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Adaptive Quantized Control for Linear Uncertain Discrete-Time Systems

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Abstract

A direct adaptive control framework for linear uncertain systems for the use of communication channels is developed. Specifically, the control signals are to be quantized and sent over a communication channel to the actuator. The proposed framework is Lyapunov-based and guarantees partial asymptotic stability, that is, Lyapunov stability of the closed-loop system states and attraction with respect to the plant states. The quantizers are logarithmic and characterized by sector-bound conditions with the conic sectors adjusted at each time instant by the adaptive controller in conjunction with the system response. Furthermore, we extend the scheme to the case where the logarithmic quantizer has a deadzone around the origin so that only a finite number of quantization levels is required to achieve practical stability. Finally, a numerical example is provided to demonstrate the efficacy of the proposed approach.

1 Introduction

To design control systems whose components are connected by shared networks, it is essential to consider the limitation due to the communication system and to ensure that the systems can operate appropriately within the given bandwidth [1–3]. An important aspect there is to use quantization schemes that have sufficient precision and, at the same time, require low communication rate. These views have prompted research interests on new quantization methods accounting for characteristics particular to control systems.

One such scheme is presented in [4] for stabilization of a linear discrete-time system where an optimal quantizer is obtained with respect to a certain measure on coarseness of the transmitted information. This quantizer has a unique feature that the quantization levels become finer in the region closer to the origin in a logarithmic way and is hence called the *logarithmic quantizer*. Furthermore, it has been shown that the coarseness of the optimal quantizer is determined solely by the unstable poles of the system. In [5], an alternative proof for the optimal design and more general results are given by viewing such quantizers as sector-bounded nonlinearities. This idea is extended in [6] and applied to the case of uncertain systems with additive bounded uncertainties using H_∞ techniques.

In contrast to the fixed-gain robust controllers, adaptive controllers are more appropriate in dealing with uncertain systems whose unknown uncertainty bounds are unknown. In other words, adaptive controllers can tolerate far greater system uncertainty levels by adjusting feedback gains in response to plant variation to improve system performance. However, adaptive control framework with quantization requirements has not been studied in the literature.

In this paper, we consider a stabilization problem for uncertain plants over networks via a direct adaptive control approach. Specifically, for a linear time-invariant plant whose parameters are uncertain with unknown bounds, we propose a design method for an adaptive controller and an input quantizer. The setup is depicted in Figure 1. The controller is on the sensor

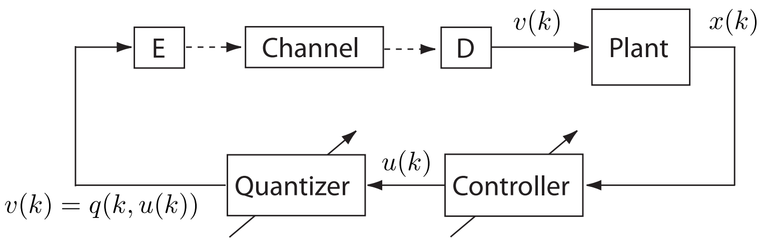


Figure 1: Adaptive control scheme with a time-varying quantizer, where E and D represent the encoder and the decoder, respectively.

side, and the control input is quantized and coded at the coder to be sent over the channel; we assume that the channel is noiseless, and hence the quantized signal is recovered at the decoder and is applied to the plant. The quantizer is time varying, and at each time instant, its parameters are determined and adjusted in response to the update in the controller gain. In this paper we employ logarithmic quantizers and follow the approach of [5] in viewing the logarithmic quantization functions as a class of sector-bounded nonlinearities, thereby aiming at maintaining the quantizer as coarse as possible at each moment.

In our adaptive control scheme, it turns out that the quantization levels must be fine while the controller gain is large, and vice versa. In general, this implies that systems that are more unstable would require more information for stabilization. This is in agreement with the implications in [4] and [5] mentioned above. Furthermore, since the coarseness of the quantization varies with time, it is necessary to send over the channel the information on the size of the sector that envelops the quantization nonlinearities. Due to this requirement, the information on the size of the sector is quantized as well. In the special case where the system matrices are known, the proposed controller and quantizer can be assumed static and the coarseness of the logarithmic quantizer reduces to the optimal ones given in [4] and [5]. Although in the adaptive case it is difficult to show optimality, we may say that our approach is nonconservative for this reason.

Finally, we emphasize that the proposed adaptive control method is Lyapunov-based and guarantees partial asymptotic stability, that is, Lyapunov stability of the closed-loop system states and attraction with respect to the plant states. (As a result, the adaptive gain states are bounded.) Note that most of the adaptive control approaches for discrete-time systems are based on recursive least squares and least mean squares algorithms [7]; the primary focus has been on state convergence rather than stability. Several notable Lyapunov-based approaches in discrete time are given in [8], [9], [10], [11], and [12]. Furthermore, the proposed scheme is extended to the case where the logarithmic quantizer has a deadzone so that only a finite number of quantization levels is required for any compact set in the state space while ultimate boundedness of closed-loop system is guaranteed. We note, nonetheless, that its proof is not straightforward. In our approach, we present a two-step proof where we first need to show that the Lyapunov-like function with respect to the adaptive gain state converges to a certain value and then, using this fact, conclude that the plant state remain bounded in the state space with a guaranteed value of the ultimate bound.

The contents of the paper are as follows. In Section 2 we present our main direct adaptive control framework for stabilization of linear uncertain systems with input quantizers. In Section 3 we show that if the system is in multivariable controllable canonical form, then we can always construct the adaptive quantized control law without knowing the system dynamics. In Section 4 we extend the results of Section 2 to linear uncertain systems with deadzone input nonlinearity. An illustrative numerical example is presented in Section 5 to demonstrate the efficacy of the proposed direct adaptive control framework with input quantizers. Finally, in Section 6 we draw some conclusions.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{I}$ denotes the set of integers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $(\cdot)^{\mathrm{T}}$ denotes transpose, and $(\cdot)^\dagger$ denotes the Moore-Penrose generalized inverse. Furthermore, we write $\mathrm{tr}(\cdot)$ for the trace operator, $\ln(\cdot)$ for the natural log, $\lambda_{\max}(M)$ (resp., $\lambda_{\min}(M)$) for the maximum (resp., minimum) eigenvalue of the symmetric matrix M, $\sigma_{\max}(M)$ for the maximum singular value of the matrix M, and $\mathrm{row}_i(X)$ for the ith row of the matrix X.

2 Adaptive Control for Linear Uncertain Systems with Input Quantizers

In this section we introduce an adaptive feedback control problem for linear uncertain discrete-time dynamical systems with input quantizers. Specifically, consider the linear uncertain discrete-time system $\mathcal{G}$ given by

$$x(k+1) = Ax(k) + Bv(k), \quad x(0) = x_0, \quad k \in \mathbb{N}_0, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $v(k) \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m}$. Here we assume that the input vector $v(k)$ takes quantized values. In particular, we assume that $v(k)$ is given by

$$v(k) = q(k, u(k)), \tag{2}$$

where $u(k) \in \mathbb{R}^m$ is the control input signal to be quantized at the encoder side and is given in the form

$$u(k) = H(k)x(k), \tag{3}$$

and $q(\cdot,\cdot)$ represents the *time-varying* logarithmic quantization function of the form

$$q_i(k, u_i) = \begin{cases} a_i(k)\rho_i^{-j}(k), & \text{if } u_i \in (a_i(k)\rho_i^{-j+1}(k), a_i(k)\rho_i^{-j}(k)], \\ -a_i(k)\rho_i^{-j}(k), & \text{if } u_i \in [-a_i(k)\rho_i^{-j}(k), -a_i(k)\rho_i^{-j+1}(k)), \\ 0, & \text{if } u_i = 0, \end{cases}$$
$$j \in \mathbb{I}, \quad i = 1, \ldots, m, \tag{4}$$

where $a_i(k) > 0$, $i = 1, \ldots, m$, $0 < \rho_i(k) < 1$, $i = 1, \ldots, m$, and $q_i(\cdot,\cdot)$ and u_i denote the ith components of $q(\cdot,\cdot)$ and u, respectively. Note that $\rho_i(\cdot)$ determines coarseness of the quantizer $q_i(\cdot,\cdot)$ for each u_i.

It is important to note that the logarithmic quantizer (4) can be characterized as a class of time-varying sector-bounded memoryless input nonlinearities $\mathcal{Q}$ which is given by

$$\begin{aligned}\mathcal{Q} \ &\triangleq \ \{q : \mathbb{N}_0 \times \mathbb{R}^m \to \mathbb{R}^m : q(\cdot, 0) = 0, \\ &\quad [q(k,u) - M_1(k)u]^{\mathrm{T}}[q(k,u) - M_2(k)u] \leq 0, \\ &\quad u \in \mathbb{R}^m, k \in \mathbb{N}_0\},\end{aligned} \tag{5}$$

where $M_1 \triangleq \mathrm{diag}[M_{1_1}, \ldots, M_{1_m}] > 0$ and $M_2 \triangleq \mathrm{diag}[M_{2_1}, \ldots, M_{2_m}] > 0$ are such that $\rho_i = M_{1_i}/M_{2_i}$, $i = 1, \ldots, m$, and $M_2 - M_1$ is positive definite (Figure 2(a)). Note that the sector condition characterizing $\mathcal{Q}$ is implied by the scalar sector conditions

$$M_{1_i}(k)u_i^2 \leq q_i(k, u_i)u_i \leq M_{2_i}(k)u_i^2, \quad u_i \in \mathbb{R}, \quad k \in \mathbb{N}_0, \quad i = 1, \ldots, m. \tag{6}$$

Since

$$\rho_i(\cdot) = \frac{M_{1_i}(\cdot)}{M_{2_i}(\cdot)} = \frac{1 - 2\delta_i(\cdot)}{1 + 2\delta_i(\cdot)}, \quad i = 1, \ldots, m, \tag{7}$$

where $\delta_i(\cdot) \triangleq \frac{1}{2}(M_{2_i}(\cdot) + M_{1_i}(\cdot))^{-1}(M_{2_i}(\cdot) - M_{1_i}(\cdot))$, the coarseness of the quantizer $q_i(\cdot, \cdot)$ is determined by $\delta_i(\cdot)$ for each $i = 1, \ldots, m$. Even though the time variation of $q(k, \cdot)$ is due solely to the variation of $\Delta(k) \triangleq \mathrm{diag}[\delta_1(k), \ldots, \delta_m(k)]$, we write $q(k, u(k))$ instead of $q(\Delta(k), u(k))$ for simplicity of exposition.

To design an adaptive feedback controller for (1) we decompose the quantization function $q(\cdot, \cdot)$ into a linear part and a nonlinear part so that

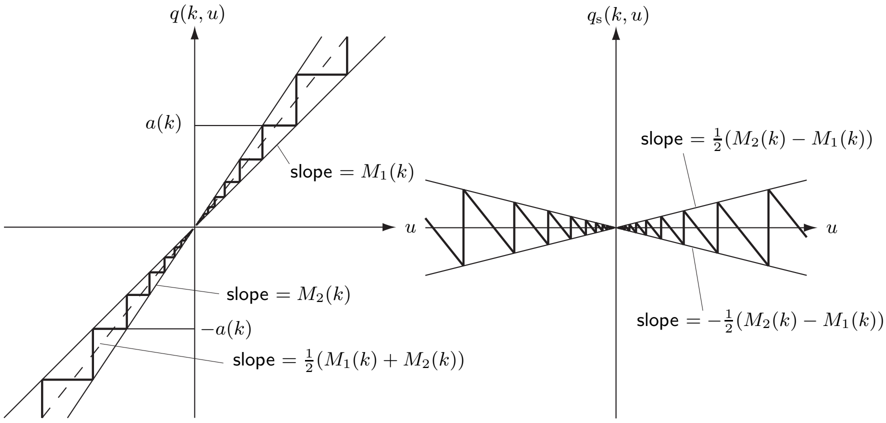
$$q(k, u) = M(k)u + q_{\mathrm{s}}(k, u), \tag{8}$$

where $q_{\mathrm{s}} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ and $M(k) \triangleq \frac{1}{2}(M_1(k) + M_2(k))$. Note that the transformed nonlinearity $q_{\mathrm{s}}(\cdot, \cdot)$ belongs to the set $\mathcal{Q}_{\mathrm{s}}$ given by

$$\begin{aligned}\mathcal{Q}_{\mathrm{s}} \ &\triangleq \ \{q_{\mathrm{s}} : \mathbb{N}_0 \times \mathbb{R}^m \to \mathbb{R}^m : q_{\mathrm{s}}(\cdot, 0) = 0, \\ &\quad q_{\mathrm{s}}^{\mathrm{T}}(k,u)q_{\mathrm{s}}(k,u) - \tfrac{1}{4}u^{\mathrm{T}}(M_2(k) - M_1(k))^2 u \leq 0, \\ &\quad u \in \mathbb{R}^m, k \in \mathbb{N}_0\}, \}\end{aligned} \tag{9}$$

(see Figure 2(b)). As discussed in the Introduction, we assume that $M_1(\cdot)$ and $M_2(\cdot)$ also take quantized (discrete) values with the aim of using network channels in the face of system uncertainties. For the guideline of choosing $M_1(k)$ and $M_2(k)$ at each $k \in \mathbb{N}_0$, see Remark 2.2 below.

Now we state the main theorem of this paper. Our objective is to design an adaptive controller in the form of (3) and a quantization rule for $u(k)$ to reduce bit rates to be sent over the communication channel. The following result, Theorem 2.1, provides a control architecture that ensures stability of the closed-loop system in the case where the system matrix A is unknown



(a) Logarithmic quantizer $q(\cdot,\cdot)$ (b) Transformed nonlinearity $q_s(\cdot,\cdot)$

Figure 2: Decomposition of a quantization function for $m = 1$

but the input matrix B is known. The case where B is also unknown is addressed in Corollary 3.1 below. For the statement of the following results define $\mathcal{A} \triangleq \{\tilde{A} \in \mathbb{R}^{n\times n} : \tilde{A} = A + BK_g^I,\ K_g^I \in \mathbb{R}^{m\times n}\}$.

Theorem 2.1 *Consider the linear uncertain discrete-time system $\mathcal{G}$ given by (1) where $A \in \mathbb{R}^{n\times n}$ is an unknown matrix, $B \in \mathbb{R}^{n\times m}$ is such that* $\operatorname{rank} B = m$, *and the pair (A,B) is stabilizable. Let $P \in \mathbb{R}^{n\times n}$ be the positive-definite solution of the Riccati equation*

$$P = \tilde{A}^{\mathrm{T}} P \tilde{A} + R - \tilde{A}^{\mathrm{T}} PB(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P\tilde{A}, \tag{10}$$

with $P \geq I_n$, where $\tilde{A} \in \mathcal{A}$ and $R \in \mathbb{R}^{n\times n}$ is positive definite. Furthermore, let $A_s \triangleq \tilde{A} + BK_g^{II}$, where $K_g^{II} \triangleq -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P\tilde{A}$, and let $Q \in \mathbb{R}^{m\times m}$ and $\varepsilon \in \mathbb{R}$ be such that $0 < Q < 2I_m$ and $\varepsilon > 0$ satisfy

$$\tfrac{1}{\varepsilon}(2I_m - Q) - 2B^{\mathrm{T}}PB \geq 0. \tag{11}$$

Then the adaptive feedback control law

$$u(k) = M^{-1}(k)K(k)x(k), \tag{12}$$

where $K(k) \in \mathbb{R}^{m\times n}$, $M_1(k)$ and $M_2(k)$ satisfy

$$R - 2K^{\mathrm{T}}(k)\Delta(k)B^{\mathrm{T}}PB\Delta(k)K(k) \geq \gamma I_n > 0, \tag{13}$$

at each time $k \in \mathbb{N}_0$, and $\gamma \in \mathbb{R}$ is an arbitrary constant, with the quantizer (2) and the update law

$$\begin{aligned} K(k+1) \;&=\; K(k) - \tfrac{1}{1+x^{\mathrm{T}}(k)Px(k)}QB^{\dagger}[x(k+1) \\ &\quad - A_s x(k) - Bq_s(k,u(k))]x^{\mathrm{T}}(k),\ K(0) = K_0, \end{aligned} \tag{14}$$

guarantees that the solution $(x(k), K(k)) \equiv (0, K_{\rm g})$*, where* $K_{\rm g} \triangleq -(B^{\rm T}PB)^{-1}B^{\rm T}PA$*, of the closed-loop system given by (1), (2), (12), and (14) is Lyapunov stable and* $x(k) \to 0$ *as* $k \to \infty$ *for all* $x_0 \in \mathbb{R}^n$*.*

Proof. First, note that from (10)

$$P = A_{\rm s}^{\rm T}PA_{\rm s} + R, \tag{15}$$

and

$$\begin{aligned} A_{\rm s}^{\rm T}PB &= (\tilde{A} - B(B^{\rm T}PB)^{-1}B^{\rm T}P\tilde{A})^{\rm T}PB \\ &= \tilde{A}^{\rm T}PB - \tilde{A}^{\rm T}PB(B^{\rm T}PB)^{-1}B^{\rm T}PB \\ &= 0. \end{aligned} \tag{16}$$

Next, define $\tilde{K}(k) \triangleq K(k) - K_{\rm g}$ and $\tilde{u}(k) \triangleq \tilde{K}(k)x(k)$, and let $K_{\rm g}^I$ be such that $\tilde{A} = A + BK_{\rm g}^I$. Note that

$$\begin{aligned} K_{\rm g} &= -(B^{\rm T}PB)^{-1}B^{\rm T}PA \\ &= K_{\rm g}^I - (B^{\rm T}PB)^{-1}B^{\rm T}P(A + BK_{\rm g}^I) \\ &= K_{\rm g}^I + K_{\rm g}^{II}. \end{aligned} \tag{17}$$

Furthermore, with $u(k)$ given by (12) it follows from (8) that

$$\begin{aligned} x(k+1) &= A_{\rm s}x(k) + B\tilde{K}(k)x(k) + Bq_{\rm s}(k, u(k)) \\ &= A_{\rm s}x(k) + B\tilde{u}(k) + Bq_{\rm s}(k, u(k)), \quad x(0) = x_0, \quad k \in \mathbb{N}_0. \end{aligned} \tag{18}$$

In addition, note that by subtracting $K_{\rm g}$ from both sides of (14) and using (18) it follows that

$$\begin{aligned} \tilde{K}(k+1) &= \tilde{K}(k) - \tfrac{1}{1+x^{\rm T}(k)Px(k)}QB^{\dagger}B\tilde{K}(k)x(k)x^{\rm T}(k) \\ &= \tilde{K}(k) - \tfrac{1}{1+x^{\rm T}(k)Px(k)}Q\tilde{K}(k)x(k)x^{\rm T}(k). \end{aligned} \tag{19}$$

To show Lyapunov stability of the closed-loop system (18) and (19), consider the Lyapunov function candidate given by

$$V(x, K) = V_1(x) + \tfrac{1}{\varepsilon}V_2(K), \tag{20}$$

where

$$V_1(x) = \ln(1 + x^{\rm T}Px), \tag{21}$$

$$V_2(K) = \mathrm{tr}[(K - K_{\rm g})^{\rm T}Q^{-1}(K - K_{\rm g})]. \tag{22}$$

Note that $V(0, K_{\rm g}) = 0$ and, since P and Q are positive definite and $\varepsilon > 0$, $V(x, K) > 0$ for all $(x, K) \neq (0, K_{\rm g})$. Furthermore, $V(x, K)$ is radially unbounded. In addition, note that

$$\begin{aligned}
\Delta V_2(x(k),K(k)) &\triangleq V_2(K(k+1)) - V_2(K(k)) \\
&= \mathrm{tr}\Big[\Big(\tilde{K}(k) - \tfrac{1}{1+x^{\mathrm{T}}(k)Px(k)}Q\tilde{K}(k)x(k)x^{\mathrm{T}}(k)\Big)^{\mathrm{T}}Q^{-1} \\
&\quad \cdot \Big(\tilde{K}(k) - \tfrac{1}{1+x^{\mathrm{T}}(k)Px(k)}Q\tilde{K}(k)x(k)x^{\mathrm{T}}(k)\Big)\Big] \\
&\quad - \mathrm{tr}[\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k)] \\
&= \mathrm{tr}[\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k)] + \tfrac{1}{(1+x^{\mathrm{T}}(k)Px(k))^2} \\
&\quad \cdot \mathrm{tr}\Big[x(k)x^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)Q\tilde{K}(k)x(k)x^{\mathrm{T}}(k)\Big] \\
&\quad - \tfrac{2}{1+x^{\mathrm{T}}(k)Px(k)}\mathrm{tr}\Big[\tilde{K}^{\mathrm{T}}(k)\tilde{K}(k)x(k)x^{\mathrm{T}}(k)\Big] \\
&\quad - \mathrm{tr}[\tilde{K}^{\mathrm{T}}(k)Q^{-1}\tilde{K}(k)] \\
&\leq -\tfrac{1}{1+x^{\mathrm{T}}(k)Px(k)}x^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I_m - Q)\tilde{K}(k)x(k),
\end{aligned} \tag{23}$$

where in (23) we used $\frac{x^{\mathrm{T}}x}{1+x^{\mathrm{T}}Px} < 1$ since $P \geq I_n$. Now, let $x(k)$ denote the solution of the closed-loop system (18). Then, using (15), (16), (23), and the fact that $2q_{\mathrm{s}}^{\mathrm{T}}B^{\mathrm{T}}PB\tilde{u} \leq \tilde{u}^{\mathrm{T}}B^{\mathrm{T}}PB\tilde{u} + q_{\mathrm{s}}^{\mathrm{T}}B^{\mathrm{T}}PBq_{\mathrm{s}}$, the Lyapunov difference along the closed-loop system trajectories is given by

$$\begin{aligned}
\Delta V(x(k),K(k)) &\triangleq V(x(k+1),K(k+1)) - V(x(k),K(k)) \\
&= \ln\Big(1 + (A_{\mathrm{s}}x(k) + B\tilde{u}(k) + Bq_{\mathrm{s}}(k,u(k)))^{\mathrm{T}}P \\
&\quad \cdot (A_{\mathrm{s}}x(k) + B\tilde{u}(k) + Bq_{\mathrm{s}}(k,u(k)))\Big) \\
&\quad - \ln(1 + x^{\mathrm{T}}(k)Px(k)) + \tfrac{1}{\varepsilon}\Delta V_2(x(k),K(k)) \\
&= \ln\Big(1 + [1 + x^{\mathrm{T}}(k)Px(k)]^{-1}\Big[x^{\mathrm{T}}(k)A_{\mathrm{s}}^{\mathrm{T}}PA_{\mathrm{s}}x(k) \\
&\quad + \tilde{u}^{\mathrm{T}}(k)B^{\mathrm{T}}PB\tilde{u}(k) + 2\tilde{u}^{\mathrm{T}}(k)B^{\mathrm{T}}PBq_{\mathrm{s}}(k,u(k)) \\
&\quad + q_{\mathrm{s}}^{\mathrm{T}}(k,u(k))B^{\mathrm{T}}PBq_{\mathrm{s}}(k,u(k)) - x^{\mathrm{T}}(k)Px(k)\Big]\Big) \\
&\quad - \tfrac{1}{\varepsilon(1+x^{\mathrm{T}}(k)Px(k))}x^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I_m - Q)\tilde{K}(k)x(k) \\
&\leq [1 + x^{\mathrm{T}}(k)Px(k)]^{-1}\Big[-x^{\mathrm{T}}(k)Rx(k) \\
&\quad + 2\tilde{u}^{\mathrm{T}}(k)B^{\mathrm{T}}PB\tilde{u}(k) + 2q_{\mathrm{s}}^{\mathrm{T}}(k,u(k))B^{\mathrm{T}}PBq_{\mathrm{s}}(k,u(k)) \\
&\quad - \tfrac{1}{\varepsilon}x^{\mathrm{T}}(k)\tilde{K}^{\mathrm{T}}(k)(2I_m - Q)\tilde{K}(k)x(k)\Big] \\
&= [1 + x^{\mathrm{T}}(k)Px(k)]^{-1}\Big[-x^{\mathrm{T}}(k)Rx(k) \\
&\quad - \tilde{u}^{\mathrm{T}}(k)\Big(\tfrac{1}{\varepsilon}(2I_m - Q) - 2B^{\mathrm{T}}PB\Big)\tilde{u}(k) \\
&\quad + 2q_{\mathrm{s}}^{\mathrm{T}}(k,u(k))B^{\mathrm{T}}PBq_{\mathrm{s}}(k,u(k))\Big], \quad k \in \mathbb{N}_0,
\end{aligned} \tag{24}$$

where in (24) we used $\ln a - \ln b = \ln\frac{a}{b}$ and $\ln(1+c) \leq c$ for $a, b > 0$ and $c > -1$, respectively. Now, using (13) and the fact that $q_{\mathrm{s}}(\cdot,\cdot)$ belongs to

$\mathcal{Q}_{\rm s}$ given by (9), it further follows from (11) and (24) that

$$\begin{aligned}\Delta V(x(k),K(k)) &\leq [1+x^{\rm T}(k)Px(k)]^{-1}\Big[-x^{\rm T}(k)Rx(k)\\ &\quad +\tfrac{1}{2}x^{\rm T}(k)K^{\rm T}(k)M^{-1}(k)(M_2(k)-M_1(k))B^{\rm T}PB\\ &\quad \cdot(M_2(k)-M_1(k))M^{-1}(k)K(k)x(k)\Big]\\ &\leq -\gamma[1+x^{\rm T}(k)Px(k)]^{-1}x^{\rm T}(k)x(k)\\ &\leq 0,\quad k\in\mathbb{N}_0, \end{aligned}\tag{25}$$

which proves that the solution $(x(k),K(k))\equiv(0,K_{\rm g})$ to (18) and (19) is Lyapunov stable. Furthermore, it follows from (the discrete-time version of) Theorem 8.4 of [13] that $x(k)\to 0$ as $k\to\infty$ for all $x_0\in\mathbb{R}^n$. □

Remark 2.1 *The conditions in Theorem 2.1 imply partial asymptotic stability; that is, the solution $(x(k),K(k))\equiv(0,K_{\rm g})$ of the overall closed-loop system is Lyapunov stable and $x(k)\to 0$ as $k\to\infty$. Hence, it follows from (14) that $K(k+1)-K(k)\to 0$ as $k\to\infty$. Furthermore, it can also be shown that the closed-loop system is partially stable with respect to the gain state K. Specifically, since $V_2(K)$ is a class-$\mathcal{K}$ function of K, it follows from (the discrete-time version of) Theorem 1 of [14] that the solution $(x(k),K(k))\equiv(\,\cdot\,,K_{\rm g})$ to (1), (44), (46) is Lyapunov stable with respect to K uniformly in x_0.*[1] ♢

Remark 2.2 *Note that the choice of $M_1(k)$ and $M_2(k)$ is arbitrary so long as (13) holds for a given ε that satisfies (11). To construct a coarse quantizer, we obviously need to take $M_1(k)$ and $M_2(k)$ such that $\Delta(k)$ is as large as possible at each time instant. Furthermore, it follows from (11) that the smaller the maximum eigenvalue of Q is, the larger ε can be and hence, by (13), $\Delta(k)$ can be taken to be large.*

We note however that the information regarding $M_1(k)$ and $M_2(k)$ must be known on both sides of the communication channel and hence must be quantized. There are several simple ways to determine and quantize $M_1(k)$ and $M_2(k)$. For example, let $M_1(k)\equiv I_m$ and $M_{2_i}(k)$, $i=1,\ldots,m$, be given by $M_{2_i}(k)\in\{1+\hat{a}\mu_i^j : j\in\mathbb{I}\}$, where $\hat{a}>0$ and $\mu_i>0$, $i=1,\ldots,m$. This implies that the smaller $\Delta(k)$ needs to be, the closer $M_{2_i}(k)$ becomes to $M_{1_i}(k)$ in a logarithmic manner for each $i=1,\ldots,m$ (see Figure 3). (Note that it is realistic in practice to impose an upper bound for $\Delta(k)$ (i.e., upper bound for $\rho^{-1}(\cdot)$ in (4)) even while $K(\cdot)$ stays close to the zero matrix.) Alternatively, another simple way to determine $M_1(\cdot)$ and $M_2(\cdot)$ is to set $M_{1_i}(k)\in\{1-\hat{a}\mu_i^j : j\in\mathbb{I}\}$ and $M_{2_i}(k)\in\{1+\hat{a}\mu_i^j : j\in\mathbb{I}\}$

[1]The dynamical system (1), (44), and (46) is said to be Lyapunov stable with respect to K uniformly in x_0 if, for every $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)$ such that $\|K_0\|<\delta$ implies that $\|K(k)\|<\varepsilon$ for all $k\in\mathbb{N}_0$ and for all $x_0\in\mathbb{R}^n$.

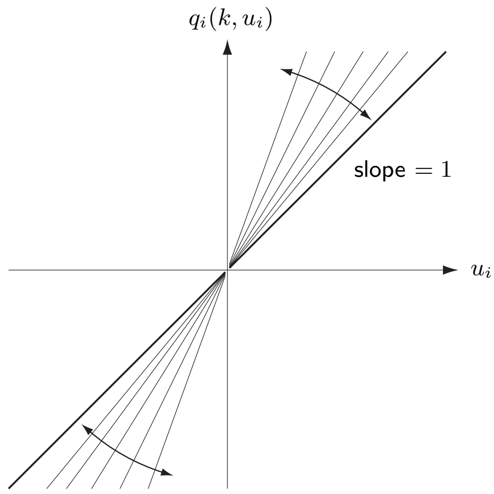


Figure 3: An example of sector bounds for the time-varying logarithmic quantizer ($M_{2_i}(k) \in \{1 + \hat{a}\mu_i^j : j \in \mathbb{I}\}$, $M_{1_i}(k) \equiv 1$)

so that $M(k) = I_m$, $k \in \mathbb{N}_0$, *and* $M_{2_i}(k) - M_{1_i}(k) = 2\hat{a}\mu_i^j < 2$, $k \in \mathbb{N}_0$, $i = 1, \ldots, m$, *since* $M_1(k) > 0$ *for all* $k \in \mathbb{N}_0$. *In either case above, there always exist* $M_1(\cdot)$ *and* $M_2(\cdot)$ *such that (13) is satisfied since* $\Delta(\cdot)$ *can be arbitrarily small. On the other hand, since* (x, K) *is Lyapunov stable with respect to* K *uniformly in* x_0 *(see Remark 2.1 above), it follows that the bounded gain state* $K(\cdot)$ *only requires the bounded* $\Delta(\cdot)$ *irrespective of* x_0 *in order for (13) to hold. This implies that only a finite number of quantized values is needed with respect to* $\Delta(\cdot)$ *and the number of quantized values depends solely on the initial value* K_0 *of the gain state* $K(\cdot)$. $\diamondsuit$

Remark 2.3 *In the single input case (i.e.,* $m = 1$*) with an unstable system matrix* A*, Theorem 2.1 has a close connection with the results given in [4]. In fact, if we have the perfect knowledge of the system dynamics, then the sector condition used in Theorem 2.1 for the largest possible conic sector reduces to the results in Theorem 2.1 of [4].*

To see this, suppose that we have the explicit knowledge of the system matrices A *and* B *so that we do not have to take adaptive control strategy. In particular, let*

$$K(k) \equiv K_{\mathrm{g}} = -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA, \tag{26}$$

where P *is the solution of the Riccati equation (10) with* $\tilde{A} = A$ *(i.e.,* $K_{\mathrm{g}}^I = 0$*). In this case, the update law (14) is superfluous by letting* $Q = 0$ *and hence it follows from (11) that the upper bound of* ε *is given by*

$$\varepsilon \leq 1/(B^{\mathrm{T}}PB). \tag{27}$$

Furthermore, take $M_1(k) \equiv 1 - \delta$ and $M_2(k) \equiv 1 + \delta$, where $0 < \delta < 1$, so that $M(k) \equiv 1$. Then it follows from (13) and (26) that

$$4\delta^2 (B^{\mathrm{T}} P B)^{-2} R^{-1/2} A^{\mathrm{T}} P B B^{\mathrm{T}} P A R^{-1/2} < 4\varepsilon I_n, \tag{28}$$

which, with (27), further implies that

$$\delta^2 (B^{\mathrm{T}} P B)^{-1} B^{\mathrm{T}} P A R^{-1} A^{\mathrm{T}} P B < 1. \tag{29}$$

Therefore, the upper bound $\delta_{\max}$ of δ is given by

$$\delta_{\max} = \sqrt{\frac{B^{\mathrm{T}} P B}{B^{\mathrm{T}} P A R^{-1} A^{\mathrm{T}} P B}}. \tag{30}$$

This is precisely the result given in [4] that characterizes the coarsest possible quantizer for the given matrices A, B, and R. In particular, [4] showed that properly choosing R in (30) further leads to the coarsest possible quantizer which is determined solely by the unstable eigenvalues of A. $\diamondsuit$

In Theorem 2.1 we assume that P is the solution to (10) which constitutes the optimal gain K_{g}^{II} for the pair $(\tilde{A}, B)$ with the quadratic cost function to be minimized [15] given by

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} x^{\mathrm{T}}(k) R x(k). \tag{31}$$

This construction yields the condition (13) that results in the identical sector bound for the case of static (non-adaptive) feedback control given in the literature [4] (see also Remark 2.3 for details). Furthermore, the matrix A_{s} defined in Theorem 2.1 does not depend on the choice of $\tilde{A}$. In fact, as long as stability is concerned, the matrix P can be replaced by the solution of the Lyapunov equation (15) with A_{s} being an arbitrary Schur (asymptotically stable) matrix that is constructed in the form of $A+BK_{\mathrm{g}}$, where $K_{\mathrm{g}} \in \mathbb{R}^{m \times n}$. In this case, closed-loop stability can be shown in a similar way to the proof of Theorem 2.1 with a new condition (instead of (13)). Finally, we note that in the case where there is no quantization requirement (i.e., $\Delta(k) \equiv 0$), the condition (13) is automatically satisfied and the adaptive control law (12), (14) specializes to the results given in [12] with $M(k) \equiv I_m$ and $q_{\mathrm{s}}(k, u) \equiv 0$.

At the end of the section we emphasize that no specific structure on the system matrices A, B is required to apply Theorem 2.1 as long as the pair (A, B) is stabilizable and $\tilde{A}$ can be given as a known matrix. However, if (1) is in controllable canonical form, then we can always construct the adaptive feedback control law without requiring knowledge of the system dynamics. These facts are exploited in the next section and generalized for the case where the input matrix B is also unknown.

3 Adaptive Quantized Control for Uncertain Systems in Canonical Form

For the design of the adaptive control law (12), (14) and the quantization rule (4), (13), Theorem 2.1 does not require the knowledge of the system matrix A nor the gain matrix K_{g} $(= K_{\mathrm{g}}^{I} + K_{\mathrm{g}}^{II})$ even though Theorem 2.1 requires that the pair (A, B) be stabilizable so that there exists a stabilizing solution to the Riccati equation (10). In this section, we show that if in particular (1) is in controllable canonical form [16] (with asymptotically stable zero dynamics), then we can *always* construct matrices A_{s} and P without requiring knowledge of the system dynamics. Furthermore, we extend the framework to the case where the input matrix B has a class of uncertainty below.

Suppose that the linear uncertain system $\mathcal{G}$ is generated by the difference model

$$\begin{aligned} z_i(k+\tau_i) + a_{i,\tau_i-1} z_i(k+(\tau_i-1)) + \cdots + a_{i,0} z_i(k) \\ = \sum_{j=1}^{m} B_{\mathrm{s}(i,j)} u_j(k), \quad k \in \mathbb{N}_0, \quad i = 1, \ldots, m, \end{aligned} \tag{32}$$

where $\tau_i \in \mathbb{N}_0$ denotes the time delay (or the relative degree) with respect to the output z_i. Here, we assume that the square matrix B_{s} composed of the entries $B_{\mathrm{s}(i,j)}$, $i, j = 1, \ldots, m$, is such that $\det B_{\mathrm{s}} \neq 0$. Furthermore, since (32) is in a form where it does not possess internal dynamics, it follows that $\tau_1 + \cdots + \tau_m$ is the dimension of the system (32). The case where (32) possesses stable internal dynamics can be handled using partial stability theory as shown in [17].

Next, define $x_i(k) \triangleq [\, z_i(k), \ldots, z_i(k+\tau_i-2)]^{\mathrm{T}}$, $i = 1, \ldots, m$, $x_{m+1}(k) \triangleq [\, z_1(k+\tau_1-1), \ldots, z_m(k+\tau_m-1)]^{\mathrm{T}}$, and $x(k) \triangleq [x_1^{\mathrm{T}}(k), \ldots, x_{m+1}^{\mathrm{T}}(k)]^{\mathrm{T}}$ so that (32) can be described by (1) with

$$A = \begin{bmatrix} A_0 \\ \Theta \end{bmatrix}, \quad B = \begin{bmatrix} 0_{(n-m)\times m} \\ B_{\mathrm{s}} \end{bmatrix}, \tag{33}$$

where $A_0 \in \mathbb{R}^{(n-m)\times n}$ is a known matrix of zeros and ones capturing the multivariable controllable canonical form representation [16] and $\Theta \in \mathbb{R}^{m\times n}$ is a matrix of uncertain constant parameters. Next, to apply Theorem 2.1 to the uncertain system (1), let $K_{\mathrm{g}}^{I} \in \mathbb{R}^{m\times s}$ be given by $K_{\mathrm{g}}^{I} = B_{\mathrm{s}}^{-1}[\Theta_{\mathrm{n}1} - \Theta]$, where $\Theta_{\mathrm{n}1} \in \mathbb{R}^{m\times n}$ is an arbitrary matrix so that $\tilde{A} = A + BK_{\mathrm{g}}^{I}$ is a known matrix (not necessarily stable). Now, since stabilizability is invariant under feedback, the pair $(\tilde{A}, B)$ is also stabilizable and hence there exists a stabilizing solution to the Riccati equation (11) so that A_{s} can be computed and used in the update law (14). Specifically, if the positive-definite matrix R is diagonal, the resulting positive-definite solution P to (10) is also diagonal and K_{g}^{II} is calculated to be $K_{\mathrm{g}}^{II} = B_{\mathrm{s}}^{-1}\Theta_{\mathrm{n}}$. In this case, it follows that $A_{\mathrm{s}} = \begin{bmatrix} A_0 \\ 0_{m\times n} \end{bmatrix}$ and hence the update law (14) is simplified as

$$K(k+1) = K(k) - \tfrac{1}{1+x^{\mathrm{T}}(k)Px(k)} QB^{\dagger}x(k+1)x^{\mathrm{T}}(k), \quad K(0) = K_0, \tag{34}$$

since $B^{\dagger}A_{\mathrm{s}} = 0$.

Next, we consider the case where A and B are *both* uncertain. Specifically, we assume that the system matrices A and B are given in the form of (33) and B is such that B_{s}, with $\sigma_{\max}(B_{\mathrm{s}}) \leq \alpha$, $\alpha > 0$, is an unknown symmetric sign-definite matrix but an upper bound α of the maximum singular value of B_{s} and the sign definiteness of B_{s} are known; that is, $0 < B_{\mathrm{s}} \leq \alpha I_m$ or $-\alpha I_m \leq B_{\mathrm{s}} < 0$. For the statement of the next result define $B_0 \triangleq \left[0_{m\times(n-m)}, I_m\right]^{\mathrm{T}}$ for $B_{\mathrm{s}} > 0$, and $B_0 \triangleq \left[0_{m\times(n-m)}, -I_m\right]^{\mathrm{T}}$ for $B_{\mathrm{s}} < 0$.

Proposition 3.1 *Consider the linear uncertain discrete-time system $\mathcal{G}$ given by (1) with A and B given by (33), where B_{s}, with $\sigma_{\max}(B_{\mathrm{s}}) < \alpha$, $\alpha > 0$, is an unknown symmetric sign-definite matrix and the sign definiteness of B_{s} is known. Let $P \in \mathbb{R}^{n\times n}$ be the positive-definite solution of the Riccati equation*

$$P = \tilde{A}^{\mathrm{T}}P\tilde{A} + R - \tilde{A}^{\mathrm{T}}PB_0(B_0^{\mathrm{T}}PB_0)^{-1}B_0^{\mathrm{T}}P\tilde{A}, \tag{35}$$

with $P \geq I_n$, where $\tilde{A} \in \mathcal{A}$ and $R \in \mathbb{R}^{n\times n}$ is positive definite. Furthermore, let $A_{\mathrm{s}} \triangleq A + B_0K_{\mathrm{g}}^{II}$, where $K_{\mathrm{g}}^{II} \triangleq -(B_0^{\mathrm{T}}PB_0)^{-1}B_0P\tilde{A}$, and let $\varepsilon, \hat{\gamma} \in \mathbb{R}$ be such that $\varepsilon > 0$ and $\hat{\gamma} > 2$ satisfy

$$\tfrac{1}{\varepsilon}(1 - \tfrac{2}{\hat{\gamma}})I_m - 2\alpha^2 B_0^{\mathrm{T}}PB_0 \geq 0. \tag{36}$$

Then the adaptive feedback control law

$$u(k) = M^{-1}(k)K(k)x(k), \tag{37}$$

where $K(k) \in \mathbb{R}^{m\times n}$ and $M_1(k)$ and $M_2(k)$ satisfy

$$R - K^{\mathrm{T}}(k)\Delta(k)\Big(\tfrac{1}{\varepsilon}(1 + \tfrac{2}{\hat{\gamma}})I_m + \alpha^2 B_0^{\mathrm{T}}PB_0\Big)\Delta(k)K(k) \geq \gamma I_n > 0, \tag{38}$$

at each $k \in \mathbb{N}_0$, with the quantizer (4) and the update law

$$K(k+1) = K(k) - \tfrac{\alpha^{-1}\hat{\gamma}^{-1}}{1+x^{\mathrm{T}}(k)Px(k)} B_0^{\mathrm{T}}[x(k+1) - A_{\mathrm{s}}x(k)]x^{\mathrm{T}}(k), \quad K(0) = K_0, \tag{39}$$

guarantees that the solution $(x(k), K(k)) \equiv (0, K_{\mathrm{g}})$, where $K_{\mathrm{g}} \in \mathbb{R}^{m\times n}$, of the closed-loop system given by (1), (37), and (39) is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$.

Proof. The proof is similar to the proof of Theorem 2.1. Specifically, note that since $A_{\rm s}^{\rm T}PB_0 = 0$, it follows that $A_{\rm s}^{\rm T}PB = A_{\rm s}^{\rm T}PB_0B_{\rm s} = 0$. Furthermore, note that

$$\begin{aligned}\tilde{K}(k+1) &= \tilde{K}(k) - \tfrac{\alpha^{-1}\hat{\gamma}^{-1}}{1+x^{\rm T}(k)Px(k)}B_0^{\rm T}[B\tilde{K}(k)x(k) + Bq_{\rm s}(k,u(k))]x^{\rm T}(k) \\ &= \tilde{K}(k) - \tfrac{\alpha^{-1}\hat{\gamma}^{-1}}{1+x^{\rm T}(k)Px(k)}|B_{\rm s}|\tilde{K}(k)x(k)x^{\rm T}(k) \\ &\quad - \tfrac{\alpha^{-1}\hat{\gamma}^{-1}}{1+x^{\rm T}(k)Px(k)}|B_{\rm s}|q_{\rm s}(k,u(k))x^{\rm T}(k). \end{aligned} \tag{40}$$

To show Lyapunov stability of the closed-loop system (18) and (40), consider the Lyapunov function candidate given by (20) with $Q = \alpha^{-1}\hat{\gamma}^{-1}|B_{\rm s}|$ $(< I_m/\gamma)$. Note that $0 < Q < I_m/2$. Now, letting $x(k)$ denote the solution of the closed-loop system (18) and using (36) and (38), the Lyapunov difference along the closed-loop system trajectories is given by

$$\begin{aligned}\Delta V(x(k),K(k)) &\le [1+x^{\rm T}(k)Px(k)]^{-1}\Big[-x^{\rm T}(k)Rx(k) \\ &\quad + 2\tilde{u}^{\rm T}(k)B^{\rm T}PB\tilde{u}(k) + 2q_{\rm s}^{\rm T}(k,u(k))B^{\rm T}PBq_{\rm s}(k,u(k)) \\ &\quad - \tfrac{1}{\varepsilon}x^{\rm T}(k)\tilde{K}^{\rm T}(k)(I_m - 2Q)\tilde{K}(k)x(k)\Big] \\ &\le [1+x^{\rm T}(k)Px(k)]^{-1}\Big[-x^{\rm T}(k)Rx(k) \\ &\quad - \tilde{u}^{\rm T}(k)\Big(\tfrac{1}{\varepsilon}(I_m - 2Q) - 2\alpha^2B_0^{\rm T}PB_0\Big)\tilde{u}(k) \\ &\quad + 2q_{\rm s}^{\rm T}(k,u(k))\Big(\tfrac{1}{\varepsilon}(I_m + 2Q) + \alpha^2B_0^{\rm T}PB_0\Big)q_{\rm s}(k,u(k))\Big] \\ &\le [1+x^{\rm T}(k)Px(k)]^{-1}\Big[-x^{\rm T}(k)Rx(k) \\ &\quad - \tilde{u}^{\rm T}(k)\Big(\tfrac{1}{\varepsilon}(I_m - \tfrac{2}{\hat{\gamma}}) - 2\alpha^2B_0^{\rm T}PB_0\Big)\tilde{u}(k) \\ &\quad + \tfrac{1}{4}x^{\rm T}(k)K^{\rm T}(k)M^{-1}(k)(M_2(k) - M_1(k)) \\ &\quad \cdot \Big(\tfrac{1}{\varepsilon}(1+\tfrac{2}{\hat{\gamma}})I_m + \alpha^2B_0^{\rm T}PB_0\Big) \\ &\quad \cdot (M_2(k) - M_1(k))M^{-1}(k)K(k)x(k)\Big] \\ &\le -\gamma[1+x^{\rm T}(k)Px(k)]^{-1}x^{\rm T}(k)x(k) \\ &\le 0, \quad k \in \mathbb{N}_0, \end{aligned} \tag{41}$$

which proves that the solution $(x(k),K(k)) \equiv (0,K_{\rm g})$ to (18) and (19) is Lyapunov stable and $x(k) \to 0$ as $k \to \infty$ for all $x_0 \in \mathbb{R}^n$. □

4 Adaptive Quantized Control with Deadzone

In practice, the input quantization methodology proposed in the preceding section cannot be applied in the sense that when the system trajectories converge to the origin, the control input necessarily takes infinite number of levels of quantized values around $u_i = 0$, $i = 1, \ldots, m$. In this section, we circumvent this problem by assuming that for each $i \in \{1, \ldots, m\}$, the control input produces no control effort when $u_i(k) \approx 0$ so that for a given compact set that contains the origin in its interior, finite levels of quantized values partition the compact set with a finite number of regions. In particular, we assume that for each $i \in \{1, \ldots, m\}$, the quantization functions $q_i(\cdot, u_i)$ lie in the sector characterized by (5) (or, equivalently, (6)) for all $u_i \notin (-\bar{u}, \bar{u})$, while $q_i(\cdot, u_i)$ vanishes for $u_i \in (-\bar{u}, \bar{u})$, where $\bar{u} > 0$; that is, $q_i(\cdot, u_i)$ in this section is given by

$$v_i(k) = q_i(k, u_i(k))\zeta(u_i(k)), \quad i = 1, \ldots, m, \tag{42}$$

where $\zeta(u_i) = 1$ for $u_i \notin (-\bar{u}, \bar{u})$ and $\zeta(u_i) = 0$ otherwise. In this case, partial asymptotic stability with respect to the plant states (cf., Remark 2.1) cannot be guaranteed due to the null physical input to the plant in the neighborhood of the equilibrium point. However, as in the following theorem, ultimate boundedness of the plant states as well as the adaptive gains is ensured. For the statement of the following results, let $B_i^\dagger \triangleq \mathrm{row}_i(B^\dagger)$, $\mathcal{N}_k \triangleq \{i \in \{1, \ldots, m\} : u_i(k) \in [-\bar{u}, \bar{u}]\}$, $k \in \mathbb{N}_0$, and recall the notions of ultimate boundedness [13] and partial ultimate boundedness [18] for discrete-time dynamical systems. Furthermore, for simplicity of exposition we assume that the matrix $M(k)$ in (8) is taken to be the identity matrix so that $q_{\mathrm{s}}(k, u) = q(k, u) - u$.

Theorem 4.1 *Consider the linear uncertain discrete-time system $\mathcal{G}$ given by (1) where $A \in \mathbb{R}^{n \times n}$ is an unknown matrix, $B \triangleq [B_1, \ldots, B_m] \in \mathbb{R}^{n \times m}$ is such that* $\mathrm{rank}\, B = m$, *and the pair (A, B) is stabilizable. Let $P \in \mathbb{R}^{n \times n}$ be the positive-definite solution of the Riccati equation (10) with $P \geq I_n$, where $\tilde{A} \in \mathcal{A}$ and $R \in \mathbb{R}^{n \times n}$ is positive definite. Furthermore, let $A_{\mathrm{s}} \triangleq \tilde{A} + BK_{\mathrm{g}}^{II}$, where $K_{\mathrm{g}}^{II} \triangleq -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P\tilde{A}$, and let $Q \triangleq \mathrm{diag}[Q_1, \ldots, Q_m] \in \mathbb{R}^{m \times m}$ and $\varepsilon \in \mathbb{R}$ be such that $0 < Q < 2I_m$ and $\varepsilon > 0$ satisfy*

$$\tfrac{1}{\varepsilon}(2 - Q_i) - 4B_i^{\mathrm{T}}PB_i \geq 0, \quad i = 1, \ldots, m. \tag{43}$$

Then the adaptive feedback control law

$$u_i(k) = K_i(k)x(k), \quad i = 1, \ldots, m, \tag{44}$$

where $K_i(k) \in \mathbb{R}^{1 \times n}$, $i = 1, \ldots, m$, and $M_1(\cdot)$ and $M_2(\cdot)$ satisfy $\frac{1}{2}(M_2(k) + M_1(k)) \equiv I_m$ and

$$R - \sum_{i \notin \mathcal{N}_k} (M_{2i}(k) - M_{1i}(k))^2 K_i^{\mathrm{T}}(k)B_i^{\mathrm{T}}PB_iK_i(k) \geq \gamma I_n > 0, \quad i = 1, \ldots, m, \tag{45}$$

at each time $k \in \mathbb{N}_0$, and $\gamma \in \mathbb{R}$ is an arbitrary constant, with the quantizer (2) and the update laws

$$K_i(k+1) = \begin{cases} K_i(k) - \frac{Q_i}{1+x^{\mathrm{T}}(k)Px(k)} B_i^{\dagger}[x(k+1) - A_{\mathrm{s}}x(k)]x^{\mathrm{T}}(k) \\ \quad -\frac{Q_i}{1+x^{\mathrm{T}}(k)Px(k)} K_i(k)x(k)x^{\mathrm{T}}(k), \quad \text{if } u_i(k) \in [-\bar{u}, \bar{u}], \\ K_i(k) - \frac{Q_i}{1+x^{\mathrm{T}}(k)Px(k)} B_i^{\dagger}[x(k+1) - A_{\mathrm{s}}x(k) \\ \quad -B_i q_{\mathrm{s}i}(k, u_i(k))]x^{\mathrm{T}}(k), \quad \text{otherwise,} \end{cases}$$

$$K_i(0) = K_{i0}, \quad i = 1, \ldots, m, \tag{46}$$

guarantees that the solution $(x(k), K(k)) \equiv (0, K_{\mathrm{g}})$, where $K(k) \triangleq [K_1^{\mathrm{T}}(k), \ldots, K_m^{\mathrm{T}}(k)]^{\mathrm{T}}$ and $K_{\mathrm{g}} \triangleq -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA$, of the closed-loop system given by (1), (44), and (46) is ultimately bounded with the ultimate bound η for x given by

$$\eta \geq \left[\frac{\exp\left[(\frac{2}{\varepsilon\gamma}\bar{u} \sum_{i=1}^{m} (2 - Q_i)\|\mathrm{row}_i(K_{\mathrm{g}})\|)^2 (\gamma + \lambda_{\max}(P)) \right] - 1}{\lambda_{\min}(P)} \right]^{\frac{1}{2}}. \tag{47}$$

Proof. First, note that (16) holds and hence $A_{\mathrm{s}}^{\mathrm{T}}PB_i = 0$, $i = 1, \ldots, m$. Next, define $\tilde{K}_i(k) \triangleq K_i(k) - K_{\mathrm{g}i}$ and $K_{\mathrm{g}i} \triangleq \mathrm{row}_i(K_{\mathrm{g}})$, $i = 1, \ldots, m$. Then, with $u_i(k)$ given by (44), it follows that

$$x(k+1) = A_{\mathrm{s}}x(k) - \sum_{i \in \mathcal{N}_k} B_i K_{\mathrm{g}i} x(k) + \sum_{i \notin \mathcal{N}_k} B_i(\tilde{K}_i(k)x(k) + q_{\mathrm{s}i}(k, u_i(k))),$$

$$x(0) = x_0, \quad k \in \mathbb{N}_0. \tag{48}$$

In addition, note that since $B_i^{\dagger}B_j = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta, by adding and subtracting $K_{\mathrm{g}i}$ to and from (46) and using (48) it follows that for $i \in \mathcal{N}_k$, i.e., for $u_i(k) \in [-\bar{u}, \bar{u}]$,

$$\begin{aligned} \tilde{K}_i(k+1) &= \tilde{K}_i(k) - \tfrac{1}{1+x^{\mathrm{T}}(k)Px(k)} Q_i B_i^{\dagger}[-B_i K_{\mathrm{g}i} x(k)]x^{\mathrm{T}}(k) \\ &\quad - \tfrac{1}{1+x^{\mathrm{T}}(k)Px(k)} Q_i K(k)x(k)x^{\mathrm{T}}(k) \\ &= \tilde{K}_i(k) - \tfrac{1}{1+x^{\mathrm{T}}(k)Px(k)} Q_i \tilde{K}_i(k)x(k)x^{\mathrm{T}}(k), \end{aligned} \tag{49}$$

and for $i \notin \mathcal{N}_k$, i.e., for $u_i(k) \notin [-\bar{u}, \bar{u}]$,

$$\begin{aligned} \tilde{K}_i(k+1) &= \tilde{K}_i(k) - \tfrac{1}{1+x^{\mathrm{T}}(k)Px(k)} Q_i B_i^{\dagger}[B_i \tilde{K}_i(k)x(k)]x^{\mathrm{T}}(k) \\ &= \tilde{K}_i(k) - \tfrac{1}{1+x^{\mathrm{T}}(k)Px(k)} Q_i \tilde{K}_i(k)x(k)x^{\mathrm{T}}(k). \end{aligned} \tag{50}$$

Hence, $\tilde{K}_i(k+1)$ takes the same form for all $k \in \mathbb{N}_0$ whether or not $u_i(k)$ belongs to the deadzone. Furthermore, consider the function

$$V_2(K) = \sum_{i=1}^{m} \tfrac{1}{Q_i}(K_i - K_{\mathrm{g}i})(K_i - K_{\mathrm{g}i})^{\mathrm{T}}, \tag{51}$$

then it follows that $\Delta V_2(x(k),K(k))$ along the closed-loop system trajectories is given by

$$\begin{aligned}
\Delta V_2(x(k),K(k)) &= \sum_{i=1}^{m}\Big(\tilde K_i(k)-\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}Q_i\tilde K_i(k)x(k)x^{\mathrm T}(k)\Big)Q_i^{-1}\\
&\quad\cdot\Big(\tilde K_i(k)-\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}Q_i\tilde K_i(k)x(k)x^{\mathrm T}(k)\Big)^{\mathrm T}\\
&\quad-\sum_{i=1}^{m}\tilde K_i^{\mathrm T}(k)Q_i^{-1}\tilde K_i(k)\\
&= \sum_{i=1}^{m}\Big[\tilde K_i^{\mathrm T}(k)Q_i^{-1}\tilde K_i(k)\\
&\quad+\tfrac{x^{\mathrm T}(k)x(k)}{(1+x^{\mathrm T}(k)Px(k))^2}x^{\mathrm T}(k)\tilde K_i^{\mathrm T}(k)Q_i\tilde K_i(k)x(k)\\
&\quad-\tfrac{2}{1+x^{\mathrm T}(k)Px(k)}x^{\mathrm T}(k)\tilde K_i^{\mathrm T}(k)\tilde K_i(k)x(k)\\
&\quad-\sum_{i=1}^{m}\tilde K_i^{\mathrm T}(k)Q_i^{-1}\tilde K_i(k)\Big]\\
&\leq -\tfrac{1}{1+x^{\mathrm T}(k)Px(k)}\sum_{i=1}^{m}x^{\mathrm T}(k)\tilde K_i^{\mathrm T}(k)(2-Q_i)\tilde K_i(k)x(k),\\
&\leq 0,\quad k\in\mathbb{N}_0, \qquad (52)
\end{aligned}$$

since by assumption $Q<2I_m$. Hence, $V_2(K(k))$ is a nonincreasing and bounded function of k and thus $K(\cdot)$ is (ultimately) bounded (see also Remark 4.1). Furthermore, it follows from the monotone convergence theorem (Theorem 8.6 of [19]) that $\lim_{k\to\infty}V_2(K(k))$ exists, which implies that $\Delta V_2(x(k),K(k))\to 0$ as $k\to\infty$.

Next, to show that $x(k)$ is ultimately bounded, consider the Lyapunov-like function (20). Now, using (15), (18), (48), and the fact that $|K_i(k)x(k)|\leq \bar u$, $i\in\mathcal{N}_k$, the Lyapunov difference along the closed-loop system trajectories is given by

$$\begin{aligned}
\Delta V(x(k),K(k)) &= \ln\Big(1+\Big(A_{\mathrm s}x(k)-\sum_{i\in\mathcal{N}_k}B_iK_{\mathrm{g}i}x(k)\\
&\quad+\sum_{i\notin\mathcal{N}_k}B_i(\tilde K_i(k)x(k)+q_{\mathrm{s}i}(k,u_i(k)))\Big)^{\mathrm T}P\\
&\quad\cdot\Big(A_{\mathrm s}x(k)-\sum_{i\in\mathcal{N}_k}B_iK_{\mathrm{g}i}x(k)+\sum_{i\notin\mathcal{N}_k}B_i(\tilde K_i(k)x(k)\\
&\quad+q_{\mathrm{s}i}(k,u_i(k)))\Big)\Big)+\tfrac{1}{\varepsilon}\Delta V_2(x(k),K(k))\\
&\quad-\ln(1+x^{\mathrm T}(k)Px(k))\\
&\leq -[1+x^{\mathrm T}(k)Px(k)]^{-1}\|x(k)\|\Big[\gamma\|x(k)\|
\end{aligned}$$

$$- \tfrac{2}{\varepsilon}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}i}\|\Big], \quad k \in \mathbb{N}_0, \tag{53}$$

where the step-by-step derivation of (53) is shown in Appendix A. Now, for each $k \in \mathbb{N}_0$ such that

$$\|x(k)\| > \tfrac{2}{\varepsilon\gamma}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}i}\|, \tag{54}$$

it follows that $\Delta V(x(k), K(k)) \leq 0$; that is, $\Delta V(x(k), K(k)) \leq 0$ for all $(x(k), K(k)) \notin \mathcal{D}_\xi$ (see Figure 4), where

$$\begin{aligned} \mathcal{D}_\xi \ \triangleq \ & \{(x, K) \in \mathbb{R}^n \times \mathbb{R}^{m\times n} : \\ & \|x\| \leq \tfrac{2}{\varepsilon\gamma}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}i}\| + \xi\}, \end{aligned} \tag{55}$$

and ξ is an arbitrarily small positive scalar. Furthermore, since $\Delta V_2(x(k), K(k)) \to 0$ as $k \to \infty$, for any $\epsilon' > 0$ there exists $k^* \in \mathbb{N}_0$ such that $\Delta V_2(x(k), K(k)) > -\epsilon'\varepsilon$, $k > k^*$, and hence from (20) and (53)

$$\begin{aligned} \Delta V_1(x(k), K(k)) \ &= \ \Delta V(x(k), K(k)) - \tfrac{1}{\varepsilon}\Delta V_2(x(k), K(k)) \\ &< \ \Delta V(x(k), K(k)) + \epsilon' \\ &\leq \ -[1 + x^{\mathrm{T}}(k)Px(k)]^{-1}\|x(k)\|\Big[\gamma\|x(k)\| \\ &\quad - \tfrac{2}{\varepsilon}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}i}\|\Big] + \epsilon', \quad k > k^*. \end{aligned} \tag{56}$$

Since $\gamma\|x(k)\| - \frac{2}{\varepsilon}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}i}\|$ is uniformly positive for all $(x(k), K(k)) \notin \mathcal{D}_\xi$, by taking a sufficiently small $\epsilon' > 0$ it follows that there exists $k^* < \infty$ such that $\Delta V_1(x(k), K(k)) \leq 0$ for any $(x(k), K(k)) \notin \mathcal{D}_\xi$, $k > k^*$.

Now, since from (56)

$$\begin{aligned} \Delta V_1(x(k), K(k)) \ &\leq \ -[1 + x^{\mathrm{T}}(k)Px(k)]^{-1}\|x(k)\|\Big[\gamma\|x(k)\| \\ &\quad - \tfrac{2}{\varepsilon}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}i}\|\Big] + \epsilon' \\ &\leq \ [1 + x^{\mathrm{T}}(k)Px(k)]^{-1}\tfrac{2}{\varepsilon}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}i}\|\|x(k)\| + \epsilon' \\ &\leq \ \tfrac{2}{\varepsilon}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}i}\|\|x(k)\| + \epsilon', \quad k > k^*, \end{aligned} \tag{57}$$

it follows that

$$\sup_{(x,K)\in\mathcal{D}}(V_1(x) + \Delta V_1(x, K)) \ \leq \ \ln(1 + \lambda_{\max}(P)(\tfrac{2}{\varepsilon\gamma}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}i}\|)^2)$$

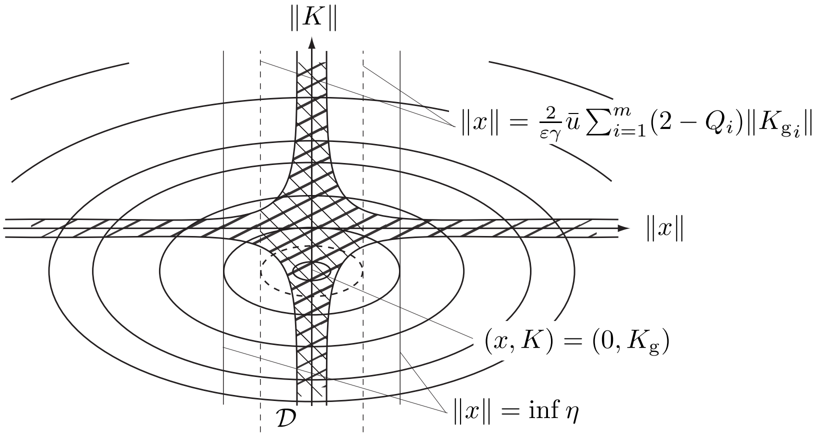


Figure 4: Visualization of the Lyapunov level sets. The shaded area indicates the region in which the quantizer produces null control effort ($K(\cdot)x(\cdot) \leq \bar{u}$).

$$
\begin{aligned}
& + \gamma(\tfrac{2}{\varepsilon\gamma}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}_i}\|)^2 \\
\leq \;& (\tfrac{2}{\varepsilon\gamma}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}_i}\|)^2(\gamma + \lambda_{\max}(P)) \\
< \;& \infty, \qquad (58)
\end{aligned}
$$

where $\mathcal{D} \triangleq \bigcap_{\xi>0}\mathcal{D}_\xi$. This proves that the solution $(x(k), K(k))$ to (1), (44), and (46) is ultimately bounded with the ultimate bound for the plant state given by (47). □

Remark 4.1 *Note that the partial Lyapunov function $V_2(K)$ given by (52) guarantees partial stability of (1), (44), and (46) with respect to the gain state K. For the details, see Remark 2.1.* ♢

Remark 4.2 *In the case where we make $\bar{u} \to 0$ in Theorem 4.1, it follows that the right-hand side of the inequality (54) converges to 0. This implies that the ultimate bound with respect to x can be made arbitrarily small, which corresponds to the case of partial asymptotic stability with respect to the plant states. As can be seen in the proof of Theorem 4.1, even though the case where $\bar{u} = 0$ was considered in Section 2 and shown to be asymptotically stable with respect to the plant state, its extension to the case where $\bar{u} > 0$ with ultimate boundedness guarantees is not straightforward. Specifically, in the proof of Theorem 4.1 we prove ultimate boundedness of the closed-loop system by showing that $\Delta V_1(x(k), K(k))$ is nonincreasing outside the set $\mathcal{D}$ and by using the that $\Delta V_2(x(k), K(k)) \to 0$ as $k \to \infty$.* ♢

In Theorem 4.1 above, we assumed that the linear uncertain system $\mathcal{G}$ is regulated with *multiple* inputs. With this assumption, the conditions (43) and (45) in Theorem 4.1 are stronger than the corresponding conditions (11) and (13) in Theorem 2.1, respectively. This is due to the fact that in the Lyapunov-like analysis (53) there appear cross terms between the control signals that belong to the deadzone and the others, which give rise to possible conservativeness in bounding the Lyapunov-like difference function $\Delta V(x(k), K(k))$. If we restrict our attention on single-input systems, the conditions (43) and (45) in Theorem 4.1 can be relaxed to achieve the identical *form* of the guaranteed ultimate bound to the one given by (47). For more discussions on this issue, see Remark 4.3.

Proposition 4.1 *Consider the linear uncertain single-input ($m = 1$) system $\mathcal{G}$ given by (1) where $A \in \mathbb{R}^{n\times n}$ is an unknown matrix, $B \in \mathbb{R}^{n\times 1}$, and the pair (A, B) is stabilizable. Let $P \in \mathbb{R}^{n\times n}$ be the positive-definite solution of the Riccati equation (10) with $P \geq I_n$, where $\tilde{A} \in \mathcal{A}$ and $R \in \mathbb{R}^{n\times n}$ is positive definite. Furthermore, let $A_{\mathrm{s}} \triangleq \tilde{A} + BK_{\mathrm{g}}^{II}$, where $K_{\mathrm{g}}^{II} \triangleq -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}P\tilde{A}$, and let $Q, \gamma, \varepsilon \in \mathbb{R}$ be such that $0 < Q < 2$ and $\gamma, \varepsilon > 0$ satisfy (11). Then the adaptive feedback control law*

$$u(k) = K(k)x(k), \tag{59}$$

where $K(k) \in \mathbb{R}^{1\times n}$ and $M_1(\cdot)$ and $M_2(\cdot)$ satisfy $\frac{1}{2}(M_2(k)+M_1(k)) \equiv 1$ and (13) at each time $k \in \mathbb{N}_0$, with the quantizer (2) and the update law

$$K(k+1) = \begin{cases} K(k) - \frac{Q}{1+x^{\mathrm{T}}(k)Px(k)}B^{\dagger}[x(k+1) - A_{\mathrm{s}}x(k)]x^{\mathrm{T}}(k) \\ \quad -\frac{Q}{1+x^{\mathrm{T}}(k)Px(k)}K(k)x(k)x^{\mathrm{T}}(k), \quad \text{if } u(k) \in [-\bar{u}, \bar{u}], \\ K(k) - \frac{Q}{1+x^{\mathrm{T}}(k)Px(k)}B^{\dagger}[x(k+1) - A_{\mathrm{s}}x(k) \\ \quad -Bq_{\mathrm{s}}(k, u(k))]x^{\mathrm{T}}(k), \quad \text{otherwise}, \end{cases} \quad K(0) = K_0, \tag{60}$$

guarantees that the solution $(x(k), K(k)) \equiv (0, K_{\mathrm{g}})$, where $K(k) \triangleq [K_1^{\mathrm{T}}(k), \ldots, K_m^{\mathrm{T}}(k)]^{\mathrm{T}}$ and $K_{\mathrm{g}} \triangleq -(B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA$, of the closed-loop system given by (1), (59), and (60) is ultimately bounded with ultimate bound η given by

$$\eta \geq \left[\frac{\exp\left[(\frac{2}{\varepsilon\gamma}\bar{u}(2-Q)\|K_{\mathrm{g}}\|)^2(\gamma + \lambda_{\max}(P))\right] - 1}{\lambda_{\min}(P)}\right]^{\frac{1}{2}}. \tag{61}$$

Proof. The proof is similar to the proof of Theorem 4.1 and hence omitted. $\square$

Remark 4.3 *As discussed in Remark 2.2 the largest attainable value of ε that satisfies (11) is twice as large as the one for the corresponding condition*

(43) for the multi-input case. In this case, the guaranteed ultimate bound (47) can become smaller. Furthermore, in order to achieve the ultimate bound (47), the condition (13) allows $\Delta(\cdot)$ twice as large as the one for (45). $\diamondsuit$

Finally, it is important to note that the case where (32) is expressed in controllable canonical form and the input matrix has a class of uncertainty as discussed in Section 3 can be similarly handled as in Proposition 3.1 to achieve the ultimate bound for x given by (47) for the multi-input case and (61) for the single-input case.

5 Illustrative Numerical Example

In this section we present a numerical example to demonstrate the utility of the proposed discrete-time adaptive control framework in the face of input quantization. Specifically, consider the linear uncertain system given by

$$\begin{aligned} z(k+2) + a_1 z(k+1) + a_0 z(k) &= bv(k), \\ z(0) = z_0, \quad z(1) = z_1, &\quad k \in \mathbb{N}_0, \end{aligned} \tag{62}$$

where $a_0, a_1, b \in \mathbb{R}$ are unknown constants, $z(k) \in \mathbb{R}$, and $v(k) \in \mathbb{R}$ is to be quantized. Note that with $x_1(k) = z(k)$ and $x_2(k) = z(k+1)$, (62) can be written in state space form (1) with $x = [x_1, x_2]^{\mathrm{T}}$, $A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$, and $B = [0, b]^{\mathrm{T}}$. Next, let $K_{\mathrm{g}}^{I} = \frac{1}{b}\left[\theta_{\mathrm{n}_1} + a_0, \theta_{\mathrm{n}_2} + a_1\right]$, where θ_{n_1}, θ_{n_2} are arbitrary scalars, so that $\tilde{A} = \begin{bmatrix} 0 & 1 \\ \theta_{\mathrm{n}_1} & \theta_{\mathrm{n}_2} \end{bmatrix}$. Now, it follows from Theorem 2.1 that the adaptive feedback controller (37) along with the quantizer (2) and the update law (39) guarantees that $x(k) \to 0$ as $k \to \infty$. Specifically, here we choose $R = I_2$ so that P satisfying (35) is given by $P = \mathrm{diag}[1, 2]$ ($> I_2$) (irrespective of θ_{n_1} and θ_{n_2} since R is diagonal). With $a_0 = 1.06$, $a_1 = -0.25$, $b = 0.4$, $Q = 0.2$, $M_1(k) \equiv 1$, $M_2(k) \in \{1 + 3 \cdot 1.3^j, j \in \mathbb{I}\}$, and initial conditions $x(0) = [-1, 3]^{\mathrm{T}}$ and $K(0) = [0, 0]$, Figure 5 shows the phase portrait of the controlled and uncontrolled system. Note that the adaptive controller is switched on at $k = 30$. Figure 6 shows the state trajectory versus time and the control signal versus time (solid lines). Furthermore, Figure 7 shows the adaptive gain history and the profile of $M_2(k)$ (solid lines). It can be seen from Figure 7 that $M_2(k)$ remains at the original value of 10 for several time steps after the controller is switched on. This implies that the required communication bit rates for control are low while the values of the adaptive gains are small.

Next, we consider the same situation as above with the difference being the deadzone assumption. In this case, it follows from Theorem 4.1 that the adaptive feedback controller (44) along with the quantizer (2) and the

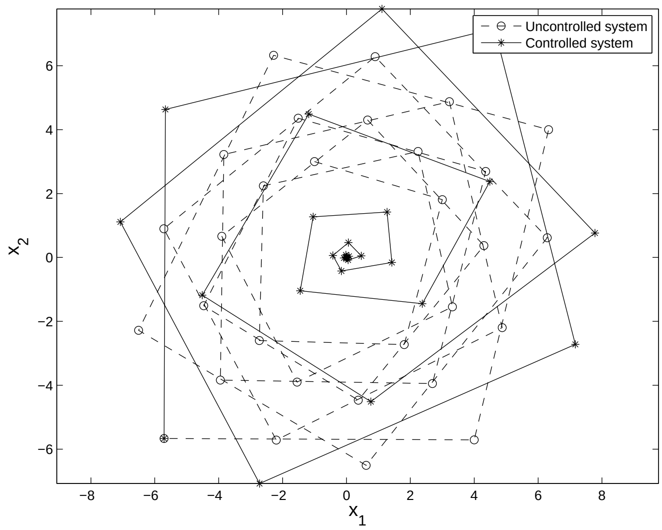


Figure 5: Phase portrait of controlled and uncontrolled system

update law (46) guarantees that $\|x(k)\| < \eta = 3.8823$ for sufficiently large k. With $\bar{u} = 1$, Figure 6 shows the state trajectory versus time and the control signal versus time (dashed lines). Furthermore, Figure 7 shows the adaptive gain history and the profile of $M_2(k)$ (dashed lines).

6 Conclusion

A discrete-time direct adaptive control framework for adaptive stabilization of multivariable linear uncertain dynamical systems with input logarithmic quantizers was developed. The proposed framework was shown to guarantee partial asymptotic stability of the closed-loop system; that is, overall closed-loop stability and attraction with respect to the plant states. Furthermore, in the case where the system is represented in controllable canonical form, the adaptive controllers can be simplified without knowledge of the system dynamics. Our control approach was not conservative in the sense that the required quantization fineness for non-uncertain linear systems coincides with the results presented in [4] which provides the coarsest quantizer. Future research will include extending the discrete-time adaptive control results to the case where output quantization is required.

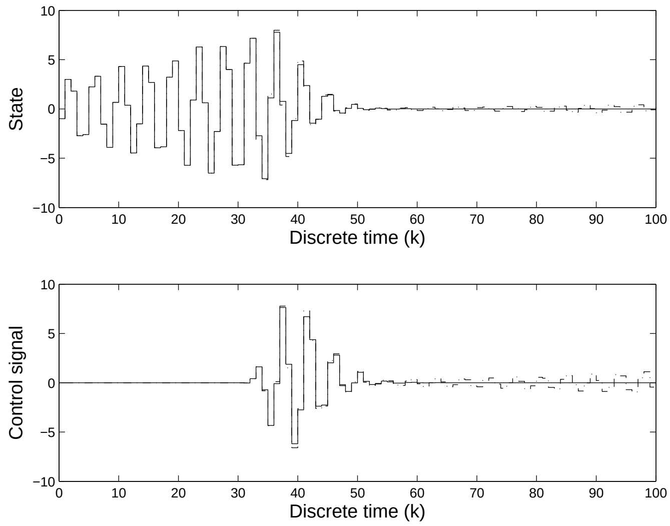


Figure 6: State trajectory and control signal versus time for the case of $\bar{u} = 0$ (solid line) and $\bar{u} = 1$ (dash-dot line)

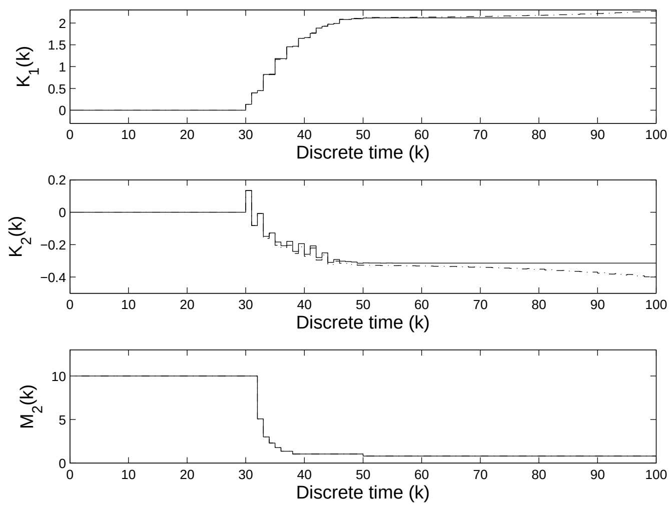


Figure 7: Adaptive gain history and profile of $M_2(k)$ for the case of $\bar{u} = 0$ (solid line) and $\bar{u} = 1$ (dash-dot line)

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A Derivation of (53)

$$\begin{aligned}
\Delta V(x(k),K(k)) &= \ln\Big(1+\Big(A_{\mathrm{s}}x(k)-\sum_{i\in\mathcal{N}_k}B_iK_{\mathrm{g}i}x(k)\\
&\quad +\sum_{i\notin\mathcal{N}_k}B_i(\tilde{K}_i(k)x(k)+q_{\mathrm{s}i}(k,u_i(k)))\Big)^{\mathrm{T}}\\
&\quad \cdot P\Big(A_{\mathrm{s}}x(k)-\sum_{i\in\mathcal{N}_k}B_iK_{\mathrm{g}i}x(k)\\
&\quad +\sum_{i\notin\mathcal{N}_k}B_i(\tilde{K}_i(k)x(k)+q_{\mathrm{s}i}(k,u_i(k)))\Big)\Big)\\
&\quad +\tfrac{1}{\varepsilon}\Delta V_2(x(k),K(k))-\ln(1+x^{\mathrm{T}}(k)Px(k))\\
&\leq \ln\Big(1+[1+x^{\mathrm{T}}(k)Px(k)]^{-1}\Big[x^{\mathrm{T}}(k)A_{\mathrm{s}}^{\mathrm{T}}PA_{\mathrm{s}}x(k)\\
&\quad +\Big(\sum_{i\in\mathcal{N}_k}B_iK_{\mathrm{g}i}x(k)\Big)^{\mathrm{T}}P\Big(\sum_{i\in\mathcal{N}_k}B_iK_{\mathrm{g}i}x(k)\Big)\\
&\quad +\Big(\sum_{i\notin\mathcal{N}_k}B_i\tilde{K}_ix(k)\Big)^{\mathrm{T}}P\Big(\sum_{i\notin\mathcal{N}_k}B_i\tilde{K}_ix(k)\Big)\\
&\quad +\Big(\sum_{i\notin\mathcal{N}_k}B_iq_{\mathrm{s}i}(k,u_i(k))\Big)^{\mathrm{T}}P\Big(\sum_{i\notin\mathcal{N}_k}B_iq_{\mathrm{s}i}(k,u_i(k))\Big)\\
&\quad -2\Big(\sum_{i\in\mathcal{N}_k}B_iK_{\mathrm{g}i}x(k)\Big)^{\mathrm{T}}P\Big(\sum_{i\notin\mathcal{N}_k}B_i\tilde{K}_i(k)x(k)\Big)\\
&\quad -2\Big(\sum_{i\in\mathcal{N}_k}B_iK_{\mathrm{g}i}x(k)\Big)^{\mathrm{T}}P\Big(\sum_{i\notin\mathcal{N}_k}B_iq_{\mathrm{s}i}(k,u_i(k))\Big)\\
&\quad +2\Big(\sum_{i\notin\mathcal{N}_k}B_i\tilde{K}_ix(k)\Big)^{\mathrm{T}}P\Big(\sum_{i\notin\mathcal{N}_k}B_iq_{\mathrm{s}i}(k,u_i(k))\Big)\\
&\quad -\tfrac{1}{\varepsilon}\sum_{i=1}^{m}(2-Q_i)(\tilde{K}_i(k)x(k))^2-x^{\mathrm{T}}(k)Px(k)\Big]\Big)
\end{aligned}$$

$$
\begin{aligned}
\leq\;& [1+x^{\mathrm{T}}(k)Px(k)]^{-1}\Big[-x^{\mathrm{T}}(k)Rx(k) \\
& + 2\Big(\sum_{i\in\mathcal{N}_k} B_i K_{\mathrm{g}_i} x(k)\Big)^{\mathrm{T}} P\Big(\sum_{i\in\mathcal{N}_k} B_i K_{\mathrm{g}_i} x(k)\Big) \\
& + 2\Big(\sum_{i\notin\mathcal{N}_k} B_i \tilde{K}_i x(k)\Big)^{\mathrm{T}} P\Big(\sum_{i\notin\mathcal{N}_k} B_i \tilde{K}_i x(k)\Big) \\
& + 2\Big(\sum_{i\notin\mathcal{N}_k} B_i q_{\mathrm{s}i}(k,u_i(k))\Big)^{\mathrm{T}} P\Big(\sum_{i\notin\mathcal{N}_k} B_i q_{\mathrm{s}i}(k,u_i(k))\Big) \\
& - \tfrac{1}{\varepsilon}\sum_{i\in\mathcal{N}_k}(2-Q_i)(\tilde{K}_i(k)x(k))^2 \\
& - \tfrac{1}{\varepsilon}\sum_{i\notin\mathcal{N}_k}(2-Q_i)(\tilde{K}_i(k)x(k))^2\Big] \\
\leq\;& [1+x^{\mathrm{T}}(k)Px(k)]^{-1}\Big[-x^{\mathrm{T}}(k)Rx(k) \\
& + 4\sum_{i\in\mathcal{N}_k} x^{\mathrm{T}}(k){K_{\mathrm{g}_i}}^{\mathrm{T}} B_i^{\mathrm{T}} P B_i K_{\mathrm{g}_i} x(k) \\
& + 4\sum_{i\notin\mathcal{N}_k} x^{\mathrm{T}}(k)\tilde{K}_i^{\mathrm{T}} B_i^{\mathrm{T}} P B_i \tilde{K}_i x(k) \\
& + 4\sum_{i\notin\mathcal{N}_k} {q_{\mathrm{s}i}}^{\mathrm{T}}(k,u_i(k)) B_i^{\mathrm{T}} P B_i q_{\mathrm{s}i}(k,u_i(k)) \\
& - \tfrac{1}{\varepsilon}\sum_{i\in\mathcal{N}_k}(2-Q_i)(K_i(k)x(k))^2 \\
& - \tfrac{1}{\varepsilon}\sum_{i\in\mathcal{N}_k}(2-Q_i)(K_{\mathrm{g}_i}x(k))^2 \\
& + \tfrac{2}{\varepsilon}\sum_{i\in\mathcal{N}_k}(2-Q_i)(K_i(k)x(k))(K_{\mathrm{g}_i}x(k)) \\
& - \tfrac{1}{\varepsilon}\sum_{i\notin\mathcal{N}_k}(2-Q_i)(\tilde{K}_i(k)x(k))^2\Big] \\
\leq\;& [1+x^{\mathrm{T}}(k)Px(k)]^{-1}\Big[-x^{\mathrm{T}}(k)Rx(k) \\
& - \sum_{i\in\mathcal{N}_k} x^{\mathrm{T}}(k){K_{\mathrm{g}_i}}^{\mathrm{T}}[\tfrac{1}{\varepsilon}(2-Q_i) - 4B_i^{\mathrm{T}} P B_i] K_{\mathrm{g}_i} x(k) \\
& - \sum_{i\notin\mathcal{N}_k} x^{\mathrm{T}}(k)\tilde{K}_i^{\mathrm{T}}[\tfrac{1}{\varepsilon}(2-Q_i) - 4B_i^{\mathrm{T}} P B_i] \tilde{K}_i x(k) \\
& + 4\sum_{i\notin\mathcal{N}_k} {q_{\mathrm{s}i}}^{\mathrm{T}}(k,u_i(k)) B_i^{\mathrm{T}} P B_i q_{\mathrm{s}i}(k,u_i(k)) \\
& + \tfrac{2}{\varepsilon}\sum_{i\in\mathcal{N}_k}(2-Q_i)(K_i(k)x(k))(K_{\mathrm{g}_i}x(k))\Big]
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & [1+x^{\mathrm{T}}(k)Px(k)]^{-1}\Big[-x^{\mathrm{T}}(k)Rx(k) \\
& + \sum_{i\notin\mathcal{N}_k} x^{\mathrm{T}}(k)K^{\mathrm{T}}(k)(M_2(k)-M_1(k))B_i^{\mathrm{T}}PB_i \\
& \cdot (M_2(k)-M_1(k))K(k)x(k) \\
& + \tfrac{2}{\varepsilon}\bar{u}\sum_{i\in\mathcal{N}_k}(2-Q_i)\|K_{\mathrm{g}i}\|\|x(k)\|\Big] \\
\leq\ & -[1+x^{\mathrm{T}}(k)Px(k)]^{-1}\|x(k)\|\Big[\gamma\|x(k)\| \\
& - \tfrac{2}{\varepsilon}\bar{u}\sum_{i=1}^{m}(2-Q_i)\|K_{\mathrm{g}i}\|\Big], \quad k\in\mathbb{N}_0. \qquad (63)
\end{aligned}
$$