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Characterization of Easily Controllable Plants Based on the Finite Frequency Phase/Gain Property: A Magic Number $\sqrt{4+2\sqrt{2}}$ in $\mathscr{H}_\infty$ Loop Shaping Design

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Characterization of Easily Controllable Plants Based on the Finite Frequency Phase/Gain Property: A Magic Number $\sqrt{4+2\sqrt{2}}$ in $\mathscr{H}_\infty$ Loop Shaping Design

Masaaki KANNO*, Shinji HARA†, and Masahiko ONISHI†

March 30th

Abstract

This paper introduces a phase/gain condition for (marginally) stable systems for characterization of easily controllable systems, and investigates the relationship between the condition and the optimal performance $\gamma_{\rm opt}$ in $\mathscr{H}_\infty$ loop shaping design. More specifically it is shown that there is a close relationship between the condition and a magic number $\sqrt{4+2\sqrt{2}}$, for both continuous-time and discrete-time systems. Furthermore a simple design procedure for robust control based on the obtained knowledge is proposed.

1 Introduction

Recently there has been an increased interest in the characterization of easily controllable plants. One of current approaches is to formulate a certain optimal control problem and to express the best achievable performance (or its bound) in terms of plant properties. There are abundant results making use of the $\mathscr{H}_2$ tracking/regulation formulation, e.g., [1]. Also several results are available which are based on $\mathscr{H}_\infty$ loop shaping design [2, 9]. Such results relate intrinsic performance limitations to unstable poles, non-minimum phase zeros, gain and time-delay of the plant to be controlled, and it is now well understood that unstable poles etc. prevent good performance from being achieved. Nevertheless little attention has been paid to the

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case of (marginally) stable systems in spite of the fact that a plant is sometimes difficult to control even if it is stable.

This paper focuses on single-input-single-output (SISO) (marginally) stable systems and tries to show some relationship between a phase/gain property and the optimal performance level $\gamma_{\rm opt}$ in $\mathscr{H}_\infty$ loop shaping design [8] for both continuous-time and discrete-time systems. More specifically a strong connection is revealed between Conditions ($\pi_{\rm c}$) and ($\pi_{\rm d}$), which are formally defined in Subsection 3.1, and a particular value $\sqrt{4+2\sqrt{2}} \simeq 2.6131$. Condition ($\pi_{\rm c}$) is a reformulation of the finite frequency positive realness (FFPR) property proposed in [6, 7] for characterizing a class of easily controllable mechanical systems, and requires a small gain in addition to the FFPR property. The purpose of this paper is threefold.

- For continuous-time systems, the relationship between Condition ($\pi_{\rm c}$) and $\sqrt{4+2\sqrt{2}}$ is proven for the class of all (marginally) stable 2nd order systems and some classes of 3rd order systems.
- Investigations into a similar relationship are made for some classes of 1st order and 2nd order discrete-time systems.
- A simple design procedure for robust control is proposed based on the obtained knowledge about the relationship for possibly high order (marginally) stable systems. In the procedure a designer would only have to decide the control bandwidth based on the phase/gain condition and could rely on the $\mathscr{H}_\infty$ loop shaping design procedure for achieving stability and robustness.

Among few results dealing with (marginally) stable plants, work relevant to this work is [2, Section 5]; it derives a lower bound for $\gamma_{\rm opt}$ implicitly based on the rolloff rate of the gain at the gain crossover frequency. (Note that, around gain crossover, the rolloff rate is closely related to the phase there if the plant is stable and minimum phase.) While the work has some significance from the theoretical point of view, it is too general to provide information useful in practice; a plant which yields $\gamma_{\rm opt} = 20$ is as difficult to control in practice as one with $\gamma_{\rm opt} = 30$. The focal point of this paper lies in the range of the optimal performance level $\gamma_{\rm opt}$ most practical engineers would be interested in. In other words the target range here is in $\gamma_{\rm opt} \leq 2\sqrt{2} \simeq 2.828$, as suggested in [8].

The paper is organized as follows. Section 2 reviews the problem formulation of $\mathscr{H}_\infty$ loop shaping design and its analytic solution. Then Section 3 recalls the concept of finite frequency positive realness, defines phase/gain property conditions, called Conditions ($\pi_{\rm c}$) and ($\pi_{\rm d}$), and reviews some preliminary results. In Section 4, some classes of 2nd order and 3rd order continuous-time systems are dealt with, and the relationship between Condition ($\pi_{\rm c}$) and $\gamma_{\rm opt} \leq \sqrt{4+2\sqrt{2}}$ is shown. Section 5 then considers the discrete-time case, and some classes of 1st order and 2nd order systems are examined. In Section 6, a simple robust design procedure based on the obtained knowledge is proposed and also a numerical example is provided. Some concluding remarks are made in Section 7.

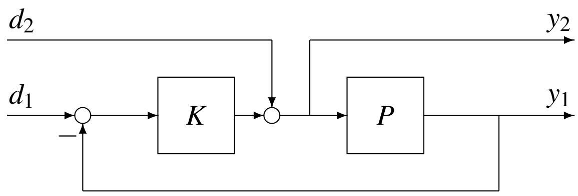

Figure 1: $\mathscr{H}_\infty$ loop shaping design formulation.

Some of the results presented in this paper are in fact already reported, e.g., in [5], but without proofs. Such results are included to make this paper self-contained, and furthermore complete proofs are provided here.

Notation: For a matrix $A \in \mathbb{C}$, its complex conjugate transpose is denoted by A^*. The $\mathscr{H}_\infty$-norm of a system is denoted by $\|\cdot\|_\infty$. The largest eigenvalue of a matrix is denoted by $\lambda_{\max}(\cdot)$ (when all the eigenvalues are real).

2 $\mathscr{H}_\infty$ Loop Shaping Design

2.1 Problem Formulation and Solution

This paper utilizes, as the index of how easy the plant is to control, the optimal performance level given by the so called $\mathscr{H}_\infty$ loop shaping design [8]. This design methodology blends the classical loop shaping technique and $\mathscr{H}_\infty$ control of modern control well. The problem formulation is stated as follows. In the feedback configuration in Figure 1, let P be a given plant or a weighted plant to be controlled. The aim is to find a stabilizing controller K that minimizes the $\mathscr{H}_\infty$-norm of the transfer function matrix from $(d_1\ d_2)^{\mathrm{T}}$ to $(y_1\ y_2)^{\mathrm{T}}$. Namely the design procedure finds K that internally stabilizes the closed-loop system and also achieves

$$\gamma_{\mathrm{opt}} := \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty . \tag{1}$$

This is an $\mathscr{H}_\infty$ optimal control problem, but it is well-known that computation of the optimal performance level γ_{opt} does not require iteration, unlike ordinary $\mathscr{H}_\infty$ control problems. More specifically, if P is strictly proper and is given in minimal state-space realization form

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] ,$$

then γ_{opt} can be written by using the solutions to an algebraic Riccati equation and a Lyapunov equation:

$$\gamma_{\mathrm{opt}} = \frac{1}{\sqrt{1-\lambda_{\max}(YQ)}} ,$$

where Y, Q are the solutions to

$$AY + YA^* - YC^*CY + BB^* = 0 \,, \tag{2}$$
$$Q(A - YC^*C) + (A - YC^*C)^*Q + C^*C = 0 \,, \tag{3}$$

in the continuous-time case [8], and

$$AYA^* - Y - AYC^*(I + CYC^*)^{-1}CYA^* + BB^* = 0 \,,$$
$$A^*QA - (I + C^*CY)Q(I + YC^*C) + (I + C^*CY)C^*C = 0 \,,$$

in the discrete-time case [4]. This feature is extremely useful when investigating the best achievable performance level in terms of plant properties.

2.2 γ_{opt} as the Easy Controllability Index

This subsection discusses the suitability of γ_{opt} as the index of easy controllability. The performance index γ_{opt} in (1) in fact indicates the robustness of the resulting closed-loop system against coprime factor uncertainty of the plant (the smaller, the better) [8]. However, since the transfer function matrix of the closed-loop system contains both the sensitivity and complementary sensitivity functions, it is also considered that the design attempts to balance the sizes of both functions. In addition, in the case of SISO systems, there is a strong relationship between γ_{opt} and the achieved gain/phase margins. These facts justify $\mathscr{H}_\infty$ loop shaping design from the point of view of classical control.

Proposition 1 ([9]) *When P is SISO, the gain margin* (GM[dB]) *and phase margin* (PM) *attained by the optimal controller $K_{\mathrm{opt}}(s)$ that achieves γ_{opt} are bounded as follows:*

$$\mathrm{GM} \geq 20\log_{10}\left(\frac{\gamma_{\mathrm{opt}}+1}{\gamma_{\mathrm{opt}}-1}\right) \,, \quad \mathrm{PM} \geq 2\arcsin\left(\frac{1}{\gamma_{\mathrm{opt}}}\right) \,.$$

This proposition is stated and proven in [9] for the continuous-time case. However the proof in [9] can be readily modified for the discrete-time case, and the proposition holds for both continuous-time and discrete-time cases.

Furthermore, in the case of continuous-time systems, some knowledge has been established based on experience. A positive real system is known to be very easy to control, and this fact is theoretically assured by the fact that γ_{opt} for such a system is guaranteed to be less than or equal to $\sqrt{2}$ [8]. However the result is of little practical significance since few realistic systems possess such a property. As a practical indication it is known from experience that $\gamma_{\mathrm{opt}} \lesssim 2\sqrt{2}$ implies a good closed-loop performance and thus that the plant is easily controllable. Note that, when $\gamma_{\mathrm{opt}} \leq 2\sqrt{2}$, it is guaranteed by Proposition 1 that

$$\mathrm{GM} \gtrsim 2.09 \ (\simeq 3.21[\mathrm{dB}]) \,, \quad \mathrm{PM} \gtrsim 41.41[\mathrm{degree}] \,.$$

Moreover the optimal controller synthesized by the $\mathscr{H}_\infty$ loop shaping design procedure shapes the open loop transfer function (to achieve stability and also to improve robustness) without drastically changing the gain crossover frequency of P, if γ_{opt} is approximately less than $2\sqrt{2}$ [8]. In other words large γ_{opt} means either a small or large gain of the controller at the gain crossover frequency of the plant, which means that the original gain crossover frequency of the plant is not suited for achieving a robust closed-loop system.

The facts stated above are deemed sufficient to conclude that γ_{opt} can be used as an indicator of how 'good' the closed-loop system can be if the crossover frequency of P is chosen to be the control bandwidth.

Remark 1 *A small difference between the gain crossover frequencies of P and PK_{opt} indicates that the lower bound for phase margin in Proposition 1 is a tight bound. The following proposition can immediately be deduced from the proof of Proposition 1 given in [9].*

Proposition 2 *Suppose that the optimal controller K_{opt} is obtained for a given (marginally) stable system P. If the gain crossover frequency of P is identical to that of PK_{opt}, then the lower bound for the phase margin in Proposition 1 is tight, i.e., is identical to the phase margin achieved by K_{opt}.*

Proof: The continuous-time case is proven. Let the gain crossover frequency of P be ω_{gc}. The assumption implies that

$$|P(j\omega_{\text{gc}})| = |K_{\text{opt}}(j\omega_{\text{gc}})| = |P(j\omega_{\text{gc}})K_{\text{opt}}(j\omega_{\text{gc}})| = 1 \ . \tag{4}$$

Also let the phase margin achieved by K_{opt} be $\phi_{\text{PM}}(>0)$. Then,

$$P(j\omega_{\text{gc}})K_{\text{opt}}(j\omega_{\text{gc}}) = e^{j(\phi_{\text{PM}}-\pi)} = -e^{j\phi_{\text{PM}}} \ . \tag{5}$$

When P is SISO, the following relationship holds [9]:

$$\gamma_{\text{opt}} = \frac{\sqrt{1+|P(j\omega)|^2}\sqrt{1+|K_{\text{opt}}(j\omega)|^2}}{|1+P(j\omega)K_{\text{opt}}(j\omega)|} \tag{6}$$

for all $\omega \in \mathbb{R}$. By noting that

$$\left|1+P(j\omega_{\text{gc}})K_{\text{opt}}(j\omega_{\text{gc}})\right| = \left|1-e^{j\phi_{\text{PM}}}\right| = 2\sin\left(\frac{\phi_{\text{PM}}}{2}\right) ,$$

and also using (4), (5), and (6), it can be deduced that

$$\phi_{\text{PM}} = 2\arcsin\left(\frac{1}{\gamma_{\text{opt}}}\right) ,$$

which is the desired result.

The discrete-time case can be proven exactly in the same manner by replacing $j\omega$ etc. with $e^{j\theta}$ etc. $\square$

3 Characterization of Easily Controllable Plants

3.1 Finite Frequency Positive Realness and Condition (π_c)/(π_d)

It is recognized that some of (marginally) stable oscillatory systems are easy to control while others are difficult. In order to characterize easily controllable plants, the concept of 'finite frequency positive realness (FFPR)' was introduced [6, 7], and the following knowledge has been obtained for the class of plants

$$\begin{aligned}\mathscr{P}_{\mathrm{c}} := \{P(s)\,|\,P(s) &\text{ is a strictly proper transfer function} \\ &\text{with all the poles in the closed complex left half plane}\} \ .\end{aligned}$$

> **FFPR** [7]: Let $P(s) \in \mathscr{P}_{\mathrm{c}}$ be the transfer function of the plant to be controlled. Then the maximum control bandwidth $\omega_b^\star$ achievable by a dynamic feedback controller is approximately the same as the maximum frequency ϖ that satisfies
>
> $$G(j\omega)+G^*(j\omega) \geq 0 \quad (\forall\ |\omega| \leq \varpi)\ ,\ \ G(s) := sP(s)\ .$$

The knowledge quoted above implies that there is some connection between a phase property (phase delay of 180[degree]) and the achievable control bandwidth. This paper clarifies that there is a strong relationship between a finite frequency phase/gain property and the achievable performance level γ_{opt}. To this end the following condition defining a phase/gain property is introduced where a small gain requirement is added to the FFPR property [5].

> **Condition (π_c)**: Either one of the following conditions holds:
> (i) $\forall\omega \in \mathbb{R}$, $\angle P(j\omega) > -\pi$
> (ii) $\exists\,\omega_\pi > 0$ such that
>
> $$\begin{aligned} 0 \leq \omega < \omega_\pi :&\ \ \angle P(j\omega) \geq -\pi\ , & & \\ \omega = \omega_\pi :&\ \ \angle P(j\omega) = -\pi\ , & &|P(j\omega)| \leq 1\ , \\ \omega > \omega_\pi :& & &|P(j\omega)| < 1\ . \end{aligned}$$

That is, ω_π is the frequency at which the phase of the plant reaches -180[degree] and from which the gain of the plant is always less than 0[dB]. Here a class of plants satisfying this condition is defined:

$$\mathscr{P}_{\mathrm{c}}^{\mathrm{g}} := \{P(s) \in \mathscr{P}_{\mathrm{c}}\,|\,P(s) \text{ satisfies Condition } (\pi_c)\}\ .$$

This paper also addresses discrete-time systems, and the corresponding condition for the discrete-time case is defined.

Condition (π_d): The following condition holds:
• $\exists\, \theta_\pi \in [0,\, \pi]$ such that

$$\begin{aligned} 0 \le \theta < \theta_\pi &: \ \angle P(e^{j\theta}) \ge -\pi \ , \\ \theta = \theta_\pi &: \ \angle P(e^{j\theta}) = -\pi \ , \qquad |P(e^{j\theta})| \le 1 \ , \\ \theta > \theta_\pi &: \qquad\qquad\qquad\qquad\quad |P(e^{j\theta})| < 1 \ . \end{aligned}$$

Similar to the continuous-time case, for a class of (marginally) stable systems

$$\mathscr{P}_\mathrm{d} := \{P(z) \,|\, P(z) \text{ is a strictly proper transfer function with all the poles in the closed unit disk}\} \ ,$$

a class of systems satisfying Condition (π_d) is defined:

$$\mathscr{P}_\mathrm{d}^\mathrm{g} := \{P(z) \in \mathscr{P}_\mathrm{d} \,|\, P(z) \text{ satisfies Condition } (\pi_\mathrm{d})\} \ .$$

Remark 2 *Even though the appearances are different, Conditions (π_c) and (π_d) are equivalent in that (i) of Condition (π_c) covers the case where ω_π does not exist. Note that, for any $P(z) \in \mathscr{P}_\mathrm{d}$, the phase always delays at least* 180[degree] *and that (i) of Condition (π_c) is not needed in the discrete-time case.*

3.2 Multiple Integrators

Before stating main results in the subsequent sections, this subsection investigates some plants with distinct characteristics and observes the validity of Condition (π_c)/(π_d). Firstly some continuous-time plants with constant phase are examined. Systems under investigation are multiple integrators:

$$P_n(s) = \left(\frac{k}{s}\right)^n \quad (k > 0) \ .$$

It is noted that the systems have phase delay of $90n$[degree] for all frequencies. The achievable control performance levels and the controllers that achieve them for $n = 1, \ldots, 4$ are given in Table 1. It should be noted that the values of γ_opt are independent of the system gain k. Also the poles and zeros of the optimal controllers (for $k = 1$) are depicted in Figure 2.

Since $P_1(s)$ is positive real, it is known that $\gamma_\mathrm{opt} \le \sqrt{2}$; the equality holds in this case. When $n = 2$, the system is a double integrator and the phase is always -180[degree]; in fact the system lies just on the boundary of Condition (π_c). Observe that the optimal controller is a phase-lead compensator which achieves at the crossover frequency of the system $P_2(s)$ the maximum phase-lead, which is 45[degree] (Figure 3). Consequently the design is considered sensible when interpreted in the context of classical control. This is another justification of using γ_opt of $\mathscr{H}_\infty$ loop shaping design as the index of easy controllability. The achieved performance γ_opt for this system is [3]

$$\gamma_\mathrm{opt} \le \gamma_\mathrm{DI} := \sqrt{4 + 2\sqrt{2}} \simeq 2.6131 \ ,$$

Table 1: Achievable Control Performance Levels and Optimal Controllers for Multiple Integrators ($\xi := \sqrt{2}+1$) [5].

$P_n(s)$	$\gamma_{\rm opt}$	Optimal Controller
$\frac{k}{s}$	$\sqrt{2}=\sqrt{1+\xi^0}\simeq 1.4142$	1
$\left(\frac{k}{s}\right)^2$	$\sqrt{4+2\sqrt{2}}=\sqrt{1+\xi^2}\simeq 2.6131$	$\frac{\xi s/k+1}{s/k+\xi}$
$\left(\frac{k}{s}\right)^3$	$2\sqrt{3}+\sqrt{6}=\sqrt{1+\xi^4}\simeq 5.9135$	$\frac{\xi^2(s/k)^2+\sqrt{2}\xi s/k+1}{(s/k)^2+\sqrt{2}\xi s/k+\xi^2}$
$\left(\frac{k}{s}\right)^4$	$\simeq 15.2898$	(too lengthy to include)

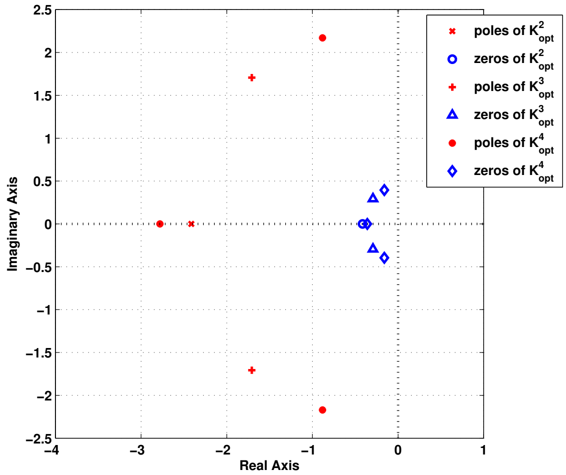

Figure 2: Locations of poles and zeros of $K_{\rm opt}(s)$ of $P_n(s) = \frac{1}{s^n}$ for $n = 2,3,4$ [5].

which is smaller than $2\sqrt{2} \simeq 2.8284$. It is therefore judged that the system is easily controllable. It is also noted that $\gamma_{\rm opt} \le \gamma_{\rm DI}$ guarantees the phase margin of at least 45[degree]. This can be seen from Proposition 1 and

$$2\arcsin\left(\frac{1}{\gamma_{\rm DI}}\right) = 45[\text{degree}] .$$

As for $n = 3$, the system is a triple integrator and the phase is -270[degree] for all frequencies. The optimal controller is again a phase-lead compensator which achieves at the crossover frequency of the system $P_3(s)$ the maximum phase-lead, which is

$$2\arctan\sqrt{2} \simeq 109.471[\text{degree}]$$

(Figure 4). This yields phase margin

$$180 - 2\arcsin\frac{\sqrt{2}+1}{\sqrt{6}} \simeq 19.471[\text{degree}] .$$

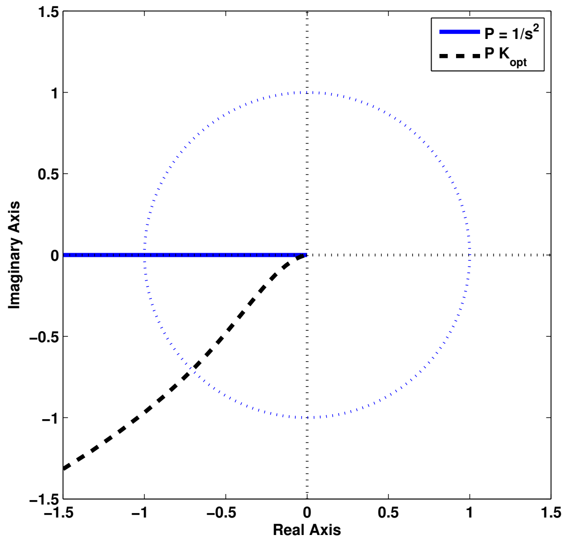

Figure 3: Nyquist plots of $P(s) = \frac{1}{s^2}$ and $P(s)K_{\mathrm{opt}}(s)$ [5].

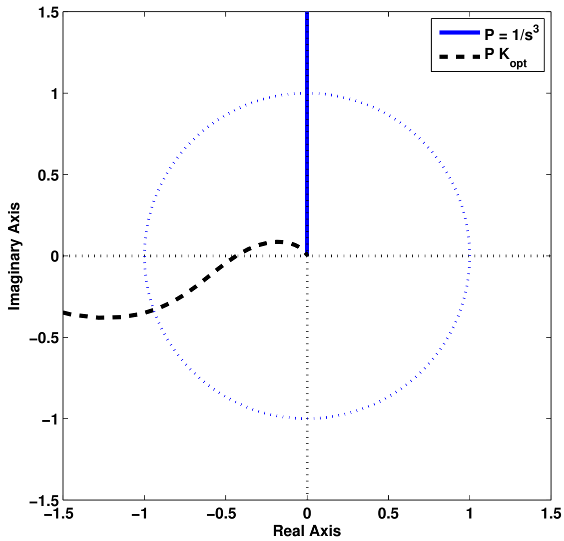

Figure 4: Nyquist plots of $P(s) = \frac{1}{s^3}$ and $P(s)K_{\mathrm{opt}}(s)$ [5].

For this system, γ_{opt} is much larger than γ_{DI} (Table 1), which implies that the system is not easy to control.

Judging from the values of γ_{opt}, it is desirable for a system to have some property similar to $P_2(s)$, rather than $P_3(s)$. It is therefore considered meaningful to identify characteristics leading to $\gamma_{\mathrm{opt}} \leq \gamma_{\mathrm{DI}}$. Also, $P_2(s)$ is on the boundary of Condition (π_{c}) as is stated. Therefore, in the continuous-time case, it is desired to establish a strong connection between Condition (π_{c}) and the relationship $\gamma_{\mathrm{opt}} \leq \gamma_{\mathrm{DI}}$.

The situation is slightly different in the case of discrete-time systems. As an example, consider a discrete-time integrator

$$P(z) = \frac{k}{z-1} \quad (k > 0) .$$

Unlike an integrator in continuous-time, the phase of this integrator changes from -90[degree] to -180[degree]; in fact no dynamical system in discrete-time has a constant phase. Also this shows that a first order discrete-time systems can have phase delay of 180[degree]. For this plant the achievable performance level can be written as

$$\gamma_{\text{opt}} = \frac{\sqrt{8 + 2k^2 + 2k\sqrt{k^2+4}}}{2} .$$

Note that γ_{opt} depends on the system gain k, unlike the continuous-time counterpart. It is more important to observe that γ_{opt} is an increasing function in k and that, when $k = 2$, the plant is on the boundary of Condition (π_{d}) and also $\gamma_{\text{opt}} = \gamma_{\text{DI}}$. Even a simple integrator exhibits different characteristics in the continuous-time and discrete-time cases. However it is pointed out that γ_{opt} shows the identical feature when seen through Conditions (π_{c}) and (π_{d}). This motivates to clarify how strong the connection between Condition (π_{c})/(π_{d}) and $\gamma_{\text{opt}} \leq \gamma_{\text{DI}}$ is.

4 Continuous-time Systems

In this section an investigation is made into the relationship between the achievable performance level γ_{opt} and the phase/gain property of the system, namely, Condition (π_{c}), for some classes of 2nd and 3rd order *continuous-time* systems.

4.1 2nd Order Systems

This subsection focuses on the 2nd order case. All the 2nd order continuous-time systems belonging to $\mathscr{P}_{\text{c}}$ can be parametrized as follows:

$$\mathscr{P}_{\text{c2}} := \left\{ P(s) = k\frac{\beta s + 1}{s^2 + \alpha_1 s + \alpha_0} \,\middle|\, \alpha_0 \geq 0,\, \alpha_1 \geq 0,\, k > 0 \right\} \subset \mathscr{P}_{\text{c}} .$$

Depending on the sign of β, $P(s) \in \mathscr{P}_{\text{c2}}$ exhibits a distinct property; P is minimum phase if $\beta \geq 0$, while P is non-minimum phase if $\beta < 0$. For a detailed investigation, $\mathscr{P}_{\text{c2}}$ is divided into two classes:

Minimum Phase:	$\mathscr{P}_{\text{c2+}} := \left\{ P(s) \in \mathscr{P}_{\text{c2}} \,\middle|\, \beta \geq 0 \right\}$,
Non-Minimum Phase:	$\mathscr{P}_{\text{c2}-} := \left\{ P(s) \in \mathscr{P}_{\text{c2}} \,\middle|\, \beta < 0 \right\}$.

The phase delay of a system in $\mathscr{P}_{\text{c2+}}$ is at most 180[degree], and thus all systems in $\mathscr{P}_{\text{c2+}}$ belong to $\mathscr{P}_{\text{c}}^{\text{g}}$. On the other hand the phase delay of any system $P(s)$

in $\mathscr{P}_{c2-}$ exceeds 180[degree]. The phase delay of $P(s)$ reaches 180[degree] at $\omega = \omega_\pi := \sqrt{\alpha_0 + \alpha_1/(-\beta)}$ (which depends on all α_0, α_1 and β), and the gain at that frequency is $|P(j\omega_\pi)| = k(-\beta)/\alpha_1$ (which is independent of α_0). Therefore, $P(s)$ belongs to $\mathscr{P}_c^g$ if and only if $k \le \alpha_1/(-\beta)$ (which is also independent of α_0). Namely the class of systems that belong to both $\mathscr{P}_{c2-}$ and $\mathscr{P}_c^g$ can be written as

$$\mathscr{P}_{c2-}^g := \mathscr{P}_{c2-} \cap \mathscr{P}_c^g = \left\{ P(s) \in \mathscr{P}_{c2-} \,\middle|\, 0 < k \le \frac{\alpha_1}{-\beta} \right\} .$$

The following theorem states that, for any $P(s)$ in

$$\mathscr{P}_{c2}^g := \mathscr{P}_{c2} \cap \mathscr{P}_c^g = \mathscr{P}_{c2+} \cap \mathscr{P}_{c2-}^g ,$$

γ_{opt} is bounded by γ_{DI} from above.

Theorem 3 ([5]) *For any system in $\mathscr{P}_{c2}^g$, it holds that $\gamma_{\text{opt}} \le \gamma_{\text{DI}}$. The equality holds when*

- $\alpha_0 = \alpha_1 = \beta = 0$ *(for $P(s) \in \mathscr{P}_{c2+}$);*
- $\alpha_0 = 0$*,* $k = \alpha_1/(-\beta)$ *(for $P(s) \in \mathscr{P}_{c2-}^g$).*

The proof is given in Appendix A.1.

For $P(s) \in \mathscr{P}_{c2-}$, as k increases, the control bandwidth gets wider, but the phase delay exceeds 180[degree]. Under this situation the control performance measured by γ_{opt} becomes worse without bound. In other words the bandwidth that can be achievable in practice has some limitation, which also agrees with the knowledge implied by FFPR. For example consider the following system:

$$P(s) = \frac{k(-\frac{1}{10}s + 1)}{s(s+1)} \in \mathscr{P}_{c2-} .$$

When $0 < k \le 10$, $P(s)$ satisfies Condition (π_c). It can be confirmed that $\gamma_{\text{opt}} = \gamma_{\text{DI}}$ when $k = 10$, i.e., when $P(s)$ is just on the boundary of Condition (π_c). The Bode and Nyquist plots of $P(s)$, $K_{\text{opt}}(s)$ and $P(s)K_{\text{opt}}(s)$ are shown in Figures 5 and 6, respectively. It can also be shown that, for large k,

$$\gamma_{\text{opt}} \simeq \frac{1}{55}k + \frac{40}{11} .$$

Notice that γ_{opt} tends to $+\infty$ as $k \to +\infty$ (Figure 7).

In this subsection it is shown that any 2nd order (marginally) stable system satisfying Condition (π_c) has γ_{opt} less than or equal to γ_{DI} and thus that it is easy to control. The equality condition suggests that the bound is tight. Furthermore, in the case of a system whose phase delay exceeds 180[degree] at the gain crossover frequency, γ_{opt} can be arbitrarily large and the system is in general difficult to control.

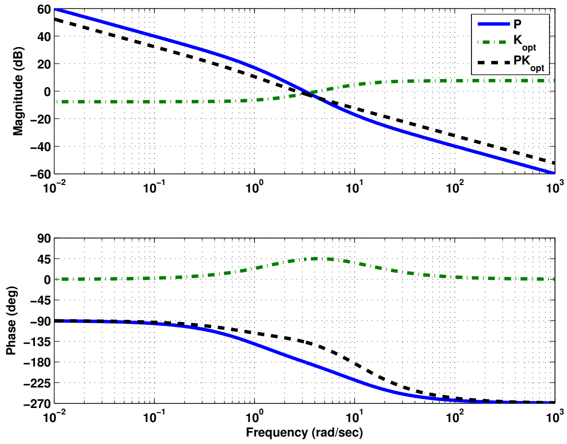

Figure 5: Bode plots of $P(s) = \frac{10(-\frac{1}{10}s+1)}{s(s+1)}$, $K_{\text{opt}}(s)$, and $P(s)K_{\text{opt}}(s)$.

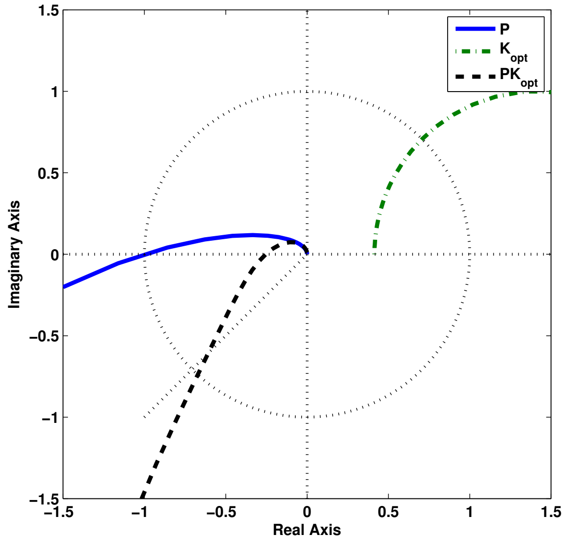

Figure 6: Nyquist plots of $P(s) = \frac{10(-\frac{1}{10}s+1)}{s(s+1)}$, $K_{\text{opt}}(s)$, and $P(s)K_{\text{opt}}(s)$.

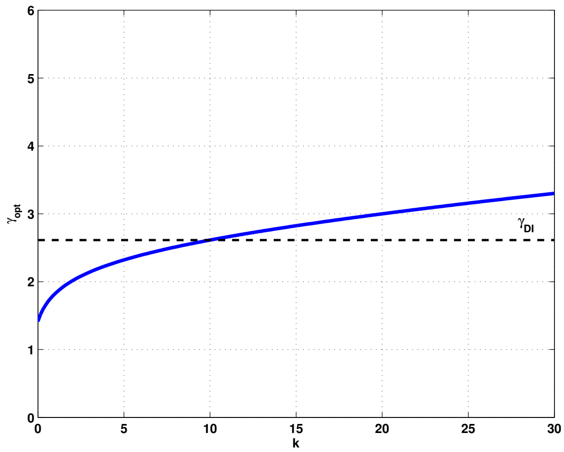

Figure 7: γ_{opt} of $P(s) = \frac{k(-\frac{1}{10}s+1)}{s(s+1)}$.

4.2 3rd Order Systems

It is no straightforward to derive a result for general higher order systems. Hence this subsection only treats two particular classes of 3rd order systems.

4.2.1 Integrator + a particular 2nd order (marginally) stable, minimum phase system

The first class to be investigated is a subclass of marginally stable, minimum phase systems:

$$\mathscr{P}_{\text{c3f2}} := \left\{ P(s) = \frac{1}{s}\frac{\alpha_1 s+1}{s+\alpha_1}\frac{\alpha_2 s+1}{s+\alpha_2} \;\middle|\; \alpha_1 \geq 0,\, \alpha_2 \geq 0 \right\} \subset \mathscr{P}_{\text{c}} \,.$$

Even though any system in $\mathscr{P}_{\text{c3f2}}$ is neither unstable nor non-minimum phase, the phase delay of some systems in $\mathscr{P}_{\text{c3f2}}$ goes beyond 180[degree]. When $0 \leq \alpha_i < 1$, the $(\alpha_i s+1)/(s+\alpha_i)$ part retards the phase; the maximum phase delay occurs at $\omega = 1$ and the phase there is $-\arcsin\big((1-\alpha_i^2)/(1+\alpha_i^2)\big)$. It is expected and also confirmed numerically that the smaller α_i becomes, the larger γ_{opt} is, i.e., the worse the achievable performance level is (Figure 8).

For this class of systems, there is again a relationship between Condition (π_{c}) and $\gamma_{\text{opt}} \leq \gamma_{\text{DI}}$, which is stated in the following proposition. The proof is provided in Appendix A.2.

Proposition 4 *For $P(s) \in \mathscr{P}_{\text{c3f2}}$, the phase delay at the gain crossover frequency $\omega = 1$ is* 180[degree] *if $(\alpha_1+1)(\alpha_2+1) = 2$ ($\alpha_2 = (1-\alpha_1)/(1+\alpha_1)$). Also, under this condition, $\gamma_{\text{opt}} \leq \gamma_{\text{DI}}$, and the equality holds when $\alpha_1 = 0$, $\alpha_2 = 1$ or $\alpha_1 = 1$, $\alpha_1 = 0$.*

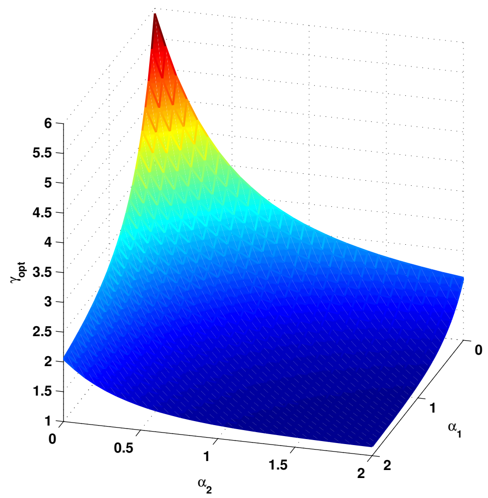

Figure 8: $\gamma_{\rm opt}$ of $P(s) = \frac{1}{s}\frac{\alpha_1 s+1}{s+\alpha_1}\frac{\alpha_2 s+1}{s+\alpha_2}$.

The result above together with the monotonicity property illustrated in Figure 8 leads to the following fact: $\gamma_{\rm opt} \le \gamma_{\rm DI}$ holds for all $P(s)$ in $\mathscr{P}_{\rm c3f2}$ with α_1 and α_2 satisfying $(\alpha_1+1)(\alpha_2+1) \ge 2$.

4.2.2 Integrator + 1st order system with time-delay

This subsection considers a system consisting of a constant gain, an integrator, a first order delay and a time-delay:

$$P(s) = k\frac{e^{-Ls}}{s(1+Ts)} \quad (k>0,\ L\ge 0,\ T>0).$$

Time/frequency scaling ($\tilde{s} = Ts$) and Padé approximation are applied to this plant:

$$P(\tilde{s}) = k\frac{Te^{-\frac{L}{T}\tilde{s}}}{\tilde{s}(1+\tilde{s})} \simeq k\frac{T}{\tilde{s}(1+\tilde{s})}\frac{1-\frac{1}{2}\frac{L}{T}\tilde{s}}{1+\frac{1}{2}\frac{L}{T}\tilde{s}} =: P_{\rm approx}(\tilde{s}) \ ,$$

and investigation is made on the class of approximated plants

$$\mathscr{P}_{\rm c,approx} := \left\{ P_{\rm approx}(\tilde{s}) \,\middle|\, k>0,\ L\ge 0,\ T>0 \right\} .$$

Let $h \equiv L/T$. Then the frequency $\tilde{\omega}_\pi$ at which the phase delay reaches 180[degree] and the gain at $\tilde{\omega} = \tilde{\omega}_\pi$ can be written as

$$\tilde{\omega}_\pi = \frac{2}{\sqrt{h(4+h)}} \ , \quad |P_{\rm approx}(j\tilde{\omega}_\pi)| = k\frac{L(4+h)}{2(2+h)} \ , \tag{7}$$

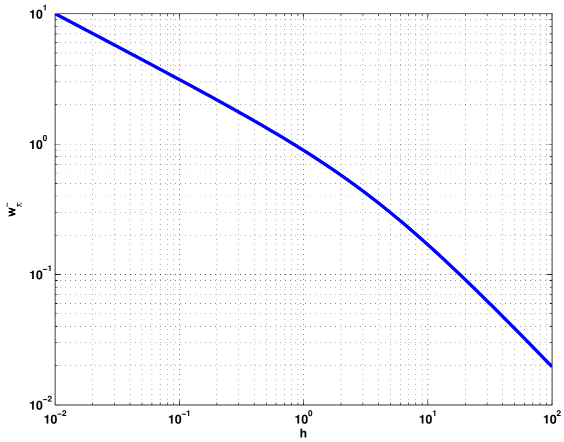

Figure 9: Achievable crossover frequency $\tilde{\omega}_\pi$ of $P_{\mathrm{approx}}(\tilde{s})$.

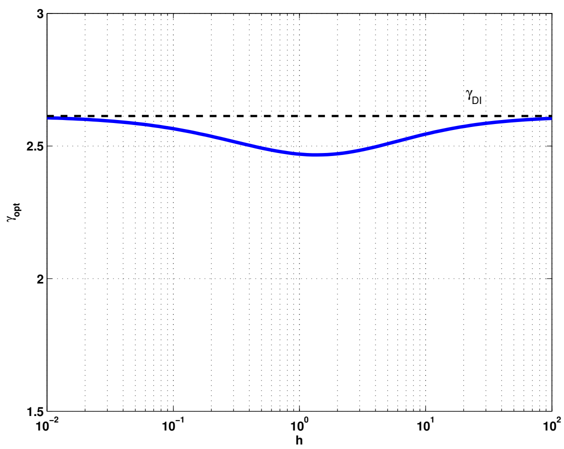

Figure 10: Achievable performance level γ_{opt} of $P_{\mathrm{approx}}(\tilde{s})$ with $k = \tilde{k}_\pi$.

respectively. In Figure 9, $\tilde{\omega}_\pi$ is plotted against h. Equation (7) implies that

$$P_{\mathrm{approx}}(s) \in \mathscr{P}_{\mathrm{c}}^{\mathrm{g}} \quad \text{if} \quad (0 <) \, k \leq \tilde{k}_\pi := \frac{2(2+h)}{L(4+h)} \, .$$

When $L = 0$ (i.e., when there is no time delay), $P_{\mathrm{approx}}(s)$ belongs to $\mathscr{P}_{c2+}$ for arbitrary k and T. This can be observed from $\tilde{k}_\pi$, since $\tilde{k}_\pi \to +\infty$ as $L \to 0$. It is a direct consequence of Theorem 3 that $\gamma_{\mathrm{opt}} \leq \gamma_{\mathrm{DI}}$ when $L = 0$.

Now it is numerically demonstrated that, if k is chosen so that $P_{\mathrm{approx}}(s) \in \mathscr{P}_{\mathrm{c}}^{\mathrm{g}}$, the best achievable performance γ_{opt} does not exceed γ_{DI}. Figure 10 depicts the values of γ_{opt} against h, when k is taken to be $\tilde{k}_\pi$, and it can be observed that γ_{opt} is smaller than γ_{DI} (γ_{opt} approaches γ_{DI} as h tends to 0 or ∞). It should be pointed out that a good performance level can be achieved even when the delay is large (i.e., for large L), but that it is at the cost of smaller bandwidth (i.e., smaller $\tilde{\omega}_\pi$).

5 Discrete-time Systems

This section investigates the relationship between the phase/gain property and the achievable performance level for some classes of *discrete-time* systems. Although the situation of the discrete-time case is different from that of the continuous-time case, it will be shown that Condition ($\pi_{\rm d}$) and $\gamma_{\rm DI}$ have a tight connection as well.

5.1 1st Order Systems

If a 1st order continuous-time system is strictly proper and (marginally) stable, then the phase only delays up to 90[degree] (at $\omega = \infty$). This means that the system is positive real and that $\gamma_{\rm opt}$ is always less than or equal to $\sqrt{2} \simeq 1.4142$ [8]. However the situation is different in the case of discrete-time systems. For a 1st order strictly proper and (marginally) stable discrete-time system, the phase always reaches -180[degree] at $\theta = \pi$ ($z = -1$). Judging from the results for continuous-time systems, one may guess that $\gamma_{\rm opt}$ for such a discrete-time system becomes larger than $\sqrt{2}$. In this section it is seen that this conjecture is in fact the case. The class of discrete-time systems which are 1st order strictly proper and (marginally) stable is defined:

$$\mathscr{P}_{\rm d1} := \left\{ P(z) = \frac{k}{z-\alpha} \;\middle|\; |\alpha| \le 1, k > 0 \right\} \subset \mathscr{P}_{\rm d} \, .$$

For a system belonging to this class, the following proposition holds.

Proposition 5 *For $P(z) \in \mathscr{P}_{\rm d1}$, a necessary and sufficient condition for $\gamma_{\rm opt} \le \gamma_{\rm DI}$ to hold is*

$$k \le 1 + \sqrt{2-\alpha^2} \, . \tag{8}$$

The proof is given in Appendix B.1.

As is pointed out in Subsection 3.2, there is no upper bound for $\gamma_{\rm opt}$ for this class of systems. Take another example. In the case of $\alpha = 0$,

$$\gamma_{\rm opt} = \sqrt{k^2+1} \, ,$$

which suggests that $\gamma_{\rm opt}$ tends to ∞ as $k \to \infty$.

It is already mentioned that the phase delay of any system belonging to $\mathscr{P}_{\rm d1}$ reaches 180[degree] at $\theta = \pi$ ($z = -1$). The gain there is $|P(-1)| = k/(1+\alpha)$. Thus the class of 1st order stable systems belonging to $\mathscr{P}_{\rm d}^{\rm g}$ can be written as

$$\mathscr{P}_{\rm d1}^{\rm g} := \mathscr{P}_{\rm d1} \cap \mathscr{P}_{\rm d}^{\rm g} = \left\{ P(z) = \frac{k}{z-\alpha} \;\middle|\; |\alpha| \le 1, 0 < k \le \alpha + 1 \right\} .$$

Since systems in this class satisfy the condition stated in Proposition 5, the following theorem is immediately obtained.

Theorem 6 *For any $P(z) \in \mathscr{P}_{\mathrm{d1}}^{\mathrm{g}}$, it holds that $\gamma_{\mathrm{opt}} \leq \gamma_{\mathrm{DI}}$. The equality holds when $\alpha = 1$, $k = 2$.*

As a special case of first order systems, discretized 1st order continuous-time (marginally) stable systems are considered and the effects of the pole location and the sampling time on the achievable performance level is investigated. Write the 1st order continuous-time system as

$$P(s) = \frac{b}{s+a} \quad (a \geq 0) \ .$$

Its zero-order hold discrete-time equivalent system (with sampling time T) is

$$P(z) = \begin{cases} \dfrac{bT}{z-1} & \text{if } a = 0 \ , \\ \dfrac{b}{a}\dfrac{1-e^{-aT}}{z-e^{-aT}} & \text{if } a > 0 \ . \end{cases} \tag{9}$$

It is noted that the expression for $a = 0$ is not a singular case in that the expression for $a > 0$ converges to that for $a = 0$ when a tends to 0 since $e^{-aT} \to 1$ and $\frac{b}{a}\big(1 - e^{-aT}\big) \to bT$ as $a \to 0$.

The following is immediate from Proposition 5.

Proposition 7 *When $a > 0$, $\gamma_{\mathrm{opt}} \leq \gamma_{\mathrm{DI}}$ holds if*

$$b \leq \frac{a}{1-e^{-aT}}\left(1+\sqrt{2-e^{-2aT}}\right) \simeq \frac{2}{T} + 2a - \frac{6}{5}a^2T + \frac{3}{2}a^3T^2 - \cdots \ . \tag{10}$$

In the case of $a = 0$, the condition is simply

$$b \leq \frac{2}{T} \ ,$$

which implies that the approximation in (10) *also holds for $a = 0$.*

The result can be interpreted as follows. When the sampling time T is small, good performance may be achieved even when the gain of the original continuous-time plant, b, is large. Also, if the original system is more stable (i.e., larger a), then b can be made larger without losing good performance.

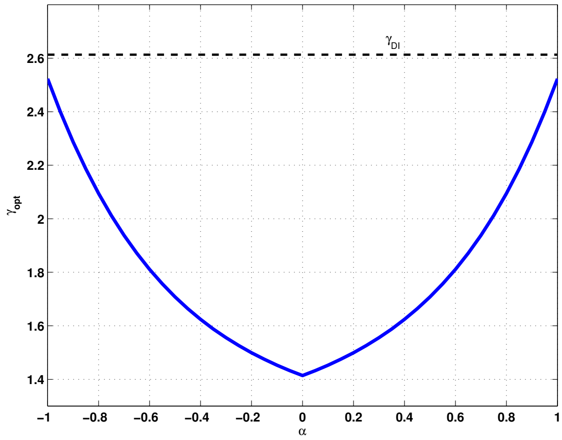

Figure 11: $\gamma_{\rm opt}$ of $P(z) = \frac{1}{z(z-\alpha)}$.

5.2 2nd Order Systems

An investigation is further made on some classes of 2nd order discrete-time systems and it is shown that, when a system belongs to $\mathscr{P}_{\rm d}^{\rm g}$, $\gamma_{\rm opt}$ is bounded by $\gamma_{\rm DI}$ from above.

5.2.1 One-step delay + first order system

Here the following class of systems is considered:

$$\mathscr{P}_{\rm d2d} := \left\{ P(z) = \frac{k}{z(z-\alpha)} \,\middle|\, |\alpha| \leq 1, k > 0 \right\} \subset \mathscr{P}_{\rm d} \, .$$

A system in this class has the form of a first order system plus one-step delay. At $\theta = \theta_\pi := \arccos(\alpha/2)$, the phase is -180[degree] and the gain is $\left|P(e^{j\theta_\pi})\right| = k$. Therefore,

$$\mathscr{P}_{\rm d2d}^{\rm g} := \mathscr{P}_{\rm d2d} \cap \mathscr{P}_{\rm d}^{\rm g} = \left\{ P(z) = \frac{k}{z(z-\alpha)} \,\middle|\, |\alpha| \leq 1, 0 < k \leq 1 \right\} .$$

For this class of systems, $\gamma_{\rm DI}$ is again an upper bound for $\gamma_{\rm opt}$.

Theorem 8 *For any $P(z) \in \mathscr{P}_{\rm d2d}^{\rm g}$, it holds that $\gamma_{\rm opt} \lneqq \gamma_{\rm DI}$.*

The proof is given in Appendix B.2.

Figure 11 depicts $\gamma_{\rm opt}$ against α when k is fixed to be 1. Also it can be deduced that $\gamma_{\rm opt}$ increases monotonically as k increases, when α is fixed. This figure therefore shows that the upper bound $\gamma_{\rm DI}$ is not a tight bound in this case.

5.2.2 Integrator + first order system

Another class of 2nd order systems is dealt with:

$$\mathscr{P}_{\rm d2i} := \left\{ P(z) = \frac{k}{(z-1)(z-\alpha)} \,\middle|\, |\alpha| \le 1, k > 0 \right\} \subset \mathscr{P}_{\rm d}\ .$$

That is, systems under consideration are in the form of a first order system plus an integrator. It is shown that systems in $\mathscr{P}_{\rm d2i}$ that satisfy Condition ($\pi_{\rm d}$) have $\gamma_{\rm opt}$ smaller than or equal to $\gamma_{\rm DI}$, but that the upper bound $\gamma_{\rm DI}$ is tight in this case, unlike the class considered just above. For $P(z) \in \mathscr{P}_{\rm d2i}$, the phase reaches -180[degree] at $\theta = \theta_\pi := \arccos\big((\alpha+1)/2\big)$. The gain there is $\left|P(e^{j\theta_\pi})\right| = k/(1-\alpha)$. Hence,

$$\mathscr{P}^{\rm g}_{\rm d2i} := \mathscr{P}_{\rm d2i} \cap \mathscr{P}^{\rm g}_{\rm d} = \left\{ P(z) = \frac{k}{(z-1)(z-\alpha)} \,\middle|\, |\alpha| \le 1, 0 < k \le 1-\alpha \right\}.$$

The following propositions are proven for two classes of extreme cases in $\mathscr{P}_{\rm d2i}$, namely, the cases of $\alpha = \pm 1$:

$$\mathscr{P}_{\rm d2ip} := \left\{ P(z) = \frac{k}{(z-1)^2} \,\middle|\, k > 0 \right\},$$
$$\mathscr{P}_{\rm d2im} := \left\{ P(z) = \frac{k}{(z-1)(z+1)} \,\middle|\, k > 0 \right\}.$$

The proofs are given in Appendices B.3 and B.4, respectively.

Proposition 9 *None of $P(z) \in \mathscr{P}_{\rm d2ip}$ satisfies Condition ($\pi_{\rm d}$). Also, for any $P(z)$ belonging to $\mathscr{P}_{\rm d2ip}$, $\gamma_{\rm opt}$ is strictly greater than $\gamma_{\rm DI}$ (i.e., $\gamma_{\rm opt} \gneq \gamma_{\rm DI}$). Furthermore, $\gamma_{\rm opt} \to \gamma_{\rm DI}$ as $k \to 0$.*

Unlike the class $\mathscr{P}_{\rm d2ip}$, the class $\mathscr{P}_{\rm d2im}$ contains some systems satisfying Condition ($\pi_{\rm d}$). More specifically,

$$\mathscr{P}^{\rm g}_{\rm d2im} := \mathscr{P}_{\rm d2im} \cap \mathscr{P}^{\rm g}_{\rm d} = \left\{ P(z) = \frac{k}{(z-1)(z+1)} \,\middle|\, 0 < k \le 2 \right\}.$$

Systems in $\mathscr{P}^{\rm g}_{\rm d2im}$ show a nice property, just like other systems that satisfy Condition ($\pi_{\rm d}$).

Proposition 10 *For any $P(z) \in \mathscr{P}^{\rm g}_{\rm d2im}$, it holds that $\gamma_{\rm opt} \le \gamma_{\rm DI}$. The equality holds when $k = 2$.*

Propositions 9 and 10 state that $\gamma_{\rm DI}$ is a tight upper bound for $\gamma_{\rm opt}$ when $\alpha = \pm 1$ and the system is in $\mathscr{P}^{\rm g}_{\rm d2i}$, unlike the case of $\mathscr{P}^{\rm g}_{\rm d2d}$. For general $\alpha \in (-1,\, 1)$, this is not the case, but it can be shown numerically that the relationship between Condition ($\pi_{\rm d}$) and $\gamma_{\rm opt} \le \gamma_{\rm DI}$ holds. Systems $P(z) = (1-\alpha)/\big((z-1)(z-\alpha)\big)$, $|\alpha| \le 1$ are on the boundary of the class $\mathscr{P}^{\rm g}_{\rm d2i}$, and the values of $\gamma_{\rm opt}$ for these systems are plotted in Figure 12. It is observed that $\gamma_{\rm opt}$ is smaller than $\gamma_{\rm DI}$, but that these values are fairly close.

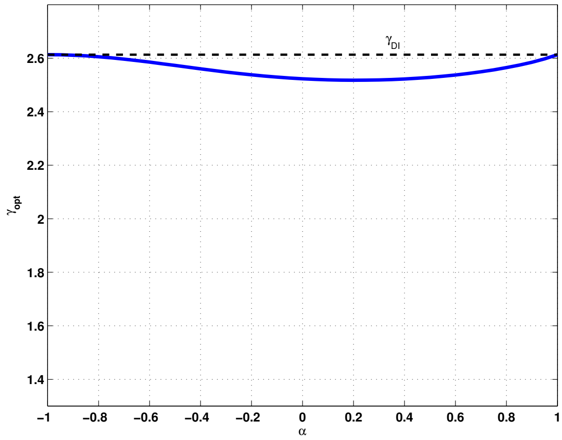

Figure 12: γ_{opt} of $P(z) = \frac{1-\alpha}{(z-1)(z-\alpha)}$.

6 $\mathscr{H}_\infty$ Loop Shaping Design Based on Condition $(\pi_{\mathrm{c}})/(\pi_{\mathrm{d}})$ with Design Example

The results presented so far are for low order systems. Nevertheless the relationship between Condition $(\pi_{\mathrm{c}})/(\pi_{\mathrm{d}})$ and the value of γ_{opt} is expected to hold approximately for higher order systems. This is because the ν-gap between two systems having a similar phase/gain property around the gain crossover frequency is small, and moreover, if the ν-gap between two systems is small, then the two systems have similar performance limitations in terms of γ_{opt} [9]. This section proposes a robust design approach based on this observation that Condition $(\pi_{\mathrm{c}})/(\pi_{\mathrm{d}})$ has a strong relationship to the achievable performance level even for higher order systems.

In the classical loop shaping design approach, a designer would shape the open-loop transfer function so that design requirements may be achieved as well as the closed-loop stability and some degree of robustness. In order to attain robustness, a controller would be designed so that the Nyquist plot of the open-loop may stay away from the point $-1 + j0$. When the $\mathscr{H}_\infty$ loop shaping design procedure is employed, a designer can rely on the procedure for accomplishing stability and robustness and the success level is indicated by the value of γ_{opt}.

Consequently, based on the results in the preceding sections and the above observation, a designer can concentrate on achieving a wider bandwidth by shaping the open-loop transfer function by means of a weight; the key point is to identify the frequency where the phase delay reaches 180[degree] and to adjust the gain etc. so that the gain at that frequency may be (close to) 1 and the plant may satisfy Condition $(\pi_{\mathrm{c}})/(\pi_{\mathrm{d}})$.

Thus the following two-step procedure for robust control design for a given plant P_0 can be suggested.

Step 1: Choose a weight W so that the weighted plant $P := WP_0$ may satisfy Condition (π_c)/(π_d).

Step 2: Apply the $\mathscr{H}_\infty$ loop shaping design procedure to P and get the (sub)optimal controller K_w. The controller to be implemented is $K = WK_w$.

It should be re-emphasized that the designer can concentrate on the first step which determines the control bandwidth, since $\mathscr{H}_\infty$ loop shaping design then automatically guarantees a good closed-loop robustness property.

The procedure is demonstrated on a torsion disk system whose transfer function is written as

$$P_0(s) = \frac{6.27\times 10^5 s^2 + 4.69\times 10^5 s + 8.72\times 10^8}{s^6 + 5.56s^5 + 5386s^4 + 2.20\times 10^4 s^3 + 5.26\times 10^6 s^2 + 1.05\times 10^7 s} .$$

The Bode and Nyquist plots of this system are plotted in Figures 13 and 14, respectively. The system is composed of an integrator, a first-order system, and two oscillatory modes: a reversed-phase mode (35.7[rad/sec]) and an in-phase mode (64.0[rad/sec]). Since there is a reversed-phase oscillatory mode, the plant is in general considered difficult to control.

The plant is shaped as follows so that the weighted plant may meet Condition (π_c) and the bandwidth may be expanded, and the $\mathscr{H}_\infty$ loop shaping design procedure is then applied to the shaped plant. The approach taken here is to reduce the peak of the gain caused by the in-phase oscillatory mode in order for the resulting controller not to excite the mode and also for Condition (π_c) to be easily met. Note that the same idea is not applicable to a reversed-phase mode; such a mode has to be controlled so that a good performance can be accomplished.

- Use the following notch filter to suppress the in-phase oscillatory mode:

$$F(s) = \frac{s^2 + 2\zeta\omega_N s + \omega_N^2}{s^2 + 2\eta\omega_N s + \omega_N^2} \quad (\zeta > 0,\ \eta > 0,\ \omega_N = 64.0) .$$

- Adjust the constant gain k so that the gain of $P(s) := kF(s)P_0(s)$ at the phase crossover frequency may be 0[dB].

There are therefore three parameters in $W(s) := kF(s)$ that can be adjusted, namely, k, η, and ζ. The following values are used here:

- $k = 1.33$, $\zeta = 0.1$, $\eta = 0.3$.

See Figures 13 and 14, respectively, for the Bode and Nyquist plots of the weighted plant $W(s)P_0(s)$. It can be observed that $W(s)P_0(s)$ satisfies Condition (π_c).

The optimal controller $K_{opt}(s)$ is designed for $W(s)P_0(s)$, and the Bode and Nyquist plots of $W(s)P_0(s)$ and $W(s)P_0(s)K_{opt}(s)$ are depicted in Figures 15 and 16, respectively. The achieved performance level is $\gamma_{opt} = 2.6797$, and the gain crossover frequency is $\omega_c = 13.84$[rad/s]. Also, GM $= 7.72$[dB] and PM $= 44.6$[degree], and the resulting closed-loop system is deemed a good design.

This clearly shows the effectiveness of Condition (π_c)/(π_d) to characterize a set of good plants even for higher order systems and thus the efficacy of the proposed approach.

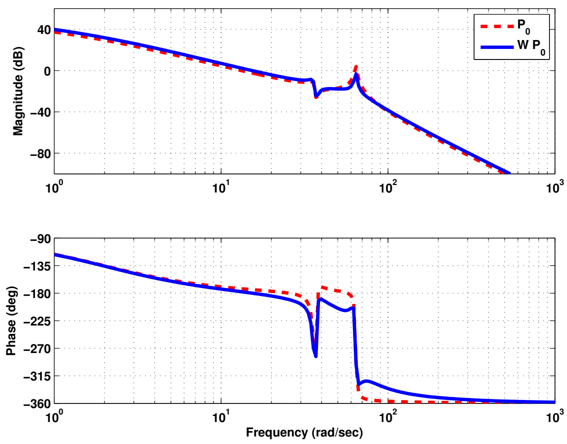

Figure 13: Bode plots of the torsion disk and the weighted plant.

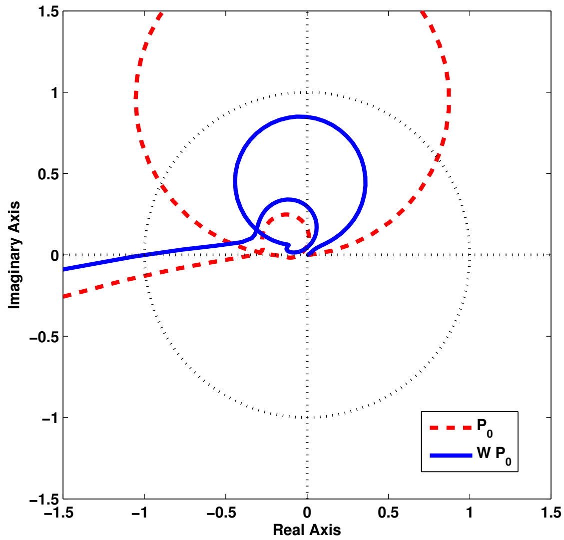

Figure 14: Nyquist plots of the torsion disk and the weighted plant.

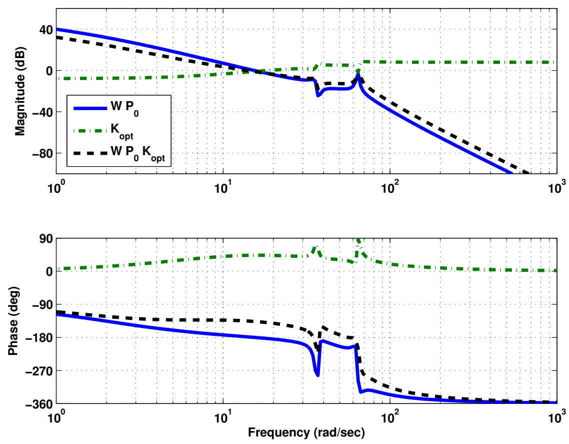

Figure 15: Bode plots of the weighted plant + controller.

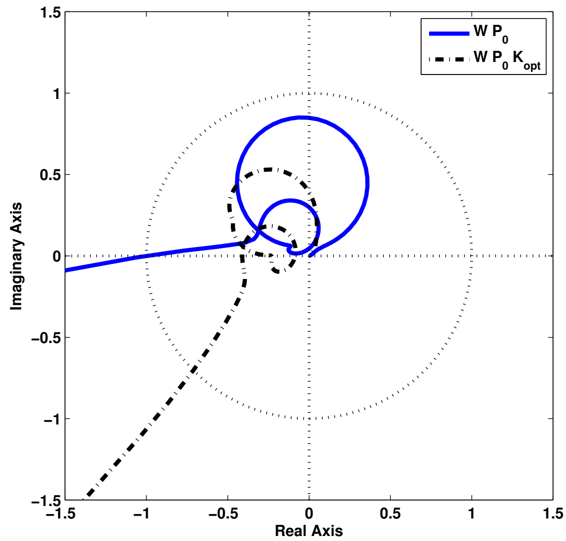

Figure 16: Nyquist plots of the weighted plant + controller.

7 Conclusion

In this paper a phase/gain condition, Condition (π_c)/(π_d), has been introduced to characterize easily controllable (marginally) stable plants, both in the continuous-time and discrete-time cases. It has been shown that there is a strong connection between Condition (π_c)/(π_d) and the value of the best achievable performance level γ_{opt} in $\mathscr{H}_\infty$ loop shaping design. Furthermore a simple design procedure for robust control is proposed which makes use of the relationship and then relies on $\mathscr{H}_\infty$ loop shaping design in order to achieve stability and robustness.

More specifically,

- A phase/gain condition, Condition (π_c)/(π_d), are proposed for (marginally) stable continuous-time/discrete-time systems for characterization of easily controllable plants, which reformulates the notion of 'finite frequency positive realness'.
- **Continuous-time systems:** It is proven that all the 2nd order (marginally) stable plants satisfying Condition (π_c) has γ_{opt} less than or equal to $\sqrt{4+2\sqrt{2}}$. Two classes of 3rd order systems are further investigated and a close relationship is also observed.
- **Discrete-time systems:** All the 1st order (marginally) stable systems that satisfy Condition (π_d) are shown to have $\gamma_{opt} \leq \sqrt{4+2\sqrt{2}}$. The same relationship is seen for some classes of 2nd order systems.
- A simple design procedure for robust control is proposed, where the plant is shaped with a weight so that the weighted plant may satisfy Condition (π_c)/(π_d) and then $\mathscr{H}_\infty$ loop shaping design is applied. The crucial point is
  - to shape the plant so that the condition may easily be satisfied;
  - to find the frequency where the phase delay of the weighted plant reaches 180[degree];
  - to adjust the gain so that the gain at that frequency may be 0[dB].

  If Condition (π_c)/(π_d) is met, the achievable performance level γ_{opt} will be *approximately* less than $\sqrt{4+2\sqrt{2}}$ even for high order systems, and thus a good performance will result.

Acknowledgements

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A Proofs for the Continuous-time Case

A.1 Proof of Theorem 3

The proof is carried out for three cases: $\beta = 0$, $\beta > 0$, and $\beta < 0$. In any case, by solving the Riccati equation (2) and the Lyapunov equation (3), it is derived that

$$\gamma_{\text{opt}} = \sqrt{1 + \left(A + \sqrt{A^2 + B}\right)^2} \ ,$$

where expressions for A and B are different in each case. It is noted that $\gamma_{\text{opt}} \le \gamma_{\text{DI}}$ is equivalent to $\left|A + \sqrt{A^2 + B}\right| \le 1 + \sqrt{2}$. To show the latter condition, it is proven that

$$0 < A \le 1 \ , \tag{11}$$

$$0 < B \le 1 \ . \tag{12}$$

The equality holds when $A = B = 1$. Since γ_{opt} is always real, the condition $A^2 + B \ge 0$ is automatically satisfied.

- Case 1: $\beta = 0$: In this case,

  $$A = \frac{Z(Z - \alpha_1)}{2k} \ ,$$

  $$B = \frac{k^2 + \alpha_0\alpha_1^2 + \alpha_1^2\sqrt{k^2 + \alpha_0^2} - Z\left(\alpha_0\alpha_1 + \alpha_1\sqrt{k^2 + \alpha_0^2}\right)}{k^2} \ ,$$

  $$Z = \sqrt{\alpha_1^2 - 2\alpha_0 + 2\sqrt{k^2 + \alpha_0^2}} \ .$$

  Firstly, since $\sqrt{k^2 + \alpha_0^2} > \alpha_0$, it is straightforward to derive that

  $$Z > \alpha_1 \ (\ge 0) \ , \tag{13}$$

  which then implies that $A > 0$. Next write A as

  $$A = \underbrace{\frac{\sqrt{k^2 + \alpha_0^2} - \alpha_0}{k}}_{=:A_1} + \underbrace{\frac{\alpha_1(\alpha_1 - Z)}{2k(\alpha_0 + \alpha_1 + 1)}}_{=:A_2}$$

  It is easy to see that $A_1 \le 1$ and that the equality holds if $\alpha_0 = 0$. Also, (13) implies that $A_2 \le 0$ where the equality holds when $\alpha_1 = 0$. As a result, (11) is proven. Moreover,

  $$1 - B = \frac{\alpha_1\left(\sqrt{k^2 + \alpha_0^2} + \alpha_0\right)(Z - \alpha_1)}{k^2} \ge 0$$

  leads to (12), and the equality holds when $\alpha_1 = 0$. This concludes the proof for the case $\beta = 0$.

- Case 2: $\beta > 0$: It is firstly noted that, by frequency conversion, β can be set to 1 without loss of generality. So it is assumed here that $\beta = 1$.

  Before the main proof, a singular case is considered. When $\alpha_0 - \alpha_1 + 1 = 0$, both the numerator and the denominator of the transfer function have factor $(s+1)$, that is, a pole/zero cancellation occurs at $s = -1$. The cancellation happens in the stable region and, after eliminating the factor, the plant can be written as

$$P(s) = \frac{k}{s+\alpha_0} .$$

  This plant is obviously a positive real one and thus $\gamma_{\mathrm{opt}} \leq \sqrt{2} < \gamma_{\mathrm{DI}}$ [8]. Therefore the theorem holds for this singular case.

  Now consider the generic case $\alpha_0 - \alpha_1 + 1 \neq 0$. In this case,

$$A = \frac{Z\left(Z - \alpha_1 + \alpha_0 - \sqrt{k^2+\alpha_0^2}\right)}{2k(\alpha_0 - \alpha_1 + 1)} ,$$

$$B = \frac{k^2(\alpha_1+1) + \alpha_0\alpha_1^2 + (k^2+\alpha_1^2)\sqrt{k^2+\alpha_0^2} - Z\left(k^2 + \alpha_0\alpha_1 + \alpha_1\sqrt{k^2+\alpha_0^2}\right)}{k^2(\alpha_0 - \alpha_1 + 1)} ,$$

$$Z = \sqrt{k^2 + \alpha_1^2 - 2\alpha_0 + 2\sqrt{k^2+\alpha_0^2}} .$$

  Similar to Case 1 ($\beta = 0$), it can be shown that $Z > \alpha_1$ (≥ 0) and that $\alpha_1 - \alpha_0 + \sqrt{k^2+\alpha_0^2} > 0$. Moreover, since

$$Z^2 - \left(\alpha_1 + \sqrt{k^2+\alpha_0^2} - \alpha_0\right)^2 = 2(\alpha_0 - \alpha_1 + 1)\left(\sqrt{k^2+\alpha_0^2} - \alpha_0\right) ,$$

  the sign of the numerator of A is the same as that of the denominator, which leads to $A > 0$. Also, A can be written as

$$A = \underbrace{\frac{\sqrt{k^2+\alpha_0^2} - \alpha_0}{k}}_{=:\,A_1} + \underbrace{\frac{-\left(\sqrt{k^2+\alpha_0^2} - \alpha_0 + \alpha_1\right)\left(Z - \alpha_1 + \alpha_0 - \sqrt{k^2+\alpha_0^2}\right)}{2k(\alpha_0 - \alpha_1 + 1)}}_{=:\,A_2} .$$

  Notice that A_1 here is identical to A_1 for Case 1 and thus that $A_1 \leq 1$. Also, $A_2 < 0$ can be proven by the same method used to show that $A > 0$. (It is noted here that the equality ($A_2 = 0$) does not hold in the case of $\beta > 0$.) These together prove $A < 1$ and, as a result, (11) is shown.

  Next, in order to show (12), the following is proven:

$$1 - B = \frac{B_{\mathrm{n}}}{k^2(\alpha_0 - \alpha_1 + 1)} > 0 , \tag{14}$$

where

$$B_\mathrm{n} = \underbrace{Z\left(\alpha_0\alpha_1 + \alpha_1\sqrt{k^2+\alpha_0^2} + k^2\right)}_{=:\,B_{\mathrm{n}1}} + \underbrace{\left(-2k^2\alpha_1 - k^2\left(\sqrt{k^2+\alpha_0^2} - \alpha_0\right) - \alpha_1^2\left(\sqrt{k^2+\alpha_0^2} + \alpha_0\right)\right)}_{=:\,B_{\mathrm{n}2}} .$$

It is immediate that $B_{\mathrm{n}1} > 0$ and that $B_{\mathrm{n}2} < 0$. Also,

$$B_{\mathrm{n}1}^2 - B_{\mathrm{n}2}^2 = 2k^2(\alpha_0 - \alpha_1 + 1)\left(2k^2\alpha_1 + \alpha_1^2\left(\sqrt{k^2+\alpha_0^2} + \alpha_0\right) + k^2\left(\sqrt{k^2+\alpha_0^2} - \alpha_0\right)\right) .$$

This means that the sign of $B_{\mathrm{n}1}^2 - B_{\mathrm{n}2}^2$ is the same as that of $\alpha_0 - \alpha_1 + 1$. This further implies that the signs of B_n and $\alpha_0 - \alpha_1 + 1$ are identical, and thus (14) or, equivalently, (12) is proven. Now the proof for $\beta > 0$ is complete. Note that the equality does not hold in this case.

- Case 3: $\beta < 0$: The proof is similar to those for Cases 1 and 2, but is slightly more complicated. Again, by frequency conversion, β can be set to -1 without loss of generality. So it is assumed here that $\beta = -1$. Note the assumption $k \le \alpha_1$.

  Firstly,

$$A = \frac{Z\left(Z + \sqrt{k^2+\alpha_0^2} - \alpha_0 - \alpha_1\right)}{2k(\alpha_0 + \alpha_1 + 1)} ,$$

$$B = \frac{k^2(1-\alpha_1) + \alpha_0\alpha_1^2 + (k^2+\alpha_1^2)\sqrt{k^2+\alpha_0^2} - Z\left(\alpha_0\alpha_1 + \alpha_1\sqrt{k^2+\alpha_0^2} - k^2\right)}{k^2(\alpha_0+\alpha_1+1)} ,$$

$$Z = \sqrt{k^2 - 2\alpha_0 + \alpha_1^2 + 2\sqrt{k^2+\alpha_0^2}} ,$$

  in this case. It can easily be seen that

$$Z \ge \max(k, \alpha_1)\ (>0) ,$$

  and thus that $A > 0$. Next, A can also be written as

$$A = \underbrace{\frac{\sqrt{k^2+\alpha_0^2} - \alpha_0}{k}}_{=:\,A_1} + \underbrace{\frac{\left(\sqrt{k^2+\alpha_0^2} - \alpha_0 - \alpha_1\right)\left(Z + \sqrt{k^2+\alpha_0^2} - \alpha_0 - \alpha_1\right)}{2k(\alpha_0+\alpha_1+1)}}_{=:\,A_2} .$$

  It is straightforward to see that $A_1 \le 1$ (the equality holds when $\alpha_0 = 0$). Since $\alpha_0 + \alpha_1 \ge \sqrt{k^2+\alpha_0^2}$, it is found that $A_2 \le 0$ (the equality holds when

$\alpha_0 = 0$ and $k = \alpha_1$). It is thus concluded that (11) holds and that the equality holds when $\alpha_0 = 0$ and $k = \alpha_1$.

Next, in order to show (12), the following relationship is proven:

$$1 - B = \frac{B_{\mathrm{n}}}{k^2(\alpha_0 + \alpha_1 + 1)} \geq 0 ,$$

where

$$B_{\mathrm{n}} = \underbrace{Z\Big(\alpha_0\alpha_1 + \alpha_1\sqrt{k^2 + \alpha_0^2} - k^2\Big)}_{=:\, B_{\mathrm{n}1}} + \underbrace{2k^2\alpha_1 - (k^2 + \alpha_1^2)\Big(\sqrt{k^2 + \alpha_0^2} - \alpha_0\Big)}_{=:\, B_{\mathrm{n}2}} .$$

Since $k^2(\alpha_0 + \alpha_1 + 1) > 0$, it is sufficient to prove that $B_{\mathrm{n}} \geq 0$. When $k \leq \alpha_1$, it holds that $B_{\mathrm{n}1} \geq 0$. Therefore, $B_{\mathrm{n}1}^2 - B_{\mathrm{n}2}^2 \geq 0$ implies $B_n \geq 0$. A straightforward calculation shows that

$$\begin{aligned} &B_{\mathrm{n}1}^2 - B_{\mathrm{n}2}^2 \\ &= 2k^2(\alpha_0 + \alpha_1 + 1)\Big(\underbrace{(k^2 + \alpha_1^2)\sqrt{k^2 + \alpha_0^2}}_{=:\, B_{\mathrm{n}3}} - \underbrace{\big(k^2(\alpha_0 + 2\alpha_1) - \alpha_0\alpha_1^2\big)}_{=:\, B_{\mathrm{n}4}}\Big) . \end{aligned}$$

It is easy to show that $B_{\mathrm{n}3} > 0$ and that

$$B_{\mathrm{n}3}^2 - B_{\mathrm{n}4}^2 = k^2\big(k^2 - 2\alpha_0\alpha_1 - \alpha_1^2\big)^2 ,$$

which concludes $B_{\mathrm{n}3} \geq B_{\mathrm{n}4}$. This implies that $B_{\mathrm{n}1}^2 - B_{\mathrm{n}2}^2 > 0$, which further leads to $B_{\mathrm{n}} \geq 0$. They altogether conclude (12). Also, B becomes 1 when $\alpha_0 = 0$ and $k = \alpha_1/(-\beta)$.

Finally it is noted that, in the original frequency, the equality condition is $\alpha_0 = 0$, $k = \alpha_1/(-\beta)$, and this concludes the proof for $\beta < 0$.

A.2 Proof of Proposition 4

Exact computation shows that

$$\gamma_{\mathrm{opt}} := \sqrt{1 + (X_1 + X_2)^2} ,$$

where

$$\begin{aligned} X_1 &:= \frac{\alpha_1^2 + \sqrt{2}\alpha_1 + 1 + \sqrt{2}}{2(\alpha_1 + 1)} , \\ X_2 &:= \frac{\sqrt{\alpha_1^4 + \big(6\sqrt{2} - 4\big)\alpha_1^3 + \big(4 + 2\sqrt{2}\big)\alpha_1^2 + \big(8 - 2\sqrt{2}\big)\alpha_1 + 3 + 2\sqrt{2}}}{2(\alpha_1 + 1)} . \end{aligned}$$

Since $X_1 > 0$ and $X_2 > 0$ for $\alpha_1 \in [0,1)$, it is sufficient to prove that $X_1 \leq \frac{1+\sqrt{2}}{2}$ and $X_2 \leq \frac{1+\sqrt{2}}{2}$. To show the former, observe that

$$\frac{1+\sqrt{2}}{2} - X_1 = \frac{\alpha_1(1-\alpha_1)}{2(\alpha_1+1)} \geq 0$$

and that the equality holds when $\alpha_1 = 0,1$. Furthermore,

$$\left(\frac{1+\sqrt{2}}{2}\right)^2 - X_2^2 = \frac{\alpha_1(1-\alpha_1)\Big(\alpha_1^2 + \big(6\sqrt{2}-3\big)\alpha_1 - 2 + 6\sqrt{2}\Big)}{4(\alpha_1+1)^2} \geq 0 \ ,$$

where the equality holds again when $\alpha_1 = 0,1$. To complete the proof, it is noted that $X_1 = X_2 = \frac{1+\sqrt{2}}{2}$ when $\alpha_1 = 0,1$, which leads to $\gamma_{\text{opt}} = \gamma_{\text{DI}}$.

B Proofs for the Discrete-time Case

B.1 Proof of Proposition 5

Exact computation yields

$$\lambda_{\max}(YQ) = \frac{(\alpha^2+k^2)y + k^2}{(\alpha^2+k^2+1)y - \alpha^2 + k^2 + 1} \ ,$$

where

$$y = \frac{1}{2}\Big(\alpha^2 + k^2 - 1 + \sqrt{(\alpha^2+k^2-1)^2 + 4k^2}\Big) \ \ (>0) \ . \tag{15}$$

The condition $\gamma_{\text{opt}} \leq \gamma_{\text{DI}}$ is equivalent to $\lambda_{\max}(YQ) \leq \frac{2+\sqrt{2}}{4}$, which leads to another equivalent condition:

$$\big\{(1+\sqrt{2})^2 - \alpha^2 - k^2\big\}y - k^2 + (1+\sqrt{2})^2(1-\alpha^2) \geq 0 \ . \tag{16}$$

If $(1+\sqrt{2})^2 - \alpha^2 - k^2 \leq 0$, then (16) does not hold since $y > 0$. Therefore, $(1+\sqrt{2})^2 - \alpha^2 - k^2 \geq 0$ is a necessary condition for (16) to hold. By replacing y in (16) with (15) and taking squares to eliminate the square root in y, a condition equivalent to $\gamma_{\text{opt}} \leq \gamma_{\text{DI}}$ is shown to be $(k-1)^2 + \alpha^2 \leq 2$. Under the assumptions on k and α, this condition is the same as (8).

B.2 Proof of Theorem 8

After a lengthy calculation it can be deduced that $\gamma_{\text{opt}} \leq \gamma_{\text{DI}}$ is equivalent to

$$\begin{cases} (6\sqrt{2}-8)k^3 + \big(7-4\sqrt{2}\big)\alpha k^2 + 2\Big(\big(\sqrt{2}-1\big)\alpha^2 - \sqrt{2}\Big)k \\ \qquad\qquad + \alpha\big(\alpha^2-1\big) \leq 0 \quad (\text{for } 0 \leq \alpha \leq 1,\, k > 0) \ , \\ (6\sqrt{2}-8)k^3 - \big(7-4\sqrt{2}\big)\alpha k^2 + 2\Big(\big(\sqrt{2}-1\big)\alpha^2 - \sqrt{2}\Big)k \\ \qquad\qquad - \alpha\big(\alpha^2-1\big) \leq 0 \quad (\text{for } -1 \leq \alpha \leq 0,\, k > 0) \ . \end{cases}$$

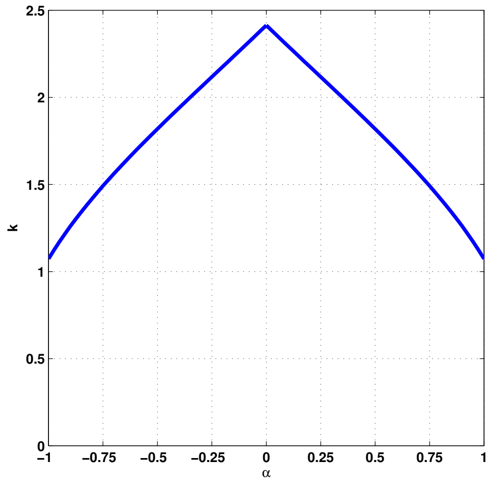

Figure 17: Region $(\alpha,\, k)$ for $\gamma_{\rm opt} \le \gamma_{\rm DI}$ (below the curve).

The sets $(\alpha,\, k)$ satisfying this condition are depicted in Figure 17. It is clear from the figure that $\{(\alpha,\, k) \,|\, |\alpha| \le 1, 0 < k \le 1\}$ is strictly included in that region. Alternatively it can be formally proven algebraically. Consider the left hand side of the condition for $0 \le \alpha \le 1$, $k > 0$. It can be deduced that, for any $\alpha \in [0,\, 1]$, it has only one strictly positive zero if seen as a polynomial in k. Therefore it has to be only shown that this zero of the polynomial is strictly greater than 1. For $\alpha = 0$, the positive zero is $k = 1 + \sqrt{2} > 1$. Also, substitute $k = 1$ in the polynomial, then

$$\alpha^3 + 2\left(\sqrt{2} - 1\right)\alpha^2 + 2\left(3 - 2\sqrt{2}\right)\alpha + 4\left(\sqrt{2} - 2\right)$$

is obtained. It can be confirmed that this polynomial has no zero between 0 and 1. Consequently there is no zero between $k = 0$ and $k = 1$ for any $\alpha \in [0,\, 1]$. By repeating the same computation for the condition for $-1 \le \alpha \le 0$, $k > 0$, the theorem is proven.

B.3 Proof of Proposition 9

Firstly the phase of any system belonging to $\mathscr{P}_{\rm d2ip}$ is already -180[degrees] at $\theta = 0$, and the phase further delays as θ increases. Also the gain at $\theta = 0$ is $+\infty$. As a consequence none of systems in $\mathscr{P}_{\rm d2ip}$ satisfies Condition ($\pi_{\rm d}$).

Also a direct calculation shows that

$$\gamma_{\rm opt} = \sqrt{4 + 2\sqrt{2} + \left(3 + 2\sqrt{2}\right)y}\,,$$

where y is the (real) positive root of

$$y^4 - k^2 y^3 - 5k^2 y^2 - 8k^2 y - 4k^2 = 0\,.$$

Since this is a quartic polynomial, the solution can be obtained in a closed-form expression (which is too lengthy to include here). From the expression it can be deduced that the (real) positive root y increases monotonically as k increases and also that $y \to 0$ as $k \to 0$, which concludes the proof.

B.4 Proof of Proposition 10

A straightforward calculation yields

$$\gamma_{\text{opt}} = \sqrt{2 + \frac{k}{2}\left(k + \sqrt{k^2+4}\right)} \ .$$

It is obvious that γ_{opt} is a monotonically increasing function with respect to k, and that $\gamma_{\text{opt}} = \gamma_{\text{DI}}$ when $k = 2$.