tau_x(\Gamma)$ be the first exit time of x_t from Γ. Then for all $T \geq 0, x \in E$,*

$$\mathrm{E}[\tau_x(\Gamma)] \leq \frac{T}{1 - \sup_{x\in E} \Pr\{\tau_x(\Gamma) > T\}}. \tag{35}$$

Proof of Lemma 3.3 At first, we claim that the support of $V_{\rho_{\rm f}}^{\rm I}(\rho_t)$ contains $[0,\gamma]$ when $V_{\rho_{\rm f}}^{\rm I}(\rho_0) = \gamma$ by using Proposition 3.1.

The Stratonovich form of (3) is given as [14]

$$\begin{aligned} d\rho_t &= \mathcal{D}_{F_z}(\rho_t)dt \\ &\quad - \frac{1}{2}\eta\left(-2\mathcal{E}_{F_z}(\rho_t)\mathcal{H}_{F_z}(\rho_t) + \mathcal{K}_{F_z}(\rho_t)\right)dt \\ &\quad + u_t\mathcal{G}_{F_y}(\rho_t)dt + \sqrt{\eta}\mathcal{H}_{F_z}(\rho_t) \circ dW \end{aligned} \tag{36}$$

where

$$
\begin{aligned}
\mathcal{D}_{F_z}(\rho) &= -\frac{1}{2}[F_z,[F_z,\rho]] \\
\mathcal{E}_{F_z}(\rho) &= 2\mathrm{tr}\,(F_z\rho) \\
\mathcal{H}_{F_z}(\rho) &= F_z\rho+\rho F_z-2\mathrm{tr}\,(F_z\rho)\rho \\
\mathcal{K}_{F_z}(\rho) &= F_z^2\rho+2F_z\rho F_z^*+\rho(F_z^*)^2 \\
&\quad -\mathrm{tr}\,(F_z^2\rho+2F_z\rho F_z^*+\rho(F_z^*)^2)\rho \\
\mathcal{G}_{F_y}(\rho) &= -i[F_y,\rho]
\end{aligned}
\tag{37}
$$

and the corresponding deterministic differential equation is

$$
\begin{aligned}
\frac{d}{dt}\rho_t &= \mathcal{D}_{F_z}(\rho_t) \\
&\quad -\frac{1}{2}\eta\left(-2\mathcal{E}_{F_z}(\rho_t)\mathcal{H}_{F_z}(\rho_t)+\mathcal{K}_{F_z}(\rho_t)\right) \\
&\quad +u\mathcal{G}_{F_y}(\rho_t)+\sqrt{\eta}\mathcal{H}_{F_z}(\rho_t)\xi
\end{aligned}
\tag{38}
$$

where ξ is an associated input. With this solution, we get

$$
\begin{aligned}
&\frac{d}{dt}V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho) \\
&= -\mathrm{tr}\left(\frac{d\rho}{dt}\rho_{\mathrm{f}}\right) \\
&= -\mathrm{tr}\left(\left\{-\frac{1}{2}\eta(-2\mathcal{E}_{F_z}(\rho)\mathcal{H}_{F_z}(\rho)+\mathcal{K}_{F_z}(\rho))\right.\right. \\
&\qquad \left.\left.+u\mathcal{G}_{F_y}(\rho)+\sqrt{\eta}\mathcal{H}_{F_z}(\rho)\xi\right\}\rho_{\mathrm{f}}\right).
\end{aligned}
\tag{39}
$$

The term which includes ξ in (39) is

$$
\begin{aligned}
&\mathrm{tr}\,(\mathcal{H}_{F_z}(\rho)\xi\rho_{\mathrm{f}}) \\
&= \mathrm{tr}\,((F_z\rho+\rho F_z-2\mathrm{tr}\,(F_z\rho))\rho_{\mathrm{f}})\xi \\
&= 2\left\{\mathrm{tr}\,(\rho F_z\rho_{\mathrm{f}})-\mathrm{tr}\,(F_z\rho)\mathrm{tr}\,(\rho\rho_{\mathrm{f}})\right\}\xi \\
&= 2\{J\rho_{11}-(J\rho_{11}+(J-1)\rho_{22}+\cdots \\
&\qquad +(-J)\rho_{NN})\rho_{11}\}\,\xi \\
&= 2\rho_{11}\{J-(J\rho_{11}+(J-1)\rho_{22}+\cdots \\
&\qquad +(-J)\rho_{NN})\}\,\xi.
\end{aligned}
\tag{40}
$$

The case (40) = 0 is when $\rho_{11}=0$ or $\rho_{11}=1$ (in this case $\rho=\rho_{\mathrm{f}}$ and $\rho_{22}=\rho_{33}=\cdots=\rho_{NN}=0$).

When $\rho_{11}=0$, $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho)=1$ and $u=1$, and from [11], it is known that $\{\rho\,|\,V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho)=1\}$ is not an invariant set with $u_t\equiv 1$. On the other hand, when $\rho_{11}=1$, $\rho=\rho_{\mathrm{f}}$ and $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho)=0$, and it is the target point. In the other case $0<\rho_{11}=\mathrm{tr}\,(\rho\rho_{\mathrm{f}})<1$, (40) except for ξ is nonzero.

From above and Proposition 3.1, the assertion that the support of $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho_t)$ contains $[0,\gamma]$ when $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}(\rho_0)=\gamma$ is proved. Therefore, we can get $\min_{t\in[0,T]}\mathrm{E}[V^{\mathrm{I}}_{\rho_{\mathrm{f}}}]<1-\gamma_0$ and finally with Proposition 3.2, we can conclude the statement [11]. $\square$

By using Lemma 3.3 and employing the similar discussion of [11], we can derive the following lemma.

Lemma 3.4 *For almost every sample path of ρ_t there exists a time $T<\infty$ after which the path never exits the set $\mathcal{S}^{<1-\gamma_0}_{\rho_{\mathrm{f}}}$.*

We omit the proof.

Proof of Theorem 3.1 By unifying the results of Step 1) $\sim$ 3), we can conclude the convergence of the solution to the target point. The convergence of the expectation can be also derived by dominated convergence. $\square$

4 Numerical Example

We demonstrate the efficiency of the proposing continuous feedback by using numerical simulations. Here we consider single spin 1/2 systems where $N=2$. The initial and the target states are

$$\rho_0=\begin{bmatrix}0&0\\0&1\end{bmatrix},\ \rho_{\mathrm{f}}=\begin{bmatrix}1&0\\0&0\end{bmatrix}$$

respectively. We simulate the solution ρ_t with Case 1) $\eta=0.9$, $\alpha=1$, $\beta=1$ and Case 2) $\eta=0.1$, $\alpha=1$, $\beta=1$, 10 times respectively. The former case satisfies the condition (19), on the other hand, the latter does not satisfy it. Fig. 2 and Fig. 3 show the average of the transitions of $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}$, which indicates the gap between the target ρ_{f} and ρ_t, with the above two cases respectively.

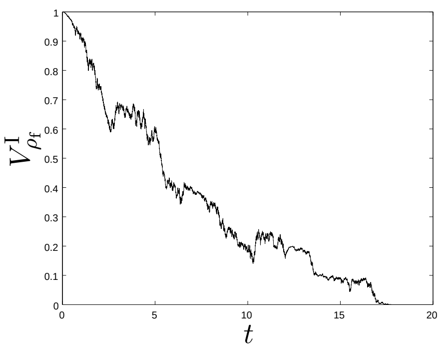

Fig.2: Average of transitions of $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}$ with $\eta=0.9$, $\alpha=1$, $\beta=1$

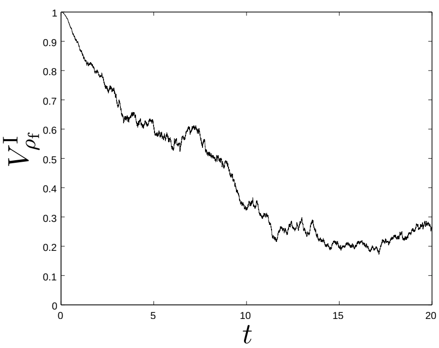

Fig.3: Average of transitions of $V^{\mathrm{I}}_{\rho_{\mathrm{f}}}$ with $\eta = 0.1$, $\alpha = 1$, $\beta = 1$

From the simulations, we can confirm the efficiency of our proposing continuous feedback. Note that (19) is a sufficient condition for the global stability, therefore, even if it is not satisfied, the system may be stable. However, we can see the significance of the condition (19) from these simulations.

5 Conclusion

In this paper, we considered control problem of N-dimensional quantum spin systems and showed that continuous feedback is possible to stochastically globally stabilize the systems. The control scheme is composed of two distinctive terms and the stability is proved by following the sample paths of the stochastic master equation strictly.

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A Appendix

The notation $C_b^{m,\delta}$ is the set $\{f \in C^{k+1}, D^\alpha f \; (|\alpha| = m) :$ δ-Hölder continuous, $\|f\|_{m+\delta} < \infty \}$ and

$$\|f\|_{m+\delta} := \|f\|_m + \sum_{|\alpha|=m} \sup \frac{D^\alpha f(x) - D^\alpha f(y)}{|x-y|^\delta}. \tag{41}$$

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