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Abstract

In this paper, we consider source location problems and their generalizations with three connectivity requirements (arc-connectivity requirements λ and two kinds of vertex-connectivity requirements κ and $\hat{\kappa}$), where the source location problems are to find a minimum-cost set $S \subseteq V$ in a given graph $G = (V, A)$ with a capacity function $u : A \to \mathbb{R}_+$ such that for each vertex $v \in V$, the connectivity from S to v (resp., from v to S) is at least a given demand $d^-(v)$ (resp., $d^+(v)$). We show that the source location problem with edge-connectivity requirements in undirected networks is strongly NP-hard, which solves an open problem posed by Arata *et al.* [2]. Moreover, we show that the source location problems with three connectivity requirements are inapproximable within a ratio of $c \ln D$ for some constant c, unless every problem in NP has an $\mathrm{O}(N^{\log\log N})$-time deterministic algorithm. Here D denotes the sum of given demands. We also devise $(1 + \ln D)$-approximation algorithms for all the extended source location problems if we have the integral capacity and demand functions. By the inapproximable results above, this implies that all the source location problems are $\Theta(\ln \sum_{v \in V}(d^+(v) + d^-(v)))$-approximable.

1 Introduction

There is vast literature on location problems in the fields of operations research, computer science, etc. (see, e.g., [13]). Location problems in networks are often formulated as optimization problems to determine the best

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location of facilities such as industrial plants or warehouses in given networks to satisfy a certain property. Location problems based on flow (i.e., connectivity) requirements, called *source location problems*, were introduced by Tamura *et al.* [18, 19], and have recently received much attention from many authors (e.g., [1, 2, 3, 8, 11, 12, 15]).

Connectivity is one of the most important factors in applications to control and design of multimedia networks. Suppose that we are asked to locate a set S of servers which can provide a certain service in a multimedia network $\mathcal{N}$. A user at vertex v can receive a service by connecting to a server in S through a path in $\mathcal{N}$. To ensure the quality of the service to v even if certain number d of links and/or vertices become out of order, we should select S so that the arc- and/or vertex-connectivity between S and v is at least $d+1$. Therefore, these kinds of fault-tolerant settings can be formulated as source location problems.

Formally, source location problems can be described as follows. Let $\mathcal{N} = (G = (V, A), u)$ be a network with a vertex set V of cardinality n, an arc set A of cardinality m, and a capacity function $u : A \to \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the set of all nonnegative reals. It has two demand functions $d^-, d^+ : V \to \mathbb{R}_+$, and a cost function $c : V \to \mathbb{R}_+$. Then the problem is given as follows.

$$\begin{array}{lll} \text{Minimize} & \displaystyle\sum_{v \in S} c(v) & \\ \text{subject to} & \psi^-(S, v) \geq d^-(v) \text{ and } \psi^+(v, S) \geq d^+(v) \quad (v \in V), & (1.1) \\ & S \subseteq V. & \end{array}$$

Here $\psi^-(X, Y)$ and $\psi^+(X, Y)$ denote certain measurements based on the connectivity from vertex set X to vertex set Y in $\mathcal{N}$. For any $v \in V$, we simply write $\psi^-(S, v)$ and $\psi^+(v, S)$ instead of $\psi^-(S, \{v\})$ and $\psi^+(\{v\}, S)$, respectively. As such measurements $\psi^\pm$, this paper studies three basic connectivity requirements: arc-connectivity λ, and two kinds of vertex-connectivity κ and $\hat{\kappa}$. Note that Problem (1.1) sometimes adopts a single measurement ψ ($= \psi^- = \psi^+$). Let us further note that in a more general problem setting we consider multiple constraints in (1.1) as $\psi_i^-(S, v) \geq d_i^-(v)$, $\psi_i^+(v, S) \geq d_i^+(v)$ for all $v \in V$ and $i = 1, 2, \cdots, \ell$. For example, we can consider the arc- and vertex-connectivity simultaneously.

Tamura *et al.* [18] first considered the source location problem with edge-connectivity (i.e., arc-connectivity) requirements ψ $(= \psi^- = \psi^+) = \lambda$, when $\mathcal{N}$ is undirected, and both the cost c and demand d are uniform (i.e., $c(v) = 1$ and $d(v) = k$ for all $v \in V$), and gave a polynomial time algorithm for it. Since then, Tamura *et al.* [19, 20], Ito *et al.* [11] and Arata *et al.* [2] have investigated the source location problem with edge-connectivity requirements in undirected networks. They provided polynomial time algorithms when the cost function or the demand function is uniform. On the other hand, it was shown that the problem is in general weakly NP-hard [2]. But, it remained open to prove the NP-hardness in the strong sense or to devise a pseudo-polynomial time algorithm. For directed networks, Ito *et al.* [12] showed that the problem is strongly NP-hard, even if either the cost function c or the demand functions d^- and d^+ are uniform, and Bárász *et al.* [3] and Heuvel *et al.* [8] provided a polynomial time algorithm if c, d^- and d^+ are all uniform. Tables (a) and (b) in Figure 1 summarize the best known bounds for the source location problems with the arc-connectivity requirements.

The source location problem with vertex-connectivity requirements (i.e., $\psi^- = \psi^+ = \kappa$) was investigated by Ito *et al.* [11]. They considered the case in which G is undirected, and the cost and the demand functions c and $d\,(= d^- = d^+)$ are both uniform (i.e., $c(v) = 1$ and $d(v) = k$ for all $v \in V$). They showed that the problem is polynomially solvable for $k \leq 2$, but NP-hard for $k \geq 3$. It is easy to see that the negative result also follows, even if G is directed. They also showed that the positive result for $k \leq 2$ can be extended to the case in which the edge-connectivity $\lambda(S, v) \geq \ell$ is required simultaneously.

Let us note that the vertex-connectivity requirements, say, $\kappa(S, v)$ ensures that there exist at least $\kappa(S, v)$ *internally* vertex-disjoint paths from S to v. This implicitly means that any source in S never has a breakdown. To take possible breakdowns of sources into consideration, Nagamochi *et al.* [15] introduced another kinds of vertex-connectivity requirements $\hat{\kappa}^-(S, v)$ and $\hat{\kappa}^+(v, S)$, where $\hat{\kappa}^-(S, v)$ (resp., $\hat{\kappa}^+(v, S)$) is the maximum number of paths from S to v (resp., from v to S) which are vertex-disjoint except at v. They presented a polynomial time algorithm for the source location problem when d^- and d^+ are uniform and $\psi^\pm = \hat{\kappa}^\pm$. Ishii *et al.* [9, 10] considered the problem with a non-uniform demand function in undirected networks, gave a polynomial time algorithm if $d(v)\ (= d^-(v) = d^+(v)) \leq 3$, and showed that the problem is NP-hard in general. Tables (c) $\sim$ (f) in Figure 1 summarize the best known bounds for the source location problems with the vertex-connectivity requirements.

Some other types of source location problems were also studied [1, 11, 18, 19]. For example, the single cover problem [18, 19] computes a minimum set S such that each vertex v has sources $s_1, s_2 \in S$ such that $\psi^-(s_1, v) \geq d^-(v)$ and $\psi^+(v, s_2) \geq d^+(v)$.

Let us next direct our attention to the extended source location problems.

We note that the cost function of the source location problem depends only on the fixed setup cost of the facilities at vertices. It is natural to consider the cost functions which depend not only on the setup cost, but also on the supply values. We consider this kind of generalization, called the extended source location problems, which was introduced in [16], Namely, we deal with source location problems with supply values $x(v)$ of facilities at $v \in V$, whose cost functions c_v $(v \in V)$ are the sum of opening cost and

	c :uniform		c :arbitrary	
d :uniform	$O(n(m + n \log n))$	Arata *et al.*[2]	$O(n(m + n \log n))$	Arata *et al.*[2]
d :arbitrary	$O(nM(n, m))^*$	Arata *et al.*[2]	weakly NP-hard	Arata *et al.*[2]

$^*M(n, m)$: the time complexity of computing a maximum flow in undirected networks with n vertices and m arcs.

(a) Edge-connectivity requirements in undirected networks

	c :uniform		c :arbitrary	
d :uniform	$O(n^3 m \log(n^2/m))$	Bárász *et al.* [3]	NP-hard	Ito *et al.* [12]
d :arbitrary	NP-hard	Ito *et al.* [12]	NP-hard	Ito *et al.* [12]

(b) Arc-connectivity requirements in directed networks

	c :uniform		c :arbitrary	
d :uniform	NP-hard*	Ito *et al.* [11]	NP-hard	Ito *et al.* [11]
d :arbitrary	NP-hard	Ito *et al.* [11]	NP-hard	Ito *et al.* [11]

* linearly solvable if $d(v) \leq 2$ for all $v \in V$ [11].

(c) Vertex-connectivity requirements κ in undirected networks

	c :uniform		c :arbitrary	
d :uniform	NP-hard*	Ito *et al.* [11]	NP-hard	Ito *et al.* [11]
d :arbitrary	NP-hard	Ito *et al.* [11]	NP-hard	Ito *et al.* [11]

* polynomially solvable if $d(v) \leq 1$ for all $v \in V$ [12].

(d) Vertex-connectivity requirements κ in directed networks

	c :uniform		c :arbitrary	
d :uniform ($d \equiv k$)	$O(\min\{k, \sqrt{n}\}kn^2)$	Nagamochi *et al.* [15]	$O(\min\{k, \sqrt{n}\}kn^2)$	Nagamochi *et al.* [15]
d :arbitrary	NP-hard*	Ishii *et al.* [9]	NP-hard**	Ishii *et al.* [9]

* linearly solvable if $d(v) \leq 3$ for all $v \in V$ [9].
** solvable in $O(n^4(\log n)^2)$ time if $d(v) \leq 3$ for all $v \in V$ [10].

(e) Vertex-connectivity requirements $\hat{\kappa}$ in undirected networks

	c :uniform		c :arbitrary	
d :uniform ($d \equiv k$)	$O(\min\{k, \sqrt{n}\}mn)$	Nagamochi *et al.* [15]	$O(\min\{k, \sqrt{n}\}mn)$	Nagamochi *et al.* [15]
d :arbitrary	NP-hard	Ishii *et al.* [9]	NP-hard	Ishii *et al.* [9]

(f) Vertex-connectivity requirements $\hat{\kappa}^{\pm}$ in directed networks

Figure 1: Current best results for the source location problem

monotone concave running cost for facility at v. We remark that monotonicity and concavity are natural assumptions on the cost, and are required in many network design problems (see, e.g., [6]). The extended source location problems were investigated in [16] for uniform edge-connectivity requirements in undirected networks. By modeling the problem as a laminar cover problem, it can be shown that it is solvable in $\mathrm{O}(nm + n^2(q + \log n))$ time, where q is the time required to compute $c_v(x)$ for each $x \in \mathbb{R}_+$ and $v \in V$.

In this paper we investigate the (extended) source location problems. We show that the source location problem with edge-connectivity requirements in undirected networks is strongly NP-hard. This solves an open problem posed in [2], and gives us a complete picture of the time complexity of the source location problems (see Figure 1). Moreover, we show that the source location problems with three connectivity requirements are inapproximable within a ratio of $c \ln \sum_{v \in V} d(v)$ for some constant c, unless every problem in NP has an $\mathrm{O}(N^{\log \log N})$-time deterministic algorithm. We also devise a $(1 + \ln \sum_{v \in V}(d^+(v) + d^-(v)))$-approximation algorithm for the extended source location problems if we have the integral capacity and demand functions. We remark that our approximation algorithm is applicable for all connectivity requirements. By the inapproximable result above, we can say that our algorithm is *optimal* for all the extended source location problems, i.e., they are $\Theta(\ln \sum_{v \in V}(d^+(v) + d^-(v)))$-approximable.

The rest of the paper is organized as follows. Section 2 introduces some notation and definitions of the source location problems, Section 3 shows the intractability of the source location problems, and Section 4 defines the extended source location problems and discuss their basic properties. Section 5 presents approximation algorithms for the (extended) source location problems. Finally, Section 6 concludes the paper.

2 Definitions and Preliminaries

Let $\mathcal{N} = (G = (V, A), u)$ be a directed network (or an undirected network) with a set V of n vertices, a set A of m arcs, and a capacity function $u : A \to \mathbb{R}_+$. We sometimes write A as E for an undirected network, if no confusion arises. A singleton set $\{x\}$ may simply be written as x. For a $W \subseteq V$, $\mathcal{N}[W]$ denotes the subnetwork of $\mathcal{N}$ induced by W.

For any $X, Y \subseteq V$, we denote $A(X, Y) = \{(x, y) \in A \mid x \in X, y \in Y\}$. For $X \subseteq V$, let $u^-(X)$ (resp., $u^+(X)$) denote the capacity sum of arcs entering (resp., leaving) X, i.e.,

$$u^-(X) = \sum_{a \in A(V \setminus X, X)} u(a), \quad u^+(X) = \sum_{a \in A(X, V \setminus X)} u(a),$$

and, for an undirected network, let us define $u(X)$ by $u(X) = u^-(X) + u^+(X)$, where we note that $u(X) = u(V \setminus X)$. For every $X \subseteq V$, a vertex

$v \in V \setminus X$ is called an *in-neighbor* (resp., *out-neighbor*) of X if there is an arc $(v, x) \in A$ (resp., $(x, v) \in A$) for some $x \in X$, and v is simply called a *neighbor* of X if at least one of these two conditions holds. The set of all in-neighbors (resp., out-neighbors) of X is denoted by $N^-(X)$ (resp., $N^+(X)$).

Let us now define three connectivities λ, κ, and $\hat{\kappa}$. For vertex subsets $X, Y \subseteq V$, we say that X is *k-arc-connected* to Y if there exists a feasible flow φ from X to Y whose value is at least k, where a flow $\varphi : A \to \mathbb{R}_+$ is a feasible flow from X to Y if it satisfies the following conditions:

$$\partial\varphi(v) \stackrel{\text{def}}{=} \sum_{(v,w)\in A} \varphi(v, w) - \sum_{(w,v)\in A} \varphi(w, v) = 0 \quad (v \in V - (X \cup Y)), \quad (2.1)$$

$$0 \leq \varphi(a) \leq u(a) \qquad (a \in A), \qquad (2.2)$$

and the value of φ is defined by $\sum_{v\in X} \partial\varphi(v)$. The *arc-connectivity* from X to Y, denoted by $\lambda(X, Y)$, is the maximum number k such that X is k-arc-connected to Y. Here we define $\lambda(X, Y) = +\infty$ if $X \cap Y \neq \emptyset$. For an undirected network, we use the term "edge" instead of "arc", e.g., arc-connectivity is called edge-connectivity if $\mathcal{N}$ is undirected.

For two sets $X, Y \subseteq V$, we say that X is *k-vertex-connected* to Y if there exists k internally vertex-disjoint paths from X to Y. The vertex-connectivity from X to Y, denoted by $\kappa(X, Y)$, is the maximum number k such that X is k-vertex-connected to Y. We define $\kappa(X, Y) = +\infty$ if $X \cap Y \neq \emptyset$ or $E(X, Y) \neq \emptyset$.

For two sets $X, Y \subseteq V$, $\hat{\kappa}^-(X, Y)$ (resp., $\hat{\kappa}^+(X, Y)$) denotes the maximum number of paths from X to Y such that no pair of paths contains a common vertex in $V \setminus Y$ (resp., $V \setminus X$). We define $\hat{\kappa}^-(X, Y) = +\infty$ and $\hat{\kappa}^+(X, Y) = +\infty$, if $X \cap Y \neq \emptyset$.

From the max-flow min-cut theorem, the connectivity conditions can be restated as follows.

Lemma 2.1 *Let X and Y be two subsets of V.*

(i) *$\lambda(X, Y) \geq k$ holds if and only if $u^+(W) \geq k$ holds for every vertex set W with $X \subseteq W \subseteq V \setminus Y$.*

(ii) *$\kappa(X, Y) \geq k$ if and only if $|N^+(W)| \geq k$ holds for every vertex set W with $X \subseteq W$ and $W \cup N^+(W) \subseteq V \setminus Y$.*

(iii) *$\hat{\kappa}^+(X, Y) \geq k$ (resp., $\hat{\kappa}^-(X, Y) \geq k$) holds if and only if $|N^+(W)| \geq k$ (resp., $|N^-(W)| \geq k$) holds for every vertex set W with $X \subseteq W \subseteq V \setminus Y$ (resp., $Y \subseteq W \subseteq V \setminus X$).* □

This paper studies the source location problems given by (1.1) with three basic connectivity requirements λ, κ and $\hat{\kappa}$. Formally, we consider three cases in which the constraints $\psi^-(S,v) \geq d^-(v)$ and $\psi^+(S,v) \geq d^+(v)$ are given as follows.

$$\lambda(S,v) \geq d^-(v) \text{ and } \lambda(v,S) \geq d^+(v), \tag{2.3}$$

$$\kappa(S,v) \geq d^-(v) \text{ and } \kappa(v,S) \geq d^+(v), \tag{2.4}$$

$$\hat{\kappa}^-(S,v) \geq d^-(v) \text{ and } \hat{\kappa}^+(v,S) \geq d^+(v). \tag{2.5}$$

Let us rewrite conditions (2.3), (2.4) and (2.5) in terms of deficient sets, which will be defined as follows.

For a vertex set $X \subseteq V$, $d^-(X)$ (resp., $d^+(X)$) denotes the maximum in-demand (resp., out-demand) among all vertices in X, i.e.,

$$d^-(X) = \max_{v \in X} d^-(v) \quad (\text{resp., } d^+(X) = \max_{v \in X} d^+(v)).$$

A set $W \subseteq V$ is *deficient* with respect to λ if $u^-(W) < d^-(W)$ or $u^+(W) < d^+(W)$. A deficient set W is called *minimal* if no other nonempty subset $X \subsetneq W$ is deficient. Let $\mathcal{W}_\lambda$ be a family of all minimal deficient sets with respect to λ. By Lemma 2.1 (i), the constraint is equivalent to

$$u^-(X) \geq d^-(v) \text{ and } u^+(X) \geq d^+(v) \qquad (v \in X \subseteq V \setminus S), \tag{2.6}$$

and hence the following lemma holds.

Lemma 2.2 ([20]) *A set $S \subseteq V$ is a feasible solution of the source location problem with arc-connectivity requirements λ if and only if $S \cap W \neq \emptyset$ holds for every $W \in \mathcal{W}_\lambda$.* □

We next consider vertex-connectivity requirements κ. From Lemma 2.1 (ii), (2.4) can be restated as

$$|N^-(X)| \geq d^-(v) \text{ and } |N^+(Y)| \geq d^+(v) \tag{2.7}$$

for all X with $v \in X$ and $X \cup N^-(X) \subseteq V \setminus S$, and for all Y with $v \in Y$ and $Y \cup N^+(Y) \subseteq V \setminus S$. Therefore, we define the deficiency in the following way. A set $W \subseteq V$ is deficient with respect to κ if W can be represented by $W = X \cup N^-(X)$ with $|N^-(X)| < d^-(X)$ or $W = X \cup N^+(X)$ with $|N^+(X)| < d^+(X)$. Let $\mathcal{W}_\kappa$ denote a family of all minimal deficient sets with respect to κ. Then we have the following lemma.

Lemma 2.3 *A set $S \subseteq V$ is feasible for the source location problem with vertex-connectivity requirements κ if and only if $S \cap W \neq \emptyset$ holds for every $W \in \mathcal{W}_\kappa$.* □

Finally, we define deficient sets with respect to $\hat{\kappa}$. A set $W \subseteq V$ is *deficient* with respect to $\hat{\kappa}$ if $|N^-(W)| < d^-(W)$ or $|N^+(W)| < d^+(W)$ holds. Let $\mathcal{W}_{\hat{\kappa}}$ be a family of all minimal deficient sets with respect to $\hat{\kappa}$. By Lemma 2.1 (iii), constraint (2.5) is equivalent to

$$|N^-(X)| \geq d^-(v) \text{ and } |N^+(X)| \geq d^+(v) \qquad (v \in X \subseteq V \setminus S), \tag{2.8}$$

and we thus have the following lemma.

Lemma 2.4 ([9]) *A set $S \subseteq V$ is feasible for the source location problem with vertex-connectivity requirements $\hat{\kappa}$ if and only if $S \cap W \neq \emptyset$ holds for every $W \in \mathcal{W}_{\hat{\kappa}}$.* □

In the subsequent sections, we frequently make use of the constraints based on deficient sets.

3 Intractability of the Source Location Problems

In this section, we show the hardness results for all the source location problems. Let us first show that the problem with edge-connectivity requirements in undirected networks is strongly NP-hard. Recall that the problem can be written as follows. Given an undirected network $\mathcal{N} = (G = (V, E), u)$ with a capacity function $u : E \to \mathbb{Z}_+$, a demand function $d : V \to \mathbb{R}_+$, and a cost function $c : V \to \mathbb{R}_+$, consider

$$\begin{array}{lll} \text{Minimize} & \sum\{c(v) \mid v \in S\} & (3.1) \\ \text{subject to} & \lambda(S, v) \geq d(v) \quad (v \in V), & \\ & S \subseteq V, & \end{array}$$

where $\lambda(S, v)$ denotes the edge-connectivity between S and v.

Theorem 3.1 *Problem* (3.1) *is strongly NP-hard.*

Proof. We show this by reducing to Problem (3.1) the set cover problem, which is known to be strongly NP-hard [7].

Problem SET COVER
Input. A set $U = \{1, 2, \cdots, p\}$ and a family $\mathcal{S} = \{S_1, \cdots, S_q\} \subseteq 2^U$.
Output. A subfamily $\mathcal{X} \subseteq \mathcal{S}$ such that $\bigcup_{S_i \in \mathcal{X}} S_i = U$ and $|\mathcal{X}|$ is minimum.

Let $\ell_i = |S_i|$ and $k_j = |\{S_i \mid S_i \ni j\}|$. Given a problem instance I of SET COVER, we construct the corresponding instance J of Problem (3.1) as follows.

$$\begin{array}{rcl} V & = & \{t_1, t_2\} \cup \{s_1, \cdots, s_q\} \cup \{x_1, \cdots, x_p\}, \\ E & = & \{(t_1, s_i) \mid i = 1, \cdots, q\} \cup \{(s_i, x_j) \mid j \in S_i, i = 1, \cdots, q\} \\ & & \cup \{(x_j, t_2) \mid j = 1, \cdots, p\}, \\ u(v, w) & = & \begin{cases} \ell_i & \text{if } v = t_1 \text{ and } w = s_i \ (i = 1, \cdots, q), \\ 1 & \text{if } v = s_i, w = x_j \text{ and } j \in S_i \ (i = 1, \cdots, q), \\ k_j - 1 & \text{if } v = x_j \ (j = 1, \cdots, p) \text{ and } w = t_2, \end{cases} \end{array}$$

$$d(v) = \begin{cases} \sum_{i=1}^{q} \ell_i & \text{if } v = t_1, t_2, \\ 0 & \text{otherwise,} \end{cases}$$

$$c(v) = \begin{cases} 0 & \text{if } v = t_2, \\ 1 & \text{if } v \in \{s_1, \cdots, s_q\}, \\ q+1 & \text{otherwise.} \end{cases}$$

We denote $S = \{s_i, \cdots, s_q\}$ and $X = \{x_1, \cdots, x_p\}$. Figure 2 gives an example of our reduction. Intuitively, s_i and x_j correspond to the set $S_i \in \mathcal{S}$ and the element j of U, respectively. We note that $\sum_{i=1}^{q} \ell_i = \sum_{j=1}^{p} k_j$.

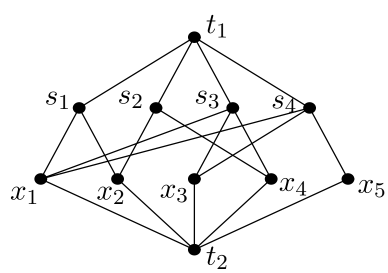

Figure 2: Reduction from SET COVER ($U = \{1, \cdots, 5\}$, $\mathcal{S} = \{\{1,2\}, \{2,4\}, \{1,3,4\}, \{1,3,5\}\}$).

For $j = 1, \cdots, p$, let $W_j = \{t_1, x_j\} \cup \{s_i \mid S_i \ni j\}$. Then we claim that the family $\mathcal{W}_\lambda$ of all minimal deficient sets in J can be represented by

$$\mathcal{W}_\lambda = \{\{t_2\}\} \cup \{W_j \mid j = 1, \cdots, p\}. \tag{3.2}$$

It is clear that $\{t_2\}$ is a minimal deficient set and W_j ($j = 1, \cdots, p$) are deficient. Let W be an arbitrary deficient set with $W \not\ni t_2$. This W satisfies $W \ni t_1$. Moreover, we have $W \cap S$, $W \cap X \neq \emptyset$, since otherwise $u(W) \geq d(t_1)$ holds, a contradiction. Then we have

$$\begin{aligned} u(W) &= \sum \{u(t_1, w) \mid w \in S \setminus W\} \\ &\quad + \sum \{u(v,w) \mid (v,w) \in E(S \cap W, X \setminus W)\} \\ &\quad + \sum \{u(v,w) \mid (v,w) \in E(S \setminus W, X \cap W)\} \\ &\quad + \sum \{u(v,t_2) \mid v \in X \cap W\} \\ &= d(t_1) - \sum \{u(v,w) \mid (v,w) \in E(S \cap W, X \cap W)\} \\ &\quad + \sum \{u(v,w) \mid (v,w) \in E(S \setminus W, X \cap W)\} \\ &\quad + \sum \{u(v,t_2) \mid v \in X \cap W\}. \end{aligned}$$

Let x_j be a vertex in $X \cap W$ such that $\{v \in S \mid (v, x_j) \in E\}$ $(= \{s_i \mid S_i \ni$

$j\}) \not\subseteq W$. Then we have

$$\sum\{u(v,x_j) \mid (v,x_j) \in E(S \setminus W, x_j)\} + u(x_j,t_2) \\ \geq \sum\{u(v,x_j) \mid (v,x_j) \in E(S \cap W, x_j)\}.$$

Hence, if all $x_j \in X \cap W$ satisfies $\{v \in S \mid (v,x_j) \in E\} \not\subseteq W$, we have

$$\sum\{u(v,w) \mid (v,w) \in E(S \setminus W, X \cap W)\} + \sum\{u(v,t_2) \mid v \in X \cap W\} \\ \geq \sum\{u(v,w) \mid (v,w) \in E(S \cap W, X \cap W)\},$$

which implies $u(W) \geq d(t_1)$ from the equation shown above. This contradicts the deficiency of W. Thus there exists a vertex $x_{j^*} \in X \cap W$ such that $\{v \in S \mid (v,x_{j^*}) \in E\} \subseteq W$. This means that W contains W_{j^*}. Therefore all W_j are deficient and there exists no other minimal deficient set W such that $W \not\ni t_2$. This proves (3.2).

We then claim that any optimal solution Y of J can be represented by

$$Y = \{t_2\} \cup \{s_i \mid S_i \in \mathcal{X}\} \tag{3.3}$$

for an optimal solution $\mathcal{X}$ of I, which completes the proof. Let Y be an optimal solution of J. It follows from (3.2) that $Y \cap W \neq \emptyset$ holds for all $W \in \mathcal{W}_\lambda$. In particular, Y must contain t_2. Note that $S \cup \{t_2\}$ is a feasible solution of J whose cost is q. Since $c(v) = q+1$ for $v \in X$, $Y \cap W_j \neq \emptyset$ implies $Y \cap \{s_i \mid S_i \ni j\} \neq \emptyset$. This means that $\mathcal{X} = \{S_i \mid s_i \in Y \cap S\}$ is a feasible solution of I, and the optimality of Y implies that $\mathcal{X}$ is an optimal solution of I. □

We remark that the reduction given above is *gap-preserving* in the following sense. If there exists a ρ-approximation algorithm for problem (3.1), then it can be turned into a ρ-approximation algorithm for Set Cover. This is due to the fact that any feasible solution $\mathcal{X}$ of I yields a feasible solution Y of J, given by (3.3), with the same cost. Moreover, any feasible solution Y of J yields a feasible solution $\mathcal{X}$ of I whose cost is not greater than the one of Y. For, if Y is not given by (3.3), it has a cost greater than q and we can replace Y by $\{t_2\} \cup \{s_1, s_2, \ldots, s_q\}$. It is known [14, 4] that problem Set Cover is not approximable within a ratio of $c \ln p$ for some constant c, unless every problem in NP has an $\mathrm{O}(N^{\log\log N})$ time deterministic algorithm. Here we can assume that q is bounded by a polynomial in p. Since we have $\sum\{d(v) \mid v \in V\} \leq 2pq$, which is polynomial in p, we have the following inapproximability result.

Theorem 3.2 *There exists a constant c such that the source location problem with arc-connectivity requirements in undirected/directed networks is not approximable within a ratio of $c \ln \sum_{v \in V} d(v)$, unless every problem in* NP *has an* $\mathrm{O}(N^{\log\log N})$ *time deterministic algorithm.*

Next, let us consider the problems with requirements κ and $\hat{\kappa}$.

Theorem 3.3 *There exists a constant c such that the source location problems with requirements κ and $\hat{\kappa}$ in undirected/directed networks is not approximable within a ratio of $c \ln \sum_{v\in V} d(v)$, unless every problem in* NP *has an* $\mathrm{O}(N^{\log\log N})$ *time deterministic algorithm.*

Proof. We only show the hardness results for the source location problems in undirected networks, since they can easily be transformed to the ones in directed networks.

(i) κ: We show this by reducing Set Cover to the source location problem with vertex-connectivity requirement κ.

Let $U = \{1, 2, \cdots, p\}$ and $\mathcal{S} = \{S_1, \cdots, S_q\} \subseteq 2^U$ be a problem instance of Set Cover. We construct the corresponding instance of the source location problem as follows.

$$
\begin{aligned}
V &= \{s_1, \cdots, s_q\} \cup \{x_1, \cdots, x_p\},\\
E &= \{(s_i, x_j) \mid j \in S_i, i = 1, \cdots, q\},\\
d(v) &= \begin{cases} k_j + 1 & \text{if } v \in \{x_1, \cdots, x_p\},\\ 0 & \text{otherwise,}\end{cases}\\
c(v) &= \begin{cases} 1 & \text{if } v \in \{s_1, \cdots, s_q\},\\ q+1 & \text{otherwise,}\end{cases}
\end{aligned}
$$

where $k_j = |\{S_i \mid S_i \ni j\}|$. We denote $S = \{s_1, \cdots, s_q\}$ and $X = \{x_1, \cdots, x_p\}$. Figure 3 gives an example of our reduction.

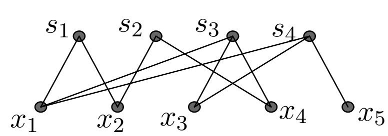

Figure 3: Reduction from Set Cover ($U = \{1, \cdots, 5\}$, $\mathcal{S} = \{\{1, 2\}, \{2, 4\}, \{1, 3, 4\}, \{1, 3, 5\}\}$).

For a $j \in U$, let $W_j = \{x_j\} \cup \{s_i \mid j \in S_i\}$. Then the family $\mathcal{W}_\kappa$ of all minimal deficient sets can be represented by

$$\mathcal{W}_\kappa = \{W_j \mid j \in U\}, \tag{3.4}$$

since $|N(\{x_j\})| = k_j < d(x_j)$. By Lemma 2.3, it is not difficult to see that Y is feasible for the instance of the source location problem whose cost is at most q if and only if $\mathcal{X} = \{S_i \mid s_i \in Y\}$ is feasible for the one of Set Cover, since $c(v) = q + 1$ for $v \in X$. This completes the proof.

(ii) $\hat{\kappa}$: We show this by reducing SET COVER to the source location problem with vertex-connectivity requirement $\hat{\kappa}$.

Let $U = \{1, 2, \cdots, p\}$ and $\mathcal{S} = \{S_1, \cdots, S_q\} \subseteq 2^U$ be a problem instance of SET COVER. We construct the corresponding instance of the source location problem as follows.

$$
\begin{aligned}
V &= \bigcup_{i=1}^{q} S^{(i)} \cup X \cup \bigcup_{j=1}^{p} X^{(j)}, \\
S^{(i)} &= \{s_1^{(i)}, \cdots, s_p^{(i)}\} \quad (i = 1, \cdots, q), \\
X &= \{x_1, \cdots, x_p\}, \\
X^{(j)} &= \{x_1^{(j)}, \cdots, x_p^{(j)}\} \quad (j = 1, \cdots, p), \\
E &= \{(s, x_j) \mid s \in S^{(i)}, j \in S_i, j = 1, \cdots, p\} \\
&\qquad\qquad \cup \{(x_j, x) \mid x \in X^{(j)}, j = 1, \cdots, p\}, \\
d(v) &= \begin{cases} |N(N(x_j))| + p & \text{if } v = x_j \,(\in X), \\ 2 & \text{if } v \in X^{(1)} \cup \cdots \cup X^{(q)}, \\ 0 & \text{otherwise,} \end{cases} \\
c(v) &= \begin{cases} 1 & \text{if } v \in S^{(1)} \cup \cdots \cup S^{(p)}, \\ q + 1 & \text{if } v \in X, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

Figure 4 gives an example of our reduction.

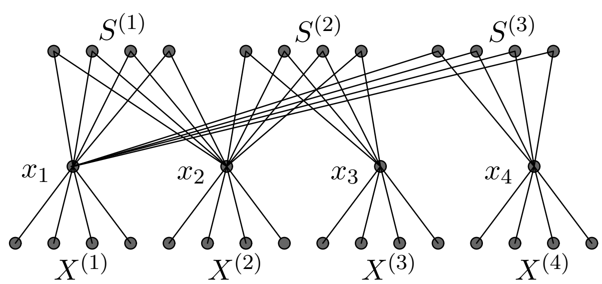

Figure 4: Reduction from SET COVER ($U = \{1, 2, 3, 4\}$, $\mathcal{S} = \{\{1, 2\}, \{2, 3\}, \{1, 4\}\}$).

For a $j \in U$, let $W_j = \{x_j\} \cup \bigcup_{S_i \ni j} S^{(i)}$, and let

$$\mathcal{F} = \{\{x\} \mid x \in X^{(j)}, j = 1, \cdots, p\} \cup \{W_j \mid j \in U\}. \tag{3.5}$$

We then claim that $\mathcal{F}$ is the family $\mathcal{W}_{\hat{\kappa}}$ of all the minimal deficient sets.

Let us first show that $\mathcal{F} \subseteq \mathcal{W}_{\hat{\kappa}}$. Let x be a vertex such that $x \in \bigcup_j X^{(j)}$. Since $d(x) = 2$ and $|N(x)| = 1$, $\{x\}$ is a minimal deficient set. For W_j given in (3.5), since $N(W_j) = (N(N(x_j)) \setminus \{x_j\}) \cup X^{(j)}$, we have

$$|N(W_j)| = (|N(N(x_j))| + p) - 1 = d(x_j) - 1,$$

which implies that W_j is deficient. Since no proper subset of W_j is deficient, W_j is a minimal deficient set. This completes the proof of $\mathcal{F} \subseteq \mathcal{W}_{\hat{k}}$.

Let F be a set such that $F \in \mathcal{W}_{\hat{k}} \setminus \mathcal{F}$. Then it is easy to see that $F \cap X \neq \emptyset$ and $F \cap X^{(j)} = \emptyset$ for $j = 1, \cdots, p$. If $F \cap X = \{x_j\}$, then F does not contain a vertex in W_j, since otherwise $F \supseteq W_j$. However, this implies that $|N(F)| \geq d(x_j)$, which contradicts the deficiency of F. On the other hand, if $|F \cap X| \geq 2$, then we have $|N(F)| \geq 2p$, since $X^{(j)} \cup X^{(j')} \subseteq N(F)$ for some j and j' with $j \neq j'$. This, together with $|N(N(x_j))| \leq p$, implies that F is not deficient, which proves that $\mathcal{F} \supseteq \mathcal{W}_{\hat{k}}$.

Let Y be a subset of V whose cost is at most q. Then by Lemma 2.4, it is not difficult to see that Y is feasible for the instance of the source location problem if and only if $\mathcal{X} = \{S_i \mid S^{(i)} \cap Y \neq \emptyset\}$ is feasible for the one of Set Cover, since $c(v) = q + 1$ for $v \in X$. This completes the proof. □

Before concluding this section, we remark that the bounds shown in Theorems 3.2 and 3.3 are tight. This will be shown in the next section by constructing $(1+\ln D)$-approximation algorithms for all the extended source location problems, where D denotes the sum of given demands.

4 The Extended Source Location Problems

This section generalizes the source location problems to the ones with supply values of source vertices. As mentioned in Section 1, we investigate the monotone concave cost $c_v : \mathbb{R}_+ \to \mathbb{R}_+$ $(v \in V)$ which models the cost depending not only on the setup cost but also on the supply value. Here a function $f : \mathbb{R} \to \mathbb{R}$ is called *monotone* if $f(x) \leq f(y)$ holds for arbitrary two reals $x, y \in \mathbb{R}$ with $x \leq y$, *concave* if $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha) f(y)$ for arbitrary two reals $x, y \in \mathbb{R}$ and α with $0 \leq \alpha \leq 1$. Let us first generalizes the arc-connectivity requirements.

Recall that $\partial\varphi(v)$ is the net out-flow at vertex $v \in V$ for a flow $\varphi : A \to \mathbb{R}_+$, i.e.,

$$\partial\varphi(v) = \sum_{(v,w)\in A} \varphi(v,w) - \sum_{(w,v)\in A} \varphi(w,v).$$

A flow $\varphi : A \to \mathbb{R}_+$ is *feasible with a supply* $x : V \to \mathbb{R}_+$ if it satisfies the following conditions:

$$-x(v) \leq \partial\varphi(v) \leq x(v) \qquad (v \in V), \tag{4.1}$$

$$0 \leq \varphi(a) \leq u(a) \qquad (a \in A). \tag{4.2}$$

Here (4.1) means that the net out-flow $\partial\varphi(v)$ and the net in-flow $-\partial\varphi(v)$ at v is at most the supply at v. For a vertex $v \in V$ and a supply $x \in \mathbb{R}_+^V$, let $\lambda^-(x; v)$ (resp., $\lambda^+(x; v)$) denote the sum of the supply $x(v)$ and the maximum net in-flow $-\partial\varphi(v)$ (resp., net out-flow $\partial\varphi(v)$) at v among all feasible flows with a supply x. In other words, $\lambda^-(x;v)$ (resp., $\lambda^+(x;v)$) denotes the maximum (s,v)-flow (resp., (v,s)-flow) value in the augmented network $\mathcal{N}^* = (G^* = (V^*, E^*), u^*)$ defined by

$$\begin{aligned} V^* &= V \cup \{s\}, \\ A^* &= A \cup \{(s,v),(v,s) \mid v \in V\}, \\ u^*(a) &= \begin{cases} u(a) & \text{if } a \in A, \\ x(v) & \text{if } a = (s,v),(v,s). \end{cases} \end{aligned} \tag{4.3}$$

The extended source location problem with arc-connectivity requirements asks for a minimum-cost supply $x \in \mathbb{R}_+^V$, i.e.,

$$\begin{aligned} &\text{Minimize} && \sum_{v \in V} c_v(x(v)) \\ &\text{subject to} && \lambda^-(x;v) \geq d^-(v) \ \text{ and } \ \lambda^+(x;v) \geq d^+(v) \quad (v \in V), \\ &&& x(v) \geq 0 \quad (v \in V). \end{aligned} \tag{4.4}$$

Here we assume that the cost $c_v : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone and concave. Note that the flows φ_v^- and φ_v^+ that respectively attain $\lambda^-(x;v) \geq d^-(v)$ and $\lambda^+(x;v) \geq d^+(v)$ in (4.4) may depend on v. It is not difficult to see that (4.4) is a generalization of the source location problem. In fact, for all $v \in V$, let

$$c_v(x(v)) = \begin{cases} 0 & \text{if } x(v) = 0, \\ c(v) & \text{otherwise,} \end{cases}$$

then we can see that (4.4) represents the source location problem.

By Lemma 2.1, we have the following lemma.

Lemma 4.1 *Problem* (4.4) *can be represented by*

$$\begin{aligned} &\text{Minimize} && \sum_{v \in V} c_v(x(v)) \\ &\text{subject to} && u^-(W) + x(W) \geq d^-(W) \quad (W \subseteq V), \\ &&& u^+(W) + x(W) \geq d^+(W) \quad (W \subseteq V), \\ &&& x(v) \geq 0 \quad (v \in V). \end{aligned} \tag{4.5}$$

Proof. Note that for a set $X \subseteq V$ $(= V^* \setminus \{s\})$, the capacity of arcs entering and leaving X in $\mathcal{N}^*$ given by (4.3) is described by

$$u^-(X) + x(X) \quad \text{and} \quad u^+(X) + x(X),$$

respectively. By the max-flow min-cut theorem in $\mathcal{N}^*$, we have

$$\begin{aligned} \lambda^-(x;v) &= \min\{u^-(W) + x(W) \mid v \in W \subseteq V\} \\ \lambda^+(x;v) &= \min\{u^+(W) + x(W) \mid v \in W \subseteq V\} \end{aligned} \tag{4.6}$$

for each $v \in V$. This immediately implies that the first two constraints of Problem (4.5) is equivalent to (4.4). $\square$

Let us next extend the source location problem with vertex-connectivity requirements κ by introducing a supply $x : V \to \mathbb{R}_+$. We first define the integral version of the extension (i.e., for integral supplies $x \in \mathbb{Z}_+^V$). Let $\kappa^-(x; v)$ (resp., $\kappa^+(x; v)$) denotes the maximum number of internally vertex-disjoint paths to v (resp., from v) such that for each $w \in V$ at most $x(w)$ paths start (resp., end) at w, where we say that paths P_1 and P_2 are *internally vertex-disjoint* if the sets of internal vertices (i.e., the vertices that are neither the starting vertex nor the end vertex) in P_1 and P_2 are disjoint. In other words, $\kappa^-(x; v)$ (resp., $\kappa^+(x; v)$) denotes the value of a maximum flow from s to v' (resp., v'' to s) in the augmented network $\mathcal{N}^* = (G^* = (V^*, A^*), u^*)$ given by

$$
\begin{aligned}
V^* &= \{v', v'' \mid v \in V\} \cup \{s\}, \\
A^* &= \{(v', v''), (v'', v') \mid v \in V\} \cup \{(v'', w') \mid (v, w) \in A\} \\
&\quad \cup \{(s, v''), (v', s) \mid v \in V\}, \\
u^*(a) &= \begin{cases} 1 & \text{if } a = (v', v'') \text{ for } v \in V, \\ +\infty & \text{if } a = (v'', v') \text{ for } v \in V, \\ +\infty & \text{if } a = (v'', w') \text{ for } (v, w) \in A, \\ x(v) & \text{if } a = (s, v''), (v', s) \text{ for } v \in V. \end{cases}
\end{aligned}
\tag{4.7}
$$

For the real version let $\kappa^-(x; v)$ denote the sum of the supply $x(v)$ and the maximum net in-flow $-\partial\varphi(v)$ at v among all feasible flows in $\mathcal{N}$ with a supply x such that

$$\sum_{(w_1, w_2) \in A:\, w_2 = t} \varphi(w_1, w_2) \leq 1 \tag{4.8}$$

holds for all t $(\neq v)$, where we assume without loss of generality that $u(a) = +\infty$ for all $a \in A$. Similarly, let $\kappa^+(x; v)$ denote the sum of the supply $x(v)$ and the maximum net out-flow $\partial\varphi(v)$ at v among all feasible flows with a supply x such that

$$\sum_{(w_1, w_2) \in A:\, w_1 = t} \varphi(w_1, w_2) \leq 1 \tag{4.9}$$

holds for all t $(\neq v)$. Similarly to the integral version, these correspond to the maximum flows in $\mathcal{N}^*$. Here we remark that (4.8) and (4.9) can naturally be extended to

$$\sum_{(w_1, w_2) \in A:\, w_2 = t} \varphi(w_1, w_2) \leq u(t), \qquad \sum_{(w_1, w_2) \in A:\, w_1 = t} \varphi(w_1, w_2) \leq u(t)$$

for a given vertex capacity $u : V \to \mathbb{R}_+$.

The extended source location problem with vertex-connectivity requirements κ can be formulated by

$$\begin{array}{lll} \text{Minimize} & \displaystyle\sum_{v \in V} c_v(x(v)) & \\ \text{subject to} & \kappa^-(x; v) \geq d^-(v) \quad \text{and} \quad \kappa^+(x; v) \geq d^+(v) \quad (v \in V), & (4.10) \\ & x(v) \geq 0 \quad (v \in V). & \end{array}$$

Similarly to (4.4), this extension also takes supply values (or capacities) of sources into account.

As can be imagined, Problem (4.10) is also represented by

$$\begin{array}{lll} \text{Minimize} & \displaystyle\sum_{v \in V} c_v(x(v)) & \\ \text{subject to} & |N^-(W)| + x(W \cup N^-(W)) \geq d^-(W) \quad (W \subseteq V), & (4.11) \\ & |N^+(W)| + x(W \cup N^+(W)) \geq d^+(W) \quad (W \subseteq V), & \\ & x(v) \geq 0 \quad (v \in V). & \end{array}$$

Lemma 4.2 *Problem* (4.10) *can be represented by Problem* (4.11).

Proof. Since $\kappa^-(x; v)$ (resp., $\kappa^+(x; v)$) denotes the value of a maximum flow from s to v' (resp., from v'' to s) in the augmented network $\mathcal{N}^* = (G^* = (V^*, A^*), u^*)$ given by (4.7),

$$\kappa^-(x; v) \geq d^-(v) \quad \text{and} \quad \kappa^+(x; v) \geq d^+(v)$$

are respectively equivalent to

$$(u^*)^-(W^*) \geq d^-(v) \qquad (v' \in W^* \subseteq V^* \setminus \{s\}), \tag{4.12}$$
$$(u^*)^+(W^*) \geq d^+(v) \qquad (v'' \in W^* \subseteq V^* \setminus \{s\}). \tag{4.13}$$

We only show that (4.12) can be represented by

$$|N^-(W)| + x(W \cup N^-(W)) \geq d^-(W),$$

since it can similarly be shown that (4.13) can be represented by $|N^+(W)| + x(W \cup N^+(W)) \geq d^+(W)$.

Let W^* be a set such that $v' \in W^* \subseteq V^* \setminus \{s\}$. If W^* has a finite $(u^*)^-(W^*)$, then we have

(i) $w' \in W^*$ implies $w'' \in W^*$,

(ii) $w' \in W^*$ implies $\{z'' \in V^* \mid (z, w) \in A\} \subseteq W^*$, and hence $(u^*)^-(W^*)$ is finite and given by

$$(u^*)^-(W^*) = \big|\{v \in V \mid v' \notin W^*, v'' \in W^*\}\big| + \sum\{x(v) \mid v'' \in W^*\}. \quad (4.14)$$

Moreover, any W^* with minimum $(u^*)^-(W^*)$ satisfies

(iii) $w'' \in W^*$ implies $w' \in W^*$ or $z' \in W^*$ for some $(w, z) \in A$.

For, if there exists a vertex $w \in V$ satisfying $w' \notin W^*$, $w'' \in W^*$, and $z' \notin W^*$ for any $(w, z) \in A$, then we have $(u^*)^-(W^* \setminus \{w''\}) < (u^*)^-(W^*)$, which contradicts the minimality of $(u^*)^-(W^*)$. From (i),(ii) and (iii) there exists $W \subseteq V$ such that

$$W^* = \{v', v'' \in V^* \mid v \in W\} \cup \{v'' \in V^* \mid v \in N^-(W)\}.$$

By (4.14), this implies that a minimum $(u^*)^-(W^*)$ can be represented by

$$(u^*)^-(W^*) = |N^-(W)| + x(W \cup N^-(W)).$$

This completes the proof. □

We remark that the generalization of the source location problem with $\hat{\kappa}$ does not make much sense, since a supply of each vertex is already bounded by 1. Hence we do not deal with the generalization.

5 Approximation Algorithms for the Extended Source Location Problems

In this section, we introduce the submodular cover problem as a natural common generalization of set cover problem [4, 14] and submodular set cover problem [5, 21], and show that the extended source location problems can be regarded as the submodular cover problem. We then show that the extended source location problems are all approximable within a ratio of $(1 + \ln \sum_{v \in V}(d^-(v) + d^+(v)))$ by producing a simple greedy algorithm for the submodular cover problem.

5.1 The Submodular Cover Problem

Before defining the submodular cover problem, let us first recall the submodular set cover problem.

Let V be a finite set. A set function $f : 2^V \to \mathbb{R}$ is *monotone nondecreasing* (simply *monotone*) if $f(X) \leq f(Y)$ holds for arbitrary two subsets $X \subseteq Y \subseteq V$, and *submodular* if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \quad (5.1)$$

holds for arbitrary two subsets $X, Y \subseteq V$. A monotone submodular function with $f(\emptyset) = 0$ is called a *polymatroid function.* A set $S \subseteq V$ is called *spanning* if $f(S) = f(V)$. Given a cost function $c : V \to \mathbb{R}_+$ and a polymatroid function $f : 2^V \to \mathbb{R}$, the *submodular set cover problem* is to compute a spanning set of minimum cost, i.e.,

$$\begin{aligned} \text{Minimize} \quad & \sum_{i \in S} c(i) \\ \text{subject to} \quad & f(S) = f(V), \\ & S \subseteq V. \end{aligned} \tag{5.2}$$

It is known [5, 21] that the problem contains the matroid base problem, the set cover problem, the partial cover problem, and so on, and it admits a simple greedy algorithm whose approximation ratio is $1 + \ln \max_{j \in V} f(\{j\})$.

We now extend the submodular set cover problem.

A function $f : \mathbb{R}_+^V \to \mathbb{R}$ is *submodular* if

$$f(x) + f(y) \geq f(x \wedge y) + f(x \vee y) \tag{5.3}$$

holds for arbitrary two vectors $x, y \in \mathbb{R}_+^V$. Here $(x \wedge y)(v) = \min\{x(v), y(v)\}$ and $(x \vee y)(v) = \max\{x(v), y(v)\}$.

The submodular cover problem to be considered in this paper can be described as follows. Given a finite set V, monotone concave cost functions $c_v : \mathbb{R}_+ \to \mathbb{R}_+$ $(v \in V)$, a monotone submodular function $f : \mathbb{R}_+^V \to \mathbb{R}_+$ and a real M, the problem asks for a minimum-cost vector $x \in \mathbb{R}_+^V$ that satisfies $f(x) \geq M$, i.e.,

$$\begin{aligned} \text{(SC)} \qquad \text{Minimize} \quad & \sum_{v \in V} c_v(x(v)) \\ \text{subject to} \quad & f(x) \geq M, \\ & x \in \mathbb{R}_+^V. \end{aligned} \tag{5.4}$$

Here we can assume without loss of generality that $f(0) = 0$ and $f(x) \leq M$ for all $x \in \mathbb{R}_+^V$, since we can modify f and M by

$$f(x) := \min\{f(x) - f(0), M - f(0)\}$$

and $M := M - f(0)$, respectively, so that $f(0) = 0$, $f(x) \leq M$ $(x \in \mathbb{R}_+^V)$, and f is still monotone submodular. This paper also considers its integral version, which is obtained from (5.4) by replacing $x \in \mathbb{R}_+^V$ with $x \in \mathbb{Z}_+^V$, where $\mathbb{Z}_+$ denotes the set of all nonnegative integers.

We remark that Wolsey [21] considered a similar generalization, but he assumed that the cost function c is linear and f is piecewise linear and concave as well as monotone and submodular.

5.2 SC Formulation of the Extended Source Location Problems

We show that all the extended source location problems can be formulated as the submodular cover problems. Let us first consider the extended source location problem (4.4) with arc-connectivity requirements λ.

Given a directed network $\mathcal{N} = (G = (V, A), u)$ and two demand functions d^-, d^+, we define a function f and a real M by

$$\begin{aligned} f(x) &= \sum_{v \in V} \left(\min\{\lambda^-(x; v), d^-(v)\} + \min\{\lambda^+(x; v), d^+(v)\} \right), \quad (5.5) \\ M &= \sum_{v \in V} (d^-(v) + d^+(v)). \quad (5.6) \end{aligned}$$

Then we note that the extended source location problem can be formulated as (5.4). Moreover, we can see that f is monotone and submodular, which proves that the problem can be formulated as a submodular cover problem.

Lemma 5.1 *A function f defined by* (5.5) *is monotone and submodular.*

Proof. Recall that $\lambda^-(x; v)$ and $\lambda^+(x; v)$ can be expressed as

$$\begin{aligned} \lambda^-(x; v) &= \min\{u^-(W) + x(W) \mid v \in W \subseteq V\}, \quad (5.7) \\ \lambda^+(x; v) &= \min\{u^+(W) + x(W) \mid v \in W \subseteq V\} \quad (5.8) \end{aligned}$$

for $x \in \mathbb{R}_+^V$ and $v \in V$ (see (4.6)). This immediately implies that f is monotone. We next show that f is submodular. Note first that the sum of submodular functions is also submodular, and for a constant h,

$$h(x) = \min\{g(x), h\}$$

is submodular if g is monotone submodular. Since the submodularity of $\lambda^+(x; v)$ can be proved similarly, we only show the submodularity of $\lambda^-(x; v)$, i.e.,

$$\lambda^-(x; v) + \lambda^-(y; v) \geq \lambda^-(x \wedge y; v) + \lambda^-(x \vee y; v) \quad (5.9)$$

holds for arbitrary two vectors $x, y \in \mathbb{R}^V$.

For two vectors $x, y \in \mathbb{R}^V$ and a vertex $v \in V$, let W_1 and W_2 be sets that contain v and satisfy $\lambda^-(x; v) = u^-(W_1) + x(W_1)$ and $\lambda^-(y; v) = u^-(W_2) + y(W_2)$. Since u^- is submodular and $x \wedge y$ is modular, we have

$$\begin{aligned} & u^-(W_1) + (x \wedge y)(W_1) + u^-(W_2) + (x \wedge y)(W_2) \\ & \quad \geq \; u^-(W_1 \cap W_2) + (x \wedge y)(W_1 \cap W_2) \\ & \qquad + u^-(W_1 \cup W_2) + (x \wedge y)(W_1 \cup W_2). \quad (5.10) \end{aligned}$$

Moreover, from $W_1 \cap W_2 \subseteq W_1, W_2$, it holds that

$$
\begin{aligned}
&x(W_1) - (x \wedge y)(W_1) + y(W_2) - (x \wedge y)(W_2) \\
&\qquad \geq (x \vee y)(W_1 \cap W_2) - (x \wedge y)(W_1 \cap W_2).
\end{aligned} \tag{5.11}
$$

By summing (5.10) and (5.11) up, we have

$$
\begin{aligned}
\lambda^-(x; v) + \lambda^-(y; v) &= u^-(W_1) + x(W_1) + u^-(W_2) + y(W_2) \\
&\geq u^-(W_1 \cap W_2) + (x \vee y)(W_1 \cap W_2) \\
&\quad + u^-(W_1 \cup W_2) + (x \wedge y)(W_1 \cup W_2) \\
&\geq \lambda^-(x \vee y; v) + \lambda^-(x \wedge y; v),
\end{aligned}
$$

where the last inequality follows from $W_1 \cap W_2$, $W_1 \cup W_2 \ni v$. □

We next consider the problem with vertex-connectivity requirements κ. Let

$$
\begin{aligned}
f(x) &= \sum_{v \in V} \left(\min\{\kappa^-(x; v), d^-(v)\} + \min\{\kappa^+(x; v), d^+(v)\} \right), \quad (5.12) \\
M &= \sum_{v \in V} (d^-(v) + d^+(v)).
\end{aligned}
$$

Then similarly to the case of arc-connectivity requirements λ, problem (4.10) can be formulated as (5.4). Thus it remains to show that f is monotone submodular.

Lemma 5.2 *A function f defined by* (5.12) *is monotone and submodular.*

Proof. Recall that $\kappa^-(x; v)$ and $\kappa^+(x; v)$ can be represented by

$$
\begin{aligned}
\kappa^-(x; v) &= \min\{|N^-(W)| + x(W) + x(N^-(W)) \mid v \in W \subseteq V\}, \\
\kappa^+(x; v) &= \min\{|N^+(W)| + x(W) + x(N^+(W)) \mid v \in W \subseteq V\}
\end{aligned}
$$

for $x \in \mathbb{R}^V_+$ and $v \in V$ (see Lemma 4.2). This immediately implies that f is monotone. We next show that $\kappa^-(x; v)$ $(v \in V)$ is submodular in x. This completes the proof, since the submodularity of $\kappa^+(x; v)$ is similarly shown.

For two nonnegative vectors $x, y \in \mathbb{R}^V_+$ and a vertex $v \in V$, let W_1 and W_2 be sets that contain v and $\kappa^-(x; v) = |N^-(W_1)| + x(W_1) + x(N^-(W_1))$, $\kappa^-(y; v) = |N^-(W_2)| + y(W_2) + y(N^-(W_2))$. By the modularity of $x \wedge y$, it holds that

$$
(x \wedge y)(W_1) + (x \wedge y)(W_2) = (x \wedge y)(W_1 \cap W_2) + (x \wedge y)(W_1 \cup W_2). \tag{5.13}
$$

From the definition of in-neighbor N^-, we have

$$
\begin{aligned}
N^-(X \cap Y) &\subseteq (N^-(X) \cap Y) \cup (N^-(Y) \cap X) \cup (N^-(X) \cap N^-(Y)), \quad (5.14) \\
N^-(X \cup Y) &= (N^-(X) \setminus Y) \cup (N^-(Y) \setminus X) \quad (5.15)
\end{aligned}
$$

for two subsets $X, Y \subseteq V$. Note that $N^-(X) \cap Y$, $N^-(Y) \cap X$ and $N^-(X) \cap N^-(Y)$ are pairwise disjoint, and $(N^-(X) \setminus Y) \cap (N^-(Y) \setminus X) = N^-(X) \cap N^-(Y)$. Therefore, from (5.14) and (5.15), we have

$$z(N^-(X)) + z(N^-(Y)) \geq z(N^-(X \cap Y)) + z(N^-(X \cup Y))$$

for any two subsets $X, Y \subseteq V$ and any nonnegative $z \in \mathbb{R}_+^V$. In particular, this shows the submodularity of $|N^-(\cdot)|$, i.e.,

$$|N^-(X)| + |N^-(Y)| \geq |N^-(X \cap Y)| + |N^-(X \cup Y)|. \tag{5.16}$$

Therefore we have the following two inequalities:

$$\begin{aligned} |N^-(W_1)| + |N^-(W_2)| &\geq |N^-(W_1 \cap W_2)| + |N^-(W_1 \cup W_2)|, && (5.17) \\ (x \wedge y)(N^-(W_1)) + (x \wedge y)(N^-(W_2)) & \\ &\geq (x \wedge y)(N^-(W_1 \cap W_2)) + (x \wedge y)(N^-(W_1 \cup W_2)). && (5.18) \end{aligned}$$

Moreover, we can see that

$$\begin{aligned} (x - x \wedge y)(W_1 \setminus N^-(W_2)) &+ (y - x \wedge y)(W_2 \setminus N^-(W_1)) \\ &\geq (x - x \wedge y)(W_1 \cap W_2) + (y - x \wedge y)(W_1 \cap W_2) \\ &= (x \vee y - x \wedge y)(W_1 \cap W_2), \end{aligned} \tag{5.19}$$

where the inequality follows from $W_1 \cap W_2 \subseteq W_1 \setminus N^-(W_2), W_2 \setminus N^-(W_1)$, and

$$\begin{aligned} &(x - x \wedge y)(N^-(W_1)) + (x - x \wedge y)(N^-(W_2) \cap W_1) \\ &\quad + (y - x \wedge y)(N^-(W_2)) + (y - x \wedge y)(N^-(W_1) \cap W_2) \\ &\geq (x - x \wedge y)\left((N^-(W_1) \cap W_2) \cup (N^-(W_2) \cap W_1) \cup (N^-(W_1) \cap N^-(W_2))\right) \\ &\quad + (y - x \wedge y)\left((N^-(W_1) \cap W_2) \cup (N^-(W_2) \cap W_1) \cup (N^-(W_1) \cap N^-(W_2))\right) \\ &= (x \vee y - x \wedge y)\left((N^-(W_1) \cap W_2) \cup (N^-(W_2) \cap W_1) \cup (N^-(W_1) \cap N^-(W_2))\right) \\ &\geq (x \vee y - x \wedge y)(N^-(W_1 \cap W_2)). \end{aligned} \tag{5.20}$$

Here the last inequality follows from (5.14). By summing (5.13) and (5.17)–(5.20) up, we have

$$\begin{aligned} &\kappa^-(x; v) + \kappa^-(y; v) \\ &\quad = |N^-(W_1)| + x(W_1) + x(N^-(W_1)) + |N^-(W_2)| + y(W_2) + y(N^-(W_2)) \\ &\quad \geq |N^-(W_1 \cap W_2)| + (x \vee y)(W_1 \cap W_2) + (x \vee y)(N^-(W_1 \cap W_2)) \\ &\qquad + |N^-(W_1 \cup W_2)| + (x \wedge y)(W_1 \cup W_2) + (x \wedge y)(N^-(W_1 \cup W_2)) \\ &\quad \geq \kappa^-(x \vee y; v) + \kappa^-(x \wedge y; v). \end{aligned}$$

This proves the lemma. □

Finally, we show that the source location problem with vertex-connectivity requirements $\hat{\kappa}$ can be formulated as the submodular set cover problem (5.2).

Let

$$\begin{aligned} f(S) &= \sum_{v \in V} \left(\min\{\hat{\kappa}^-(S,v), d^-(v)\} + \min\{\hat{\kappa}^+(S,v), d^+(v)\} \right), \quad (5.21) \\ M &= \sum_{v \in V} (d^-(v) + d^+(v)). \end{aligned}$$

Then similarly to the cases above, the source location problem with vertex-connectivity requirements $\hat{\kappa}$ can be formulated as (5.2). Thus it remains to show that $f : 2^V \to \mathbb{R}_+$ is monotone submodular.

Lemma 5.3 *A function f defined by* (5.21) *is monotone and submodular.*

Proof. Recall that $\hat{\kappa}^-(S,v)$ and $\hat{\kappa}^+(v,S)$ can be represented by

$$\begin{aligned} \hat{\kappa}^-(S,v) &= \min\{|N^-(W)| \mid v \in W \subseteq V \setminus S\}, \\ \hat{\kappa}^+(v,S) &= \min\{|N^+(W)| \mid v \in W \subseteq V \setminus S\} \end{aligned}$$

for $S \subseteq V$ and $v \in V$ (see Lemma 2.1 (iii)). This immediately implies that f is monotone. We next show that $\hat{\kappa}^-(S,v)$ $(v \in V)$ is submodular in S, i.e.,

$$\hat{\kappa}^-(S_1,v) + \hat{\kappa}^-(S_2,v) \geq \hat{\kappa}^-(S_1 \cap S_2, v) + \hat{\kappa}^-(S_1 \cup S_2, v)$$

holds for any two sets $S_1, S_2 \subseteq V$.

For two sets $S_1, S_2 \subseteq V$ and a vertex $v \in V$, let W_i, $i = 1, 2$ be sets that satisfy

$$v \in W_i \subseteq V \setminus S_i \quad \text{and} \quad \hat{\kappa}^-(S_i, v) = |N^-(W_i)|.$$

Since $|N^-(\cdot)|$ is submodular, we have

$$\begin{aligned} \hat{\kappa}^-(S_1,v) + \hat{\kappa}^-(S_2,v) &= |N^-(W_1)| + |N^-(W_2)| \\ &\geq |N^-(W_1 \cap W_2)| + |N^-(W_2 \cup W_2)| \\ &\geq \hat{\kappa}^-(S_1 \cup S_2, v) + \hat{\kappa}^-(S_1 \cap S_2, v), \end{aligned}$$

where the last inequality follows from $W_1 \cap W_2 \subseteq V \setminus (S_1 \cup S_2)$, $W_1 \cup W_2 \subseteq V \setminus (S_1 \cap S_2)$, and $W_1 \cap W_2$, $W_1 \cup W_2 \ni v$.

Similarly, we can prove the submodularity of $\hat{\kappa}^+(S,v)$. This completes the proof. □

By Lemmas 5.1, 5.2 and 5.3, it is proved that the (extended) source location problems can be formulated as the submodular cover problems.

5.3 An Approximation Algorithm for Submodular Cover Problem

In this section, we propose a simple greedy algorithm for the submodular cover problem defined in Section 5.1.

The algorithm starts with $x = \mathbf{0}$ and repeatedly increases x until it becomes a feasible solution for the problem. For $x \in \mathbb{R}^V$, $v \in V$, and $\delta\,(> 0)$, let

$$g_x(v;\delta) \;=\; \frac{c_v(x(v)+\delta) - c_v(x(v))}{f(x+\delta\chi_v) - f(x)}, \tag{5.22}$$

where χ_v is the vth unit vector, i.e., $\chi_v(w) = 1$ if $w = v$, and 0 otherwise. Note that each x with $f(x) < M$ has $v \in V$ and $\delta\,(> 0)$ such that the denominator in (5.22) is positive, if $f(y) \geq M$ for some y, which follows from the monotonicity and the submodularity of f. In each iteration, it finds an element $v^* \in V$ and positive real $\delta^* > 0$ that is *the most cost-effective*, i.e., that attains

$$g_x(v^*;\delta^*) \;=\; \min_{v\in V,\delta>0} \{g_x(v;\delta)\}, \tag{5.23}$$

where we assume the existence of the minimum in (5.23).

This algorithm is formally described as follows.

Algorithm Greedy_SC

Input: A finite set V, a monotone submodular function $f : \mathbb{R}^V_+ \to \mathbb{R}_+$, monotone concave cost functions $c_v : \mathbb{R}_+ \to \mathbb{R}_+$ $(v \in V)$, and a real $M\ (> 0)$.
/* Let us assume that $f(0) = 0$ and $f(x) \leq M$ for all $x \in \mathbb{R}^V_+$ */

Output: A feasible solution $x \in \mathbb{R}^V_+$ for Problem SC.

Step 0. $x(v) := 0$ for all $v \in V$.

Step 1. **While** $f(x) < M$ **do**
(I) Find an element $v^* \in V$ and a real $\delta^* > 0$ that satisfies (5.23).
(II) $x(v^*) := x(v^*) + \delta^*$.

Step 2. Output x and halt. □

We remark that, even if $v^* \in V$ and $\delta^* > 0$ always exist in Step 1, the algorithm might not halt in general since for v^* and δ^* in Step 1, $f(x + \delta^*\chi_{v^*}) - f(x)$ might be $f(x + \delta^*\chi_{v^*}) - f(x) \to +0$, and hence infinitely many iterations are necessary to attain $f(x) = M$.

However, this does not occur for many problem instances, including the ones constructed from the (extended) source location problems. As for the integral version of the problem, we modify algorithm GREEDY_SC by replacing " a real $\delta^* > 0$ " in Step 1 with " an integer $\delta^* > 0$ ".

We show that the algorithm given above computes a feasible solution of problem SC with a ratio of $(1 + \ln(M/\varepsilon))$ if v^* and δ^* in Step 1 always satisfy $f(x + \delta^* \chi_{v^*}) - f(x) \geq \varepsilon$ for some positive real $\varepsilon > 0$.

For a given instance of the submodular cover problem, let OPT be the minimum cost and let $x^{(p)}$ denote the vector obtained by the pth iteration of GREEDY_SC (e.g., $x^{(0)} = 0$). Suppose that GREEDY_SC chooses an element $v^{(p)} \in V$ and a real $\delta^{(p)} > 0$ at the tth iteration (i.e., $x^{(p+1)} = x^{(p)} + \delta^{(p)} \chi_{v^{(p)}}$). Let $price(k) = g_{x^{(p)}}(v; \delta)$ for every real $k > 0$ with $f(x^{(p)}) < k \leq f(x^{(p+1)})$. Note that, by the definition of $price(k)$, $\int_0^M price(k) dk$ is equal to the cost of the solution which GREEDY_SC outputs. Then we have the following lemma.

Lemma 5.4 *$price(k) < \frac{OPT}{M-k}$ holds for any k with $0 < k \leq M$.*

Proof. Let us arbitrarily take an optimal solution x^*. We denote $\{v_1, \cdots, v_\ell\} = \{v \in V \mid x^{(p)}(v) < x^*(v)\}$. Let $\delta_i = x^*(v_i) - x^{(p)}(v_i)$ and $x_i = x_{i-1} + \delta_i \chi_{v_i}$ $(i = 1, \cdots, \ell)$ with $x_0 = x^{(p)}$. We remark that x_ℓ is a feasible solution for the problem instance. Since c_v $(v \in V)$ is nonnegative, we have

$$
\begin{aligned}
& OPT \\
& \geq \sum_{i=1}^{\ell} c_{v_i}(x^*(v_i)) \\
& \geq \sum_{i=1}^{\ell} \big(c_{v_i}(x_{i-1}(v_i) + \delta_i) - c_{v_i}(x_{i-1}(v_i)) \big) \\
& = \sum_{i=1}^{\ell} \left(\frac{c_{v_i}(x_{i-1}(v_i) + \delta_i) - c_{v_i}(x_{i-1}(v_i))}{f(x_{i-1} + \delta_i \chi_{v_i}) - f(x_{i-1})} \big(f(x_{i-1} + \delta_i \chi_{v_i}) - f(x_{i-1}) \big) \right) \quad (5.24)
\end{aligned}
$$

It clearly holds that

$$x_{i-1} \wedge (x_0 + \delta_i \chi_{v_i}) = x_0 \quad \text{and} \quad x_{i-1} \vee (x_0 + \delta_i \chi_{v_i}) = x_{i-1} + \delta_i \chi_{v_i}$$

for all $i = 1, \cdots, \ell$. Hence, by the submodularity of f, we have

$$f(x_0 + \delta_i \chi_{v_i}) - f(x_0) \geq f(x_{i-1} + \delta_i \chi_{v_i}) - f(x_{i-1}). \quad (5.25)$$

From (5.24) and (5.25),

$$
\begin{aligned}
OPT & \geq \sum_{i=1}^{\ell} \left(\frac{c_{v_i}(x_{i-1}(v_i) + \delta_i) - c_{v_i}(x_{i-1}(v_i))}{f(x_0 + \delta_i \chi_{v_i}) - f(x_0)} \big(f(x_{i-1} + \delta_i \chi_{v_i}) - f(x_{i-1}) \big) \right) \\
& = \sum_{i=1}^{\ell} g_{x^{(p)}}(v_i; \delta_i) \big(f(x_{i-1} + \delta_i \chi_{v_i}) - f(x_{i-1}) \big) \\
& \geq g_{x^{(p)}}(v^{(p)}; \delta^{(p)}) \left(M - f(x^{(p)}) \right), \quad (5.26)
\end{aligned}
$$

where the equality follows from $x^{(p)}(v_i) = x_{i-1}(v_i)$ and $x^{(p)} = x_0$, and the last inequality follows from the optimal choice of algorithm GREEDY_SC, and

$$\sum_{i=1}^{\ell}\big(f(x_{i-1}+\delta_i\chi_{v_i}) - f(x_{i-1})\big) = f(x_\ell) - f(x_0).$$

Hence $price(k)$ $(= g_{x^{(p)}}(v^{(p)};\delta^{(p)})) < \frac{OPT}{M-k}$ holds, since $M - f(x^{(p)}) > M - k$ holds for any k with $f(x^{(p)}) < k \leq f(x^{(p+1)})$. □

We are now ready to show the following theorem for the approximation ratio of GREEDY_SC.

Theorem 5.5 *Let ε be a nonnegative real such that v^* and δ^* in Step 1 always satisfy $f(x+\delta^*\chi_{v^*}) - f(x) \geq \varepsilon$. Then* GREEDY_SC *computes a solution whose cost is at most $\left(1+\ln\frac{M}{\varepsilon}\right)$ times OPT. Moreover, it is polynomial if*

(i) *v^* and δ^* in Step 1 can be computed in polynomial time, and*

(ii) *the number of iterations is bounded by a polynomial in the input size.*

Proof. Since the second statement is easily shown, we only prove the first statement. From the discussion before Lemma 5.4, the cost of a solution computed by GREEDY_SC is equal to $\int_0^M price(k)\,dk$. By Lemma 5.4, we have $price(k) < \frac{OPT}{M-k}$ for $k \leq M - \varepsilon$. Moreover, since v^* and δ^* in Step 1 always satisfy $f(x+\delta^*\chi_{v^*}) - f(x) \geq \varepsilon$, (5.26) implies

$$price(k) = price(M-\varepsilon) < \frac{OPT}{\varepsilon}$$

for any $k > M - \varepsilon$. Hence, we have

$$\int_0^M price(k)dk < \varepsilon\cdot\frac{OPT}{\varepsilon} + \int_0^{M-\varepsilon}\frac{OPT}{M-k}\,dk = \left(1+\ln\frac{M}{\varepsilon}\right)OPT.$$

□

As for the integral version, we have the following theorem. Here $H(k) = \sum_{i=1}^{k} 1/k$ is the kth harmonic number, which is bounded by $1+\ln k$.

Theorem 5.6 GREEDY_SC *computes a solution whose cost is at most $H(M)$ times OPT. Moreover, it is polynomial if*

(i) *v^* and δ^* in Step 1 can be computed in polynomial time, and*

(ii) *the number of iterations is bounded by a polynomial in the input size.*

Proof. This follows from $\varepsilon \geq 1$. $\square$

In the next section, we show the conditions in Theorems 5.5 and 5.6 hold for the extended source location problems. Namely, we show that the extended source location problems are $(1 + \ln \sum_{v \in V}(d^-(v) + d^+(v)))$-approximable in polynomial time.

5.4 Applying a Greedy Algorithm to the Source Location Problems

This section applies algorithm GREEDY_SC to the extended source location problems. Let us first consider the problem with arc-connectivity requirements λ.

Recall that the denominator of $g_x(v;\delta)$ is given by

$$\begin{aligned}
f(x+\delta\chi_v)-f(x) & \\
&= \sum_{w\in V}\left(\min\{\lambda^-(x+\delta\chi_v;w), d^-(w)\} + \min\{\lambda^+(x+\delta\chi_v;w), d^+(w)\}\right) \\
&\quad - \sum_{w\in V}\left(\min\{\lambda^-(x;w), d^-(w)\} + \min\{\lambda^+(x;w), d^+(w)\}\right) \\
&= \sum_{w\in V}\left(\min\{\lambda^-(x+\delta\chi_v;w), d^-(w)\} - \min\{\lambda^-(x;w), d^-(w)\}\right) \\
&\quad + \sum_{w\in V}\left(\min\{\lambda^+(x+\delta\chi_v;w), d^+(w)\} - \min\{\lambda^+(x;w), d^+(w)\}\right).
\end{aligned}$$

We first show that this denominator has the following properties.

Lemma 5.7 *For a vector $x \in \mathbb{R}_+^V$, $f(x+\delta\chi_v)-f(x)$ is concave and piecewise linear in δ. Moreover, the number of line segments is at most $2n+1$.*

Proof. For a vertex $w \in V$, let

$$\begin{aligned}
\lambda_w^-(\delta) &= \min\{\lambda^-(x+\delta\chi_v;w), d^-(w)\} - \min\{\lambda^-(x;w), d^-(w)\}, \qquad (5.27)\\
\mu_w &= \min\{\min\{u^-(W)+x(W) \mid w \in W \subseteq V \setminus \{v\}\}, d^-(w)\} \\
&\quad - \min\{\lambda^-(x;w), d^-(w)\}.
\end{aligned}$$

Then since $\lambda^-(x;w) = \min\{u^-(W)+x(W) \mid w \in W \subseteq V\}$, $\lambda_w^-(\delta)$ is given by

$$\lambda_w^-(\delta) = \begin{cases} \delta & \text{if } \delta \leq \mu_w, \\ \mu_w & \text{otherwise.} \end{cases} \qquad (5.28)$$

Similarly, if we define

$$\begin{aligned}
\lambda_w^+(\delta) &= \min\{\lambda^+(x+\delta\chi_v;w), d^+(w)\} - \min\{\lambda^+(x;w), d^+(w)\}, \\
\nu_w &= \min\{\min\{u^+(W)+x(W) \mid w \in W \subseteq V \setminus \{v\}\}, d^+(w)\} \\
&\quad - \min\{\lambda^+(x;w), d^+(w)\},
\end{aligned}$$

we have

$$\lambda_w^+(\delta) = \begin{cases} \delta & \text{if } \delta \le \nu_w, \\ \nu_w & \text{otherwise.} \end{cases} \tag{5.29}$$

Note that each $\lambda_w^{\pm}$ is concave and consists of at most two linear segments. Since $f(x+\delta\chi_v) - f(x) = \sum_{w\in V}(\lambda_w^-(\delta) + \lambda_w^+(\delta))$, it is concave and consists of at most $2n+1$ line segments. □

Lemma 5.8 *For a vector $x \in \mathbb{R}_+^V$, $f(x+\delta\chi_v) - f(x)$ can be computed in polynomial time.*

Proof. From (5.28) and (5.29), we can see that the set of breakpoints in $f(x+\delta\chi_v)-f(x)$ is $\{\mu_w, \nu_w \mid w \in V\}$. Moreover, μ_w and ν_w can be obtained by computing maximum (s,w)- and (w,s)-flow in the network obtained from $\mathcal{N}^*$ given in (4.3) by identifying s with v. This completes the proof. □

Lemma 5.9 *For a vertex $v \in V$ and a real $x \in \mathbb{R}_+^V$,*

$$\min_{\delta>0} g_x(v;\delta) = \min\{g_x(v;\mu_w), g_x(v;\nu_w) \mid \mu_w, \nu_w > 0, w \in V\}. \tag{5.30}$$

Proof. Recall that

$$g_x(v;\delta) \quad = \quad \frac{c_v(x(v)+\delta) - c_v(x(v))}{f(x+\delta\chi_v) - f(x)}.$$

Since c_v is monotone and concave, so is the numerator of $g_x(v;\delta)$. Moreover the denominator is piecewise linear and concave. Thus the lemma follows. More precisely, let δ_i, $i = 1, \cdots, \ell$ be positive reals such that $\delta_i < \delta_{i+1}$, $i = 1, \cdots, \ell-1$ and $\{\delta_i \mid i = 1, \cdots, \ell\} = \{\mu_w, \nu_w > 0 \mid w \in V\}$. Then from Lemma 5.8, we can assume that $f(x+\delta\chi_v) - f(x)$ can be described by

$$f(x+\delta\chi_v) - f(x) \quad = \quad \begin{cases} \alpha_i\delta + \beta_i & \text{if } \delta_{i-1} < \delta \le \delta_i,\ i = 1, \cdots, \ell, \\ \alpha_{\ell+1}\delta + \beta_{\ell+1} & \text{if } \delta_\ell < \delta, \end{cases}$$

where $\delta_0 = 0$, $\alpha_1 > \cdots > \alpha_{\ell+1} (= 0)$, $\beta_1 (= 0) < \cdots < \beta_{\ell+1}$ and $\alpha_i\delta_i + \beta_i = \alpha_{i+1}\delta_i + \beta_{i+1}$ for $i = 1, \cdots, \ell$. It is easy to see that $g_x(v;\delta_\ell) = \min_{\delta\ge\delta_\ell} g_x(v;\delta)$, since $\alpha_{\ell+1} = 0$ and $c_v(x(v)+\delta) - c_v(x(v))$ is monotone. We show that

$$\begin{aligned} g_x(v;\delta_1) \quad &= \quad \min_{0\le\delta\le\delta_1} g_x(v;\delta) \text{ and} \\ \min\{g_x(v;\delta_{i-1}), g_x(v;\delta_i)\} \quad &= \quad \min_{\delta_{i-1}\le\delta\le\delta_i} g_x(v;\delta) \quad (i = 2, \cdots, \ell). \end{aligned}$$

Let

$$\begin{aligned} h_i &= \frac{c_v(x(v)+\delta_i) - c_v(x(v)+\delta_{i-1})}{\delta_i - \delta_{i-1}} \ (\geq 0) \\ k_i &= \frac{\big(c_v(x(v)+\delta_{i-1}) - c_v(x(v))\big)\delta_i - \big(c_v(x(v)+\delta_i) - c_v(x(v))\big)\delta_{i-1}}{\delta_i - \delta_{i-1}}, \end{aligned}$$

where we note that $k_1 = 0$. Then we have

$$\begin{aligned} h_i\,\delta + k_i &\leq c_v(x(v)+\delta) - c_v(x(v)) \quad \text{if} \quad \delta_{i-1} < \delta < \delta_i, \\ h_i\,\delta + k_i &= c_v(x(v)+\delta) - c_v(x(v)) \quad \text{if} \quad \delta = \delta_{i-1}, \delta_i, \end{aligned} \tag{5.31}$$

since c_v is concave. Hence it holds that

$$\begin{aligned} \frac{c_v(x(v)+\delta) - c_v(x(v))}{f(x+\delta\chi_v) - f(x)} &\geq \frac{h_i\,\delta + k_i}{\alpha_i\delta + \beta_i} \\ &= \frac{h_i}{\alpha_i} + \frac{\alpha_i k_i - h_i\beta_i}{\alpha_i(\alpha_i\delta + \beta_i)} \end{aligned} \tag{5.32}$$

for $\delta_{i-1} \leq \delta \leq \delta_i$. Since $\alpha_i > 0$ and $\beta_i \geq 0$ for $i = 1, \cdots, \ell$, δ_{i-1} attains the minimum of (5.32) for a real δ such that $\delta_{i-1} \leq \delta \leq \delta_i$ if $\alpha_i k_i - h_i\beta_i < 0$, and δ_i attains the minimum if $\alpha_i k_i - h_i\beta_i \geq 0$. This, together with (5.31), implies that δ_{i-1} or δ_i attains the minimum of $g_x(v;\delta)$ for $\delta_{i-1} \leq \delta \leq \delta_i$ ($i = 2, \cdots, \ell$), and δ_1 attains a minimum of $g_x(v;\delta)$ for $0 \leq \delta \leq \delta_1$, where the second sentence follows from $k_1 = 0$. This proves the lemma. □

Lemmas 5.8 and 5.9 imply the following result.

Lemma 5.10 *For the extended source location problem with arc-connectivity requirements λ, each iteration of Step 1 in* GREEDY_SC *computes an element $v^* \in V$ and a real $\delta^* > 0$ that satisfy* (5.23) *in polynomial time.*

Let us assume that v^* and δ^* are chosen in some iteration of Step 1 in GREEDY_SC. Then the number of line segments of $f(x+\delta\chi_v) - f(x)$ decreases at least one. Since we have at most $2n+1$ line segments of $f(x+\delta\chi_v) - f(x)$, $v \in V$, the following lemma follows.

Lemma 5.11 *For the extended source location problem with arc-connectivity requirements λ,* GREEDY_SC *requires at most $n(2n+1)$ iterations.*

Theorem 5.12 *For the extended source location problem with arc-connectivity requirements λ,* GREEDY_SC *computes in polynomial time a feasible solution whose cost is at most $1 + \ln\sum_{v\in V}(d^-(v) + d^+(v))$ times the optimal cost if d^-, d^+ and u are integral.*

Proof. The polynomiality of GREEDY_SC follows from Theorem 5.5 and Lemmas 5.10 and 5.11. Since μ_w and ν_w are integers if d^-, d^+ and u are integral, we have $\varepsilon \geq 1$ in Theorem 5.5, and hence the approximation ratio is at most $1 + \ln\sum_{v\in V}(d^-(v) + d^+(v))$. □

It follows from the discussion above that the integral version of the extended source location problem is also solvable.

Theorem 5.13 *For the integral version of the extended source location problem with arc-connectivity requirements* λ*,* GREEDY_SC *computes in polynomial time a feasible solution whose cost is at most* $1+\ln\sum_{v\in V}(d^-(v)+d^+(v))$ *times the optimal cost if* d^-*,* d^+ *and* u *are integral.*

Let us move to the problem with vertex-connectivity requirements κ. Let

$$\begin{aligned}\mu_w &= \min\{\min\{|N^-(W)|+x(W\cup N^-(W)) \mid w\in W\subseteq V,\\ &\qquad v\notin W\cup N^-(W)\}, d^-(w)\} - \min\{\kappa^-(x;w), d^-(w)\}\\ \nu_w &= \min\{\min\{|N^+(W)|+x(W\cup N^+(W)) \mid w\in W\subseteq V,\\ &\qquad v\notin W\cup N^+(W)\}, d^+(w)\} - \min\{\kappa^+(x;w), d^+(w)\}.\end{aligned}$$

Then for vertex-connectivity requirements κ, we have results similar to the ones in Lemmas 5.7 $\sim$ 5.11. For example, μ_w and ν_w can respectively be obtained in polynomial time by computing maximum (s,w')- and (w'',s)-flows in the network obtained from $\mathcal{N}^*$ given in (4.7) by identifying s with v.

Theorem 5.14 *For* (*the integral version of*) *the extended source location problem with vertex-connectivity requirements* κ*,* GREEDY_SC *computes in polynomial time a feasible solution whose cost is at most* $1+\ln\sum_{v\in V}(d^-(v)+d^+(v))$ *times the optimal cost if* d^- *and* d^+ *are integral.*

Similarly, the extended source location problem with $\hat{\kappa}$ is approximable. As a corollary, we have the following result, where it can also be obtained by formulating the problems as submodular set cover problems.

Corollary 5.15 *For the source location problems with vertex-connectivity requirements* κ *and* $\hat{\kappa}$*,* GREEDY_SC *computes in polynomial time a feasible solution whose cost is at most* $k+2\ln n$ *times the optimal cost, where* $k=1+\ln 2$*.*

Proof. This follows from $d^-(v)$, $d^+(v)\le n$. □

6 Conclusion

In this paper, we have considered the source location problems and their generalizations. We have showed that the source location problem with edge-connectivity requirements in undirected networks is strongly NP-hard, which solves an open problem posed by Arata *et al.* [2], and that there exists a constant c such that no source location problems with three connectivity requirements in undirected/directed networks are approximable within a ratio of $c \ln D$, unless every problem in NP has an $O(N^{\log\log N})$-time deterministic algorithm. Here D denotes the sum of given demands. We have also proposed $(1 + \ln D)$-approximation algorithms for all the extended source location problems if we have the integral capacity and demand functions. By combining our negative results, this shows that our approximation algorithms for all the problems are optimal.

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References

[1] K. Andreev, C. Garrod, and B. Maggs: Simultaneous source location, *SCS Technical Report* CMU-CS-03-162, Carnegie Mellon University, Pittsburgh.

[2] K. Arata, S. Iwata, K. Makino, and S. Fujishige: Locating sources to meet flow demands in undirected networks, *J. Algorithms*, **42** (2002), 54–68.

[3] M. Bárász, J. Becker, and A. Frank: An algorithm for source location in directed graphs, *Operations Research Letters*, **33** (2005), 221–230.

[4] U. Feige: A threshold of $\ln n$ for approximating set cover, *J. ACM*, **45** (1998), 634–652.

[5] T. Fujito: Approximation algorithms for submodular set cover with applications, *IEICE Trans.*, **E83-D** (2000), 480–487.

[6] A. Goel and D. Estrin: Simultaneous optimization for concave costs: Single sink aggregation or single source buy-at-bulk, *Proceedings of 14th Annual ACM-SIAM Symposium on Discrete Algorithms* (2003), 499–505.

[7] M. R. Garey and D. S. Johnson: *Computers and Intractability: A Guide to the Theory NP-Completeness*, Freeman, New York, 1979.

[8] J. van den Heuvel and M. Johnson: Transversals of subtree hypergraphs and the source location problem in digraphs, *CDAM Research Report*, LSE-CDAM-2004-10, London School of Economics.

[9] T. Ishii, H. Fujita, and H. Nagamochi: Source location problem with local 3-vertex-connectivity requirements, *Proceedings of 3rd Hungarian-Japanese Symposium on Discrete Mathematics and Its Applications*, (2003), 368–377.

[10] T. Ishii, H. Fujita, and H. Nagamochi: Minimum cost source location problem with local 3-vertex-connectivity requirements, *Computing Theory: The Australian Theory Symposium*, (2005), 97-105.

[11] H. Ito, M. Ito, Y. Itatsu, H. Uehara, and M. Yokoyama: Source location problems considering vertex-connectivity and edge-connectivity simultaneously, *Networks*, **40** (2002), 63–70.

[12] H. Ito, K. Makino, K. Arata, S. Honami, Y. Itatsu, and S. Fujishige: Source location problem with flow requirements in directed networks, *Optimization Methods and Software*, **18** (2003), 427–435.

[13] M. Labbe, D. Peeters, and J.-F. Thisse: Location on networks, In M. O. Ball et al. (eds.), *Handbooks in OR & MS*, **8**, North-Holland (1995), 551–624.

[14] C. Lund and M. Yannakakis: On the hardness of approximating minimization problems, *J. ACM*, **41** (1994), 960–981.

[15] H. Nagamochi, T. Ishii, and H. Ito: Minimum cost source location problem with vertex-connectivity requirements in digraphs, *Information Processing Letters*, **80** (2001), 287–294.

[16] M. Sakashita, K. Makino, and S. Fujishige: Minimizing a monotone concave function with laminar covering constraints, *ISAAC 2005*, LNCS **3827** (2005), 71–81.

[17] M. Sakashita, K. Makino, and S. Fujishige: Minimum cost source location problems with flow requirements. *LATIN 2006*, LNCS **3887** (2006), 769-780.

[18] H. Tamura, M. Sengoku, S. Shinoda, and T. Abe: Location problems on undirected flow networks, *IEICE Trans.*, **E73** (1990), 1989–1993.

[19] H. Tamura, M. Sengoku, S. Shinoda, and T. Abe: Some covering problems in location theory on flow networks, *IEICE Trans.*, **E75-A** (1992), 678–683.

[20] H. Tamura, H. Sugawara, M. Sengoku, and S. Shinoda: Plural cover problem on undirected flow networks, *IEICE Trans.*, **J81-A** (1998), 863–869 (in Japanese).

[21] L.A. Wolsey: An analysis of the greedy algorithm for the submodular set covering problem, *Combinatorica*, **2** (1982), 385–393.