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Ken'ichiro TANAKA   Masaaki SUGIHARA
Kazuo MUROTA   Masatake MORI

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.i.u-tokyo.ac.jp/mi/mi-e.htm

Function Classes for Double Exponential Integration Formulas

Ken'ichiro Tanaka* ‡, Masaaki Sugihara* §, Kazuo Murota* ¶, Masatake Mori† ||

*Department of Mathematical Informatics,
Graduate School of Information Science and Technology, University of Tokyo

†Department of Mathematical Sciences, Tokyo Denki University

‡kenitiro@misojiro.t.u-tokyo.ac.jp, §m_sugihara@mist.i.u-tokyo.ac.jp,
¶murota@mist.i.u-tokyo.ac.jp, ||mmori@r.dendai.ac.jp

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Abstract

The double exponential (DE) formulas for numerical integration are known to be highly efficient, more efficient than the single exponential (SE) formulas in many cases. Function classes suited to the SE formulas have already been investigated in the literature through rigorous mathematical analysis, whereas this is not the case with the DE formulas. This paper identifies function classes suited to the DE formulas in a way compatible with the existing theoretical results for the SE formulas. The DE formulas are good for more restricted classes of functions, but more efficient for such functions. Two concrete examples demonstrate the subtlety in the behavior of the DE formulas that is revealed by our theoretical analysis.

1 Introduction

The double exponential (DE) formulas for numerical integration [1], proposed by Takahasi and Mori [13], are known to be highly efficient. The idea is to transform a given problem to

$$\int_{-\infty}^{\infty} f(\psi(t))\psi'(t)\mathrm{d}t$$

through a change of variable $x = \psi(t)$ and then apply the trapezoidal formula to the transformed integral above. For the transformation function $\psi(t)$ the DE formulas employ an appropriate DE transformation [4] such as

$$\psi_{\mathrm{DE1}} : (-\infty,\infty) \to (-1,1), \quad \psi_{\mathrm{DE1}}(t) := \tanh((\pi/2)\sinh t), \tag{1.1}$$

$$\psi_{\mathrm{DE2}} : (-\infty,\infty) \to (-\infty,\infty), \quad \psi_{\mathrm{DE2}}(t) := \sinh((\pi/2)\sinh t), \tag{1.2}$$

$$\psi_{\mathrm{DE3}} : (-\infty,\infty) \to (0,\infty), \quad \psi_{\mathrm{DE3}}(t) := \exp((\pi/2)\sinh t), \tag{1.3}$$

$$\psi_{\mathrm{DE4}} : (-\infty,\infty) \to (0,\infty), \quad \psi_{\mathrm{DE4}}(t) := \exp(t - \exp(-t)), \tag{1.4}$$

$$\psi_{\mathrm{DE5}} : (-\infty,\infty) \to (0,\infty), \quad \psi_{\mathrm{DE5}}(t) := \log(\exp((\pi/2)\sinh t) + 1), \tag{1.5}$$

where ψ_{DE5} is proposed recently in [5]. More explicitly, the formulas with these transformations are as follows:

$$\begin{aligned}
\int_{-1}^{1} f(x)\mathrm{d}x &\approx h \sum_{k=-N}^{N} f(\psi_{\mathrm{DE1}}(kh))\psi'_{\mathrm{DE1}}(kh) \\
&= h \sum_{k=-N}^{N} f(\tanh((\pi/2)\sinh(kh)))\,\frac{\pi\cosh(kh)}{2\cosh^2((\pi/2)\sinh(kh))}, \qquad (1.6)
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\mathrm{d}x &\approx h \sum_{k=-N}^{N} f(\psi_{\mathrm{DE2}}(kh))\psi'_{\mathrm{DE2}}(kh) \\
&= h \sum_{k=-N}^{N} f(\sinh((\pi/2)\sinh(kh)))\,(\pi/2)\cosh(kh)\cosh((\pi/2)\sinh(kh)), \qquad (1.7)
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} f(x)\mathrm{d}x &\approx h \sum_{k=-N}^{N} f(\psi_{\mathrm{DE3}}(kh))\psi'_{\mathrm{DE3}}(kh) \\
&= h \sum_{k=-N}^{N} f(\exp((\pi/2)\sinh(kh)))\,(\pi/2)\cosh(kh)\exp((\pi/2)\sinh(kh)), \qquad (1.8)
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} f(x)\mathrm{d}x &\approx h \sum_{k=-N}^{N} f(\psi_{\mathrm{DE4}}(kh))\psi'_{\mathrm{DE4}}(kh) \\
&= h \sum_{k=-N}^{N} f(\exp(kh-\exp(-kh)))\,(1+\exp(-kh))\exp(kh-\exp(-kh)), \qquad (1.9)
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} f(x)\mathrm{d}x &\approx h \sum_{k=-N}^{N} f(\psi_{\mathrm{DE5}}(kh))\psi'_{\mathrm{DE5}}(kh) \\
&= h \sum_{k=-N}^{N} f(\log(\exp((\pi/2)\sinh(kh))+1))\,\frac{\pi\cosh(kh)\exp((\pi/2)\sinh(kh))}{2(\exp((\pi/2)\sinh(kh))+1)}. \qquad (1.10)
\end{aligned}$$

Besides the DE formulas there are a number of efficient integration formulas based on the same idea of a change of variable, but using different transformation functions [2, 3, 6, 7, 12]. Among them are the single exponential (SE) formulas, by which we mean those formulas advocated by Stenger [8, 9]; the explicit forms of the SE formulas are given in Section 2. Generally, the DE formulas are more efficient than the SE formulas.

As a theoretical result on the DE formulas, an error estimate of the form $\exp(-cN/\log N)$ is given by Takahasi and Mori [13] by means of the saddle point method. More rigorous mathematical analysis is done by Sugihara [11], with an observation that implies a certain optimality of the DE formulas. For the SE formulas, on the other hand, some classes of functions suited to the formulas have been identified by Stenger [9] through rigorous mathematical analysis, whereas this is not the case with the DE formulas.

This paper identifies the function classes suited to the DE formulas in a way compatible with the existing results for the SE formulas. The DE formulas are good for more restricted classes of functions, but more efficient for such functions. It may be said that the essence of the present results is already implicit in [11], and the main contribution of this paper is to tailor the implicit observation there to explicit statements that are compatible with the corresponding results for the SE formulas.

This paper is organized as follows. In Section 2, we review Stenger's theorems for the SE formulas by way of comparison with our results. In Section 3, we present our theorems for the DE formulas as the main result of this paper. In Section 4, we show two concrete examples with numerical results that demonstrate the subtlety in the behavior of the DE formulas revealed by our theoretical analysis. In Section 5 we give the proofs of the theorems.

2 Function Classes for SE Formulas

This section is a review of some relevant results on the integration formulas based on single exponential transformations.

The single exponential transformations are given by the following functions:

$$\psi_{\mathrm{SE1}} : (-\infty,\infty) \to (-1,1), \quad \psi_{\mathrm{SE1}}(t) := \tanh(t/2), \tag{2.1}$$
$$\psi_{\mathrm{SE2}} : (-\infty,\infty) \to (-\infty,\infty), \quad \psi_{\mathrm{SE2}}(t) := \sinh t, \tag{2.2}$$
$$\psi_{\mathrm{SE3}} : (-\infty,\infty) \to (0,\infty), \quad \psi_{\mathrm{SE3}}(t) := \exp t, \tag{2.3}$$
$$\psi_{\mathrm{SE4}} : (-\infty,\infty) \to (0,\infty), \quad \psi_{\mathrm{SE4}}(t) := \operatorname{arcsinh}(\exp t). \tag{2.4}$$

Accordingly, the integration formulas with these transformations are given as follows:

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx h \sum_{k=-N}^{N} f(\psi_{\mathrm{SE1}}(kh))\psi'_{\mathrm{SE1}}(kh) = h \sum_{k=-N}^{N} f(\tanh(kh/2))\, \frac{1}{2\cosh^2(kh/2)}, \tag{2.5}$$

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x \approx h \sum_{k=-N}^{N} f(\psi_{\mathrm{SE2}}(kh))\psi'_{\mathrm{SE2}}(kh) = h \sum_{k=-N}^{N} f(\sinh(kh))\, \cosh(kh), \tag{2.6}$$

$$\int_{0}^{\infty} f(x)\mathrm{d}x \approx h \sum_{k=-N}^{N} f(\psi_{\mathrm{SE3}}(kh))\psi'_{\mathrm{SE3}}(kh) = h \sum_{k=-N}^{N} f(\exp(kh))\, \exp(kh), \tag{2.7}$$

$$\int_{0}^{\infty} f(x)\mathrm{d}x \approx h \sum_{k=-N}^{N} f(\psi_{\mathrm{SE4}}(kh))\psi'_{\mathrm{SE4}}(kh) = h \sum_{k=-N}^{N} f(\operatorname{arcsinh}(\exp(kh)))\frac{\exp(kh)}{\sqrt{1+\exp(2kh)}}. \tag{2.8}$$

These formulas are called the SE formulas.

In the theorems below, integrands suited to the SE formulas are specified with reference to complex regions. For $d > 0$ we define a strip region $\mathcal{D}_d$ as

$$\mathcal{D}_d := \{z \in \mathbf{C} \mid |\operatorname{Im} z| < d\}. \tag{2.9}$$

Then we define $\mathcal{D}_{\mathrm{SE}i}(d)$ as the image of $\mathcal{D}_d$ through $\psi_{\mathrm{SE}i}$; that is,

$$\mathcal{D}_{\mathrm{SE}i}(d) := \{z = \psi_{\mathrm{SE}i}(w) \mid w \in \mathcal{D}_d\} \quad (i = 1, \ldots, 4).$$

Figures 1 to 4 illustrate these regions together with their boundaries $\partial\mathcal{D}_{\mathrm{SE}i}(d)$.

Theorems 2.1 to 2.4 below give asymptotic error estimates for the SE formulas with mathematical rigor.

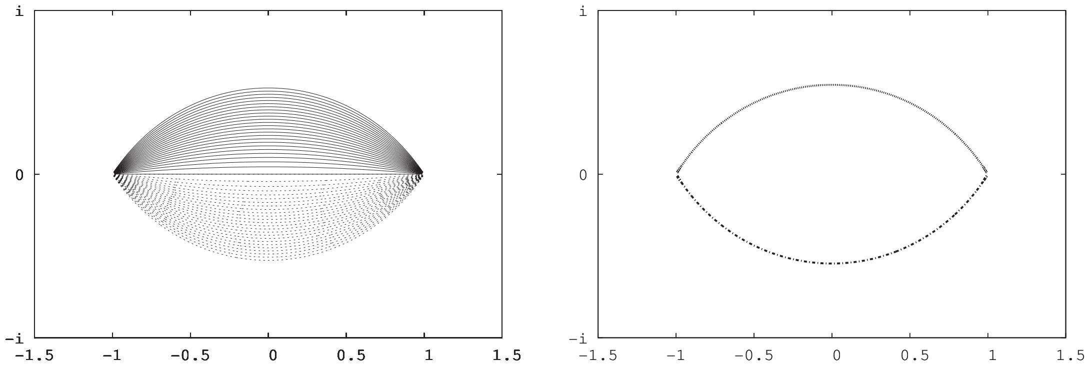


Figure 1: Region $\mathcal{D}_{\mathrm{SE1}}(1)$ and its boundary $\partial\mathcal{D}_{\mathrm{SE1}}(1)$

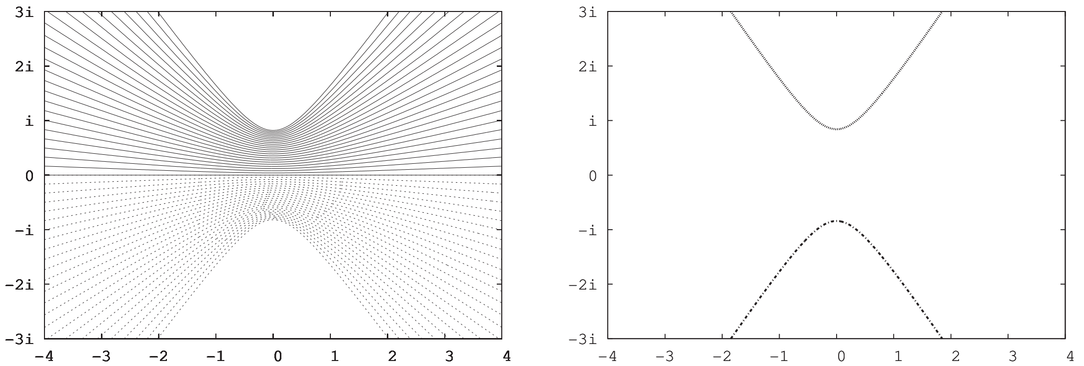


Figure 2: Region $\mathcal{D}_{\mathrm{SE2}}(1)$ and its boundary $\partial\mathcal{D}_{\mathrm{SE2}}(1)$

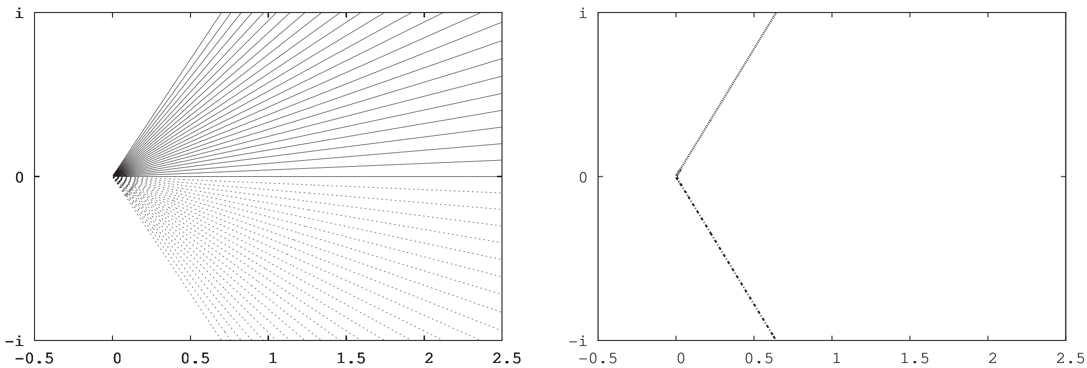


Figure 3: Region $\mathcal{D}_{\mathrm{SE3}}(1)$ and its boundary $\partial\mathcal{D}_{\mathrm{SE3}}(1)$

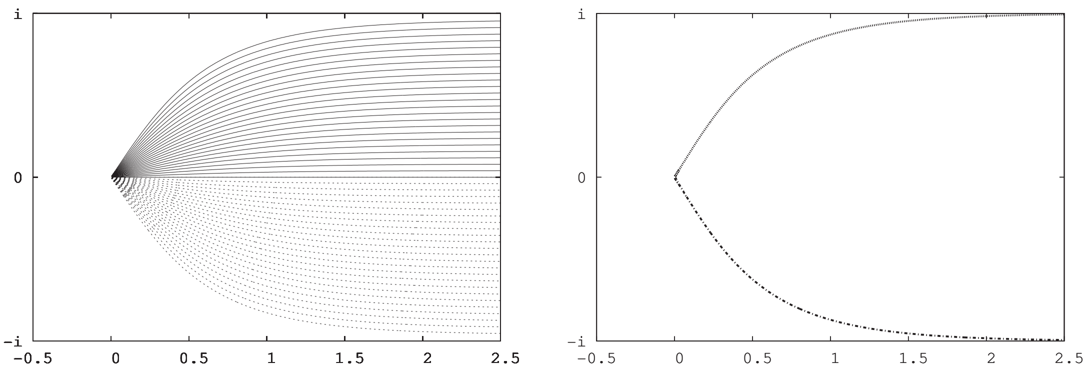


Figure 4: Region $\mathcal{D}_{\mathrm{SE4}}(1)$ and its boundary $\partial\mathcal{D}_{\mathrm{SE4}}(1)$

Theorem 2.1 (Stenger [9]). Assume that f is holomorphic on $\mathcal{D}_{\mathrm{SE1}}(d)$ for d with $0 < d < \pi$ and satisfies

$$\forall z \in \mathcal{D}_{\mathrm{SE1}}(d): \ |f(z)| \leq C_1 |(1-z^2)^{\beta-1}| \tag{2.10}$$

for constants $C_1 > 0$ and $\beta > 0$. Then there exists a constant C, independent of N, such that

$$\left| \int_{-1}^{1} f(x)\,\mathrm{d}x - h \sum_{k=-N}^{N} f(\psi_{\mathrm{SE1}}(kh))\psi'_{\mathrm{SE1}}(kh) \right| \leq C \exp\left(-\sqrt{2\pi d \beta N}\right),$$

where

$$h = \sqrt{\frac{2\pi d}{\beta N}}.$$

Theorem 2.2 (Stenger [9]). Assume that f is holomorphic on $\mathcal{D}_{\mathrm{SE2}}(d)$ for d with $0 < d < \pi/2$ and satisfies

$$\forall z \in \mathcal{D}_{\mathrm{SE2}}(d): \ |f(z)| \leq C_1 \left| \frac{1}{(1+z^2)^{\beta/2+1/2}} \right| \tag{2.11}$$

for constants $C_1 > 0$ and $\beta > 0$. Then there exists a constant C, independent of N, such that

$$\left| \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - h \sum_{k=-N}^{N} f(\psi_{\mathrm{SE2}}(kh))\psi'_{\mathrm{SE2}}(kh) \right| \leq C \exp\left(-\sqrt{2\pi d \beta N}\right),$$

where

$$h = \sqrt{\frac{2\pi d}{\beta N}}.$$

Theorem 2.3 (Stenger [9]). Assume that f is holomorphic on $\mathcal{D}_{\mathrm{SE3}}(d)$ for d with $0 < d < \pi/2$ and satisfies

$$\forall z \in \mathcal{D}_{\mathrm{SE3}}(d): \ |f(z)| \leq C_1 \left|\frac{z^{\beta-1}}{(1+z^2)^{\beta}}\right| \tag{2.12}$$

for constants $C_1 > 0$ and $\beta > 0$. Then there exists a constant C, independent of N, such that

$$\left|\int_0^{\infty} f(x)\,\mathrm{d}x - h\sum_{k=-N}^{N} f(\psi_{\mathrm{SE3}}(kh))\psi'_{\mathrm{SE3}}(kh)\right| \leq C\exp\left(-\sqrt{2\pi d\beta N}\right),$$

where

$$h = \sqrt{\frac{2\pi d}{\beta N}}.$$

Theorem 2.4 (Stenger [9]). Assume that f is holomorphic on $\mathcal{D}_{\mathrm{SE4}}(d)$ for d with $0 < d < \pi/2$ and satisfies

$$\forall z \in \mathcal{D}_{\mathrm{SE4}}(d): \ |f(z)| \leq C_1 \left|\left(\frac{z}{1+z}\right)^{\beta-1}\exp(-\beta z)\right| \tag{2.13}$$

for constants $C_1 > 0$ and $\beta > 0$. Then there exists a constant C, independent of N, such that

$$\left|\int_0^{\infty} f(x)\,\mathrm{d}x - h\sum_{k=-N}^{N} f(\psi_{\mathrm{SE4}}(kh))\psi'_{\mathrm{SE4}}(kh)\right| \leq C\exp\left(-\sqrt{2\pi d\beta N}\right),$$

where

$$h = \sqrt{\frac{2\pi d}{\beta N}}.$$

3 Function Classes for DE Formulas

In this section, we present our theorems for the error estimate of the DE formulas. Recall the transformation functions $\psi_{\mathrm{DE}i}$ $(i = 1, \ldots, 5)$ given in (1.1)–(1.5):

$$\begin{aligned}
&\psi_{\mathrm{DE1}} : (-\infty,\infty) \to (-1,1), \quad \psi_{\mathrm{DE1}}(t) := \tanh((\pi/2)\sinh t),\\
&\psi_{\mathrm{DE2}} : (-\infty,\infty) \to (-\infty,\infty), \quad \psi_{\mathrm{DE2}}(t) := \sinh((\pi/2)\sinh t),\\
&\psi_{\mathrm{DE3}} : (-\infty,\infty) \to (0,\infty), \quad \psi_{\mathrm{DE3}}(t) := \exp((\pi/2)\sinh t),\\
&\psi_{\mathrm{DE4}} : (-\infty,\infty) \to (0,\infty), \quad \psi_{\mathrm{DE4}}(t) := \exp(t - \exp(-t)),\\
&\psi_{\mathrm{DE5}} : (-\infty,\infty) \to (0,\infty), \quad \psi_{\mathrm{DE5}}(t) := \log(\exp((\pi/2)\sinh t) + 1).
\end{aligned}$$

To state our theorems we need to introduce complex regions $\mathcal{D}_{\mathrm{DE}i}(d)$ that are defined as the images of $\mathcal{D}_d$ in (2.9) through the transformation functions $\psi_{\mathrm{DE}i}$; that is,

$$\mathcal{D}_{\mathrm{DE}i}(d) := \{z = \psi_{\mathrm{DE}i}(w) \mid w \in \mathcal{D}_d\} \quad (i = 1, \ldots, 5).$$

Figures 5 to 9 illustrate these regions together with their boundaries $\partial\mathcal{D}_{\mathrm{DE}i}(d)$. We regard $\mathcal{D}_{\mathrm{DE}i}(d)$ as a region on the Riemann surface.

We are now in the position to state the main theorems. The proofs are shown in Section 5.

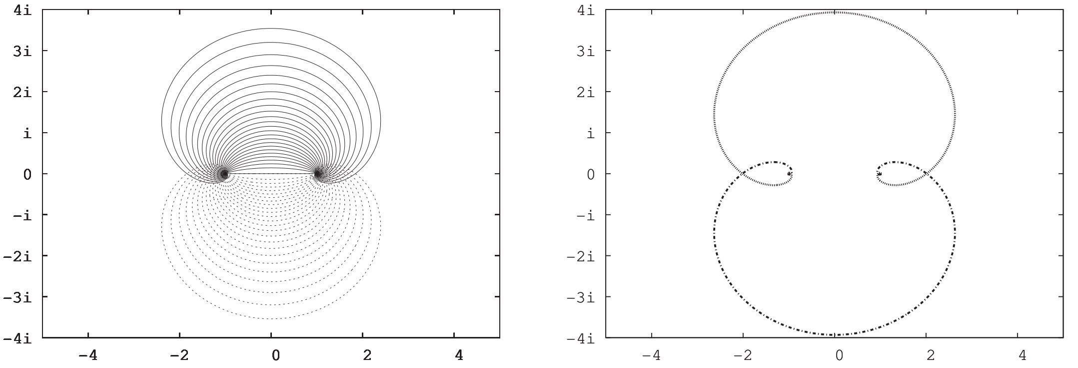


Figure 5: Region $\mathcal{D}_{\mathrm{DE1}}(1)$ and its boundary $\partial\mathcal{D}_{\mathrm{DE1}}(1)$

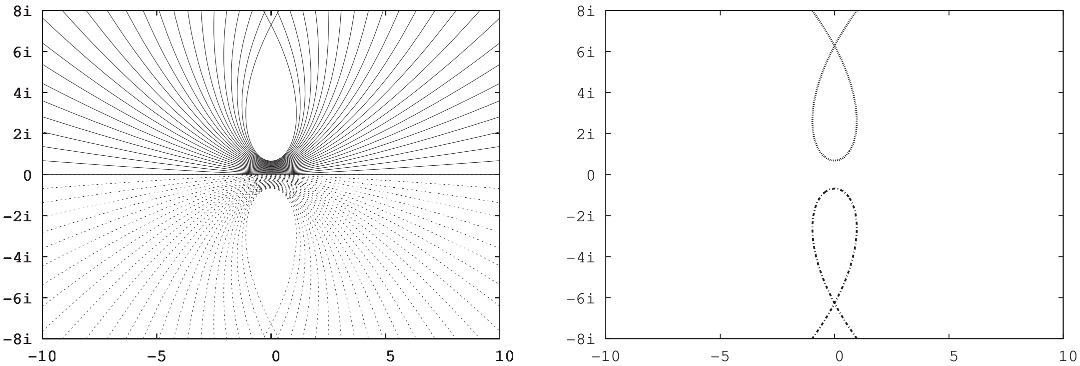


Figure 6: Region $\mathcal{D}_{\mathrm{DE2}}(1/2)$ and its boundary $\partial\mathcal{D}_{\mathrm{DE2}}(1/2)$

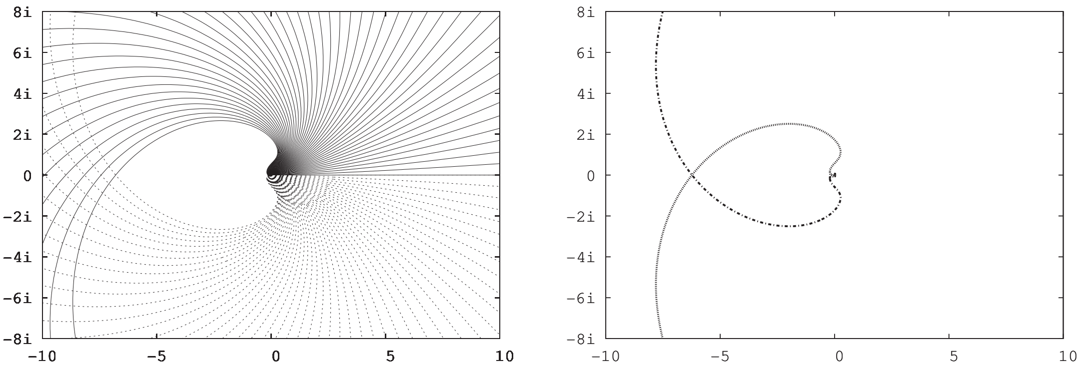


Figure 7: Region $\mathcal{D}_{\mathrm{DE3}}(1)$ and its boundary $\partial\mathcal{D}_{\mathrm{DE3}}(1)$

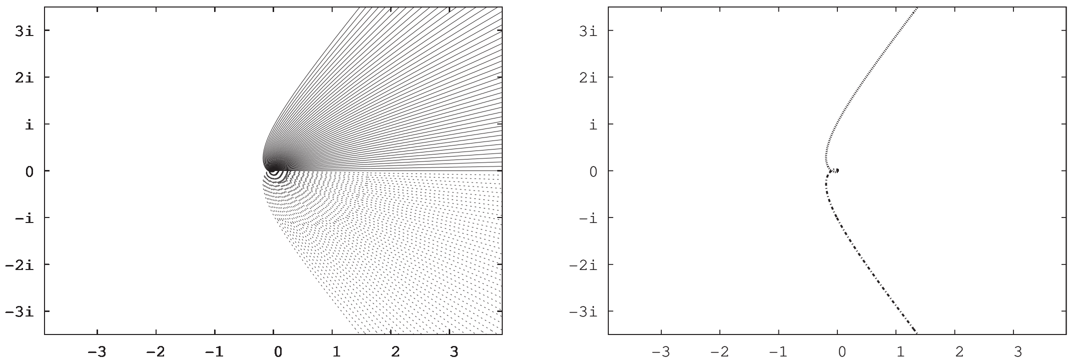


Figure 8: Region $\mathcal{D}_{\mathrm{DE4}}(1)$ and its boundary $\partial\mathcal{D}_{\mathrm{DE4}}(1)$

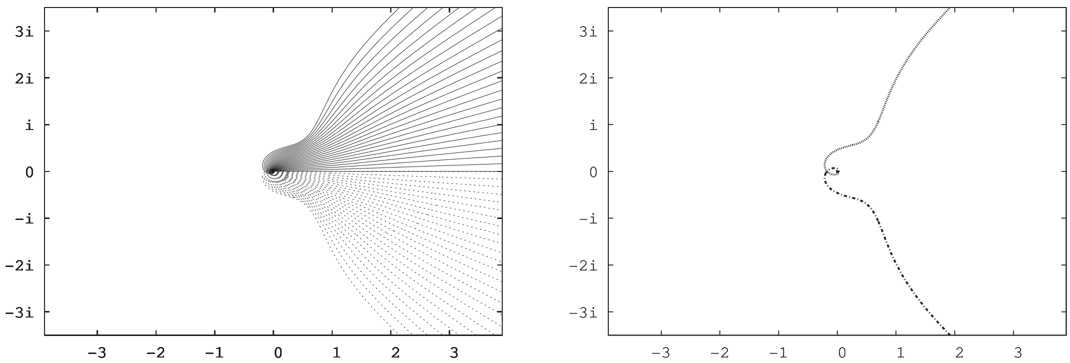


Figure 9: Region $\mathcal{D}_{\mathrm{DE5}}(1)$ and its boundary $\partial\mathcal{D}_{\mathrm{DE5}}(1)$

Theorem 3.1. Assume that f is holomorphic on $\mathcal{D}_{\mathrm{DE1}}(d)$ for d with $0 < d < \pi/2$ and satisfies

$$\forall z \in \mathcal{D}_{\mathrm{DE1}}(d):\ |f(z)| \leq C_1 |(1-z^2)^{\beta-1}| \tag{3.1}$$

for constants $C_1 > 0$ and $\beta > 0$. Then there exists a constant C, independent of N, such that

$$\left| \int_{-1}^{1} f(x)\,\mathrm{d}x - h \sum_{k=-N}^{N} f(\psi_{\mathrm{DE1}}(kh))\psi'_{\mathrm{DE1}}(kh) \right| \leq C \exp\left(-\frac{2\pi d N}{\log(4dN/\beta)} \right),$$

where

$$h = \frac{\log(4dN/\beta)}{N}.$$

Theorem 3.2. Assume that f is holomorphic on $\mathcal{D}_{\mathrm{DE2}}(d)$ for d with $0 < d < \pi/2$ and satisfies

$$\forall z \in \mathcal{D}_{\mathrm{DE2}}(d):\ |f(z)| \leq C_1 \left| \frac{1}{(1+z^2)^{\beta/2+1/2}} \right| \tag{3.2}$$

for constants $C_1 > 0$ and $\beta > 0$. Then there exists a constant C, independent of N, such that

$$\left| \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - h \sum_{k=-N}^{N} f(\psi_{\mathrm{DE2}}(kh))\psi'_{\mathrm{DE2}}(kh) \right| \leq C \exp\left(-\frac{2\pi d N}{\log(8dN/\beta)} \right),$$

where

$$h = \frac{\log(8dN/\beta)}{N}.$$

Theorem 3.3. Assume that f is holomorphic on $\mathcal{D}_{\mathrm{DE3}}(d)$ for d with $0 < d < \pi/2$ and satisfies

$$\forall z \in \mathcal{D}_{\mathrm{DE3}}(d):\ |f(z)| \leq C_1 \left| \frac{z^{\beta-1}}{(1+z^2)^{\beta}} \right| \tag{3.3}$$

for constants $C_1 > 0$ and $\beta > 0$. Then there exists a constant C, independent of N, such that

$$\left| \int_{0}^{\infty} f(x)\,\mathrm{d}x - h \sum_{k=-N}^{N} f(\psi_{\mathrm{DE3}}(kh))\psi'_{\mathrm{DE3}}(kh) \right| \leq C \exp\left(-\frac{2\pi d N}{\log(8dN/\beta)} \right),$$

where

$$h = \frac{\log(8dN/\beta)}{N}$$

Theorem 3.4. Assume that f is holomorphic on $\mathcal{D}_{\mathrm{DE4}}(d)$ for d with $0 < d < \pi/2$ and satisfies

$$\forall z \in \mathcal{D}_{\mathrm{DE4}}(d):\ |f(z)| \leq C_1 \left| \left(\frac{z}{1+z} \right)^{\beta-1} \exp(-\beta z) \right| \tag{3.4}$$

for constants $C_1 > 0$ and $\beta > 0$. Then there exists a constant C, independent of N, such that

$$\left| \int_{0}^{\infty} f(x)\,\mathrm{d}x - h \sum_{k=-N}^{N} f(\psi_{\mathrm{DE4}}(kh))\psi'_{\mathrm{DE4}}(kh) \right| \leq C \exp\left(-\frac{2\pi d N}{\log(2\pi dN/\beta)} \right)$$

where

$$h = \frac{\log(2\pi dN/\beta)}{N}.$$

Theorem 3.5. Assume that f is holomorphic on $\mathcal{D}_{\mathrm{DE5}}(d)$ for d with $0 < d < \pi/2$ and satisfies

$$\forall z \in \mathcal{D}_{\mathrm{DE5}}(d): \ |f(z)| \leq C_1 \left| \left(\frac{z}{1+z} \right)^{\beta - 1} \exp(-\beta z) \right| \tag{3.5}$$

for constants $C_1 > 0$ and $\beta > 0$. Then there exists a constant C, independent of N, such that

$$\left| \int_0^\infty f(x)\,\mathrm{d}x - h \sum_{k=-N}^{N} f(\psi_{\mathrm{DE5}}(kh))\psi'_{\mathrm{DE5}}(kh) \right| \leq C \exp\left(-\frac{2\pi d N}{\log(8dN/\beta)} \right),$$

where

$$h = \frac{\log(8dN/\beta)}{N}.$$

4 Examples Not Suited to DE Formulas

The DE formula works excellently under fairly general functions, as specified in Theorems 3.1 to 3.5. This does not mean, however, that the formula works for any integrands. In this section we show two examples for which the DE formula is not so efficient as the naive intuition expects. In so doing we intend to indicate the sharpness of our theoretical results. In practical applications, however, there will be no doubt that the DE formula is one of the most reliable methods for numerical integration.

4.1 Jacobi's elliptic function

As the first example we apply the DE formula to the numerical integration of

$$f(x) = (1 - x^2)\ \mathrm{cn}(2 \operatorname{arctanh} x, \sqrt{0.5}) \tag{4.1}$$

on the interval $(-1, 1)$, where "cn" is Jacobi's elliptic function (so-called "cn" function). We are to employ the DE formula in (1.6).

This integrand f does not meet the assumptions of Theorem 3.1. In fact, there is no $d > 0$ such that $f(\psi_{\mathrm{DE1}}(\cdot))\psi'_{\mathrm{DE1}}(\cdot)$ is holomorphic on $\mathcal{D}_d$. This is because the poles of the transformed integrand

$$f(\psi_{\mathrm{DE1}}(z))\psi'_{\mathrm{DE1}}(z) = \frac{\pi \cosh z\ \mathrm{cn}(\pi \sinh z, \sqrt{0.5})}{2\cosh^4((\pi/2)\sinh z)}$$

arising from $\mathrm{cn}(\pi \sinh z, \sqrt{0.5})$ are located (see Figure 10) at

$$\operatorname{arcsinh}\left(\frac{K}{\pi}(2n \pm \mathrm{i}) \right) \quad (n \in \mathbf{Z})$$

with

$$\lim_{n \to \pm\infty} \operatorname{Im} \operatorname{arcsinh}\left(\frac{K}{\pi}(2n \pm \mathrm{i}) \right) = 0,$$

where

$$K = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - 0.5 \sin^2 \theta}} = 1.85407 \cdots.$$

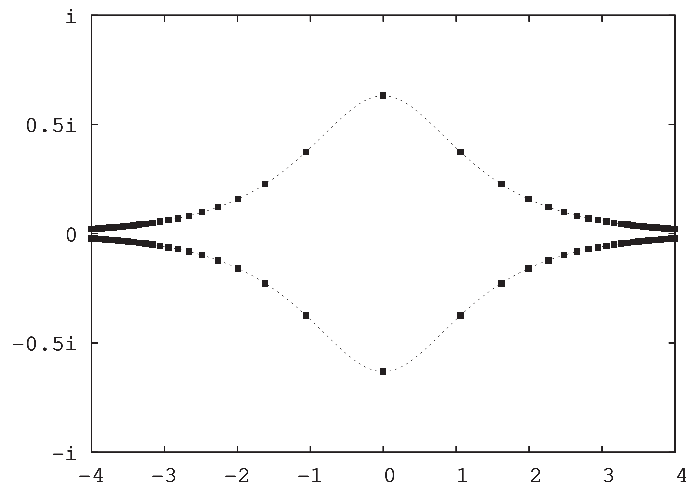


Figure 10: Poles at $\operatorname{arcsinh}(K(2n \pm \mathrm{i})/\pi)$

The errors of the DE formula with

$$h = \frac{\log(2^p \pi N)}{N} \quad (p = -5, -4, -3, -2, -1, 0, 1, 2, 3),$$

as observed in our numerical experiments, are depicted in Figure 11. The error does not decay as $\exp(-cN/\log N)$, but seemingly as $\exp(-c\sqrt{N})$. Note that there is no theoretical recipe for the choice of h.

For comparison, the SE formula (2.5) is applied to f, with the results shown in Figure 12. Theorem 2.1 is applicable to this function and the theory indicates the choice of $h = \sqrt{\pi K/N}$. We have also tried with

$$h = \sqrt{\frac{2^p \pi K}{N}} \quad (p = -3, -2, -1, 0, 1, 2, 3),$$

where $p = 0$ corresponds to the theoretical value. The computational results show that the theoretical choice of h gives the highest accuracy.

Finally we mention that we used

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = 0.8194555274929631681197057029 71$$

as the "true" value in computing the errors, whereas we computed the above value with *Mathematica* by executing

```
NIntegrate[f[x], {x, -1, 1}, WorkingPrecision -> 40, MaxRecursion -> 100]
```

where the number of the significant digits set by the command "`NIntegrate`" is usually smaller than the value of "`WorkingPrecision`" by 10. Other computations are done with double-precision floating point numbers.

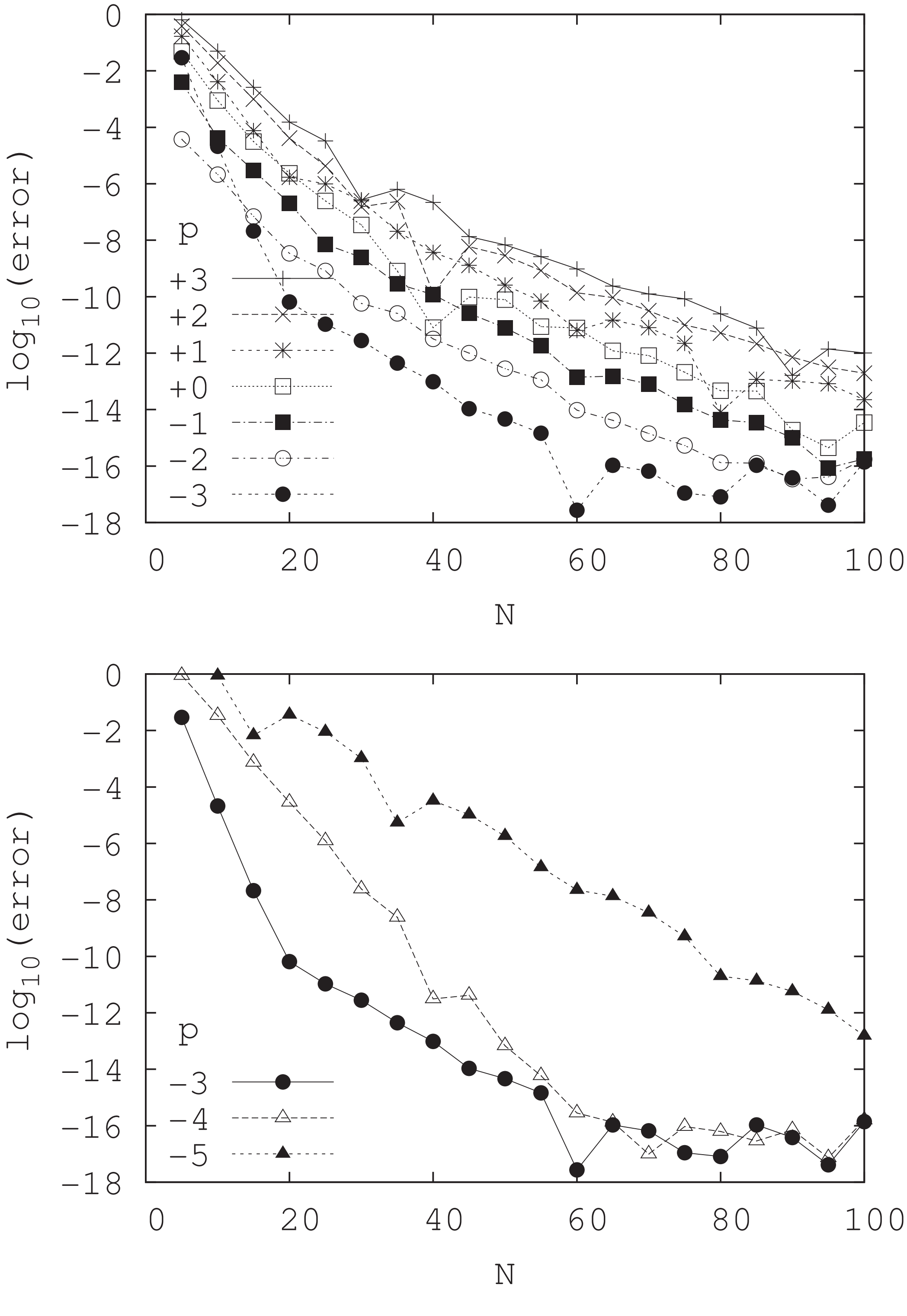


Figure 11: Errors of the DE formula for $f(x) = (1 - x^2)\ \mathrm{cn}(2\,\mathrm{arctanh}\, x, \sqrt{0.5})$

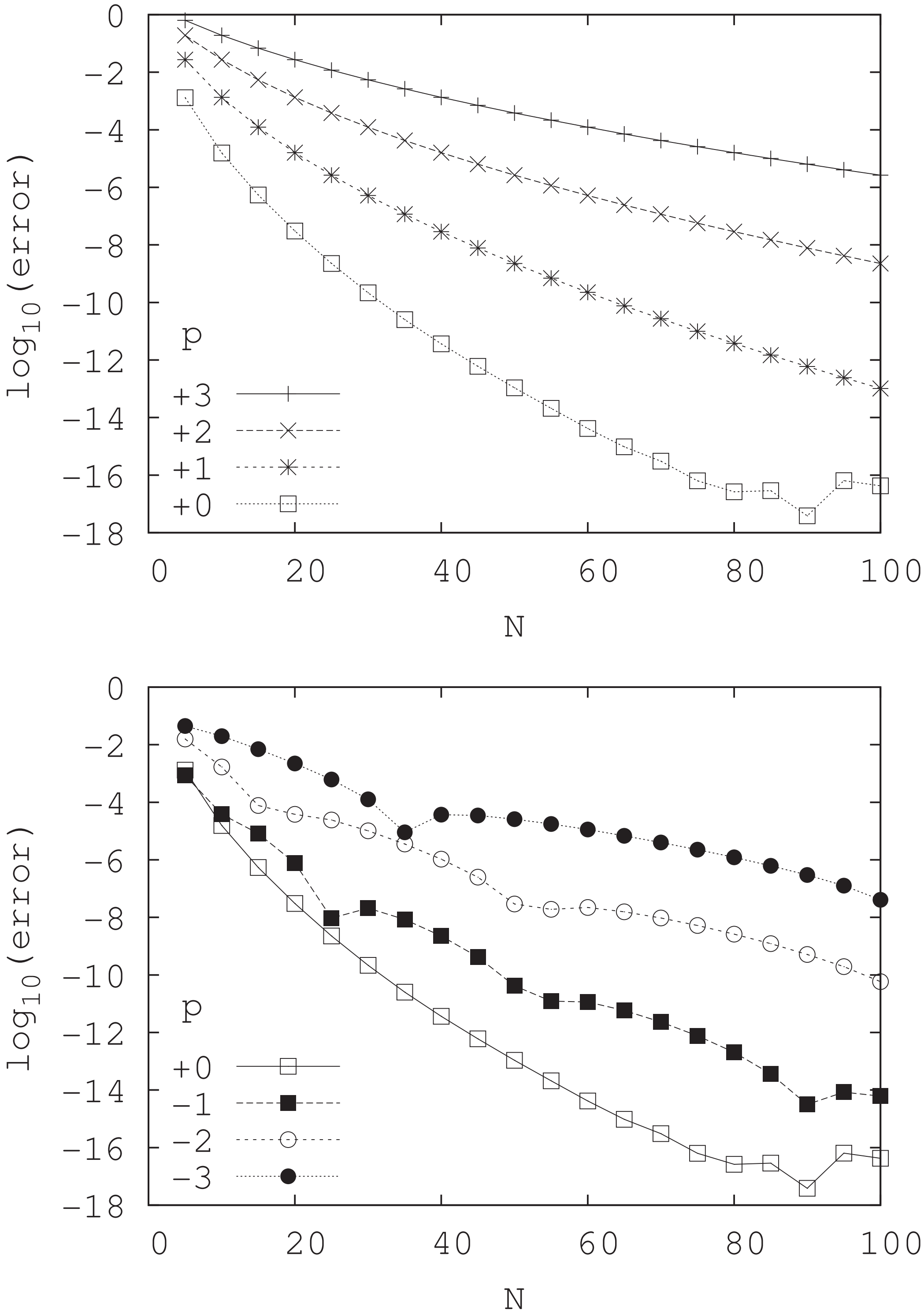


Figure 12: Errors of the SE formula for $f(x) = (1 - x^2)\ \mathrm{cn}(2 \operatorname{arctanh} x, \sqrt{0.5})$

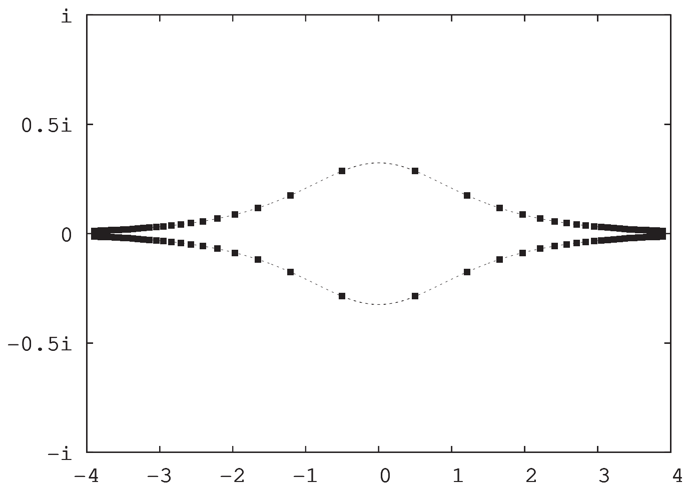


Figure 13: Poles at $\mathrm{arcsinh}\left[\frac{1}{\pi}\left\{\left(\frac{\pi}{2}+n\pi\right)\pm \mathrm{i}\right\}\right]$

4.2 An elementary function

Our second example is an elementary function

$$f(x) = \frac{2(1-x^2)}{\cos(4\,\mathrm{arctanh}\,x)+\cosh 2}, \tag{4.2}$$

which is tough to the DE formula in the same way as Jacobi's elliptic function. For the integration of f on the interval $(-1,1)$ we employ the DE formula in (1.6).

This integrand f does not meet the assumptions of Theorem 3.1 for the same reason as Jacobi's elliptic function in the previous subsection. In fact, the denominator of (4.2) is equal to

$$2\cos(s+\mathrm{i})\cos(s-\mathrm{i}) \qquad \text{with } s = 2\,\mathrm{arctanh}\,x,$$

and therefore $f(\psi_{\mathrm{DE1}}(\cdot))\psi'_{\mathrm{DE1}}(\cdot)$ has poles (see Figure 13) at

$$\mathrm{arcsinh}\left[\frac{1}{\pi}\left\{\left(\frac{\pi}{2}+n\pi\right)\pm \mathrm{i}\right\}\right] \quad (n\in\mathbf{Z}).$$

The errors of the DE formula with

$$h = \frac{\log(2^p N)}{N} \quad (p=-3,-2,-1,0,1,2,3)$$

are depicted in Figure 14. Again the error does not decay as $\exp(-cN/\log N)$, but seemingly as $\exp(-c\sqrt{N})$. Note that there is no theoretical recipe for the choice of h.

For comparison, the SE formula (2.5) is applied to f, with the results shown in Figure 15. Theorem 2.1 is applicable to this function and the theory indicates the choice of $h=\sqrt{\pi/N}$. We have also tried with

$$h = \sqrt{\frac{2^p\pi}{N}} \quad (p=-3,-2,-1,0,1,2,3),$$

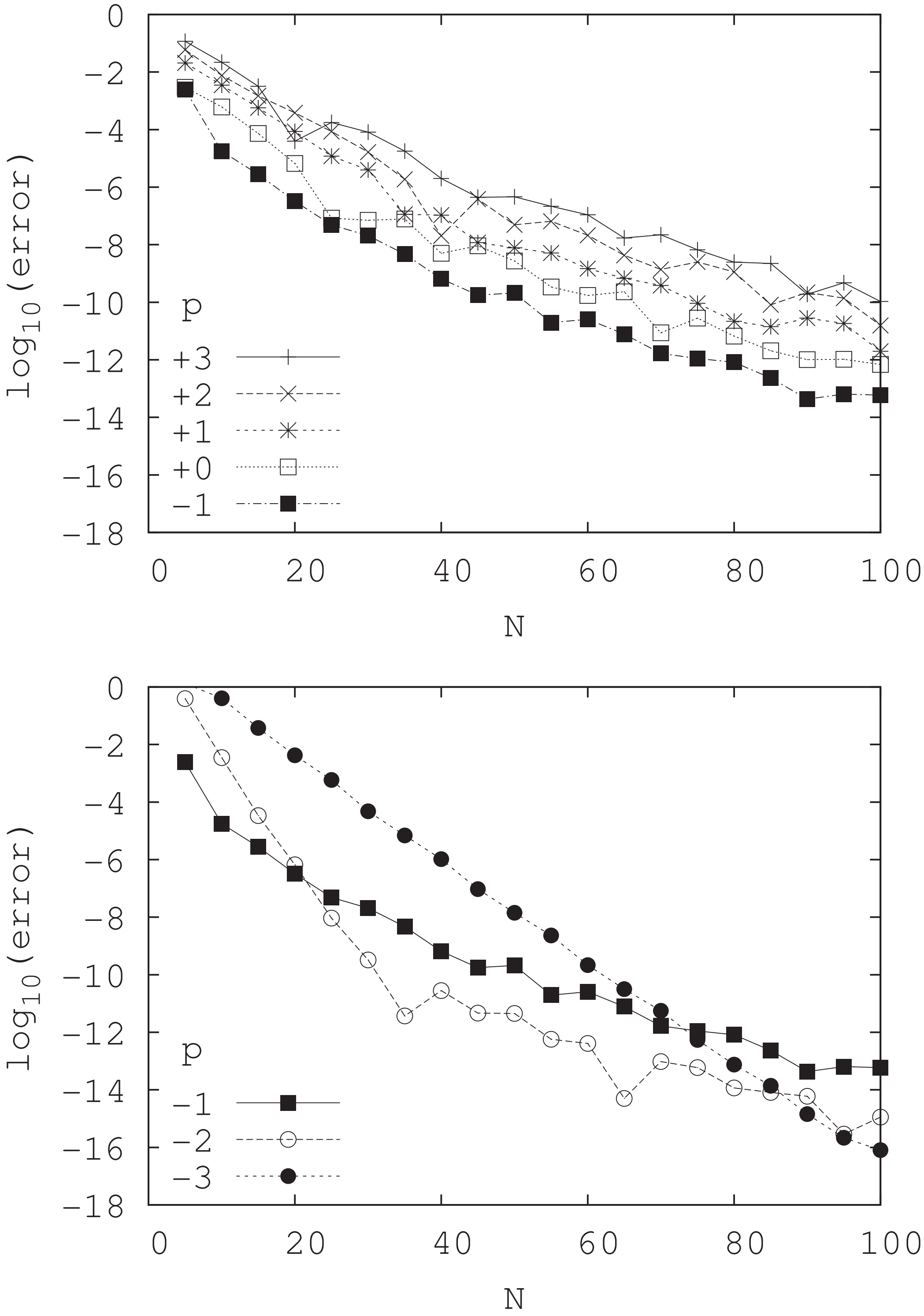


Figure 14: Errors of the DE formula for $f(x) = \dfrac{2(1-x^2)}{\cos(4\,\mathrm{arctanh}\,x) + \cosh 2}$

where $p = 0$ corresponds to the theoretical value. The computational results show that the theoretical choice of h gives the highest accuracy.

Finally we mention that in computing the errors we used

$$\int_{-1}^{1} f(x)\,\mathrm{d}x = 0.71194382297059827888000 4050315,$$

which was computed as described in the previous subsection.

5 Proofs

In this section, we prove Theorems 3.1 to 3.5 in turn. The proofs are based on a variant (Theorem 5.2) of a well-known error estimate (Theorem 5.1) of the trapezoidal formula on $(-\infty, \infty)$.

5.1 Trapezoidal Formula on $(-\infty, \infty)$

Error estimates of the trapezoidal formula on $(-\infty, \infty)$ are shown in this subsection. The following theorem is known to be fundamental.

Theorem 5.1 ([11, Theorem 3.2]). For $d > 0$, let f be a function holomorphic on $\mathcal{D}_d$ such that

$$\mathcal{N}(f, d) \equiv \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} (|f(x + \mathrm{i}\,(d - \varepsilon))| + |f(x - \mathrm{i}\,(d - \varepsilon))|)\,\mathrm{d}x < \infty, \tag{5.1}$$

$$\lim_{x \to \pm\infty} \int_{-(d-\varepsilon)}^{d-\varepsilon} |f(x + \mathrm{i}\,y)|\,\mathrm{d}y = 0 \tag{5.2}$$

for arbitrary ε with $0 < \varepsilon < d$, and

$$\forall x \in \mathbf{R}: \; |f(x)| \le A\,\exp(-B\,\exp(\gamma|x|)) \tag{5.3}$$

for constants $A, B > 0$ and $\gamma > 0$ with $\gamma d \le \pi/2$. Then there exists a constant C, independent of N, such that

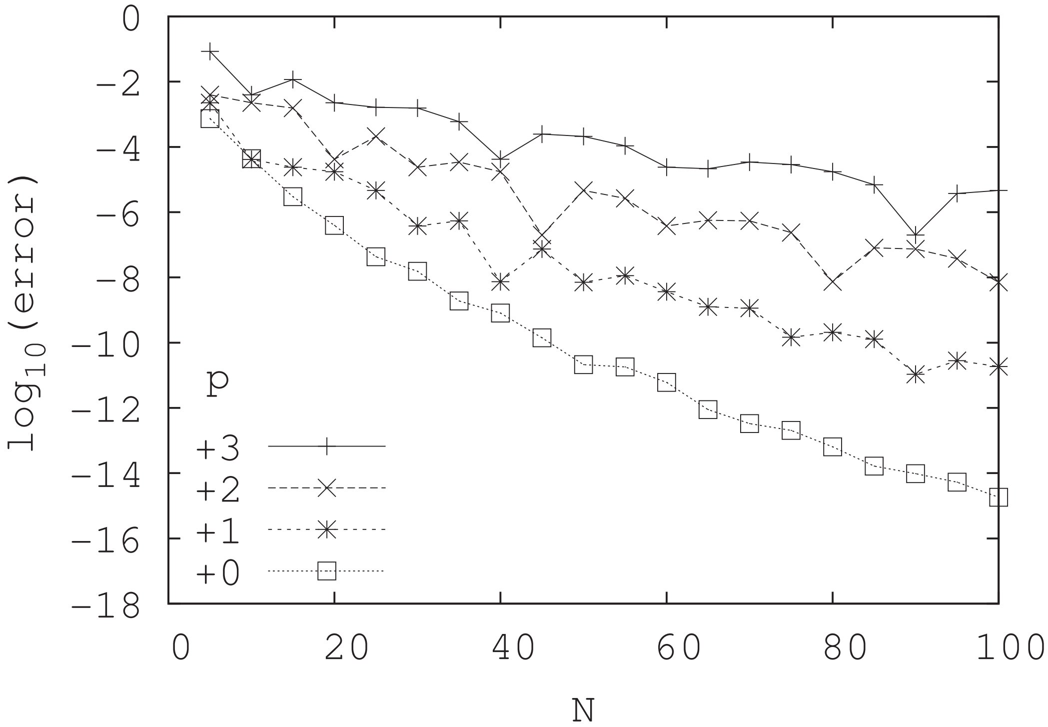
$$\left| \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - h \sum_{k=-N}^{N} f(kh) \right| \le C \exp\left(-\frac{2\pi d\gamma N}{\log(2\pi d\gamma N/B)} \right), \tag{5.4}$$

where

$$h = \frac{\log(2\pi d\gamma N/B)}{\gamma N}. \tag{5.5}$$

Proof. A sketch of the proof is given here in view of its fundamental role in subsequent arguments. We divide the error into two parts as

$$\left| \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - h \sum_{k=-N}^{N} f(kh) \right| \le \left| \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - h \sum_{k=-\infty}^{\infty} f(kh) \right| + \left| h \sum_{|k|>N} f(kh) \right|. \tag{5.6}$$



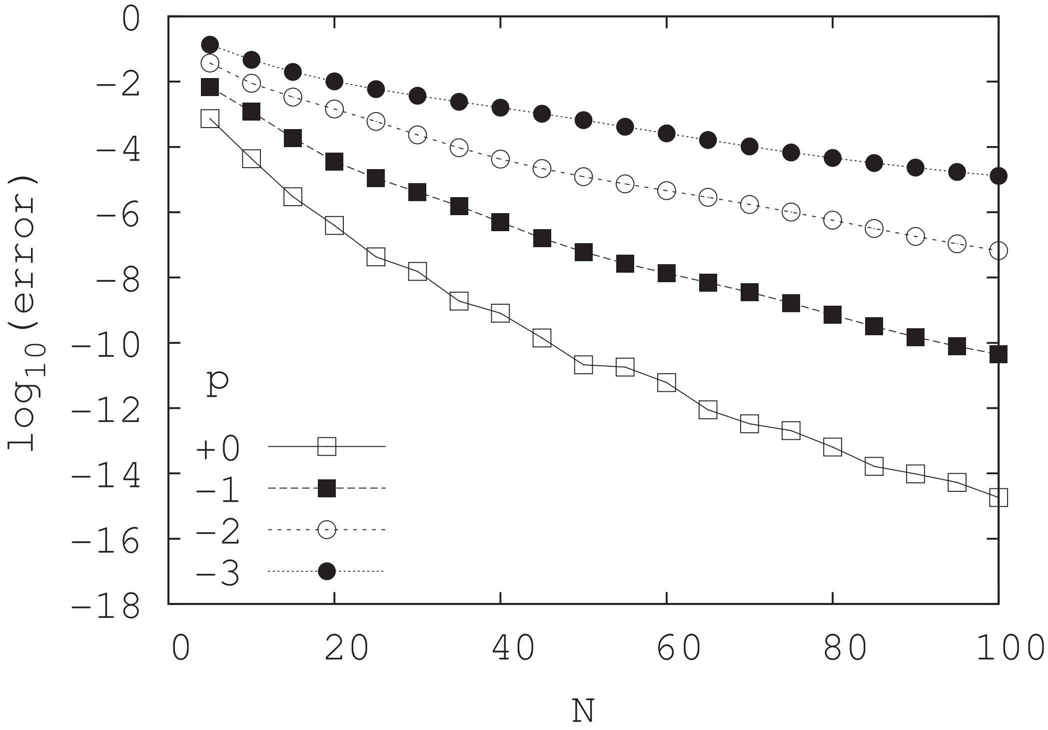


Figure 15: Errors of the SE formula for $f(x) = \dfrac{2(1-x^2)}{\cos(4\,\mathrm{arctanh}\,x) + \cosh 2}$

The first term on the right-hand side is referred to as the discretization error and the second as the truncation error. For the discretization error it follows from (5.1), (5.2), (5.5), and an estimate by a contour integral that

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - h\sum_{k=-\infty}^{\infty} f(kh)\right| &\leq \frac{\exp(-2\pi d/h)}{1-\exp(-2\pi d/h)}\mathcal{N}(f,d)\\ &\leq C_1 \exp\left(-\frac{2\pi d\gamma N}{\log(2\pi d\gamma N/B)}\right)\mathcal{N}(f,d), \end{aligned}\tag{5.7}$$

where $C_1 > 0$ is a constant. For the truncation error we have from (5.3) and (5.5) that

$$\begin{aligned}\left|h\sum_{|k|>N} f(kh)\right| &\leq h\sum_{|k|>N} A\exp(-B\exp(\gamma|kh|))\\ &= 2h\sum_{k=N+1}^{\infty} A\exp(-B\exp(\gamma kh))\\ &\leq 2\int_{Nh}^{\infty} A\exp(-B\exp(\gamma x))\,\mathrm{d}x\\ &\leq 2\int_{Nh}^{\infty} A\frac{\exp(\gamma x)}{\exp(\gamma Nh)}\exp(-B\exp(\gamma x))\,\mathrm{d}x\\ &= \left[-\frac{2A}{B\gamma\exp(\gamma Nh)}\exp(-B\exp(\gamma x))\right]_{x=Nh}^{\infty}\\ &= \frac{2A\exp(-B\exp(\gamma Nh))}{B\gamma\exp(\gamma Nh)}\\ &= \frac{2A\exp(-2\pi d\gamma N)}{2\pi d\gamma^2 N}.\end{aligned}$$

Hence follows the claim. ■

Theorem 5.1 above can be strengthened to the following theorem, in which the assumption (5.8) is weaker than (5.3). This theorem will be used as the key lemma in proving Theorems 3.1 to 3.5.

Theorem 5.2. For $d > 0$, let f be a function holomorphic on $\mathcal{D}_d$ such that (5.1), (5.2), and

$$\forall x \in \mathbf{R}:\ |f(x)| \leq A\ \exp(\gamma|x|)\ \exp(-B\exp(\gamma|x|)) \tag{5.8}$$

for constants $A, B, \gamma > 0$. Then there exists a constant C, independent of N, such that

$$\left|\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - h\sum_{k=-N}^{N} f(kh)\right| \leq C\exp\left(-\frac{2\pi d\gamma N}{\log(2\pi d\gamma N/B)}\right), \tag{5.9}$$

where

$$h = \frac{\log(2\pi d\gamma N/B)}{\gamma N}.$$

Proof. From the proof of Theorem 5.1 we see that the estimate (5.7) is still valid for the discretization error. For the truncation error we have the following:

$$
\begin{aligned}
\left| h \sum_{|k|>N} f(kh) \right| &\leq h \sum_{|k|>N} A \exp(\gamma|kh|) \exp(-B \exp(\gamma|kh|)) \\
&= 2h \sum_{k=N+1}^{\infty} A \exp(\gamma kh) \exp(-B \exp(\gamma kh)) \\
&\leq 2 \int_{Nh}^{\infty} A \exp(\gamma x) \exp(-B \exp(\gamma x)) \, \mathrm{d}x \\
&= \left[-\frac{2A}{B\gamma} \exp(-B \exp(\gamma x)) \right]_{x=Nh}^{\infty} \\
&= \frac{2A}{B\gamma} \exp(-B \exp(\gamma Nh)).
\end{aligned}
$$

Then (5.9) follows from (5.6). ■

The following lemma gives a sufficient condition for f to satisfy the assumptions of Theorem 5.2 in term of a dominating function g of f.

Lemma 5.3. A function f holomorphic on $\mathcal{D}_d$ satisfies (5.1), (5.2), and (5.8) for constants $A, B, \gamma > 0$, if there exists a function g on $\overline{\mathcal{D}_d}$ such that

$$
\forall z \in \mathcal{D}_d : |f(z)| \leq |g(z)|, \tag{5.10}
$$

$$
\forall x \in \mathbf{R},\ \forall y \in \mathbf{R}\,(|y| \leq d) : |g(x + \mathrm{i}\, y)| \leq A' \exp(\gamma'|x|) \exp(-B' \exp(\gamma'|x|)) \tag{5.11}
$$

for some constants $A', B', \gamma' > 0$, and

$$
\forall x \in \mathbf{R} : \ |g(x)| \leq A \exp(\gamma|x|) \exp(-B \exp(\gamma|x|)). \tag{5.12}
$$

Proof. (5.1) and (5.2) follow from (5.11) with (5.10), whereas (5.8) from (5.12) with (5.10). ■

5.2 Proof of Theorem 3.1

In the proofs of Theorem 3.1 to 3.5 in Subsections 5.2 to 5.6, we show that the transformed integrand function $\tilde{f} = f(\psi_{\mathrm{DE}i}(\cdot))\psi'_{\mathrm{DE}i}(\cdot)$ satisfies the assumptions of Theorem 5.2 by demonstrating a dominating function g for $\tilde{f}$ as described in Lemma 5.3.

In this subsection we deal with ψ_{DE1}. The transformed function $\tilde{f}(z) = f(\psi_{\mathrm{DE1}}(z))\psi'_{\mathrm{DE1}}(z)$ is holomorphic on $\mathcal{D}_d$. Since

$$
\forall z \in \mathcal{D}_d : \ |f(\psi_{\mathrm{DE1}}(z))\psi'_{\mathrm{DE1}}(z)| \leq C_1 \left| \frac{\cosh z}{\{\cosh^2((\pi/2)\sinh z)\}^{\beta}} \right|
$$

from (3.1), we can take

$$
g(z) = C_1 \frac{\cosh z}{\{\cosh^2((\pi/2)\sinh z)\}^{\beta}}
$$

to meet the first requirement (5.10) in Lemma 5.3. We can also show that this function $g(z)$ satisfies (5.11) by letting $B = \beta$ in Lemma 5.4 below. For the third condition (5.12), for $x \in \mathbf{R}$ we have

$$|g(x)| \leq A \exp(|x|) \exp\left(-\frac{\pi\beta}{2}\exp(|x|)\right)$$

for a constant $A > 0$. Thus g is valid in Lemma 5.3 and therefore $\tilde{f}$ satisfies the assumptions of Theorem 5.2 for $B = \pi\beta/2$ and $\gamma = 1$. Hence follows the claim of Theorem 3.1.

Lemma 5.4. Let d be a constant with $0 < d < \pi/2$ and $B > 0$ be a positive constant. Then the function

$$g(z) := \frac{\cosh z}{\{\cosh^2((\pi/2)\sinh z)\}^B}$$

satisfies (5.11).

Proof. Let $x, y \in \mathbf{R}$ and $|y| \leq d$. Using that $\delta_d := (1/2)\operatorname{arccosh}(1/\sin d) > 0$ satisfies $\cosh\delta_d \sin d < 1$, we have

$$\begin{aligned}
|\cosh((\pi/2)\sinh(x + \mathrm{i}\,y))|^2 &= \cosh^2((\pi/2)\sinh x\cos y) - \sin^2((\pi/2)\cosh x\sin y) \\
&\geq \cosh^2(((\pi/2)\cos d)\sinh x) - \sin^2((\pi/2)\cosh x\sin y) \\
&\geq \begin{cases} 1 - \sin^2((\pi/2)\cosh\delta_d\sin d) & (|x| \leq \delta_d), \\ \cosh^2(((\pi/2)\cos d)\sinh x) - 1 & (|x| > \delta_d). \end{cases}
\end{aligned}$$

It follows from this fact and

$$|\cosh(x + \mathrm{i}\,y)|^2 = \cosh^2 x - \sin^2 y \leq \cosh^2 x$$

that

$$\left|\frac{\cosh(x + \mathrm{i}\,y)}{\{\cosh^2((\pi/2)\sinh(x + \mathrm{i}\,y))\}^B}\right| \leq \begin{cases} \dfrac{\cosh\delta_d}{\{1 - \sin^2((\pi/2)\cosh\delta_d\sin d)\}^B} & (|x| \leq \delta_d), \\ \dfrac{\cosh x}{\{\cosh^2(((\pi/2)\cos d)\sinh x) - 1\}^B} & (|x| > \delta_d). \end{cases}$$

■

5.3 Proof of Theorem 3.2

First note that the transformed integrand function $f(\psi_{\mathrm{DE2}}(\cdot))\psi'_{\mathrm{DE2}}(\cdot)$ is holomorphic on $\mathcal{D}_d$. It follows from (3.2) that

$$\begin{aligned}
\forall z \in \mathcal{D}_d: \ |f(\psi_{\mathrm{DE2}}(z))\psi'_{\mathrm{DE2}}(z)| &\leq C_1 \left|\frac{1}{\{\cosh^2((\pi/2)\sinh z)\}^{\beta/2+1/2}}\right| \\
&\quad \cdot |\cosh((\pi/2)\sinh z) \cdot (\pi/2)\cosh z| \\
&= C_1 \frac{\pi}{2} \left|\frac{\cosh z}{\{\cosh^2((\pi/2)\sinh z)\}^{\beta/2}}\right|.
\end{aligned}$$

Then the rest of the proof is similar to that of Theorem 3.1. Note that we set $B = \beta/2$ in Lemma 5.4 to show (5.11).

5.4 Proof of Theorem 3.3

First note that the transformed integrand function $f(\psi_{\mathrm{DE3}}(\cdot))\psi'_{\mathrm{DE3}}(\cdot)$ is holomorphic on $\mathcal{D}_d$. It follows from (3.3) that

$$\begin{aligned}\forall z \in \mathcal{D}_d: \ |f(\psi_{\mathrm{DE3}}(z))\psi'_{\mathrm{DE3}}(z)| &\leq C_1 \frac{\pi}{2} \left| \left(\frac{\psi_{\mathrm{DE3}}(z)}{1+\psi_{\mathrm{DE3}}(z)^2} \right)^{\beta} \cosh z \right| \\ &= C_1 \frac{\pi}{2} \left| \left(\frac{1}{\psi_{\mathrm{DE3}}(z)^{-1}+\psi_{\mathrm{DE3}}(z)} \right)^{\beta} \cosh z \right| \\ &= C_1 \frac{\pi}{2^{\beta+1}} \left| \frac{\cosh z}{\{\cosh^2((\pi/2)\sinh z)\}^{\beta/2}} \right|.\end{aligned}$$

Then the rest of the proof is similar to that of Theorem 3.1. Note that we set $B = \beta/2$ in Lemma 5.4 to show (5.11).

5.5 Proof of Theorem 3.4

The transformed integrand function $f(\psi_{\mathrm{DE4}}(z))\psi'_{\mathrm{DE4}}(z)$ is holomorphic on $\mathcal{D}_d$. It follows from (3.4) that

$$\begin{aligned}|f(\psi_{\mathrm{DE4}}(z))\psi'_{\mathrm{DE4}}(z)| \leq C_1 \Bigg| & \left\{ \frac{\exp z}{\exp z + \exp(\exp(-z))} \right\}^{\beta} \cdot \exp(-\beta \exp z \cdot \exp(-\exp(-z))) \\ & \cdot (\exp(-\exp(-z)) + \exp(-z)) \cdot (1+\exp z) \Bigg|\end{aligned}$$

holds for all $z \in \mathcal{D}_d$. Accordingly we choose as $g(z)$ the right-hand side above. Then (5.10) is satisfied. This function $g(z)$ satisfies (5.11) by Lemma 5.5 below with $B = \beta$. As for (5.12), for $x \in \mathbf{R}$ we have

$$|g(x)| \leq A \exp(|x|) \exp(-\beta \exp(|x|))$$

for a constant $A > 0$. Thus g is valid in Lemma 5.3 and therefore $f(\psi_{\mathrm{DE4}}(z))\psi'_{\mathrm{DE4}}(z)$ satisfies the assumptions of Theorem 5.2 for $B = \beta$ and $\gamma = 1$. Hence follows the claim of Theorem 3.4.

Lemma 5.5. Let d be a constant with $0 < d < \pi/2$ and $B > 0$ be a positive constant. Then the function

$$\begin{aligned}g(z) := & \left\{ \frac{\exp z}{\exp z + \exp(\exp(-z))} \right\}^{B} \cdot \exp(-B \exp z \cdot \exp(-\exp(-z))) \\ & \cdot (\exp(-\exp(-z)) + \exp(-z)) \cdot (1+\exp z)\end{aligned}$$

satisfies (5.11).

Proof. Let $x, y \in \mathbf{R}$ and $|y| \leq d$. Let $x, y \in \mathbf{R}$ and $|y| \leq d$, and put

$$\begin{aligned}g_1(z) &= \frac{\exp z}{\exp z + \exp(\exp(-z))}, \\ g_2(z) &= \exp(-B \exp z \cdot \exp(-\exp(-z))), \\ g_3(z) &= (\exp(-\exp(-z)) + \exp(-z)) \cdot (1+\exp z)\end{aligned}$$

to obtain $g(z) = g_1(z)^B g_2(z) g_3(z)$. We note

$$\begin{aligned}
|g_1(x+\mathrm{i}\,y)| &= \left|\frac{1}{1+\exp(\exp(-x-\mathrm{i}\,y)-x-\mathrm{i}\,y)}\right| \\
&\le \frac{1}{|1-|\exp(\exp(-x-\mathrm{i}\,y)-x-\mathrm{i}\,y)||} \\
&= \frac{1}{|1-\exp(\mathrm{e}^{-x}\cos y - x)|}, \qquad (5.13)\\
|g_2(x+\mathrm{i}\,y)| &= |\exp(-B\exp(x+\mathrm{i}\,y)\cdot\exp(-\exp(-x-\mathrm{i}\,y)))| \\
&= \exp\{-B\cdot\mathrm{Re}\,(\exp(x+\mathrm{i}\,y)\cdot\exp(-\exp(-x-\mathrm{i}\,y)))\} \\
&= \exp\{-B\cdot\exp(x-\mathrm{e}^{-x}\cos y)\cdot\cos(y+\mathrm{e}^{-x}\sin y)\}, \qquad (5.14)\\
|g_3(x+\mathrm{i}\,y)| &= |(\exp(-\exp(-x-\mathrm{i}\,y))+\exp(-x-\mathrm{i}\,y))\cdot(1+\exp(x+\mathrm{i}\,y))| \\
&\le (|\exp(-\exp(-x-\mathrm{i}\,y))|+|\exp(-x-\mathrm{i}\,y)|)\cdot(1+|\exp(x+\mathrm{i}\,y)|) \\
&= (\exp(-\mathrm{e}^{-x}\cos y)+\mathrm{e}^{-x})\cdot(1+\exp x). \qquad (5.15)
\end{aligned}$$

It turns out to be convenient to choose a (sufficiently large) positive number $\hat{x}$ such that

$$\alpha := \hat{x} - \mathrm{e}^{-\hat{x}} > 0, \qquad \delta := \pi/2 - d - \mathrm{e}^{-\hat{x}}\sin d > 0, \tag{5.16}$$

and estimate $|g(z)|$ by dividing into three cases: (i) $x < 0$, (ii) $0 \le x \le \hat{x}$, and (iii) $x > \hat{x}$.

Case (i) with $x < 0$: By (5.13) we have

$$|g_1(x+\mathrm{i}\,y)| \le \frac{1}{\exp(\mathrm{e}^{-x}\cos y - x) - 1} \le \frac{1}{\exp(\mathrm{e}^{-x}\cos d) - 1}.$$

In (5.14) we have

$$\exp(x-\mathrm{e}^{-x}\cos y)\cdot\cos(y+\mathrm{e}^{-x}\sin y) \ge -1$$

and therefore

$$|g_2(x+\mathrm{i}\,y)| \le \exp B.$$

By (5.15) we have

$$|g_3(x+\mathrm{i}\,y)| \le 4\,\mathrm{e}^{-x}.$$

Combining the above three inequalities we obtain

$$|g(x+\mathrm{i}\,y)| = |g_1(x+\mathrm{i}\,y)|^B\cdot|g_2(x+\mathrm{i}\,y)|\cdot|g_3(x+\mathrm{i}\,y)| \le \frac{4\exp(B-x)}{(\exp(\mathrm{e}^{-x}\cos d)-1)^B},$$

from which follows the inequality in (5.11).

Case (ii) with $0 \le x \le \hat{x}$: We can regard $|g(x+\mathrm{i}\,y)|$ as a continuous function on a bounded and closed region with $0 \le x \le \hat{x}$ and $|y| \le d$. Then there exists a constant C such that

$$|g(x+\mathrm{i}\,y)| \le C,$$

which implies the inequality in (5.11).

Case (iii) with $x > \hat{x}$: Recall the definitions of α and δ in (5.16). By (5.13) we have

$$|g_1(x + \mathrm{i}\, y)| \le \frac{1}{1 - \exp(\mathrm{e}^{-x} - x)} \le \frac{1}{1 - \exp(\mathrm{e}^{-\hat{x}} - \hat{x})} = \frac{1}{1 - \exp(-\alpha)}.$$

In (5.14) we have

$$\begin{aligned}
&\exp(x - \mathrm{e}^{-x} \cos y) \ge \exp(x - \mathrm{e}^{-x}) \ge \exp(x - 1), \\
&\cos(y + \mathrm{e}^{-x} \sin y) \ge \cos(d + \mathrm{e}^{-x} \sin d) \ge \cos(d + \mathrm{e}^{-\hat{x}} \sin d) = \cos(\pi/2 - \delta),
\end{aligned}$$

and therefore,

$$|g_2(x + \mathrm{i}\, y)| \le \exp\left(-\frac{B\, \cos(\pi/2 - \delta)}{\mathrm{e}} \exp x \right).$$

By (5.15) we have

$$|g_3(x + \mathrm{i}\, y)| \le 4 \exp x.$$

It follows from the above three inequalities that

$$|g(x + \mathrm{i}\, y)| \le \frac{4 \exp x}{(1 - \exp(-\alpha))^B} \cdot \exp\left(-\frac{B\, \cos(\pi/2 - \delta)}{\mathrm{e}} \exp x \right),$$

which implies the inequality in (5.11). ■

5.6 Proof of Theorem 3.5

First, the transformed function $f(\psi_{\mathrm{DE5}}(z))\psi'_{\mathrm{DE5}}(z)$ is holomorphic on $\mathcal{D}_d$. It follows from (3.5) that

$$\begin{aligned}
&|f(\psi_{\mathrm{DE5}}(z))\psi'_{\mathrm{DE5}}(z)| \\
&\le C_1 \left| \left(\frac{\log(\exp((\pi/2)\sinh z) + 1)}{1 + \log(\exp((\pi/2)\sinh z) + 1)} \right)^{\beta - 1} \cdot \exp(-\beta \log(\exp((\pi/2)\sinh z) + 1)) \right. \\
&\qquad \left. \cdot \frac{(\pi/2)\cosh z \cdot \exp((\pi/2)\sinh z)}{\exp((\pi/2)\sinh z) + 1} \right|
\end{aligned}$$

holds for all $z \in \mathcal{D}_d$. Accordingly we choose as $g(z)$ the right-hand side above. Then (5.10) is satisfied. This function $g(z)$ satisfies (5.11) by Lemma 5.6 below with $B = \beta$. As for (5.12), on the other hand, it can be shown (cf. (5.19), (5.23) below) that

$$|g(x)| \le A\, \exp(|x|)\, \exp\left(-\frac{\pi\beta}{4} \exp(|x|) \right)$$

holds for all $x \in \mathbf{R}$ with a constant $A > 0$. Thus g is valid in Lemma 5.3 and therefore $f(\psi_{\mathrm{DE5}}(z))\psi'_{\mathrm{DE5}}(z)$ satisfies the assumptions of Theorem 5.2 for $B = \pi\beta/4$ and $\gamma = 1$. Thus we have proven Theorem 3.5.

Lemma 5.6. Let d be a constant with $0 < d < \pi/2$ and $B > 0$ be a positive constant. Then the function

$$g(z) := \left(\frac{\log(\exp((\pi/2)\sinh z) + 1)}{1 + \log(\exp((\pi/2)\sinh z) + 1)}\right)^{B-1} \cdot \exp(-B\log(\exp((\pi/2)\sinh z) + 1)) \cdot \frac{(\pi/2)\cosh z \cdot \exp((\pi/2)\sinh z)}{\exp((\pi/2)\sinh z) + 1}$$

satisfies (5.11).

Proof. Let $x, y \in \mathbf{R}$ and $|y| \leq d$, and set $z = x + \mathrm{i}\,y$. We consider asymptotic estimates as $x \to \infty$ or $x \to -\infty$.

First we consider the case of $x \to \infty$. Since $|\log(\exp((\pi/2)\sinh z) + 1)| \to \infty$, we have

$$\left|\left(\frac{\log(\exp((\pi/2)\sinh z) + 1)}{1 + \log(\exp((\pi/2)\sinh z) + 1)}\right)^{B-1}\right| \leq C_1 \tag{5.17}$$

for a constant $C_1 > 0$. Furthermore, in a similar manner, we have

$$\left|\frac{\exp((\pi/2)\sinh z)}{\exp((\pi/2)\sinh z) + 1}\right| \leq C_2 \tag{5.18}$$

for another constant $C_2 > 0$. The remaining part of g is expressed as

$$(\pi/2)\cosh z \cdot \exp(-B\log(\exp((\pi/2)\sinh z) + 1)) = \frac{(\pi/2)\cosh z}{(\exp((\pi/2)\sinh z) + 1)^B}. \tag{5.19}$$

By (5.17), (5.18), and (5.19) we obtain the desired estimate (5.11).

Next, we consider the case where $x \to -\infty$. In this case we have

$$\frac{1}{|\exp((\pi/2)\sinh z) + 1|} \leq C_3 \tag{5.20}$$

for a constant $C_3 > 0$, and

$$|\exp(-B\log(\exp((\pi/2)\sinh z) + 1))| = \frac{1}{|(\exp((\pi/2)\sinh z) + 1)^B|} \leq C_4 \tag{5.21}$$

for another constant $C_4 > 0$. The remaining part of g is

$$\left(\frac{\log(\exp((\pi/2)\sinh z) + 1)}{1 + \log(\exp((\pi/2)\sinh z) + 1)}\right)^{B-1} \cdot \exp((\pi/2)\sinh z) \cdot (\pi/2)\cosh z. \tag{5.22}$$

In general, for $\zeta \in \mathbf{C}$ with $|\zeta| < 1/2$, we have

$$|\log(\zeta + 1)| = \left|\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\zeta^n}{n}\right| \leq |\zeta|\sum_{n=1}^{\infty}|\zeta|^{n-1} = \frac{|\zeta|}{1 - |\zeta|} < 1$$

and therefore

$$\left|\frac{\log(\zeta + 1)}{1 + \log(\zeta + 1)}\right| \leq \frac{|\log(\zeta + 1)|}{1 - |\log(\zeta + 1)|} \leq \frac{|\zeta|/(1 - |\zeta|)}{1 - |\zeta|/(1 - |\zeta|)} = \frac{|\zeta|}{1 - 2|\zeta|}.$$

Here $(1-2|\zeta|)^{-1}$ is bounded for ζ with sufficiently small absolute values. and therefore the absolute value of (5.22) is bounded by

$$\begin{aligned} &C_5|\exp((\pi/2)\sinh z)|^{B-1}\cdot|\exp((\pi/2)\sinh z)|\cdot|(\pi/2)\cosh z| \\ &= C_5|\exp((\pi/2)\sinh z)|^{B}\cdot|(\pi/2)\cosh z| \end{aligned} \tag{5.23}$$

for a constant $C_5 > 0$. Taking this estimate, (5.20), and (5.21) into consideration, we obtain the desired estimate. ■

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