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Sign-Solvable Linear Complementarity Problems

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Abstract

This paper presents a connection between qualitative matrix theory and linear complementarity problems (LCPs). An LCP is said to be *sign-solvable* if the set of the sign patterns of the solutions is uniquely determined by the sign patterns of the given coefficients. We provide a characterization for sign-solvable LCPs such that the coefficient matrix has nonzero diagonals, which can be tested in polynomial time. This characterization leads to an efficient combinatorial algorithm to find the sign pattern of a solution for these LCPs. The algorithm runs in $\mathrm{O}(\gamma)$ time, where γ is the number of the nonzero coefficients.

1 Introduction

This paper deals with linear complementarity problems (LCPs) in the following form:

$$\begin{array}{lll} \mathrm{LCP}(A,b): & \text{find} & (w,z) \\ & \text{s.t.} & w = Az + b, \\ & & w^{\mathrm{T}} z = 0, \\ & & w \geq 0,\ z \geq 0, \end{array}$$

where A is a real square matrix, and b is a real vector. The LCP, introduced by Cottle [4], Cottle and Dantzig [5], and Lemke [15], is one of the most widely studied mathematical programming problems, which contains linear programming problems and convex quadratic programming problems. Solving $\mathrm{LCP}(A,b)$ for an arbitrary matrix A is NP-complete [3], while there are several classes of matrices A for which the associated LCPs can be solved efficiently. For details of the theory of LCPs, see the books of Cottle, Pang, and Stone [6] and Murty [19].

The *sign* of a real number a, denoted by $\operatorname{sgn} a$, is defined to be $+$ for $a > 0$, $-$ for $a < 0$, and 0 for $a = 0$. The *sign pattern* of a real matrix A is the $\{+, 0, -\}$-pattern matrix obtained from A by replacing each entry by its sign. Matrix analysis by sign patterns, called *qualitative matrix theory*, was originated in economics by Samuelson [22]. Various results about qualitative matrix theory are compiled in the book of Brualdi and Shader [1]. For a matrix A, we denote by $\mathcal{Q}(A)$ the set of all matrices having the same sign pattern as A, called the *qualitative class* of A. The qualitative class of a vector is defined similarly. A square matrix A is said to be *sign-nonsingular* if $\tilde{A}$ is nonsingular for any $\tilde{A} \in \mathcal{Q}(A)$. The problem of recognizing sign-nonsingular matrices has many equivalent problems in combinatorics [16, 20, 23, 25], while its time complexity had been open for a long time. In 1999, Robertson, Seymour, and Thomas [21] presented a polynomial-time algorithm for solving this problem (cf. McCuaig [17, 18]).

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For linear programming, Iwata and Kakimura [10] introduced sign-solvability in terms of qualitative matrix theory. A linear program $\max\{cx \mid Ax = b, x \geq 0\}$, denoted by $\mathrm{LP}(A, b, c)$, is *sign-solvable* if the set of the sign patterns of the optimal solutions of $\mathrm{LP}(\tilde{A}, \tilde{b}, \tilde{c})$ is the same as that of $\mathrm{LP}(A, b, c)$ for any $\tilde{A} \in \mathcal{Q}(A)$, $\tilde{b} \in \mathcal{Q}(b)$, and $\tilde{c} \in \mathcal{Q}(c)$. They showed that recognizing sign-solvability of a given LP is co-NP-complete, and gave a sufficient condition for sign-solvable linear programs, which can be tested in polynomial time. Moreover, they devised a polynomial-time algorithm to obtain the sign pattern of an optimal solution for linear programs satisfying this sufficient condition.

In this paper, we introduce sign-solvability for linear complementarity problems. We say that $\mathrm{LCP}(A, b)$ is *sign-solvable* if the set of the sign patterns of the solutions of $\mathrm{LCP}(\tilde{A}, \tilde{b})$ coincides with that of $\mathrm{LCP}(A, b)$ for any $\tilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. An $\mathrm{LCP}(A, b)$ such that all diagonal entries of A are nonzero is said to have *nonzero diagonals*. The class of LCPs with nonzero diagonals includes LCPs associated with positive semidefinite matrices, $\boldsymbol{P}$-matrices, and nondegenerate matrices, which are all well known in the theory of LCPs. This paper aims at providing a characterization for a sign-solvable $\mathrm{LCP}(A, b)$ with nonzero diagonals, and describing a polynomial-time algorithm to solve them from the sign patterns of A and b.

We first provide a sufficient condition for sign-solvable LCPs with nonzero diagonals. A square matrix A is *term-nonsingular* if the determinant of A contains at least one nonvanishing expansion term. A square matrix A is *term-singular* if it is not term-nonsingular. A matrix A is term-singular if and only if $\tilde{A}$ is singular for any $\tilde{A} \in \mathcal{Q}(A)$. An $m \times n$ matrix with $m \leq n$ is said to be *totally sign-nonsingular* if all submatrices of order m are either sign-nonsingular or term-singular, namely, if the nonsingularity of each submatrix of order m is determined uniquely by the sign pattern of the matrix. Totally sign-nonsingular matrices were investigated in the context of sign-solvability of linear systems [1, 11, 12, 24] (the terms "matrices with signed mth compound" and "matrices with signed null space" are used instead). Recognizing totally sign-nonsingular matrices can be done in polynomial time by testing sign-nonsingularity of related square matrices [10]. We show that, if the matrix $M = (A\ b)$ is totally sign-nonsingular and A has nonzero diagonals, then $\mathrm{LCP}(A, b)$ is sign-solvable.

We then present a characterization of sign-solvable LCPs with nonzero diagonals. A matrix is said to be *row-mixed* if every row has both positive and negative entries. For an $\mathrm{LCP}(A, b)$ with nonzero diagonals, we introduce the *residual row-mixed* matrix, which will be defined in Section 3. Then $\mathrm{LCP}(A, b)$ with nonzero diagonals is sign-solvable if and only if either its residual row-mixed matrix has no rows or it is totally sign-nonsingular. The residual row-mixed matrix can be obtained in polynomial time. Thus the sign-solvability of a given $\mathrm{LCP}(A, b)$ with nonzero diagonals can be recognized in polynomial time.

This characterization leads to a combinatorial polynomial-time algorithm to solve a given $\mathrm{LCP}(A, b)$ with nonzero diagonals from the sign patterns of A and b. The algorithm tests the sign-solvability, and finds the sign pattern of a solution if $\mathrm{LCP}(A, b)$ is sign-solvable. In this algorithm, we obtain a solution of $\mathrm{LCP}(\tilde{A}, \tilde{b})$ for some $\tilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. If $\mathrm{LCP}(A, b)$ is sign-solvable, then $\mathrm{LCP}(A, b)$ has a solution with the same sign pattern as the obtained one. The time complexity is $\mathrm{O}(\gamma)$, where γ is the number of nonzero entries in A and b. We note that the obtained sign pattern easily derives a solution of the given LCP by Gaussian elimination. Thus a sign-solvable LCP with nonzero diagonals is a class of LCPs which can be solved in polynomial time.

Before closing this section, we give some notations and definitions used in the following sections.

For a matrix A, the row and column sets are denoted by U and V. If A is a square matrix, suppose that U and V are both identical with N. We denote by a_{ij} the (i, j)-entry in A. Let $A[I, J]$ be the submatrix in A with row subset I and column subset J, where the orderings of the elements of I and J are compatible with those of U and V. The submatrix $A[J,J]$ is abbreviated as $A[J]$. The *support* of a row subset I, denoted by $\Gamma(I)$, is the set of columns having nonzero entries in the submatrix $A[I,V]$, that is, $\Gamma(I) = \{j \in V \mid \exists i \in I, a_{ij} \neq 0\}$. For a vector b, the jth entry of b is denoted by b_j. The vector $b[J]$ means the subvector with index subset J. The *support* of a vector b is the column index subset $\{j \mid b_j \neq 0\}$.

For a square matrix A, let π be a bijection from the row set N to the column set N. We denote by $p(A|_\pi) = \mathrm{sgn}\pi \prod_{i\in N} a_{i\pi(i)}$ the expansion term of $\det A$ corresponding to π. Then a matrix A is term-nonsingular if and only if there exists a bijection $\pi : N \to N$ with $p(A|_\pi) \neq 0$. A square matrix A is sign-nonsingular if and only if A is term-nonsingular and every nonvanishing expansion term of $\det A$ has the same sign [1]. Thus, if A is sign-nonsingular, the determinant of every matrix in $\mathcal{Q}(A)$ has the same sign. It is also shown in [1] that, if a square matrix A is sign-nonsignular, then A is not row-mixed.

This paper is organized as follows. In Section 2, we provide a sufficient condition using totally sign-nonsingular matrices. Section 3 gives a characterization for sign-solvable LCPs with nonzero diagonals. In Section 4, we describe a polynomial-time algorithm to solve sign-solvable LCPs with nonzero diagonals from the sign patterns of the given coefficients.

2 Totally Sign-Nonsingular Matrices

In this section, we give a sufficient condition for sign-solvable LCPs using totally sign-nonsingular matrices. For that purpose, we define *sign-nondegenerate* matrices. A square matrix A is *nondegenerate* if every principal minor is nonzero. A matrix A is nondegenerate if and only if $\mathrm{LCP}(A, b)$ has a finite number of solutions for any vector b [6]. Recognizing nondegenerate matrices is co-NP-complete [2, 19]. A square matrix A is said to be *sign-nondegenerate* if $\tilde{A}$ is nondegenerate for any $\tilde{A} \in \mathcal{Q}(A)$. Then the following lemma holds, which implies that sign-nondegeneracy can be tested in polynomial time.

Lemma 2.1. *A square matrix A is sign-nondegenerate if and only if A is a sign-nonsingular matrix with nonzero diagonals.*

Proof. To see the necessity, suppose that A is sign-nondegenerate. Let $\tilde{A}$ be a matrix in $\mathcal{Q}(A)$. Since all principal minors in $\tilde{A}$ are nonzero, all diagonal entries are nonzero. Moreover, $\det \tilde{A}$ is nonzero, which implies that A is sign-nonsingular. Thus A is a sign-nonsingular matrix with nonzero diagonals.

To see the sufficiency, suppose that A is a sign-nonsingular matrix with nonzero diagonals. Let $J \subseteq N$ be an index subset. Since the principal submatrix $A[J]$ has nonzero diagonals, $A[J]$ is term-nonsingular. Let σ_1 and σ_2 be bijections from J to J such that $p(A[J]|_{\sigma_1}) \neq 0$ and $p(A[J]|_{\sigma_2}) \neq 0$. Define bijections $\pi_k : N \to N$ to be $\pi_k(j) = j$ if $j \in N \setminus J$ and $\pi_k(j) = \sigma_k(j)$ if $j \in J$ for $k = 1, 2$. Since A has nonzero diagonals, $p(A|_{\pi_1})$ and $p(A|_{\pi_2})$ are both nonzero. By $p(A|_{\pi_k}) = p(A[J]|_{\sigma_k}) \prod_{i\in N\setminus J} a_{ii}$ for $k = 1, 2$, it follows from sign-nonsingularity of A that the two nonzero terms $p(A[J]|_{\sigma_1})$ and $p(A[J]|_{\sigma_2})$ have the same sign. Thus $A[J]$ is sign-nonsingular, which implies that A is sign-nondegenerate. □

We now obtain the following theorem. For $\mathrm{LCP}(A, b)$, let M be the matrix in the form of $M = (A\ b)$, where the column set is indexed by $N \cup \{g\}$.

Theorem 2.2. *For a linear complementarity problem* $\mathrm{LCP}(A, b)$ *with nonzero diagonals, if the matrix $M = (A\ b)$ is totally sign-nonsingular, then* $\mathrm{LCP}(A, b)$ *is sign-solvable.*

Proof. First assume that $\mathrm{LCP}(A,b)$ has a solution (w,z). Let J be the support of z. Then we have $A_J \begin{pmatrix} w[N\setminus J] \\ z[J] \end{pmatrix} + b = 0$, where A_J is the matrix in the form of

$$A_J = \begin{pmatrix} O & A[J] \\ -I & A[N\setminus J, J] \end{pmatrix}.$$

Since A is sign-nondegenerate by Lemma 2.1, each principal submatrix is sign-nonsingular, and hence A_J is also sign-nonsingular by $\det A_J = \pm \det A[J]$. Then it holds by Cramer's rule that

$$z_j = \begin{cases} -\det A_J^j/\det A_J, & \text{if } j \in J, \\ 0, & \text{if } j \in N\setminus J, \end{cases} \tag{1}$$

$$w_j = \begin{cases} 0, & \text{if } j \in J, \\ -\det A_J^j/\det A_J, & \text{if } j \in N\setminus J, \end{cases} \tag{2}$$

where A_J^j is the matrix obtained from A_J by replacing the jth column vector of A_J with b. The determinant of A_J^j is represented by

$$\det A_J^j = \begin{cases} \pm\det M[J, J-j+g], & \text{if } j \in J, \\ \pm\det M[J+j, J+g], & \text{if } j \in N\setminus J, \end{cases} \tag{3}$$

where $J-j+g$ means $J\setminus\{j\}\cup\{g\}$ with g being put at the position of j in J, the set $J+j$ coincides with $J\cup\{j\}$, and $J+g$ means $J\cup\{g\}$ in which g is put at the same position as that of j in $J+j$.

We show that A_J^j is either term-singular or sign-nonsingular for any $J\subseteq N$ and $j\in N$. Assume that there exists $j\in N$ such that A_J^j is term-nonsingular, but not sign-nonsingular. First suppose that $j\in J$. By (3), the submatrix $M[J, J-j+g]$ is term-nonsingular, but not sign-nonsingular. Then there exist two bijections σ_1 and σ_2 from J to $J-j+g$ such that $p(M[J,J-j+g]|_{\sigma_1})$ and $p(M[J,J-j+g]|_{\sigma_2})$ are both nonzero, and have the opposite signs. Define two bijections $\pi_k : N \to N-j+g$ to be $\pi_k(i) = i$ if $i\in N\setminus J$ and $\pi_k(i) = \sigma_k(i)$ if $i\in J$ for $k=1,2$. By $p(M[N,N-j+g]|_{\pi_k}) = p(M[J,J-j+g]|_{\sigma_k})\prod_{i\in N\setminus J} a_{ii}$ for $k=1,2$, the two nonzero terms $p(M[N,N-j+g]|_{\pi_1})$ and $p(M[N,N-j+g]|_{\pi_2})$ are both nonzero, and have the opposite signs. This contradicts the total sign-nonsingularity of M. Next suppose that $j\in N\setminus J$. Then, by (3), $M[J+j, J+g]$ is term-nonsingular, but not sign-nonsingular. Let σ_1 and σ_2 be bijections from $J+j$ to $J+g$ such that $p(M[J+j,J+g]|_{\sigma_1})$ and $p(M[J+j,J+g]|_{\sigma_2})$ are both nonzero, and have the opposite signs. Define two bijections $\pi_k : N\to N-j+g$ for $k=1,2$ to be $\pi_k(i)=i$ if $i\in N\setminus(J\cup\{j\})$ and $\pi_k(i)=\sigma_k(i)$ if $i\in J\cup\{j\}$. Then the two nonzero terms $p(M[N,N-j+g]|_{\pi_1})$ and $p(M[N,N-j+g]|_{\pi_2})$ have the opposite signs, which contradicts the total sign-nonsingularity of M.

Thus A_J^j is either term-singular or sign-nonsingular for any index j. The matrix A_J is sign-nonsingular. Therefore, it follows from (1) that the sign pattern of (w,z) is independent of the magnitudes of A and b. Hence $\mathrm{LCP}(\tilde{A},\tilde{b})$ has a solution with the same sign pattern as that of (w,z) for any $\tilde{A}\in\mathcal{Q}(A)$ and $\tilde{b}\in\mathcal{Q}(b)$. Thus $\mathrm{LCP}(A,b)$ is sign-solvable.

Next assume that $\mathrm{LCP}(A,b)$ has no solutions. Note that $\mathrm{LCP}(A,b)$ has no solutions if and only if $A_J x + b = 0$ has no nonnegative solutions for any $J\subseteq N$, that is, there exists $j\in N$ such that $(A_J^{-1}b)_j < 0$ for any $J\subseteq N$. It follows from Cramer's rule that we have $(A_J^{-1}b)_j = -\det A_J^j/\det A_J < 0$. Since $\det A_J^j \neq 0$, the matrix A_J^j is sign-nonsingular. Hence it holds that $-\det \tilde{A}_J^j/\det\tilde{A}_J < 0$ for any $\tilde{A}\in\mathcal{Q}(A)$ and $\tilde{b}\in\mathcal{Q}(b)$. Thus $\mathrm{LCP}(\tilde{A},\tilde{b})$ has no solutions for any $\tilde{A}\in\mathcal{Q}(A)$ and $\tilde{b}\in\mathcal{Q}(b)$, which means that $\mathrm{LCP}(A,b)$ is sign-solvable. □

Sign-solvable LCPs do not necessarily satisfy this sufficient condition. Indeed, consider LCP(A, b), where A and b are defined to be

$$A = \begin{pmatrix} +p_1 & +p_2 \\ +p_3 & +p_4 \end{pmatrix} \text{ and } b = \begin{pmatrix} +p_5 \\ +p_6 \end{pmatrix}$$

for positive constants $p_1, \ldots, p_6 > 0$. Then LCP(A, b) has a unique solution $w = (p_5\, p_6)^{\mathrm{T}}$ and $z = 0$, and hence LCP(A, b) is sign-solvable. However, this does not satisfy the condition of Theorem 2.2, as A is not sign-nonsingular.

We conclude this section with sign-solvability of LCPs associated with another class of matrices. A square matrix A is a ***P**-matrix* if every principal minor is positive. A $\boldsymbol{P}$-matrix is clearly nondegenerate. It is known that A is a $\boldsymbol{P}$-matrix if and only if LCP(A, b) has a unique solution for any vector b. Recognizing $\boldsymbol{P}$-matrices is co-NP-complete [7]. A matrix A is a *sign-**P**-matrix* if all matrices in $\mathcal{Q}(A)$ are $\boldsymbol{P}$-matrices. Then similar statements to Lemma 2.1 and Theorem 2.2 hold for sign-$\boldsymbol{P}$-matrices.

Corollary 2.3. *A square matrix A is a sign-**P**-matrix if and only if A is a sign-nonsingular matrix with positive diagonals.*

Corollary 2.4. *For a linear complementarity problem* LCP(A, b) *with positive diagonals, if the matrix $M = (A\ b)$ is totally sign-nonsingular, then* LCP$(\tilde{A}, \tilde{b})$ *has a unique solution with the same sign pattern as that of* LCP(A, b).

3 Characterization for Sign-Solvable LCPs with Nonzero Diagonals

In this section, we describe a characterization for a sign-solvable LCP(A, b) with nonzero diagonals.

3.1 The Residual Row-Mixed Matrix

We first introduce the *residual row-mixed* matrix of LCP(A, b) with nonzero diagonals.

For each row index i, the ith equation of LCP(A, b) is represented by

$$w_i = \sum_{j \in \Gamma(\{i\})} a_{ij} z_j + b_i. \tag{4}$$

First assume that M has a nonpositive row i, that is, $b_i \leq 0$ and $a_{ij} \leq 0$ for all $j \in N$. Suppose that $b_i < 0$. Since any solution of LCP(A, b) is nonnegative, the ith row implies that LCP(A, b) has no solutions. If $b_i = 0$, then a solution (w, z) of LCP(A, b) must satisfy that $z_j = 0$ for any $j \in \Gamma(\{i\})$.

Next assume that M has a nonnegative row i, that is, $b_i \geq 0$ and $a_{ij} \geq 0$ for all $j \in N$. Let (w, z) be a solution of LCP(A, b). If $w_i > 0$, then the complementarity implies $z_i = 0$. Suppose that $w_i = 0$. Since any solution is nonnegative, (w, z) must satisfy $z_j = 0$ for any $j \in \Gamma(\{i\})$, and hence $z_i = 0$ by $a_{ii} \neq 0$. Thus, if LCP(A, b) has a solution and M has a nonnegative row i, any solution of LCP(A, b) must satisfy that $z_i = 0$. Note that there exists $j \in \Gamma(\{i\})$ with $z_j > 0$ if and only if the left-hand side of (4) is positive, i.e., $w_i > 0$.

Therefore, if M has a nonnegative or nonpositive row, then we know that some entries of any solution must be zero. We can repeat this process as follows. Set $M^{(1)} = M$. For a positive integer ν and a matrix $M^{(\nu)}$, let $I_-^{(\nu)}$ be the set of nonpositive rows in $M^{(\nu)}$, and $I_+^{(\nu)}$ be the set of nonnegative rows that have a nonzero entry in $M^{(\nu)}$. If $\Gamma(I_-^{(\nu)})$ contains the index g, then the

LCP has no solutions. Define $I^{(\nu)} = I_+^{(\nu)} \cup I_-^{(\nu)}$ and $J^{(\nu)} = I_+^{(\nu)} \cup \Gamma(I_-^{(\nu)})$. Then any solution (w, z) of $\mathrm{LCP}(A, b)$ satisfies $z_j = 0$ for any $j \in J^{(\nu)}$. Let $M^{(\nu+1)}$ be the matrix obtained from $M^{(\nu)}$ by deleting the rows indexed by $I^{(\nu)}$ and the columns indexed by $J^{(\nu)}$. Repeat this for $\nu = 1, 2, \ldots$ until $I^{(\nu)} = J^{(\nu)} = \emptyset$, that is, until either $M^{(\nu)}$ is row-mixed or $M^{(\nu)}$ has no rows.

We call the remaining row-mixed submatrix $M' = (A' \ b')$ the *residual row-mixed* matrix of $\mathrm{LCP}(A, b)$. Note that, if $\mathrm{LCP}(A, b)$ has solutions, the column index g is not deleted in each iteration. We denote by U' and V' the row and column sets of A', respectively. Let $\bar{U}' = N \backslash U'$ and $\bar{V}' = N \backslash V'$. Since A has nonzero diagonals, $\bar{U}' \subseteq \bar{V}'$ holds, and hence we have $V' \subseteq U'$.

Suppose that the residual row-mixed matrix M' has no rows. Then $\bar{V}' = N$ holds. This means that any solution (w, z) of $\mathrm{LCP}(A, b)$ must satisfy $z = 0$. Since g is not deleted in each iteration, the vector b is nonnegative. Thus $(b, 0)$ is a unique solution of $\mathrm{LCP}(A, b)$.

Next suppose that $M' = (A' \ b')$ is row-mixed. Consider the following system:

$$\begin{aligned} & w = A'z + b', \\ & w_i^{\mathrm{T}} z_i = 0, \text{ for any } i \in V', \\ & w \geq 0, \ z \geq 0. \end{aligned} \tag{5}$$

We claim that there exists a one-to-one correspondence between solutions of $\mathrm{LCP}(A, b)$ and (5). For a solution (w, z) of $\mathrm{LCP}(A, b)$, the pair $(w[U'], z[V'])$ is a solution of (5). Conversely, let (w', z') be a solution of (5). Define (w, z) to be $z[V'] = z'$, $z[\bar{V}'] = 0$, and $w = Az + b$. Then $w[U'] = A'z' + b' = w' \geq 0$ holds. Moreover, since each row in $A[\bar{U}', V']$ is nonnegative, we have $w[\bar{U}'] = A[\bar{U}', V']z' + b[\bar{U}'] \geq 0$. By $V' \subseteq U'$, the pair (w, z) satisfies the complementarity $w^{\mathrm{T}} z = 0$. Thus (w, z) is a solution of $\mathrm{LCP}(A, b)$.

3.2 Characterization

Using the residual row-mixed matrix M' of $\mathrm{LCP}(A, b)$, we have the following theorem.

Theorem 3.1. *For a linear complementarity problem* $\mathrm{LCP}(A, b)$ *with nonzero diagonals,* $\mathrm{LCP}(A, b)$ *is sign-solvable if and only if one of the followings holds:*

- *The residual row-mixed matrix M' has no rows.*
- *The residual row-mixed matrix M' is totally sign-nonsingular.*

In order to prove this theorem, we give some definitions. A linear system $Ax = b$ has *nonnegative signed solutions* if the set of the sign patterns of nonnegative solutions of $\tilde{A}x = \tilde{b}$ is the same as that of nonnegative solutions of $Ax = b$ for any $\tilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. A matrix A is said to have *nonnegative signed null space* if $Ax = 0$ has nonnegative signed solutions. Matrices with nonnegative signed null space were examined by Fisher, Morris, and Shapiro [8]. They showed that a row-mixed matrix has nonnegative signed null space if and only if it is the matrix called *mixed dominating* in Fischer and Shapiro [9]. By the result of mixed dominating matrices, the following two lemmas hold.

Lemma 3.2 (Fischer and Shapiro [9])**.** *If a row-mixed matrix A has nonnegative signed null space, then the rows of A are linearly independent.*

A matrix A is said to have *row-full term-rank* if A has a term-nonsingular submatrix with row size. A matrix A has *column-full term-rank* if A^{T} has row-full term-rank.

Lemma 3.3 (Fischer, Morris, and Shapiro [8])**.** *An $n \times (n+1)$ row-mixed matrix has nonnegative signed null space if and only if it is a totally sign-nonsingular matrix with row-full term-rank.*

We have the following lemmas.

Lemma 3.4. *Suppose that* $(A\ b)$ *is row-mixed. If the linear system* $Ax + b = 0$ *has nonnegative signed solutions, then it has a solution all of whose entries are positive.*

Proof. Since $(A\ b)$ is row-mixed, there exist $\tilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$ such that the sum of the columns of $\tilde{A}$ and $\tilde{b}$ is zero, that is, $\tilde{A}\mathbf{1} + \tilde{b} = 0$, where $\mathbf{1}$ is the column vector whose entries are all one. This implies that $\tilde{A}x = \tilde{b}$ has a solution all of whose entries are positive for any $\tilde{A} \in \mathcal{Q}(A)$ and $\tilde{b} \in \mathcal{Q}(b)$. □

Lemma 3.5. *Suppose that* $(A\ b)$ *is row-mixed and that* A *has column-full term-rank. The linear system* $Ax + b = 0$ *has nonnegative signed solutions if and only if the matrix* $(A\ b)$ *has nonnegative signed null space.*

Proof. Suppose that the matrix $(A\ b)$ has nonnegative signed null space. Since $\{x \mid Ax + b = 0, x \geq 0\} = \{x \mid (A\ b)\binom{x}{1} = 0, x \geq 0\}$ is contained in the set of nonnegative vectors in the null space of $(A\ b)$, the linear system $Ax + b = 0$ has nonnegative signed solutions.

Next suppose that $Ax + b = 0$ has nonnegative signed solutions, but $(A\ b)$ does not have nonnegative signed null space. Since the set of the nonnegative vectors in the null space of $(A\ b)$ consists of the union of $\{x \mid Ax = 0, x \geq 0\}$ and $\{x \mid (A\ b)\binom{x}{x_g} = 0, x \geq 0, x_g > 0\}$, this assumption implies that A does not have nonnegative signed null space. Let $\tilde{A}$ be a matrix with column full rank. Then the null space of $\tilde{A}$ is empty, and $\tilde{A}x + b = 0$ has a unique solution all of whose entries are positive by Lemma 3.4. By the assumption, there exists $\hat{A} \in \mathcal{Q}(A)$ such that $\hat{A}x = 0$ has a nonnegative, nonzero solution x^*. Lemma 3.4 implies that $\hat{A}x + b = 0$ has a solution x^0 all of whose entries are positive. Then $x^0 - \mu x^*$, where $\mu = \min_{i \in N} x_i^0 / x_i^*$, is also a nonnegative solution of $\hat{A}x + b = 0$. This contradicts that $Ax + b = 0$ has nonnegative signed solutions. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. There exists a one-to-one correspondence between solutions of $\mathrm{LCP}(A, b)$ and (5). Hence $\mathrm{LCP}(A, b)$ is sign-solvable if and only if the set of sign patterns of solutions of (5) is uniquely determined by the sign patterns of A' and b'.

To show the necessity, suppose that $\mathrm{LCP}(A, b)$ is sign-solvable and that M' has a row. Let x be a nonnegative vector with $A'x + b' = 0$. Since $(0, x)$ is a solution of (5), sign-solvability of $\mathrm{LCP}(A, b)$ implies that the linear system $A'x + b' = 0$ has nonnegative signed solutions. The matrix A has nonzero diagonals and $V' \subseteq U'$, which implies that A' has column-full term-rank. It holds by Lemma 3.5 that $(A'\ b')$ has nonnegative signed null space. Since the rows of $(A'\ b')$ are linearly independent by Lemma 3.2, $U' = V'$ holds, i.e., A' is square. Therefore, by Lemma 3.3, $(A'\ b')$ is totally sign-nonsingular.

We next show the sufficiency. Suppose that M' has no rows. Then $(b, 0)$ is a unique solution of $\mathrm{LCP}(A, b)$, which means that $\mathrm{LCP}(A, b)$ is sign-solvable. Next suppose that the matrix $M' = (A'\ b')$ is totally sign-nonsingular. By $V' \subseteq U'$, it holds that $|U'| = |V'|$ or $|U'| = |V'| + 1$. If $|U'| = |V'|$, then M' is sign-nonsingular, which contradicts that M' is row-mixed. Hence we have $|U'| = |V'| + 1$. Since A' has nonzero diagonals, (5) forms the linear complementarity problem with nonzero diagonals. By Theorem 2.2, $\mathrm{LCP}(A', b')$ is sign-solvable, and hence so is $\mathrm{LCP}(A, b)$. □

If M is row-mixed, then the residual row-mixed matrix is M itself. Hence Theorem 3.1 implies the following corollary.

Corollary 3.6. *Let* A *have nonzero diagonals, and* $(A\ b)$ *be a row-mixed matrix. Then* $\mathrm{LCP}(A, b)$ *is sign-solvable if and only if the matrix* $M = (A\ b)$ *is totally sign-nonsingular.*

We close this section with an example of sign-solvable LCPs with nonzero diagonals. Consider LCP(A, b), where A and b have the sign patterns, respectively,

$$\begin{pmatrix} + & + & 0 & 0 & 0 \\ - & + & + & 0 & + \\ + & - & + & - & 0 \\ - & 0 & 0 & - & + \\ 0 & - & + & 0 & + \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ + \\ - \\ 0 \\ - \end{pmatrix}.$$

The residual row-mixed matrix is

$$\begin{pmatrix} + & - & 0 & - \\ 0 & - & + & 0 \\ + & 0 & + & - \end{pmatrix},$$

which is obtained from the matrix $(A\ b)$ by deleting the first two rows and the first two columns. This residual row-mixed matrix is totally sign-nonsingular, and hence LCP(A, b) is sign-solvable.

4 Algorithm for Sign-Solvable LCPs with Nonzero Diagonals

In this section, we describe an algorithm for a given LCP(A, b) with nonzero diagonals. The algorithm tests sign-solvability of LCP(A, b), and finds the sign pattern of a solution of LCP(A, b) if it is sign-solvable.

The algorithm starts with finding the residual row-mixed matrix $M' = (A'\ b')$ as described in the previous section. We denote by U' and V' the row and column sets of A', respectively. Let $\bar{U}' = N \setminus U'$ and $\bar{V}' = N \setminus V'$. Note that $V' \subseteq U'$ holds. If M' has a row and M' is not totally sign-nonsingular, then return that LCP(A, b) is not sign-solvable by Theorem 3.1.

Assume that M' has no rows. Then LCP(A, b) is sign-solvable, and $(b, 0)$ is a unique solution of LCP(A, b).

Next assume that M' has a row and $M' = (A'\ b')$ is totally sign-nonsingular. Then LCP(A, b) is sign-solvable by Theorem 3.1. Since M' is row-mixed, there exists $\tilde{M} \in \mathcal{Q}(M)$ such that the sum of the columns of $\tilde{M}'$ is zero, that is, $\tilde{A}'\mathbf{1} + \tilde{b}' = 0$, where $\mathbf{1}$ is the column vector whose entries are all one. Hence it follows from (5) that the pair (w, z), defined to be $z[\bar{V}'] = 0$, $z[V'] = +\mathbf{1}$, and $w = Az + b$, is a solution of LCP$(\tilde{A}, \tilde{b})$. This means that the vector w satisfies that $w_j > 0$ if $j \in \bar{U}'$ and $A[\{j\}, V']$ has nonzero entries, and $w_j = 0$ otherwise. Since LCP(A, b) is sign-solvable, (w, z) is the sign pattern of a solution of LCP(A, b).

We now summarize the algorithm description.

Algorithm: An algorithm for LCPs with nonzero diagonals.

Input: A linear complementarity problem LCP(A, b) with nonzero diagonals.

Output: The sign pattern of a solution if LCP(A, b) is sign-solvable.

Step 1: Set $M^{(1)} = M$ and $\nu = 1$. Repeat the following until $I^{(\nu)} = J^{(\nu)} = \emptyset$.

- **1-1:** Find $I_-^{(\nu)}$ and $I_+^{(\nu)}$, where $I_-^{(\nu)}$ is the set of nonpositive rows in $M^{(\nu)}$, and $I_+^{(\nu)}$ is the set of nonnegative rows that have a nonzero entry in $M^{(\nu)}$.
- **1-2:** If $g \in \Gamma(I_-^{(\nu)})$, then return that LCP$(A, b)$ has no solutions.
- **1-3:** Let $I^{(\nu)} = I_+^{(\nu)} \cup I_-^{(\nu)}$ and $J^{(\nu)} = I_+^{(\nu)} \cup \Gamma(I_-^{(\nu)})$. Define $M^{(\nu+1)}$ to be the matrix obtained by deleting the rows indexed by $I^{(\nu)}$ and the columns indexed by $J^{(\nu)}$ from $M^{(\nu)}$.

1-4: Set $\nu = \nu + 1$ and go back to Step 1.

Step 2: Let M' be the remaining submatrix, and U', V' be its row and column sets, respectively. If M' has a row and M' is not totally sign-nonsingular, then return that $\mathrm{LCP}(A, b)$ is not sign-solvable. Otherwise go to Step 3.

Step 3: Return that $\mathrm{LCP}(A, b)$ is sign-solvable and do the following.

2-1: If U' is empty, then return the sign pattern of a solution $(w, z) = (b, 0)$.

2-2: Otherwise, return the sign pattern of (w, z) defined to be

$$\operatorname{sgn} z_j = \begin{cases} +, & \text{if } j \in V' \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \operatorname{sgn} w_j = \begin{cases} +, & \text{if } j \in K \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

where K is the set of rows which have nonzero entries in $A[\bar{U}', V']$, that is, $K = \{j \in \bar{U}' \mid \Gamma(\{j\}) \cap V' \neq \emptyset\}$.

Applying this algorithm to the example at the end of Section 3, we obtain the sign pattern of a solution, $w = (\,0\ +\ \ 0\ \ 0\ \ 0)^{\mathrm{T}}$ and $z = (\,0\ \ 0\ +\ +\,+)^{\mathrm{T}}$.

Based on this algorithm, we can compute a solution of a sign-solvable $\mathrm{LCP}(A, b)$ as well as the sign pattern of a solution. Suppose that M' has a row. The solution (w, z) with the obtained sign pattern satisfies that $A'z[V'] + b' = 0$, $z[\bar{V}'] = 0$. Since A' is nonsingular by total sign-nonsingularity of M', we can compute a solution of $\mathrm{LCP}(A, b)$ by performing Gaussian elimination.

The running time bound of the algorithm is now given as follows. Note that an $n \times (n+1)$ row-mixed matrix A is a totally sign-nonsingular matrix with row-full term-rank if and only if A is the matrix called *S-matrix* in [1, 14]. S-matrices can be recognized in $\mathrm{O}(n^2)$ time [13].

Theorem 4.1. *For a linear complementarity problem* $\mathrm{LCP}(A, b)$ *with nonzero diagonals, let n be the matrix size of A, and γ the number of nonzero entries in A and b. Then the algorithm tests sign-solvability in* $\mathrm{O}(n^2)$ *time, and, if* $\mathrm{LCP}(A, b)$ *is sign-solvable, the algorithm finds the sign pattern of a solution in* $\mathrm{O}(\gamma)$ *time.*

Proof. In the νth iteration in Step 1, it requires $\mathrm{O}(\gamma_\nu)$ time to find $I^{(\nu)}$ and $J^{(\nu)}$, where γ_ν is the number of nonzero entries in the columns deleted in the νth iteration. Since each column is deleted at most once, Step 1 takes $\mathrm{O}(\gamma)$ time in total. In Step 2, if the residual row-mixed matrix M' is totally sign-nonsingular, M' has row-full term-rank and the column size is one larger than the row size. Hence testing total sign-nonsingularity of M' is equivalent to recognizing S-matrices. Thus it requires $\mathrm{O}(n^2)$ time to test sign-solvability in Step 2. Step 3 requires $\mathrm{O}(\gamma)$ time. Thus this statement holds. □

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