MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

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METR 2007–19 April 2007

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Periodicity of hyperplane arrangements with integral coefficients modulo positive integers

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April 2007

Abstract

We study central hyperplane arrangements with integral coefficients modulo positive integers q. We prove that the cardinality of the complement of the hyperplanes is a quasi-polynomial in two ways, first via the theory of elementary divisors and then via the theory of the Ehrhart quasi-polynomials. This result is useful for determining the characteristic polynomial of the corresponding real arrangement. With the former approach, we also prove that intersection lattices modulo q are periodic except for a finite number of q's.

Key words: characteristic polynomial, Ehrhart quasi-polynomial, elementary divisor, hyperplane arrangement, intersection lattice.

1 Introduction

When a linear form in $x_1, \ldots, x_m$ with integral coefficients is given, we may naturally consider its "q-reduction" for any positive integer q. The q-reduction is the image by the modulo q projection $\mathbb{Z}[x_1, \ldots, x_m] \longrightarrow \mathbb{Z}_q[x_1, \ldots, x_m]$, where $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$. In this paper, we call the kernel of the resulting linear form a "hyperplane" in $V := \mathbb{Z}_q^m$. Suppose that a finite set of nonzero linear forms with integral coefficients is given. Then it not only defines a central hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^m$, but also gives a "hyperplane arrangement" $\mathcal{A}_q$ in V through the q-reduction for each $q \in \mathbb{Z}_{>0}$. A basic fact we prove in this paper is that the cardinality of the complement $M(\mathcal{A}_q)$ of the arrangement $\mathcal{A}_q$ in V, as a function of q, is a quasi-polynomial in q. (In other words, there exist a positive integer ρ (a period) and polynomials $P_j(t)$ $(1 \le j \le \rho)$ such that $|M(\mathcal{A}_q)| = P_r(q)$ $(1 \le r \le \rho,\ q \in r + \rho\mathbb{Z}_{\ge 0})$

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for all $q \in \mathbb{Z}_{>0}$.) We provide two proofs of this fact. The first proof uses the theory of elementary divisors. The second proof is based on the theory of the Ehrhart quasi-polynomials applied to each chamber of the arrangement.

In our setting, the approach via elementary divisors is more powerful than the one via the Ehrhart theory. The former gives more information on the coefficients of the quasi-polynomials, and it also enables us to prove that the intersection lattices modulo q are themselves periodic except for a finite number of q's. Despite the advantage of the approach via elementary divisors for our setting, we also consider the connection to the Ehrhart theory an important aspect of our discussion, because many results in the Ehrhart theory can be applied to further develop the arguments in this paper.

Especially when q is a prime, the arrangement $\mathcal{A}_q$ lies in the vector space $V = \mathbb{Z}_q^m$. In this case, it is well known (e.g., [9], [16, (4.10)], [10, Thm.3.2]) that $|M(\mathcal{A}_q)|$ is equal to $\chi(\mathcal{A}_q, q)$ and that $\chi(\mathcal{A}_q, t)$ coincides with $\chi(\mathcal{A}, t)$ for a sufficiently large prime q, where $\chi(-, t)$ stands for the characteristic polynomial (e.g., [13, Def.2.52], [15, Chap.3, Ex.56]) of an arrangement. These facts provide the "finite field method" to study the real arrangement $\mathcal{A}$. The method was developed by Crapo and Rota [9], Björner and Ekedahl [7], Blass and Sagan [8] and Athanasiadis [1, 2, 3] among others, and has been used to solve problems related to hyperplane arrangements. It was used in [10] to find the characteristic polynomials of the mid-hyperplane arrangements up to a certain dimension. Athanasiadis [4] studies a problem similar to but different from the problem in the present paper. He proves that the coefficients of the characteristic polynomial of a certain deformation of a central arrangement are quasi-polynomials. A series of works by Athanasiadis on the finite field method is worth special mention as the driving force of the research on this method.

For the theory of hyperplane arrangement, the reader is referred to [13]. For the Ehrhart theory for counting lattice points in rational polytopes, see the book by Beck and Robins [5]. Beck and Zaslavsky [6] study the extension of the Ehrhart theory to counting lattice points in "inside-out polytopes".

The organization of the paper is as follows. In the rest of this section, we set up our notation. In Section 2, we prove that the cardinality of the complement $M(\mathcal{A}_q)$ is a quasi-polynomial in q, via the theory of elementary divisors (Section 2.1) and via the theory of the Ehrhart quasi-polynomials (Section 2.2). Based on this result, we consider a way of calculating the characteristic polynomial $\chi(\mathcal{A}, t)$ of the corresponding real arrangement $\mathcal{A}$ (Section 2.3). In Section 3, we prove that the intersection lattices modulo q are periodic except for a finite number of q's.

In our forthcoming paper [11], we apply the results in the present paper to the arrangements arising from root systems and the mid-hyperplane arrangements.

1.1 Setup and notation

Let $m, n \in \mathbb{Z}_{>0}$ be positive integers. In this paper, m denotes the dimension and n is the number of hyperplanes in an arrangement. Suppose we are given an $m \times n$ integer matrix

$$C = (c_1, \dots, c_n) \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$$

consisting of column vectors $c_j = (c_{1j}, \dots, c_{mj})^T \in \mathbb{Z}^m,\ 1 \le j \le n$. Here, ^{T} denotes the transpose and $\mathrm{Mat}_{m\times n}(\mathbb{Z})$ stands for the set of $m \times n$ matrices with integer elements. We assume that integral vectors c_j are nonzero:

$$c_j \neq (0, \dots, 0)^T, \quad 1 \le j \le n. \tag{1}$$

Consider a real central hyperplane arrangement

$$\mathcal{A} = \mathcal{A}_C := \{H_j : 1 \le j \le n\}$$

with

$$H_j = H_{c_j} := \{x = (x_1, \dots, x_m) \in \mathbb{R}^m : xc_j = 0\}.$$

As an example, let us take $m = 2,\ n = 3$ and

$$C = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix}, \tag{2}$$

i.e., $c_1 = (1, -1)^T,\ c_2 = (1, 1)^T,\ c_3 = (-2, 1)^T$. Then the corresponding hyperplane arrangement in $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ is $\mathcal{A} = \{H_1, H_2, H_3\}$ with

$$H_1 : x - y = 0, \quad H_2 : x + y = 0, \quad H_3 : -2x + y = 0.$$

Since the coefficient vectors $c_j = (c_{1j}, \dots, c_{mj})^T \in \mathbb{Z}^m,\ 1 \le j \le n$, defining H_j are integral, we can consider the reductions of c_j modulo positive integers $q \in \mathbb{Z}_{>0}$. Fix $q \in \mathbb{Z}_{>0}$ and let

$$[c_j]_q = ([c_{1j}]_q, \dots, [c_{mj}]_q)^T \in \mathbb{Z}_q^m$$

be the *q-reduction* of c_j, i.e., $[c_{ij}]_q = c_{ij} + q\mathbb{Z} \in \mathbb{Z}_q,\ 1 \le i \le m,\ 1 \le j \le n$. In $V = \mathbb{Z}_q^m$, let us consider

$$H_{j,q} = H_{c_j,q} := \{x = (x_1, \dots, x_m) \in V : x[c_j]_q = [0]_q\},$$

and define

$$\mathcal{A}_q = \mathcal{A}_{C,q} := \{H_{j,q} : 1 \le j \le n\}.$$

We emphasize that $\mathcal{A}_q = \mathcal{A}_{C,q}$ is determined by C and q, but not by $\mathcal{A} = \mathcal{A}_C$ and q. For a non-prime q, it may not be appropriate to call $H_{j,q}$ a hyperplane, but by abusing the terminology we call $H_{j,q}$ a hyperplane, and $\mathcal{A}_q$ an arrangement of hyperplanes. In our previous example (2), $\mathcal{A}_q = \{H_{1,q}, H_{2,q}, H_{3,q}\}$ with

$$\begin{aligned} H_{1,q} &= \{([0]_q, [0]_q), ([1]_q, [1]_q), \dots, ([q-1]_q, [q-1]_q)\}, \\ H_{2,q} &= \{([0]_q, [0]_q), ([1]_q, [q-1]_q), \dots, ([q-1]_q, [1]_q)\}, \\ H_{3,q} &= \{([0]_q, [0]_q), ([1]_q, [2]_q), ([2]_q, [4]_q), \dots, ([q-1]_q, [q-2]_q)\}. \end{aligned} \tag{3}$$

In the finite field method and its generalization in the present paper, we are interested in the cardinality of the complement of $\mathcal{A}_q$. We denote the complement by

$$M(\mathcal{A}_q) := V \setminus \bigcup_{1 \le j \le n} H_{j,q}$$

and its cardinality by $|M(\mathcal{A}_q)|$. We will prove that $|M(\mathcal{A}_q)|$ is a quasi-polynomial in q of degree m and with the leading coefficient identically equal to 1. That is, there exist *a period* $\rho \in \mathbb{Z}_{>0}$ and $\alpha_{h,s} \in \mathbb{Q},\ 0 \le h \le m-1,\ s \in \mathbb{Z}_\rho$, such that

$$|M(\mathcal{A}_q)| = q^m + \alpha_{m-1,[q]_\rho} q^{m-1} + \cdots + \alpha_{1,[q]_\rho} q + \alpha_{0,[q]_\rho}, \quad q \in \mathbb{Z}_{>0}; \tag{4}$$

in fact, $\alpha_{h,s},\ 0 \le h \le m-1,\ s \in \mathbb{Z}_\rho$, are integral: $\alpha_{h,s} \in \mathbb{Z}$. In this paper, we will call (4) the *characteristic quasi-polynomial* of $\mathcal{A}_q$, because, as we will see in Section 2.3, the value (4) coincides with $\chi(\mathcal{A}, q)$ if q and ρ are coprime, where $\chi(\mathcal{A}, t)$ denotes the characteristic polynomial (e.g., [13, Def.2.52], [15, Chap.3, Ex.56]) of the real arrangement $\mathcal{A}$. The minimum period is simply called *the period* of $|M(\mathcal{A}_q)|$. Often it is not trivial to find the period of $|M(\mathcal{A}_q)|$, although it is relatively easy to evaluate some multiple of the period, which we simply call a period.

This is because of the following. The sum $\chi_1(q) + \chi_2(q)$ of two quasi-polynomials $\chi_1(q), \chi_2(q)$ is a quasi-polynomial having as a period the least common multiple of the periods of $\chi_1(q)$ and $\chi_2(q)$. However, due to possible cancellations of terms, the period of $\chi_1(q) + \chi_2(q)$ may be smaller than this least common multiple. See McAllister and Woods [12].

For a subset $J = \{j_1, \dots, j_k\} \subseteq \{1, \dots, n\}$, write

$$H_{J,q} := \bigcap_{j \in J} H_{j,q} = H_{j_1,q} \cap \cdots \cap H_{j_k,q}. \tag{5}$$

When J is nonempty, $H_{J,q}$ in (5) is determined by the q-reduction of the $m \times k$ submatrix

$$C_J := (c_{j_1}, \dots, c_{j_k}) \in \mathrm{Mat}_{m\times k}(\mathbb{Z})$$

of C; when J is empty, we understand that $H_{\emptyset,q} = V$.

The Smith normal form of an integer matrix $G \in \mathrm{Mat}_{m\times k}(\mathbb{Z}),\ k \in \mathbb{Z}_{>0}$, is

$$SGT = \begin{pmatrix} E & O \\ O & O \end{pmatrix} \in \mathrm{Mat}_{m\times k}(\mathbb{Z}), \qquad E = \mathrm{diag}(e_1, \dots, e_\ell), \quad \ell = \mathrm{rank}\, G, \tag{6}$$
$$e_1, \dots, e_\ell \in \mathbb{Z}_{>0}, \quad e_1 | e_2 | \cdots | e_\ell,$$

where $S \in \mathrm{Mat}_{m\times m}(\mathbb{Z})$ and $T \in \mathrm{Mat}_{k\times k}(\mathbb{Z})$ are unimodular matrices. The positive integers $e_1, \dots, e_\ell$ are the *elementary divisors* of G. For simplicity, we often use the following notation

$$\mathrm{diag}(\{e_1, \dots, e_\ell\}; m, k) = \begin{pmatrix} E & O \\ O & O \end{pmatrix} \in \mathrm{Mat}_{m\times k}(\mathbb{Z}).$$

2 Characteristic quasi-polynomial

2.1 Via elementary divisors

In this subsection, we prove that $|M(\mathcal{A}_q)| = |V \setminus \bigcup_{1 \le j \le n} H_{j,q}|$ is a quasi-polynomial in $q \in \mathbb{Z}_{>0}$ using the theory of elementary divisors.

Let $I_Y(\,\cdot\,)$, $Y \subseteq V$, stand for the characteristic function (indicator function) of Y : $I_Y(x) = 1,\ x \in Y$ and $I_Y(x) = 0,\ x \in V \setminus Y$. Then for every $x \in V$,

$$\prod_{j=1}^{n}\left(1 - I_{H_{j,q}}(x)\right) = \sum_{J \subseteq \{1,\ldots,n\}} (-1)^{|J|} I_{H_{J,q}}(x) = I_V(x) + \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|} I_{H_{J,q}}(x),$$

which may be viewed as the inclusion-exclusion principle. Therefore, from the relation $x \in M(\mathcal{A}_q) \Leftrightarrow 1 = \prod_{j=1}^{n}(1 - I_{H_{j,q}}(x))$, we have

$$|M(\mathcal{A}_q)| \;=\; \sum_{x \in V} \prod_{j=1}^{n}\left(1 - I_{H_{j,q}}(x)\right) = q^m + \sum_{\emptyset \neq J \subseteq \{1,\ldots,n\}} (-1)^{|J|} \left|H_{J,q}\right|. \tag{7}$$

Hence it suffices to verify that for each nonempty subset $J = \{j_1, \ldots, j_k\}$ of $\{1, \ldots, n\}$, the cardinality $|H_{J,q}|$ is a quasi-polynomial in $q \in \mathbb{Z}_{>0}$. Actually, we can show that $|H_{J,q}|$ is a quasi-monomial with an integral coefficient.

Fix $J = \{j_1, \ldots, j_k\} \neq \emptyset$ and consider $C_J = (c_{j_1}, \ldots, c_{j_k}) \in \mathrm{Mat}_{m \times k}(\mathbb{Z})$. For each $q \in \mathbb{Z}_{>0}$, let us define $f_{J,q} : V = \mathbb{Z}_q^m \to \mathbb{Z}_q^k$ by

$$x \mapsto x[C_J]_q, \tag{8}$$

where $[C_J]_q = ([c_{j_1}]_q, \ldots, [c_{j_k}]_q) \in \mathrm{Mat}_{m \times k}(\mathbb{Z}_q)$ is the q-reduction of C_J. Then $|H_{J,q}| = |\ker f_{J,q}|$, so the problem reduces to proving that $|\ker f_{J,q}|$ is a quasi-monomial in q. This fact can be shown by using the following general lemma.

Lemma 2.1. *Let m and k be positive integers. Let $f : \mathbb{Z}^m \to \mathbb{Z}^k$ be a $\mathbb{Z}$-homomorphism. Then the cardinality of the kernel of the induced morphism $f_q : \mathbb{Z}_q^m \to \mathbb{Z}_q^k$ is a quasi-monomial of $q \in \mathbb{Z}_{>0}$. Furthermore, suppose f is represented by a matrix $G \in \mathrm{Mat}_{m \times k}(\mathbb{Z})$. Then this quasi-monomial $|\ker f_q|,\ q \in \mathbb{Z}_{>0}$, can be expressed as*

$$|\ker f_q| = (d_1(q) \cdots d_\ell(q))q^{m-\ell}, \tag{9}$$

where $\ell = \operatorname{rank} G$ and $d_j(q) := \gcd\{e_j, q\},\ 1 \leq j \leq \ell$. Here, $e_1, \ldots, e_\ell \in \mathbb{Z}_{>0},\ e_1|e_2|\cdots|e_\ell$, are the elementary divisors of G. In that case, the quasi-monomial $|\ker f_q|,\ q \in \mathbb{Z}_{>0}$, has the minimum period e_ℓ, where we consider e_0 to be one.

Proof. If f is the zero $\mathbb{Z}$-homomorphism, then $|\ker f_q| = |\mathbb{Z}_q^m| = q^m$ and the theorem is trivially true. So we may assume that f is not the zero $\mathbb{Z}$-homomorphism. Since $|\ker f_q| = q^m/|\mathrm{im} f_q|$, we will study $|\mathrm{im} f_q|$.

Suppose f is represented by an $m \times k$ integer matrix $G \in \mathrm{Mat}_{m \times k}(\mathbb{Z})$. Then, for $q \in \mathbb{Z}_{>0}$, the induced morphism $f_q : \mathbb{Z}_q^m \to \mathbb{Z}_q^k$ is given by $x \mapsto x[G]_q$.

Consider the Smith normal form of G in (6). Since unimodularity is preserved under q-reductions, we may assume that G is of the form

$$G = \mathrm{diag}(\{e_1, \ldots, e_\ell\}; m, k)$$

from the outset. Then we have

$$f_q(x) = ([e_1]_q x_1, \dots, [e_\ell]_q x_\ell, [0]_q, \dots, [0]_q) \in \mathbb{Z}_q^k$$

for $x = (x_1, \dots, x_m) \in \mathbb{Z}_q^m$. Therefore, $\mathrm{im} f_q = [e_1]_q \mathbb{Z}_q \times \cdots \times [e_\ell]_q \mathbb{Z}_q$ and hence

$$|\mathrm{im} f_q| = \frac{q}{d_1(q)} \times \cdots \times \frac{q}{d_\ell(q)} = \frac{q^\ell}{d_1(q) \cdots d_\ell(q)},$$

where $d_j(q) = \gcd\{e_j, q\}$, $1 \le j \le \ell$. Consequently, we obtain (9).

Now, for any $j = 1, \dots, \ell$, we have $d_j(q + e_\ell) = \gcd\{e_j, q + e_\ell\} = \gcd\{e_j, q\} = d_j(q)$. Therefore, (9) is a quasi-monomial in q of degree $m - \ell < m$ and with a period e_ℓ. In fact, we can show that e_ℓ is the minimum period as follows.

Let e' be the minimum period. Note $e' | e_\ell$. We have $d_j(e_\ell) = e_j \ge d_j(e') = d_j(e' + e_\ell) > 0$ for all $j = 1, \dots, \ell$. Since e' is a period, $d_1(e_\ell) \cdots d_\ell(e_\ell) = d_1(e' + e_\ell) \cdots d_\ell(e' + e_\ell)$. Therefore $e_\ell = d_\ell(e_\ell) = d_\ell(e' + e_\ell) = e'$. ∎

Now, $f_{J,q} : V = \mathbb{Z}_q^m \to \mathbb{Z}_q^k$ in (8) is induced from the $\mathbb{Z}$-homomorphism $f_J : \mathbb{Z}^m \to \mathbb{Z}^k$ represented by C_J. Thus, Lemma 2.1 implies that

$$|H_{J,q}| = |\ker f_{J,q}| = (d_{J,1}(q) \cdots d_{J,\ell(J)}(q)) q^{m-\ell(J)} \tag{10}$$

is a quasi-monomial with the period $e_{J,\ell(J)}$, where $\ell(J) := \mathrm{rank}\, C_J$ and $d_{J,j}(q) := \gcd\{e_{J,j}, q\}$, $1 \le j \le \ell(J)$. Here, $e_{J,1}, \dots, e_{J,\ell(J)} \in \mathbb{Z}_{>0}$, $e_{J,1} | e_{J,2} | \cdots | e_{J,\ell(J)}$, denote the elementary divisors of C_J. Note that $\ell(J) > 0$ for all J, $|J| \ge 1$, because of the assumption (1).

Remark 2.2. *Assume that q is prime. Then each $d_{J,j}(q) = \gcd\{e_{J,j}, q\}$, $1 \le j \le \ell(J)$, is 1 or q, and $d_{J,j}(q) = q$ if and only if $[e_{J,j}]_q = 0$. It follows from (10) that $X := H_{J,q}$ for any nonempty J satisfies $|X| = q^{m-\ell'} = q^{\dim X}$, where $\ell' = |\{j : 1 \le j \le \ell(J),\ [e_{J,j}]_q \ne 0\}|$. Note that $|X| = q^{\dim X}$ for $X = H_{J,q}$ is true also when J is empty: $|\mathbb{Z}_q^m| = q^m$.*

From the discussions so far, we reach the following conclusions. First, $|M(\mathcal{A}_q)|$, $q \in \mathbb{Z}_{>0}$, is a monic quasi-polynomial in q of degree m. Second, a period of this quasi-polynomial can be obtained in the following way. For each $m \times k$ $(1 \le k \le n)$ submatrices C_J of $C = (c_1, \dots, c_n) \in \mathrm{Mat}_{m \times n}(\mathbb{Z})$, find its largest elementary divisor $e(J) := e_{J,\ell(J)}$. Let

$$\rho_0 := \mathrm{lcm}\{e(J) : J \subseteq \{1, \dots, n\}, J \ne \emptyset\}.$$

Then ρ_0 is a period of $|M(\mathcal{A}_q)|$.

For computing ρ_0 when $m < n$, we can restrict the size of J as $|J| \le m$:

$$\rho_0 = \mathrm{lcm}\{e(J) : J \subseteq \{1, \dots, n\},\ 1 \le |J| \le \min\{m, n\}\}. \tag{11}$$

We can prove (11) in the following way. First, we note the next lemma.

Lemma 2.3. *Let $f_1, f_2 : \mathbb{Z}^n \to \mathbb{Z}^m$ be two $\mathbb{Z}$-homomorphism with $\mathrm{rank}(\mathrm{im} f_1) = \mathrm{rank}(\mathrm{im} f_2)$ and $\mathrm{im} f_2 \subseteq \mathrm{im} f_1$. Then the largest elementary divisor of f_1 divides the largest elementary divisor of f_2.*

Proof. Define $I_i = \mathrm{Ann}(\mathrm{coker} f_i) := \{p \in \mathbb{Z} : p(\mathrm{coker} f_i) = 0\}$, and the ideal I_i is generated by the largest elementary divisor of f_i $(i = 1, 2)$. Since there is a natural projection $\mathrm{coker} f_2 \to \mathrm{coker} f_1$, we have $I_2 \subseteq I_1$. This shows the lemma. ∎

Now, suppose $m < n$, and take an arbitrary $J \subseteq \{1, \ldots, n\}$ with $m < |J| \leq n$. Let $\ell = \ell(J) = \mathrm{rank}\, C_J$ $(\leq m)$. Then we can take a subset $\tilde{J} \subset J$, $|\tilde{J}| = \ell$, such that $\mathrm{rank}\, C_{\tilde{J}} = \ell$. For this $\tilde{J}$, we have $\mathrm{im} g_{\tilde{J}} \subseteq \mathrm{im} g_J$, where $g_J, g_{\tilde{J}} : \mathbb{Z}^n \to \mathbb{Z}^m$ are the $\mathbb{Z}$-homomorphisms defined by C_J and $C_{\tilde{J}}$, respectively: $g_J(x) = \sum_{j \in J} x_j c_j$, $g_{\tilde{J}}(x) = \sum_{j \in \tilde{J}} x_j c_j$, $x = (x_1, \ldots, x_n)^T \in \mathbb{Z}^n$. Then Lemma 2.3 implies that $e(J) | e(\tilde{J})$. From this observation, we obtain (11). When n is considerably larger than m, the restriction $|J| \leq m$ is computationally very useful.

Let us find a period ρ_0 for our example (2). Take $J = \{1, 2\}$. Then we have

$$C_J = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

with the Smith normal form $\mathrm{diag}(1, 2)$. Hence $e(J) = e_{J,2} = 2$. In a similar manner, we can find $e(J)$ for the other J's with $1 \leq |J| \leq 2$, and obtain $\rho_0 = \mathrm{lcm}\{1, 1, 1, 2, 1, 3\} = 6$.

Furthermore, for $|J| \geq 1,\ 1 \leq j \leq \ell(J)$,

$$d_{J,j}(q) = \gcd\{e_{J,j}, q\} = \gcd\{e_{J,j}, \rho_0, q\} = \gcd\{e_{J,j}, \gcd\{\rho_0, q\}\}. \tag{12}$$

This implies that the coefficient $d_{J,1}(q) \cdots d_{J,\ell(J)}(q)$ of each monomial $|H_{J,q}|$, $|J| \geq 1$, in (10) depends on q only through $\gcd\{\rho_0, q\}$. Therefore, the constituents of the quasi-polynomial $|M(\mathcal{A}_q)|$ in (7) coincide for all q with the same $\gcd\{\rho_0, q\}$.

We summarize the results obtained so far as follows:

Theorem 2.4. *The function $|M(\mathcal{A}_q)|$ is a monic quasi-polynomial in $q \in \mathbb{Z}_{>0}$ of degree m with a period ρ_0 given in* (11). *Furthermore, in (4) with $\rho = \rho_0$, the coefficients $\alpha_{h,[q]_{\rho_0}} \in \mathbb{Z}$, $0 \leq h \leq m-1$, depend on $[q]_{\rho_0}$ only through* $\gcd\{\rho_0, q\}$.

Let us find the characteristic quasi-polynomial for our example (2). Since $\rho_0 = 6$, we know by Theorem 2.4 that each of the sets $\{1, 5\}, \{2, 4\}, \{3, 9\}, \{6, 12\}$ of values of q determines a constituent of the characteristic quasi-polynomial $|M(\mathcal{A}_q)|$. For $q = 1$, we have $V = H_{1,1} = H_{2,1} = H_{3,1} = \{([0]_1, [0]_1)\}$ and thus $|M(\mathcal{A}_1)| = 0$. For $q = 5$, we can count $|H_{1,5} \cup H_{2,5} \cup H_{3,5}| = 13$ and get $|M(\mathcal{A}_5)| = 5^2 - 13 = 12$. By interpolation, we obtain the constituent $q^2 - 3q + 2$ for $\gcd\{6, q\} = 1$. In this way, we can get the following characteristic quasi-polynomial:

$$|M(\mathcal{A}_q)| = \begin{cases} q^2 - 3q + 2 & \text{when } \gcd\{6, q\} = 1, \\ q^2 - 3q + 3 & \text{when } \gcd\{6, q\} = 2, \\ q^2 - 3q + 4 & \text{when } \gcd\{6, q\} = 3, \\ q^2 - 3q + 5 & \text{when } \gcd\{6, q\} = 6. \end{cases} \tag{13}$$

From this characteristic quasi-polynomial, we can see that the minimum period is $6 = \rho_0$.

2.2 Via the Ehrhart theory

We want to show via the Ehrhart theory that $|M(\mathcal{A}_q)| = |V \setminus \bigcup_{1\le j\le n} H_{j,q}|$ is a quasi-polynomial in $q \in \mathbb{Z}_{>0}$. The Ehrhart theory is indeed useful for establishing that $|M(\mathcal{A}_q)|$ is a quasi-polynomial, and gives a geometric insight into its period. However, it does not seem to give information on the constituents of the quasi-polynomial.

For $j = 1, \ldots, n$, let

$$S_j := \mathbb{Z} \cap \{xc_j \mid x \in [0,1)^m\}.$$

For example, for $c_j = (1, \ldots, 1)^T \in \mathbb{Z}^m$

$$\min_{\substack{x\in[0,1)^m\\ x_1+\cdots+x_m\in\mathbb{Z}}} (x_1 + \cdots + x_m) = 0, \qquad \max_{\substack{x\in[0,1)^m\\ x_1+\cdots+x_m\in\mathbb{Z}}} (x_1 + \cdots + x_m) = m-1,$$

and $S_j = \{0, 1, \ldots, m-1\}$ for this c_j. Now define the additional "translated" hyperplanes

$$H_j^{s_j}(q) = H_{c_j}^{s_j}(q) := \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : xc_j = s_j q\} \subset \mathbb{R}^m, \quad s_j \in S_j,$$

for $j = 1, \ldots, n$, and consider the real hyperplane arrangement

$$\mathcal{A}^{\mathrm{deform}}(q) = \mathcal{A}_C^{\mathrm{deform}}(q) := \{H_j^{s_j}(q) : s_j \in S_j,\ 1 \le j \le n\}.$$

For any positive integer q, we can express $|M(\mathcal{A}_q)| = |V \setminus \bigcup_{1\le j\le n} H_{j,q}|$ as

$$(14)\quad |M(\mathcal{A}_q)| = \left|\mathbb{Z}^m \cap [0,q)^m \setminus \bigcup \mathcal{A}^{\mathrm{deform}}(q)\right| = \left|\mathbb{Z}^m \cap \left(q \times \left([0,1)^m \setminus \bigcup \mathcal{A}^{\mathrm{deform}}\right)\right)\right|,$$

where $\bigcup \mathcal{A}^{\mathrm{deform}}(q) := \bigcup_{H\in\mathcal{A}^{\mathrm{deform}}(q)} H$ and $\mathcal{A}^{\mathrm{deform}} := \mathcal{A}^{\mathrm{deform}}(1)$.

Now, let us consider $[0,1)^m \setminus \bigcup \mathcal{A}^{\mathrm{deform}}$ in (14). We see that $[0,1)^m$ is cut by the hyperplanes $H_j^{s_j}(1),\ s_j \in S_j,\ 1 \le j \le n$, into

$$P^O(s_1, \ldots, s_n) := \{x \in [0,1)^m : s_j < xc_j < s_j + 1,\ 1 \le j \le n\},$$

$(s_1, \ldots, s_n) \in S_1^* \times \cdots \times S_n^* =: S^*$, where $S_j^* := S_j \cup \{\min S_j - 1\},\ 1 \le j \le n$. Therefore

$$(15)\qquad [0,1)^m \setminus \bigcup \mathcal{A}^{\mathrm{deform}} = \bigsqcup_{(s_1,\ldots,s_n)\in S^*} P^O(s_1, \ldots, s_n)$$

is a disjoint union. From (14) and (15), we obtain

$$|M(\mathcal{A}_q)| = \sum_{(s_1,\ldots,s_n)\in S^*} \left|\mathbb{Z}^m \cap qP^O(s_1, \ldots, s_n)\right| = \sum_{(s_1,\ldots,s_n)\in S^*} i(P^O(s_1, \ldots, s_n), q),$$

where $i(P^O(s_1, \ldots, s_n), q) := |\mathbb{Z}^m \cap qP^O(s_1, \ldots, s_n)|$.

It should be noted that $P^O(s_1, \ldots, s_n),\ (s_1, \ldots, s_n) \in S^*$, are not necessarily open in $\mathbb{R}^m$. However, by applying the Ehrhart theory to some faces of each nonempty $P^O(s_1, \ldots, s_n)$, we can show that $i(P^O(s_1, \ldots, s_n), q)$ is a quasi-polynomial of $q \in \mathbb{Z}_{>0}$ with degree $d = \dim(P^O(s_1, \dots, s_n))$ and the leading coefficient equal to the normalized volume of $P^O(s_1, \dots, s_n)$. When $d = m$, the normalized volume is the same as the usual volume in $\mathbb{R}^m$. Therefore, we can conclude that the sum $\sum_{(s_1,\dots,s_n)\in S^*} i(P^O(s_1, \dots, s_n), q) = |M(\mathcal{A}_q)|$ is a quasi-polynomial of $q \in \mathbb{Z}_{>0}$ with degree m and the leading coefficient $\sum \mathrm{vol}_m(P^O(s_1, \dots, s_n)) = \mathrm{vol}_m([0, 1)^m) = 1$, where $\mathrm{vol}_m(\,\cdot\,)$ denotes the usual volume in $\mathbb{R}^m$.

Let us move on to the investigation into periods of the characteristic quasi-polynomial $|M(\mathcal{A}_q)|,\ q \in \mathbb{Z}_{>0}$. From the above discussion, we see that a common multiple of periods of $i(P^O(s_1, \dots, s_n), q),\ (s_1, \dots, s_n) \in S^*$, is a period of $|M(\mathcal{A}_q)|$. Let $\bar{P}(s_1, \dots, s_n)$ denote the closure of $P^O(s_1, \dots, s_n)$. Define the *denominator* $\mathcal{D}(\mathcal{A}^{\mathrm{deform}})$ of $\mathcal{A}^{\mathrm{deform}}$ by

$$\begin{aligned}\mathcal{D}(\mathcal{A}^{\mathrm{deform}}) \ &:= \ \min\{q \in \mathbb{Z}_{>0} : \text{ all } q\bar{P}(s_1, \dots, s_n),\ (s_1, \dots, s_n) \in S^*, \\ &\qquad \text{are integral polytopes}\}.\end{aligned}$$

The Ehrhart theory now implies that $\mathcal{D}(\mathcal{A}^{\mathrm{deform}})$ is a period of $|M(\mathcal{A}_q)|$.

Put $\tilde{C} := (C, I_m)$, where I_m is the $m \times m$ identity matrix. For $J \subseteq \{1, \dots, n+m\}$, let $\tilde{C}_J$ denote the $m \times |J|$ submatrix of $\tilde{C}$ consisting of the columns corresponding to the elements of J. Then, in view of Cramer's formula, we see that $\mathcal{D}(\mathcal{A}^{\mathrm{deform}})$ divides

$$\begin{aligned}\rho_{\mathrm{E}} := \mathrm{lcm}\{\det(\tilde{C}_J) : J \subset \{1, \dots, n+m\} \\ \text{such that } \ |J| = m \ \text{ and } \ \det(\tilde{C}_J) \neq 0\}.\end{aligned}$$

Hence, the minimum period divides ρ_{E}. For J with $|J \cap \{n+1, \dots, n+m\}| = h < m$, the determinant $\det(\tilde{C}_J)$ equals an $(m-h) \times (m-h)$ minor of C up to sign. Therefore, we can also write

$$\rho_{\mathrm{E}} = \mathrm{lcm}\{\text{nonzero } j\text{-minors of } C,\ 1 \leq j \leq m\}. \tag{16}$$

Now, recall the well-known fact that $\bar{e}(J) := e_{J,1} \times \cdots \times e_{J,\ell(J)}$ is equal to the greatest common divisor of all the (nonzero) $\ell(J)$-minors of $C_J,\ J \neq \emptyset$, and note the relation $e(J) = e_{J,\ell(J)} | \bar{e}(J),\ J \neq \emptyset$. Then we can easily see from (11) and (16) that $\rho_0 | \rho_{\mathrm{E}}$. Therefore, ρ_0 gives a tighter bound for the period of the characteristic quasi-polynomial $|M(\mathcal{A}_q)|$ than ρ_{E}.

In our working example (2), we have $\rho_{\mathrm{E}} = \mathrm{lcm}\{1, 1, -2, -1, 1, 1, 2, -1, 3\} = 6$ and thus $\rho_0 = \rho_{\mathrm{E}}$. In general, if we obtain the characteristic quasi-polynomial by interpolation using ρ_{E} as a period and find that ρ_{E} happens to be the minimum period, then we know $\rho_0 = \rho_{\mathrm{E}}$.

2.3 Characteristic polynomial of the real arrangement

Let $\chi(\mathcal{A}, t)$ be the characteristic polynomial of the real hyperplane arrangement $\mathcal{A} = \{H_j : 1 \leq j \leq n\}$, where $H_j = \{x \in \mathbb{R}^m : xc_j = 0\},\ 1 \leq j \leq n$.

Theorem 2.5. *Let ρ be a period of the quasi-polynomial $|M(\mathcal{A}_q)|$ and q be a positive integer relatively prime to ρ. Then $|M(\mathcal{A}_q)| = \chi(\mathcal{A}, q)$.*

Proof. Choose $c \in \mathbb{Z}_{\geq 0}$ and $q' \in \mathbb{Z}$ such that $q = \rho c + q'$ and $1 \leq q' \leq \rho$. By Theorem 2.4, there exist integers $\alpha_0, \ldots, \alpha_{m-1}$ such that $|M(\mathcal{A}_k)| = k^m + \alpha_{m-1}k^{m-1} + \cdots + \alpha_0$ for all $k \in q' + \rho\mathbb{Z}_{\geq 0}$. Since q' and ρ are relatively prime, then by Dirichlet's theorem on arithmetic progressions (e.g., [14]), $q' + \rho\mathbb{Z}_{\geq 0}$ contains an infinite number of primes. On the other hand, it is well known (e.g., [9] [16, (4.10)] [10, Theorem 3.2]) that, when k is a sufficiently large prime, $|M(\mathcal{A}_k)|$ coincides with $\chi(\mathcal{A}, k)$. Remember that the characteristic polynomial $\chi(\mathcal{A}, t)$ is a monic polynomial of degree $\dim(\mathbb{R}^m) = m$. This implies that $\chi(\mathcal{A}, t) = t^m + \alpha_{m-1}t^{m-1} + \cdots + \alpha_0$ and thus $\chi(\mathcal{A}, q) = q^m + \alpha_{m-1}q^{m-1} + \cdots + \alpha_0 = |M(\mathcal{A}_q)|$. ∎

The argument above implies that we can obtain the characteristic polynomial $\chi(\mathcal{A}, t)$ by counting $|M(\mathcal{A}_{q_i})| = |\mathbb{Z}_{q_i}^m \setminus \bigcup_{1 \leq j \leq n} H_{j,q_i}|$ for an arbitrary set of m distinct values $q_1, \ldots, q_m$ with $\gcd\{\rho, q_i\} = 1$ $(1 \leq i \leq m)$. Note that $q_1, \ldots, q_m$ need not be prime.

When q' and ρ are not relatively prime, $q' + \rho\mathbb{Z}_{\geq 0}$ contains at most one prime (and this prime is not necessarily "sufficiently large"), so the above argument does not hold. For $q' + \rho\mathbb{Z}_{\geq 0}$ with such q''s, we obtain different polynomials than $\chi(\mathcal{A}, t)$.

In our example (2), the constituent of the characteristic quasi-polynomial (13) for $1 + 6\mathbb{Z}_{\geq 0}$ and $5 + 6\mathbb{Z}_{\geq 0}$, i.e., $\gcd\{6, q\} = 1$, is the characteristic polynomial of $\mathcal{A}$. Thus $\chi(\mathcal{A}, t) = t^2 - 3t + 2 = (t-1)(t-2)$.

3 Periodicity of the Intersection lattice

In this section, we show that the intersection lattice $L_q = L(\mathcal{A}_q)$ (e.g., [13, 2.1], [15, Chap.3, Ex.56]) is periodic for large enough q. Let us begin with our working example to illustrate the periodicity of the intersection lattice L_q.

In our example (2), the "hyperplanes" $H_{1,q}, H_{2,q}, H_{3,q}$ were given in (3). For $q = 1$, $V = \{([0]_1, [0]_1)\} = H_{1,1} = H_{2,1} = H_{3,1}$. For $q = 2$, $H_{1,2} = H_{2,2}$ and for $q = 3$, $H_{2,3} = H_{3,3}$. These are the exceptions. From $q = 4$ on, we have the periodicity of the intersection lattice—the lattice of $H_{J,q} = \cap_{j \in J} H_{j,q}$, $J \subseteq \{1, 2, 3\}$, by reverse inclusion. First, it is easily seen that for $q \geq 4$, $H_{j,q}$, $j = 1, 2, 3$, are distinct, proper subsets of V. Furthermore, for $q \geq 4$,

$$H_{\{1,2\},q} = \begin{cases} \{([0]_q, [0]_q)\}, & q : \text{odd}, \\ \{([0]_q, [0]_q), ([\frac{q}{2}]_q, [\frac{q}{2}]_q)\}, & q : \text{even}, \end{cases}$$

$$H_{\{2,3\},q} = \begin{cases} \{([0]_q, [0]_q)\}, & 3 \not| \, q, \\ \{([0]_q, [0]_q), ([\frac{q}{3}]_q, [\frac{2q}{3}]_q), ([\frac{2q}{3}]_q, [\frac{q}{3}]_q)\}, & 3 \mid q, \end{cases}$$

and $H_{\{1,3\},q} = H_{\{1,2,3\},q} = \{([0]_q, [0]_q)\}$. We see that the intersection lattice for this example is periodic and has the period 6 for $q \geq 4$. The Hasse diagrams for the four types of the intersection lattices are illustrated in Figure 1. In Figure 1, the subscript $_q$ is omitted for simplicity.

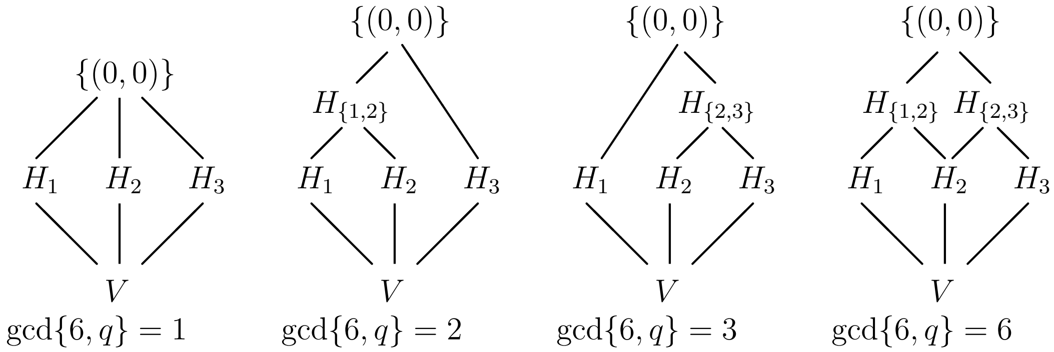

Figure 1: Hasse diagrams of intersection lattices for $q \geq 4$

Let $J = \{j_1, \dots, j_k\}$, $1 \leq j_1 < \cdots < j_k \leq n$, $1 \leq k \leq n$, be a nonempty subset of $\{1, \dots, n\}$. We write the Smith normal form of $C_J \in \mathrm{Mat}_{m \times k}(\mathbb{Z})$ as

$$(17) \qquad \begin{aligned} &S_J C_J T_J = \mathrm{diag}(\{e_{J,1}, \dots, e_{J,\ell(J)}\}; m, k) =: \tilde{E}_J, \\ &\qquad \ell(J) = \mathrm{rank}\, C_J, \quad e_{J,1}, \dots, e_{J,\ell(J)} \in \mathbb{Z}_{>0}, \quad e_{J,1} | e_{J,2} | \cdots | e_{J,\ell(J)}. \end{aligned}$$

As in Section 2.1, we write $e(J) = e_{J,\ell(J)}$, the largest elementary divisor of C_J.

Let ρ_0 be the least common multiple of all $e(J)$'s with $1 \leq |J| \leq \min\{m, n\}$ as in (11). Furthermore, define

$$q_0 := \max_{\emptyset \neq J \subseteq \{1, \dots, n\}} \min_{S_J} \max\{|u| : u \text{ is an entry of } S_J C \text{ or } C\},$$

where the minimization is over all possible choices of S_J in (17) for each fixed J.

We are now in a position to state the main theorem of this section.

Theorem 3.1. *Let J be an arbitrary nonempty subset of $\{1, \dots, n\}$. Suppose $q, q' \in \mathbb{Z}_{>0}$ satisfy $q, q' > q_0$ and $\gcd\{\rho_0, q\} = \gcd\{\rho_0, q'\}$. Then, for any $j \in \{1, \dots, n\}$, we have that $H_{j,q} \supseteq H_{J,q}$ if and only if $H_{j,q'} \supseteq H_{J,q'}$.*

When $j \in J$, the theorem is trivially true.

In proving Theorem 3.1, we need the following proposition. Regard $V = \mathbb{Z}_q^m$ as a $\mathbb{Z}_q$-module. Let V^* be the $\mathbb{Z}_q$-module consisting of the linear forms on V. For any $A \subseteq V$, we denote by $I(A)$ the set of linear forms vanishing on A:

$$I(A) = \{\alpha \in V^* : \alpha(x) = 0 \ \text{ for all } \ x \in A\}.$$

Also, for any $B \subseteq V^*$, let $V(B)$ stand for the set of points at which each linear form in B vanishes:

$$V(B) = \{x \in V : \alpha(x) = 0 \ \text{ for all } \ \alpha \in B\}.$$

Evidently, $I(A)$ and $V(B)$ are submodules of V^* and V, respectively.

Proposition 3.2. *For any $B \subseteq V^*$, we have $I(V(B)) = \langle B \rangle$, where $\langle B \rangle$ signifies the submodule of V^* spanned by B.*

Proof. It suffices to show $I(V(B)) = B$ for any submodule B of V^*. It is trivially true that $I(V(B)) \supseteq B$, so we will prove $I(V(B)) \subseteq B$.

Let $C_q(B) \in \mathrm{Mat}_{m\times k}(\mathbb{Z}_q)$, $k = |B|$, be the coefficient matrix of B. We can find an integral matrix $C(B) \in \mathrm{Mat}_{m\times k}(\mathbb{Z})$ whose q-reduction is $C_q(B)$, i.e., $[C(B)]_q = C_q(B)$. Now, let $e_1|e_2|\cdots|e_\ell$, $\ell = \operatorname{rank} C(B)$, be the elementary divisors of $C(B)$. In $\mathbb{Z}_q$, we then have $C_q(B)$ is equivalent to

$$\mathrm{diag}(\{[e_1], \ldots, [e_{\ell'}]\}; m, k) \in \mathrm{Mat}_{m\times k}(\mathbb{Z}_q), \tag{18}$$

where $[e_1], \ldots, [e_{\ell'}] \in \mathbb{Z}_q \setminus \{0\}$, $\ell' \leq \ell$. Here, we are writing $[\,\cdot\,]$ for $[\,\cdot\,]_q$ for simplicity. We can choose $C(B)$ in such a way that $\ell' = \ell$, and we decide to do so. From (18) we see that we can assume B is spanned by $[e_1]y_1, \ldots, [e_\ell]y_\ell$ after a suitable coordinate change, where $\{y_1, \ldots, y_\ell, y_{\ell+1}, \ldots, y_m\}$ is a basis of V^*. It follows that $V(B)$ is spanned by

$$p_1 := (\,[q/d_1(q)]\,, 0, \ldots, 0), \ldots,\ p_\ell := (\underbrace{0, \ldots, 0}_{\ell-1}, [q/d_\ell(q)]\,, 0, \ldots, 0),$$

$$p_{\ell+1} := (\underbrace{0, \ldots, 0}_{\ell}, 1, 0, \ldots, 0), \ldots,\ p_m := (0, \ldots, 0, 1)$$

with $d_j(q) = \gcd\{e_j, q\}$, $1 \leq j \leq \ell$.

Now, take an arbitrary $\alpha = [a_1]y_1 + \cdots + [a_m]y_m \in I(V(B)) = I(p_1, \ldots, p_m)$ with $[a_1], \ldots, [a_m] \in \mathbb{Z}_q$. Then we have $0 = \alpha(p_1) = [qa_1/d_1(q)]$, so $a_1 = r_1 d_1(q)$ for some $r_1 \in \mathbb{Z}$. This implies $[a_1] = [r_1][d_1(q)] = [r_1'][e_1]$ with $[r_1'] := [r_1][e_1/d_1(q)]^{-1} \in \mathbb{Z}_q$, where $[e_1/d_1(q)]^{-1}$ exists because $\gcd\{e_1/d_1(q), q\} = 1$. Similarly, for each $j = 2, \ldots, \ell$, we have $[a_j] = [r_j'][e_j]$ for some $[r_j'] \in \mathbb{Z}_q$. Moreover, for $j = \ell+1, \ldots, m$, we obtain $0 = \alpha(p_j) = [a_j]$. Therefore, we have $\alpha = [r_1'][e_1]y_1 + \cdots + [r_\ell'][e_\ell]y_\ell \in B$, and the proof is complete. ∎

Proof of Theorem 3.1.

Without loss of generality, we may assume $j = 1$. Let $[S_J]_q, [C_J]_q, [T_J]_q$ and $[\tilde{E}_J]_q$ be the q-reductions of S_J, C_J, T_J and $\tilde{E}_J$ in (17), respectively.

First, we know by Proposition 3.2 that $H_{1,q} \supseteq H_{J,q}$ if and only if $[c_1]_q$ lies in the column space of $[C_J]_q$ in $\mathbb{Z}_q^m$. Since S_J^{-1} and T_J^{-1} exist in $\mathrm{Mat}_{m\times m}(\mathbb{Z})$ and $\mathrm{Mat}_{k\times k}(\mathbb{Z})$, respectively, the latter condition is equivalent to $[c_1]_q$ being in the column space of $[C_J]_q[T_J]_q = [S_J^{-1}]_q[\tilde{E}_J]_q$, which in turn is equivalent to $[S_J]_q[c_1]_q$ being in the column space of $[\tilde{E}_J]_q$ in $\mathbb{Z}_q^m$.

Next, let us paraphrase the above condition in $\mathbb{Z}_q^m$ as a condition in $\mathbb{Z}^m$. The condition holds if and only if $S_J c_1 \in \mathbb{Z}^m$ is in the column space of $(\tilde{E}_J, qI_m) \in \mathrm{Mat}_{m\times(k+m)}(\mathbb{Z})$ in $\mathbb{Z}^m$. Noting that $e_{J,j}\mathbb{Z} + q\mathbb{Z} = d_{J,j}(q)\mathbb{Z}$ with $d_{J,j}(q) = \gcd\{e_{J,j}, q\}$, $1 \leq j \leq \ell(J)$, we see that the condition holds if and only if $S_J c_1$ is in the column space of $\mathrm{diag}(d_{J,1}(q), \ldots, d_{J,\ell(J)}(q), q, \ldots, q) \in \mathrm{Mat}_{m\times m}(\mathbb{Z})$. Since the absolute value of each entry of $S_J c_1 \in \mathbb{Z}^m$ is less than q, the condition is equivalent to $S_J c_1$ being in the column space of

$$\mathrm{diag}(\{d_{J,1}(q), \ldots, d_{J,\ell(J)}(q)\}; m, \ell(J)) \in \mathrm{Mat}_{m\times \ell(J)}(\mathbb{Z}). \tag{19}$$

Now, since the absolute value of each entry of $S_J c_1$ is less than q' as well, the preceding argument holds true also for q'. Moreover, we see from (12) that $d_{J,j}(q) = d_{J,j}(q')$ for $j = 1, \ldots, \ell(J)$. Thus (19) remains the same when q is replaced by q'. Therefore, we obtain the desired result. ∎

Our assumption (1) implies that $H_{j,q} \not\supseteq H_{\emptyset,q} = V,\ 1 \leq j \leq n$, for all $q > q_0$. From this observation and Theorem 3.1, it follows immediately that $L_q = L(\mathcal{A}_q)$ for $q > q_0$ is periodic in q with a period ρ_0.

Corollary 3.3. *The intersection lattice $L_q = L(\mathcal{A}_q)$ is periodic in $q > q_0$ with a period ρ_0:*

$$L_{q+s\rho_0} \simeq L_q \text{ for all } q > q_0 \text{ and } s \in \mathbb{Z}_{\geq 0}.$$

Finally, we make a remark on the coarseness of the intersection lattices for different q's. In Figure 1 we see that the intersection lattice for the case $\gcd\{6, q\} = 6$ is the most detailed and that the coarseness is nested according to the divisibility of $\gcd\{6, q\}$. This observation can be generally stated as follows.

Proposition 3.4. *Let $I, J \subseteq \{1, \ldots, n\}$ and suppose that $H_{I,q} = H_{J,q}$ for some $q > q_0$. Then $H_{I,q'} = H_{J,q'}$ for every $q' > q_0$ such that $\gcd\{\rho_0, q'\} | \gcd\{\rho_0, q\}$.*

Proof. It suffices to show that for any $i \in I$, if $[c_i]_q$ lies in the column space of $[C_J]_q$ in $\mathbb{Z}_q^m$, then $[c_i]_{q'}$ lies in the column space of $[C_J]_{q'}$ in $\mathbb{Z}_{q'}^m$. Without loss of generality, take $i = 1$ and assume that $[c_1]_q$ lies in the column space of $[C_J]_q$ in $\mathbb{Z}_q^m$. Then $S_J c_1$ is in the column space of (19). Now, because $\gcd\{\rho_0, q'\} | \gcd\{\rho_0, q\}$ by assumption, we can see from (12) that $d_{J,j}(q') | d_{J,j}(q),\ 1 \leq j \leq \ell(J)$. This implies that $S_J c_1$ is in the column space of (19) with q replaced by q'. Therefore, $[c_1]_{q'}$ lies in the column space of $[C_J]_{q'}$ in $\mathbb{Z}_{q'}^m$. ∎

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