MATHEMATICAL ENGINEERING TECHNICAL REPORTS

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METR 2007–17 March 2007

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Abstract

We derive some results on contrarian and one-sided strategies by Skeptic for the fair-coin game in the framework of the game-theoretic probability of Shafer and Vovk [8]. In particular, concerning the rate of convergence of the strong law of large numbers (SLLN), we prove that Skeptic can force that the convergence has to be slower than or equal to $O(n^{-1/2})$. This is achieved by a very simple contrarian strategy of Skeptic. This type of result, bounding the rate of convergence from below, contrasts with more standard results of bounding the rate of SLLN from above by using momentum strategies. We also derive a corresponding one-sided result.

1 Introduction

In the theory of game-theoretic probability by Shafer and Vovk [8], various "probability laws" such as SLLN are proved by constructing explicit strategies of Skeptic, who is one of the two players in a game. Construction of a clever and explicit strategy of Skeptic often leads to a remarkably simple proof of the corresponding result in the measure theoretic probability theory, even without preparations from measure theory. This is already apparent in the simple strategy used in Chapter 3 of Shafer and Vovk [8], where Skeptic always bets a fixed proportion of his capital. See also the Bayesian strategies of Skeptic in coin-tossing games in [3]. New problems and their solutions offered by the framework of the game-theoretic probability are now actively investigated in various directions. For background material and further developments of the game-theoretic probability see Vovk and Shafer [10] and Shafer and Vovk [9] and references therein. See Takeuchi [12] for some original ideas and results. Defensive forecasting, which is a new non-parametric forecasting method based on the game-theoretic probability, was initiated in [14] and [15].

We can roughly classify strategies of Skeptic into two classes, namely the class of momentum strategies and the class of contrarian strategies. This distinction was clearly demonstrated in a talk by Glenn Shafer [7]. See also [13]. Consider again the convergence in SLLN. In momentum strategies Skeptic assumes that Reality, the other player of the game, will keep deviating from the (zero) theoretical mean in the same direction and bets accordingly. In contrast, in contrarian strategies Skeptic assumes that Reality tries to decrease deviation from the mean and bets accordingly. In both of these strategies, Skeptic looks only at the absolute deviation and in this sense these strategies are two-sided strategies. A more primitive strategy of Skeptic is one-sided and bets only toward a particular direction (up or down). In Shafer and Vovk [8] one-sided strategies are treated as restrictions on the move space of Skeptic, namely Skeptic is only allowed to buy a certain "ticket". However as we discuss in Section 3, Skeptic can use one-sided strategies to force stronger unbiasedness to Reality than implied by two-sided strategies.

It is only natural to expect that stronger results require more complicated strategies by Skeptic. For example in [2] we have shown that a simple strategy of Skeptic based on the past average of the moves by Reality forces SLLN for the case of bounded Reality's moves. However if Reality's moves are unbounded, strategies for forcing SLLN are much more complicated as discussed in [4].

As another example, we mention that the proof of the law of the iterated logarithm (LIL) in Chapter 5 of Shaver and Vovk [8] is much harder than the proof of SLLN or the central limit theorem. There are two parts in the proof of LIL. In the first part the growth rate of the capital process is bounded from above by a momentum strategy and in the second part it is bounded from below by a contrarian strategy. As seen from the proof, the latter part of the proof is much more difficult. This suggests that construction of effective contrarian strategies of Skeptic is more challenging than construction of momentum strategies. In this paper we only consider the fair-coin game, which is the simplest game in the game-theoretic probability. However we believe that the contrarian strategies obtained in this paper can be generalized further and give insights on other strategies in various more general games in the game-theoretic probability. We should also mention that our results on the rate of convergence of SLLN are already implied by LIL. Therefore the merit of this paper is to clarify implications of very simple explicit contrarian strategies.

Organization of the paper is as follows. For the rest of this section, we briefly introduce necessary notations and definitions from [8]. In Section 2 we consider contrarian strategies. It is divided into two subsections. In subsection 2.1 multiplicative contrarian strategies based on the past average of Reality's moves are studied and in subsection 2.2 additive contrarian strategies based on the past sum of Reality's moves are studied. In Section 3 we study one-sided strategies to strengthen results obtained in Section 2. We end the paper with some discussions in Section 4.

1.1 Notations and definitions

Here we summarize necessary notations and definitions from [8] for our paper. We also give a formal definition of stopping times.

In this paper we consider the fair-coin game. Its protocol is given as follows.

FAIR-COIN GAME
$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \dots$:
  Skeptic announces $M_n \in \mathbb{R}$.
  Reality announces $x_n \in \{-1, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.
END FOR

A finite sequence $t = x_1 x_2 \cdots x_n$ consisting of 1 and -1 is called a situation and the set of all situations is denoted by $\Omega^\diamond$. The length of $t = x_1 x_2 \cdots x_n$ is n. The initial situation, which is the special situation of the length 0, is denoted by $\square \in \Omega^\diamond$. If a situation t is an initial segment of another situation t', we say that t precedes t' and t' follows t. For a situation $t = x_1 x_2 \cdots x_n$, we define $-t := (-x_1)(-x_2) \cdots (-x_n)$. If we consider the infinite binary tree describing the progress of the fair-coin game, a situation can be identified with a node in the tree.

An infinite sequence $\xi = x_1 x_2 \dots$ consisting of 1 and -1 is called a path and the set of all paths is denoted by Ω. An event E is a subset of Ω. For a path ξ, the situation consisting of the first n terms of ξ is written as ξ_n. Given a path ξ and a situation t, if there exists n such that $\xi_n = t$, we say that ξ goes through t. Given a situation t, we define the cylinder set $O_t \subset \Omega$ by

$$O_t := \{\xi \mid \xi \text{ goes through } t\}.$$

A process is a function $\Omega^\diamond \to \mathbb{R}$ and a variable is a function $\Omega \to \mathbb{R}$. Given a process f, a variable f_n is defined by

$$f_n(\xi) := f(\xi_n). \tag{1}$$

In this paper the symbols s and $\overline{x}$ are used for two special processes, the sum and the average. They are defined by

$$\begin{aligned} s(x_1 x_2 \cdots x_n) &:= x_1 + x_2 + \cdots + x_n, \\ \overline{x}(x_1 x_2 \cdots x_n) &:= \frac{x_1 + x_2 + \cdots + x_n}{n}, \end{aligned}$$

where $s(\square) = \overline{x}(\square) = 0$. By (1), we also write $s(x_1 x_2 \cdots x_n) = s_n(\xi)$, $\overline{x}(x_1 x_2 \cdots x_n) = \overline{x}_n(\xi)$.

The strategy $\mathcal{P}$ of Skeptic is a process. When $\mathcal{P}$ is a strategy of Skeptic, the capital process of $\mathcal{P}$ (with zero initial capital) is denoted by $\mathcal{K}^{\mathcal{P}}$ and defined by

$$\mathcal{K}^{\mathcal{P}}(x_1 x_2 \cdots x_n) := \mathcal{P}(\square) x_1 + \mathcal{P}(x_1) x_2 + \cdots + \mathcal{P}(x_1 x_2 \cdots x_{n-1}) x_n$$

and $\mathcal{K}^{\mathcal{P}}(\Box) := 0$. The capital process of $\mathcal{P}$ is the total amount Skeptic earns when he follows the strategy $\mathcal{P}$. If Skeptic uses the strategy $\mathcal{P}$ with the initial capital a and Reality chooses the path ξ, then the capital Skeptic holds at the end of the n-th round is $a + \mathcal{K}_n^{\mathcal{P}}(\xi)$. When ξ is fixed, we write simply $s_n, \overline{x}_n$ or $\mathcal{K}_n^{\mathcal{P}}$.

We say that Skeptic can weakly force an event $E \subset \Omega$, if there exists a strategy $\mathcal{P}$ of Skeptic such that

$$\limsup_n \mathcal{K}_n^{\mathcal{P}}(\xi) = \infty, \qquad \forall \xi \notin E, \tag{2}$$

under the restriction of the "collateral duty"

$$\mathcal{K}_n^{\mathcal{P}}(\xi) \geq -1, \quad \forall \xi \in \Omega, \forall n \geq 0.$$

We say that Skeptic can force E if $\limsup_n$ in (2) is replaced by $\lim_n$. By Lemma 3.1 of [8], if Skeptic can weakly force E, then he can force E.

The upper price $\overline{\mathrm{E}}[x]$ of a variable x is defined by

$$\overline{\mathrm{E}}[x] := \inf\{a \mid \exists \mathcal{P} \ \forall \xi \in \Omega : a + \mathcal{K}_n^{\mathcal{P}}(\xi) \geq x(\xi) \text{ a.a.}\}, \tag{3}$$

where a.a. (almost always) means "except for a finite number of n". We can regard a variable x as a ticket whose holder earns $x(\xi)$ if Reality chooses ξ and $\overline{\mathrm{E}}[x]$ is the infimum of the initial capital with which Skeptic can superreplicate x. When $a \geq \overline{\mathrm{E}}[x]$, we say that Skeptic can buy x for a. Given a situation t, the upper price of x on the situation t is also defined and denoted by $\overline{\mathrm{E}}_t[x]$ (Chapter 1 of [8]).

Finally we give a formal definition of a stopping time. A stopping time is a variable $f\colon \Omega \to \{1, 2, \dots\} \cup \{\infty\}$, which satisfies

$$\forall \xi, \xi' \in \Omega, \forall n \in \mathbb{N} : (f(\xi) = n \ \& \ \xi_n = \xi'_n) \Rightarrow f(\xi') = n,$$

where $\mathbb{N} = \{1, 2, \dots\}$. For a stopping time f and a path ξ, if $f(\xi) < \infty$, $\xi_{f(\xi)}$ is the situation where the value of $f(\xi)$ is determined. We let $[f]$ denote the set of situations where the value of f is determined:

$$[f] := \{t \in \Omega^{\diamond} \mid \exists \xi : \xi_{f(\xi)} = t\}. \tag{4}$$

A stopping time f can be identified with $[f]$.

2 Contrarian strategies

In this section we consider forcing the following events:

$$E_1 := \{\xi \mid \limsup_{n\to\infty} \sqrt{n}|\overline{x}_n| \geq 1\}, \tag{5}$$

$$E_2 := \{\xi \mid |s_n| > \sqrt{n} - 1 \text{ i.o.}\}, \tag{6}$$

where i.o. (infinitely often) means "for infinitely many n". In Subsection 2.1 we prove that Skeptic can force E_1 by a mixture of multiplicative contrarian strategies based on the past average of Reality's moves and in Subsection 2.2 we prove that Skeptic can force E_2 by a mixture of additive contrarian strategies based on the past sum of Reality's moves. Since $E_2 \subset E_1$, forcing E_2 is stronger than forcing E_1. However the multiplicative strategy in Subsection 2.1 is of interest, because it is a contrarian counterpart of the momentum strategy studied in [2].

2.1 Multiplicative contrarian strategy

In this section we study the following multiplicative contrarian strategy $\mathcal{P}_c$:

$$\mathcal{P}_c : \ M_n = -c\overline{x}_{n-1}\mathcal{K}_{n-1},$$

where c is an arbitrary constant satisfying $0 < c \le \frac{1}{2}$ and the initial capital is $\mathcal{K}_0 = a = 1$. The case of $-\frac{1}{2} \le c < 0$ was studied in [2]. Let

$$\mathcal{Q} = \sum_{i \in \mathbb{N}} \frac{1}{2^i}\mathcal{P}_{1/2^i}$$

denote an infinite mixture of the strategies $\mathcal{P}_c$, $c = 1/2^i$, $i = 1, 2, \ldots$. Then the following result holds.

Theorem 1 *Skeptic can force E_1 by $\mathcal{Q}$.*

The rest of this subsection is devoted to a proof of Theorem 1. Let

$$E_1^c := \left\{ \xi \ \middle| \ \limsup_{n\to\infty} (1+2c)\sqrt{n}|\overline{x}_n| \ge 1 \right\}.$$

Then E_1 can be represented as

$$E_1 = \bigcap_{i \in \mathbb{N}} E_1^{1/2^i}.$$

Thus, by Lemma 3.2 of [8], Theorem 1 is a consequence of the following lemma.

Lemma 1 *Skeptic can force E_1^c with $\mathcal{P}_c$.*

In order to prove the lemma, we need to show that for any $\xi \in \Omega$

- Skeptic's capital $\mathcal{K}_n = 1 + \mathcal{K}_n^{\mathcal{P}_c}$ never gets negative, and
- $\limsup_{n\to\infty}(1+2c)\sqrt{n}|\overline{x}_n| < 1 \ \Rightarrow \ \mathcal{K}_n \to \infty \quad (n \to \infty)$.

By definition for any ξ,

$$\begin{aligned} \mathcal{K}_n &= \mathcal{K}_{n-1} - c\overline{x}_{n-1}\mathcal{K}_{n-1}x_n \\ &= \mathcal{K}_{n-1}(1 - c\overline{x}_{n-1}x_n) \\ &= \prod_{i=2}^{n}(1 - c\overline{x}_{i-1}x_i). \end{aligned} \tag{7}$$

In the expression (7) the index i starts from 2 because $\overline{x}_0 = 0$. Also for any $i = 2, 3, \ldots, n$, $1 - c\overline{x}_{i-1}x_i > 0$. Thus the first statement on the collateral duty is trivial.

We divide the proof of the second statement into four parts.

Step 1 In step 1 and step 2, we fix an arbitrary path $\xi = x_1 x_2 \cdots \in \Omega$. Since $\log(1+t) \geq t - t^2$ whenever $|t| \leq \frac{1}{2}$, from (7)

$$\begin{aligned}\log \mathcal{K}_n &= \sum_{i=2}^{n} \log(1 - c\overline{x}_{i-1} x_i) \\ &\geq -c \sum_{i=2}^{n} \overline{x}_{i-1} x_i - c^2 \sum_{i=2}^{n} \overline{x}_{i-1}^2 x_i^2. \end{aligned} \tag{8}$$

We use the identity

$$\sum_{i=2}^{n} \overline{x}_{i-1} x_i = \frac{1}{2} \sum_{i=2}^{n} \frac{i}{i-1} \overline{x}_i^2 + \frac{n}{2} \overline{x}_n^2 - \frac{1}{2} \left(x_1^2 + \sum_{i=2}^{n} \frac{1}{i-1} x_i^2 \right) \tag{9}$$

which is shown in [2] and easily follows from

$$s_{n-1} x_n = \frac{1}{2} (s_n^2 - s_{n-1}^2 - x_n^2).$$

Substituting (9) into (8), we have

$$\begin{aligned}\log \mathcal{K}_n &\geq -c \left\{ \frac{1}{2} \sum_{i=2}^{n} \frac{i}{i-1} \overline{x}_i^2 + \frac{n}{2} \overline{x}_n^2 - \frac{1}{2} \left(x_1^2 + \sum_{i=2}^{n} \frac{1}{i-1} x_i^2 \right) \right\} - c^2 \sum_{i=2}^{n} \overline{x}_{i-1}^2 x_i^2 \\ &= -\frac{c}{2} \sum_{i=2}^{n} \frac{i}{i-1} \overline{x}_i^2 - \frac{nc}{2} \overline{x}_n^2 + \frac{c}{2} \left(x_1^2 + \sum_{i=2}^{n} \frac{1}{i-1} x_i^2 \right) - c^2 \sum_{i=2}^{n} \overline{x}_{i-1}^2 x_i^2. \end{aligned} \tag{10}$$

Now, $x_i^2 = 1$ because $x_i \in \{-1, 1\}$ and

$$1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \geq \log n = \int_1^n \frac{1}{x} \mathrm{d}x.$$

Thus we have

$$\begin{aligned}(10) &= -\frac{c}{2} \sum_{i=2}^{n} \frac{i}{i-1} \overline{x}_i^2 - \frac{nc}{2} \overline{x}_n^2 + \frac{c}{2} \left(1 + \sum_{i=2}^{n} \frac{1}{i-1} \right) - c^2 \sum_{i=2}^{n} \overline{x}_{i-1}^2 \\ &\geq -\frac{c}{2} \sum_{i=2}^{n} \frac{i}{i-1} \overline{x}_i^2 - \frac{nc}{2} \overline{x}_n^2 + \frac{c}{2} (1 + \log n) - c^2 \sum_{i=2}^{n} \overline{x}_{i-1}^2 \\ &= \frac{c}{2} (1 + \log n) - \left(\frac{c}{2} \sum_{i=2}^{n} \frac{i}{i-1} \overline{x}_i^2 + c^2 \sum_{i=2}^{n} \overline{x}_{i-1}^2 + \frac{nc}{2} \overline{x}_n^2 \right) \\ &= \frac{c}{2} \left\{ 1 + \log n - \left(\sum_{i=2}^{n} \frac{i}{i-1} \overline{x}_i^2 + 2c \sum_{i=2}^{n} \overline{x}_{i-1}^2 + n \overline{x}_n^2 \right) \right\} \\ &= \frac{c}{2} \left\{ 1 + \log n \left(1 - \frac{\sum_{i=2}^{n} \frac{i}{i-1} \overline{x}_i^2 + 2c \sum_{i=2}^{n} \overline{x}_{i-1}^2 + n \overline{x}_n^2}{\log n} \right) \right\}. \end{aligned} \tag{11}$$

By (11) we have shown that

$$\xi \in F_c \ \Rightarrow\ \lim_{n\to\infty} \mathcal{K}_n = \infty.$$

where

$$F_c = \left\{ \xi \ \middle|\ \limsup_{n\to\infty} \frac{\sum_{i=2}^n \frac{i}{i-1}\overline{x}_i^2 + 2c\sum_{i=2}^n \overline{x}_{i-1}^2 + n\overline{x}_n^2}{\log n} < 1 \right\}. \tag{12}$$

Step 2 Rewriting the numerator in (12), we have

$$\begin{aligned}
&\sum_{i=2}^n \frac{i}{i-1}\overline{x}_i^2 + 2c\sum_{i=2}^n \overline{x}_{i-1}^2 + n\overline{x}_n^2 \\
&\quad = 2c\overline{x}_1^2 + \sum_{i=2}^{n-1}\left(\frac{i}{i-1} + 2c\right)\overline{x}_i^2 + \frac{n}{n-1}\overline{x}_n^2 + n\overline{x}_n^2 \\
&\quad = 2c + \sum_{i=2}^{n-1}\left(1 + \frac{1}{i-1} + 2c\right)\overline{x}_i^2 + \left(1 + \frac{1}{n-1} + n\right)\overline{x}_n^2 \\
&\quad \le 2c + \sum_{i=2}^{n-1}\left(1 + \frac{1}{i-1} + 2c\right)\overline{x}_i^2 + n\overline{x}_n^2 + \left(1 + \frac{1}{n-1}\right),
\end{aligned} \tag{13}$$

where we used $\overline{x}_1^2 = 1$ and $\overline{x}_n \le 1$.

By (13) we can state that for any $\epsilon > 0$ there exist $N_1(\epsilon)$ and A_ϵ such that for any $\xi \in \Omega$

$$\sum_{i=2}^n \frac{i}{i-1}\overline{x}_i^2 + 2c\sum_{i=2}^n \overline{x}_{i-1}^2 + n\overline{x}_n^2 \le (1+\epsilon+2c)\sum_{i=N_1(\epsilon)}^{n-1} \overline{x}_i^2 + n\overline{x}_n^2 + A_\epsilon. \tag{14}$$

Step 3 Now we consider the following event.

$$F_{c,\epsilon} = \{\xi \mid \limsup_{n\to\infty}\ (1+\epsilon+2c)n\overline{x}_n^2 < 1\},$$

where $\epsilon > 0$ is fixed. We will show that $F_{c,\epsilon} \subset F_c$. Fix any path $\xi \in F_{c.\epsilon}$. Then there exist $\alpha = \alpha_\xi < 1$ and $N_2 = N_2(\xi)$ such that for any $n \ge N_2$

$$(1+\epsilon+2c)\overline{x}_n^2 \le \alpha\frac{1}{n}.$$

Therefore for $n > N_2$

$$\begin{aligned}(1+\epsilon+2c)\sum_{i=N_1'(\xi)}^{n-1}\overline{x}_i^2 &= (1+\epsilon+2c)\sum_{i=N_1'(\xi)}^{N_2-1}\overline{x}_i^2 + (1+\epsilon+2c)\sum_{i=N_2}^{n-1}\overline{x}_i^2 \\ &\le (1+\epsilon+2c)\sum_{i=N_1'(\xi)}^{N_2-1}\overline{x}_i^2 + \alpha\left(\frac{1}{N_2}+\cdots+\frac{1}{n-1}\right) \\ &\le \alpha\log n + B_\xi, \qquad (15)\end{aligned}$$

where $N_1'(\xi) = N_1(\epsilon(\xi))$ and B_ξ is a constant. Substituting (15) into (14), we have

$$\sum_{i=2}^{n}\frac{i}{i-1}\overline{x}_i^2 + 2c\sum_{i=2}^{n}\overline{x}_{i-1}^2 + n\overline{x}_n^2 \le \alpha\log n + n\overline{x}_n^2 + C_\xi,$$

where $C_\xi = A_{\epsilon(\xi)} + B_\xi$. Thus

$$\begin{aligned}\limsup_{n\to\infty}\frac{\sum_{i=2}^{n}\frac{i}{i-1}\overline{x}_i^2 + 2c\sum_{i=2}^{n}\overline{x}_{i-1}^2 + n\overline{x}_n^2}{\log n} &\le \limsup_{n\to\infty}\frac{\alpha\log n + n\overline{x}_n^2 + C_\xi}{\log n} \\ &= \alpha < 1,\end{aligned}$$

where we used $\limsup_{n\to\infty}\dfrac{n\overline{x}_n}{\log n} = 0$ since $\limsup_{n\to\infty}(1+\epsilon+2c)n\overline{x}_n^2 < 1$.

Step 4 By definition for any $\xi \in \Omega\backslash E_1^c$ we have

$$\limsup_{n\to\infty}\ (1+2c)n\overline{x}_n^2 < 1.$$

So we can find $\epsilon(\xi)$ such that

$$\limsup_{n\to\infty}\ (1+\epsilon(\xi)+2c)n\overline{x}_n^2 < 1.$$

Then Step 1 and Step 3 show $\mathcal{K}_n \to \infty$ $(n \to \infty)$ if Reality chooses $\xi \in \Omega\backslash E_1^c$. This completes the proof of Lemma 1.

2.2 Additive contrarian strategies

In this section we prove the following theorem which gives a somewhat stronger statement than Theorem 1:

Theorem 2 *Skeptic can weakly force E_2.*

In order to prove this theorem we need to combine various strategies. The basic ingredient is an additive contrarian strategy in (16) below. Other strategies will be studied in separate subsections. Here is the general flow of the proof. First (section 2.2.1) we will construct a strategy which makes the initial capital $\mathcal{K}_0 = 1$ increase to $\mathcal{K}_n = 1 + \frac{n}{2}\epsilon$ when $s_n = 0$. Next (section 2.2.2), we will construct a strategy weakly forcing $\{s_n = 0 \text{ i.o.}\}$. These strategies have a risk that the capital becomes negative if Reality makes $|s_n|$ as large as $\sqrt{n}$. Therefore Skeptic must stop running the strategies right before his capital becomes negative in order to observe the collateral duty. But if Reality keeps $|s_n|$ smaller than $\sqrt{n}$ forever, Skeptic can keep running strategies and then $\limsup \mathcal{K}_n = \infty$. Thus Skeptic can weakly force that $|s_n|$ becomes as large as $\sqrt{n}$ eventually. Now dividing the initial capital into countably many accounts, Skeptic can weakly force that $|s_n|$ become as large as $\sqrt{n}$ infinitely often.

2.2.1 The strategy increasing the capital when the sum process returns to the origin

Here we consider the following additive contrarian strategy $\tilde{\mathcal{P}}^\epsilon$:

$$\tilde{\mathcal{P}}^\epsilon : \ M_n = -\epsilon s_{n-1}, \tag{16}$$

where $\epsilon > 0$ is a small positive constant. If we temporarily ignore the collateral duty, the strategy $\tilde{\mathcal{P}}^\epsilon$ has a very simple explicit capital process described in the next lemma.

Lemma 2

$$\mathcal{K}_n^{\tilde{\mathcal{P}}^\epsilon} = \frac{\epsilon}{2}(n - s_n^2). \tag{17}$$

Proof: We use an induction on n. When $n = 0$, (17) holds by definition. Now assume that (17) holds for $n = k$. There are five cases depending on the signs of x_{k+1} and s_k.

1. $s_k = 0$.
2. $x_{k+1} = 1, s_k > 0$.
3. $x_{k+1} = 1, s_k < 0$.
4. $x_{k+1} = -1, s_k > 0$.
5. $x_{k+1} = -1, s_k < 0$.

case 1) $\boldsymbol{s_k = 0}$: By the assumption of induction,

$$\mathcal{K}_k^{\tilde{\mathcal{P}}^\epsilon} = \frac{\epsilon}{2}k.$$

Since $s_k = 0$, $s_{k+1}^2 = 1$ and $M_{k+1} = 0$. Thus we have

$$\begin{aligned} \mathcal{K}_{k+1}^{\tilde{\mathcal{P}}^\epsilon} &= \mathcal{K}_k^{\tilde{\mathcal{P}}^\epsilon} + M_{k+1}x_{k+1} \\ &= \frac{\epsilon}{2}k \\ &= \frac{\epsilon}{2}(k + 1 - s_{k+1}^2). \end{aligned}$$

case 2) $x_{k+1} = 1, s_k > 0$: By the assumption of induction,

$$\mathcal{K}_k^{\tilde{\mathcal{P}}^\epsilon} = \frac{\epsilon}{2}(k - s_k^2).$$

Since $M_{k+1} = -\epsilon s_k$ and $s_{k+1}^2 = (s_k+1)^2$, we have

$$\begin{aligned}
\mathcal{K}_{k+1}^{\tilde{\mathcal{P}}^\epsilon} &= \mathcal{K}_k^{\tilde{\mathcal{P}}^\epsilon} + M_{k+1}x_{k+1} \\
&= \frac{\epsilon}{2}(k - s_k^2) - \epsilon s_k \\
&= \frac{\epsilon}{2}(k - s_k^2 - 2s_k) \\
&= \frac{\epsilon}{2}(k + 1 - (s_k^2 + 2s_k + 1)) \\
&= \frac{\epsilon}{2}(k + 1 - (s_k + 1)^2) \\
&= \frac{\epsilon}{2}(k + 1 - s_{k+1}^2).
\end{aligned}$$

The other cases are proved by almost the same argument. ■

We can intuitively understand the behavior of $\mathcal{K}_n^{\tilde{\mathcal{P}}^\epsilon}$ with Figure 1. In Figure 1, the value beside a point denotes the value of $\mathcal{K}_n^{\tilde{\mathcal{P}}^\epsilon}$ at that situation and the value beside a diagonal line indicates the payoff Skeptic obtains in the next round.

As seen in (17), however small ϵ is, $1 + \mathcal{K}_n^{\tilde{\mathcal{P}}^\epsilon}$ will be negative if Reality makes $|s_n|$ large enough. Here we consider the way to avoid the bankruptcy. The condition for $\mathcal{K}_n^{\tilde{\mathcal{P}}^\epsilon}$ to be greater or equal to -1 is

$$\mathcal{K}_n^{\tilde{\mathcal{P}}^\epsilon} = \frac{2}{\epsilon}(n - s_n^2) \geq -1 \ \Leftrightarrow\ |s_n| \leq \sqrt{n + \frac{2}{\epsilon}}. \tag{18}$$

Here, suppose that Skeptic follows $\tilde{\mathcal{P}}^\epsilon$ and is going to announce the n-th move M_n at the n-th round. He can refer to s_{n-1} but not to s_n. If

$$|s_{n-1}| > \sqrt{n + \frac{2}{\epsilon}} - 1,$$

then he should stop following $\mathcal{P}^\epsilon$, or else Reality can make him bankrupt. We let $\mathcal{P}^\epsilon$ denote the strategy that follows $\tilde{\mathcal{P}}^\epsilon$ under this stopping rule:

$$\mathcal{P}^\epsilon : M_n = \begin{cases} -\epsilon s_{n-1} & \text{if } |s_{i-1}| \leq \sqrt{i + \frac{2}{\epsilon}} - 1,\ i = 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3 *The strategy $\mathcal{P}^\epsilon$ weakly forces the following E_3^ϵ:*

$$E_3^\epsilon := \left\{ \xi \;\middle|\; \exists n : |s_n| > \sqrt{n + 1 + \frac{2}{\epsilon}} - 1 \ \underline{\text{or}}\ \left(\limsup_{n\to\infty} |s_n| = \infty \ \&\ s_n \neq 0 \text{ a.a.} \right) \right\}.$$

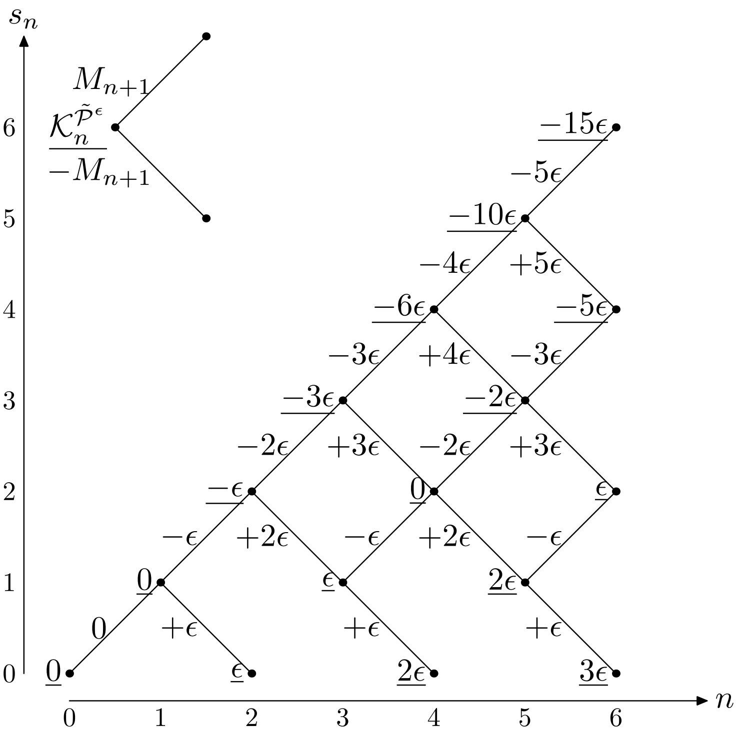

Figure 1: Behavior of $\mathcal{K}_n^{\tilde{\mathcal{P}}^\epsilon}$

Proof: Fix any path $\xi \notin E_3^\epsilon$. Then

$$|s_{n-1}| \leq \sqrt{n + \frac{2}{\epsilon}} - 1, \quad \forall n$$

holds and the capital process $\mathcal{K}_n^{\mathcal{P}^\epsilon}(\xi)$ is equal to $\mathcal{K}_n^{\tilde{\mathcal{P}}^\epsilon}(\xi)$. Furthermore at least one of the following two cases holds:

1. There exists L such that $|s_n| < L$, $\forall n$,
2. $s_n = 0$ holds for infinitely many n.

case 1) Since $s_n^2 < L^2$,

$$\mathcal{K}_n^{\mathcal{P}^\epsilon} = \frac{\epsilon}{2}(n - s_n^2) > \frac{\epsilon}{2}(n - L^2) \to \infty \quad (n \to \infty).$$

case 2) When $s_n = 0$,

$$\mathcal{K}_n^{\mathcal{P}^\epsilon} = \frac{\epsilon}{2} n.$$

Therefore if $s_n = 0$ occurs infinitely often, $\limsup_{n\to\infty} \mathcal{K}_n^{\mathcal{P}^\epsilon} = \infty$. ∎

Now we define E_3 by

$$E_3 := \left\{ \xi \;\middle|\; |s_n| > \sqrt{n} - 1 \text{ i.o. } \underline{\text{or}} \ \left(\limsup_{n\to\infty} |s_n| = \infty \ \& \ s_n \neq 0 \text{ a.a.} \right) \right\}.$$

Then

$$\bigcap_{i\in\mathbb{N}} E_3^{2^{-i}} \subset E_3$$

and by Lemma 3.2 of [8] the next corollary holds.

Corollary 1 *Skeptic can weakly force* E_3.

2.2.2 A strategy weakly forcing boundedness or two-sided unboundedness of the sum process

Here we consider weakly forcing the following event

$$E_4 := \left\{ \xi \;\middle|\; \limsup_{n\to\infty} |s_n| < \infty \ \underline{\text{or}} \ \left(\limsup_{n\to\infty} s_n = \infty \ \& \ \liminf_{n\to\infty} s_n = -\infty \right) \right\}.$$

Actually in weakly forcing E_4 we combine two one-sided strategies. Consider the following very simple additive one-sided strategy:

$$\mathcal{P}^{-N} : M_n = \begin{cases} \dfrac{1}{N} & \text{if } \min_{1\le i\le n-1} s_i > -N, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

$\mathcal{P}^{-N}$ bets the constant amount $1/N$ until s_n reaches $-N$ for the first time. Similarly define $\mathcal{P}^{+N}$ by

$$\mathcal{P}^{+N} : M_n = \begin{cases} -\dfrac{1}{N} & \text{if } \max_{1\le i\le n-1} s_i < N, \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

Corresponding to these strategies in the following lemma we consider two one-sided events.

Lemma 4 *Let N be any positive number and define E_4^{-N}, E_4^{+N} as follows:*

$$\begin{aligned} E_4^{-N} &:= \left\{ \xi \;\middle|\; \min_{i=1,2,\dots} s_i \le -N \ \underline{\text{or}} \ \limsup_{n\to\infty} s_n < \infty \right\}, \\ E_4^{+N} &:= \left\{ \xi \;\middle|\; \max_{i=1,2,\dots} s_i \ge N \ \underline{\text{or}} \ \liminf_{n\to\infty} s_n > -\infty \right\}. \end{aligned}$$

Skeptic can weakly force E_4^{+N} and E_4^{-N}.

Proof: Clearly the capital process $\mathcal{K}^{\mathcal{P}^{-N}}$ of $\mathcal{P}^{-N}$ is given by

$$\mathcal{K}_n^{\mathcal{P}^{-N}} = \begin{cases} \dfrac{s_n}{N} & \text{if } \min_{1 \le i \le n-1} s_i > -N, \\ -1 & \text{otherwise.} \end{cases}$$

This shows that $\mathcal{P}^{-N}$ weakly forces E_4^{-N}. The proof for E_4^{+N} is almost the same by using $\mathcal{P}^{+N}$. ∎

Corollary 2 *Skeptic can weakly force E_4.*

Proof: Define E_4^- and E_4^+ as follows:

$$\begin{aligned} E_4^- &:= \left\{ \xi \;\middle|\; \liminf_{n\to\infty} s_n = -\infty \text{ \underline{or} } \limsup_{n\to\infty} s_n < \infty \right\}, \\ E_4^+ &:= \left\{ \xi \;\middle|\; \liminf_{n\to\infty} s_n > -\infty \text{ \underline{or} } \limsup_{n\to\infty} s_n = \infty \right\}. \end{aligned}$$

Then we can write

$$\begin{aligned} E_4^- &= \bigcap_{N\in\mathbb{N}} E_4^{-N}, \\ E_4^+ &= \bigcap_{N\in\mathbb{N}} E_4^{N}, \end{aligned}$$

and

$$E_4 = E_4^- \cap E_4^+.$$

By the Lemma 3.2 of [8], the corollary holds. ∎

2.2.3 Proof of Theorem 2

Using Corollary 1 and Corollary 2, now we can prove Theorem 2.

Proof: From Corollary 1 and Corollary 2, Skeptic can weakly force $E_3 \cap E_4$. So we only have to show $E_2 \supset E_3 \cap E_4$. Now we set

$$\begin{aligned} A_1 &:= \left\{ \xi \;\middle|\; \limsup_{n\to\infty} |s_n| = \infty \;\&\; s_n \neq 0 \text{ a.a.} \right\}, \\ A_2 &:= \left\{ \xi \;\middle|\; \limsup_{n\to\infty} |s_n| < \infty \right\}, \\ A_3 &:= \left\{ \xi \;\middle|\; \limsup_{n\to\infty} s_n = \infty \;\&\; \liminf_{n\to\infty} s_n = -\infty \right\}. \end{aligned}$$

Then we can write

$$E_3 = E_2 \cup A_1, \qquad E_4 = A_2 \cup A_3.$$

By definition

$$\emptyset = E_2 \cap A_2 = A_1 \cap A_2 = A_1 \cap A_3$$

and therefore

$$E_3 \cap E_4 = E_2 \cap A_3 \subset E_2.$$

■

This proof also shows the next theorem.

Theorem 3 *Skeptic can weakly force A_3.*

3 One sided strategies

The statement of Theorem 2 is only concerned with the behavior of $|s_n|$ and Reality is forced to make $|s_n| > \sqrt{n} - 1$ infinitely often. But it says nothing about the sign of s_n. Hence Reality can choose a path such that $s_n > \sqrt{n} - 1$ infinitely often but $s_n < -\sqrt{n} + 1$ only finitely often. In this section we prove the following theorem which eliminates this shortcoming.

Theorem 4 *Skeptic can weakly force the following E_5 and E_6:*

$$\begin{aligned} E_5 &:= \{\xi \mid s_n > \sqrt{n} - 1 \text{ i.o.}\}, \\ E_6 &:= \{\xi \mid s_n < -\sqrt{n} + 1 \text{ i.o.}\}. \end{aligned}$$

The statement in Theorem 4 seems to be innocuous and one might expect that it can be proved by the obvious symmetry of the fair-coin game. Actually we found it difficult to prove Theorem 4 by combination of simple strategies. Recall that in the previous section, except for combining countably many strategies, the individual strategies were very simple and explicit. Furthermore it should be possible to generalize the results in the previous section to more general protocols than the fair-coin game by introducing pricing of quadratic hedges as in Chapter 4 of [8].

On the other hand, our proof of Theorem 4 uses the fact that in the fair-coin game it is conceptually very easy to determine the price of every variable. Mathematically it is the same as the pricing of options for binomial models, which is explained in standard introductory textbooks on mathematical finance (e.g. [1]). See also [11] for a game-theoretic exposition of the pricing formulas for the binomial model.

3.1 Two stopping times

For $i = 1, 2, \ldots$, we define stopping times w_i and v_i by

$$\begin{aligned} w_i &:= \min\{n > v_{i-1} \mid s_n = 0\}, \\ v_i &:= \min\left\{n > w_i \;\middle|\; |s_n| > \sqrt{n} - 1\right\}, \end{aligned}$$

where $v_0 := 0$ and if the set in the definition is empty then the value of the variable is ∞. v_i is the first hitting time of the two-sided $\sqrt{n}$-boundary after leaving the origin at w_i and w_i is the first time of returning to the origin after v_{i-1}. See Figure 2.

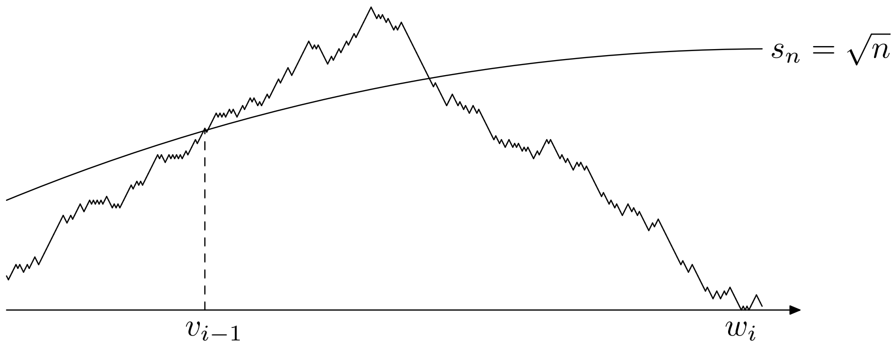

Figure 2: definition of w_i and v_i

Now for $i = 1, 2, \ldots$, we define a variable X_i by

$$X_i := \begin{cases} 1 & v_i < \infty \;\&\; s_{v_i} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

We can think of X_i as a ticket, which pays you one dollar if the sum process hits the negative boundary $-\sqrt{n}$ (rather than the positive boundary $\sqrt{n}$) at time v_i.

3.2 Proof of theorem 4

This section is devoted to the proof of Theorem 4. It is conceptually very easy. The essential point is the proof of Lemma 5 below. We begin by giving our proof other than Lemma 5.

Remember that Reality is forced the event $A_3 \cap E_2$. Therefore we can assume that she chooses $\xi \in A_3 \cap E_2$. Therefore for proving Theorem 4, it suffices to prove that Skeptic can weakly force $A_3 \cap E_2 \Rightarrow E_5$ (cf. Lemma 2.1 of [4]).

In Lemma 5, we prove $\overline{\mathrm{E}}_t[X_i] \leq \frac{1}{2}$ for any situation $t \in [w_i]$ by constructing the replicating strategy of X_i. We let $\mathcal{P}^{X_i}$ denote this strategy. Once $\mathcal{P}^{X_i}$ is constructed, the strategy weakly forcing $A_3 \cap E_2 \Rightarrow E_5$ is given as follows:

- Buy $\frac{1}{2}\mathcal{K}X_i$ when the present situation is in $[w_i]$ for $i = 1, 2, \ldots$, where $\mathcal{K}$ denotes the present capital Skeptic possesses and buying $\frac{1}{2}\mathcal{K}X_i$ actually means running $\frac{1}{2}\mathcal{K}\mathcal{P}^{X_i}$.

Suppose Reality chooses the path $\xi \notin (A_3 \cap E_2)^C \cup E_5$. Since $\xi \in A_3 \cap E_2$, $w_i < \infty$ and $v_i < \infty$ for any i. Thus Skeptic runs $\frac{1}{2}\mathcal{K}_{w_i}\mathcal{P}^{X_i}$ from the w_i-th round for each i. After the v_i-th round, his capital becomes $\frac{3}{2}\mathcal{K}_{w_i}$ if $s_{v_i} < 0$ and $\frac{1}{2}\mathcal{K}_{w_i}$ if $s_{v_i} > 0$. But $s_{v_i} < 0$ for all sufficiently large i since $\xi \notin E_5$, then Skeptic's capital increases to ∞.

Now it remains to prove the following lemma.

Lemma 5 *For any $i \in \mathbb{N}$ and any $t \in [w_i]$*

$$\overline{\mathrm{E}}_t[X_i] \leq \frac{1}{2}. \tag{21}$$

Proof: First we rephrase the lemma for simplicity. Fix any situation $t \in [w_i]$ and suppose that the length of t is l. Define a stopping time u_l and a variable Y_l as

$$u_l := \min\left\{ n \;\middle|\; |s_n| > \sqrt{n+l} - 1 \right\},$$
$$Y_l := \begin{cases} 1 & \text{if } u_l < \infty \;\&\; s_{u_l} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Considering the upper price of X_i at the situation t is equivalent to considering the upper price of Y_l at the situation $\square$, since $s(t) = 0$. Thus it suffices to show $\overline{\mathrm{E}}[Y_l] \leq \frac{1}{2}$ for the proof of $\overline{\mathrm{E}}_t[X_i] \leq \frac{1}{2}$.

The set $\{Y_l = 1\}$ can be decomposed as

$$\{\xi \mid Y_l(\xi) = 1\} = \bigcup_{i=1}^{\infty} A_i,$$
$$A_i := \{\xi \mid u_l(\xi) = i, s_i < 0\}.$$

Using A_i, Y_l can be decomposed as

$$Y_l = \sum_{i=1}^{\infty} Z_i, \qquad Z_i(\xi) := \begin{cases} 1 & \text{if } \xi \in A_i, \\ 0 & \text{otherwise.} \end{cases}$$

Whether $\xi \in A_i$ or not depends solely on ξ_i, so A_i can be decomposed into the cylinder sets defined by the situations of length i, that is, there exist t^i_j $(j = 1, 2, \ldots, a_i)$ such that

$$A_i = \bigcup_{j=1}^{a_i} O_{t^i_j},$$

where the length of t^i_j is i $(j = 1, 2, \ldots, a_i)$. We set,

$$t^i_j = y^{i,j}_1 y^{i,j}_2 \cdots y^{i,j}_i, \quad (y^{i,j}_p \in \{-1, 1\}, p = 1, 2, \ldots, i).$$

Using t^i_j, we define the strategy $\mathcal{P}^{i,j}$ by

$$\mathcal{P}^{i,j}: \; M_n = \begin{cases} y^{i,j}_n \mathcal{K}_{n-1} & \text{for } n = 1, 2, \dots, i, \\ 0 & \text{for } n > i, \end{cases}$$

where we temporarily suppose $\mathcal{K}_0 = 2^{-i}$. Intuitively speaking, this strategy prepares the amount of 2^{-i} as the initial capital and bet all the available capital on the realization of the situation t^i_j. Hence, if t^i_j realizes, that is, $x_p = y^{i,j}_p$, $(p = 1, 2, \dots, i)$, then the capital grows to 2^i times, otherwise the capital becomes zero. Thus the capital process $\mathcal{K}^{\mathcal{P}^{i,j}}$ satisfies

$$2^{-i} + \mathcal{K}^{\mathcal{P}^{i,j}}_i(\xi) = \begin{cases} 1 & \text{if } \xi \in O_{t^i_j}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we define the strategy $\mathcal{P}^i$ by

$$\mathcal{P}^i := \sum_{j=1}^{a_i} \mathcal{P}^{i,j}.$$

The strategy $\mathcal{P}^i$ requires the amount of $a_i 2^{-i}$ as the initial capital and its capital process is written as

$$a_i 2^{-i} + \mathcal{K}^{\mathcal{P}^i}_i(\xi) = \begin{cases} 1 & \text{if } \xi \in \bigcup_{j=1}^{a_i} O_{t^i_j}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus Skeptic can replicate Z_i with the initial capital $a_i 2^{-i}$. Then, Skeptic can replicate $Y_l = \sum_{i=1}^{\infty} Z_i$ with initial capital $\sum_{i=1}^{\infty} a_i 2^{-i}$, so it suffices to show

$$\sum_{i=1}^{\infty} a_i 2^{-i} \le \frac{1}{2}.$$

for the proof of $\overline{\mathrm{E}}[Y_l] \le \frac{1}{2}$

Fix an arbitrary large number k. We consider the event B_k defined by

$$B_k \;\; := \;\; \{\xi \mid u_l(\xi) \le k, s_l < 0\}.$$

Whether $\xi \in B_k$ or not depends solely on ξ_k, so B_k can be decomposed into the cylinder sets:

$$B_k = \bigcup_{q=1}^{b_k} O_{t'^k_q}, \tag{22}$$

where the length of t'^k_q is k $(q = 1, 2, \dots, b_k)$. Here we show $b_k \le 2^{k-1}$. First, remember that there are just 2^k situations of length k. If we define B^-_k as

$$B^-_k := \bigcup_{q=1}^{b_k} O_{-t'^k_q},$$

then $B_k \cap B_k^- = \emptyset$ by definition. Thus $b_k \le \frac{1}{2} \cdot 2^k = 2^{k-1}$. Furthermore B_k can be decomposed also into A_i:

$$B_k = \bigcup_{i=1}^{k} A_i = \bigcup_{i=1}^{k} \bigcup_{j=1}^{a_i} O_{t_j^i}.$$

Just 2^{k-i} situations of $\{t_q'^k\}_{q=1}^{b_k}$ follow t_j^i, so the cylinder set $O_{t_j^i}$ can be decomposed into 2^{k-i} cylinder sets:

$$O_{t_j^i} = \bigcup_{h=1}^{2^{k-i}} O_{t'^k_{c[i,j,h]}},$$

where $1 \le c[i,j,h] \le b_k$ and $c[i,j,h] \ne c[i',j',h']$ if $(i,j,h) \ne (i',j',h')$. Thus,

$$B_k = \bigcup_{i=1}^{k} \bigcup_{j=1}^{a_i} \bigcup_{h=1}^{2^{k-i}} O_{t'^k_{c[i,j,h]}}. \tag{23}$$

By (22) and (23)

$$b_k = \sum_{i=1}^{k} a_i 2^{k-i}.$$

Since $b_k \le 2^{k-1}$,

$$\sum_{i=1}^{k} a_i 2^{k-i} \le 2^{k-1} \quad \Rightarrow \quad \sum_{i=1}^{k} a_i 2^{-i} \le \frac{1}{2} \quad \Rightarrow \quad \sum_{i=1}^{\infty} a_i 2^{-i} \le \frac{1}{2}.$$

■

Lastly, we show that the inequality in the (21) is in fact an equality. Here we decompose Ω into three subsets,

$$\begin{aligned} D_1^i &:= \{\xi \mid v_i < \infty \ \& \ s_{v_i} < 0\}, \\ D_2^i &:= \{\xi \mid v_i = \infty\}, \\ D_3^i &:= \{\xi \mid v_i < \infty \ \& \ s_{v_i} > 0\}. \end{aligned}$$

Then $X_i = 1_{D_1^i}$, where $1_{D_1^i}$ is the indicator function of D_1^i. By the definition of the upper price,

$$1 = \overline{\mathrm{E}}_t[1] \le \overline{\mathrm{E}}_t[1_{D_1^i}] + \overline{\mathrm{E}}_t[1_{D_2^i}] + \overline{\mathrm{E}}_t[1_{D_3^i}] \tag{24}$$

By symmetry property,

$$\overline{\mathrm{E}}_t[1_{D_1^i}] = \overline{\mathrm{E}}_t[1_{D_1^i}]. \tag{25}$$

Since Skeptic can force $v_i < \infty$ we have

$$\overline{\mathrm{E}}_t[1_{D_2^i}] = 0. \tag{26}$$

Equations (24), (25) and (26) shows $\overline{\mathrm{E}}_t[1_{D_1^i}] \ge \frac{1}{2}$, thus $\overline{\mathrm{E}}_t[1_{D_1^i}] = \overline{\mathrm{E}}_t[X_i] = \frac{1}{2}$.

4 Some discussions

In this paper we showed that Skeptic can (weakly) force E_1, E_2, E_5 and E_6 in the fair-coin game. As mentioned in Section 1 these statements are weaker than LIL, which is shown in [8] in the game-theoretic framework. But we want to emphasize the simplicity of our strategies. Actually, Skeptic needs only to keep the value of s_n in memory in the strategies forcing E_1 and E_2.

In the proof of Lemma 5, we only proved the existence of the replicating strategy of Y_l rather than providing an explicit formula for bet (the move M_n of Skeptic) of the strategy. The bet of the replicating strategy is directly related with the price of Y_l at an arbitrary situation by the argument of "delta hedge"[11]. Let $\eta(n,s)$ denote the price of Y_l given the round n and the value of process s. Here let us consider the problem in the measure-theoretic framework rather than the game-theoretic framework. Then $\eta(n,s)$ can be written as the measure-theoretic conditional expectation:

$$\eta(n,s) = \mathrm{E}[Y_l \mid s_n = s]. \tag{27}$$

Given $\eta(n,s)$, the bet of replicating strategy by delta hedge is calculated as follows([11]):

$$M_n = \frac{\eta(n+1,s+1) - \eta(n+1,s-1)}{2}.$$

But in practice it would be difficult to express $\eta(n,s)$ analytically. For the case of Brownian motion [6] gives results on (27). However they are very complicated involving zeros of a special function.

In order to prove the existence of the replicating strategy, we used the argument of betting on specific paths. This type of argument can be found in the field of algorithmic theory of randomness, for instance Muchnik et al. uses the same idea in [5, Theorem 9.4]. We think that the idea is logically very powerful because it can be used to prove the existence of a superreplicating strategy for any ticket in the fair-coin game.

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