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The characteristic quasi-polynomials of the arrangements of root systems

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Abstract

For an irreducible root system R, consider a coefficient matrix S of the positive roots with respect to the associated simple roots. Then S defines an arrangement of "hyperplanes" modulo a positive integer q. The cardinality of the complement of this arrangement is a quasi-polynomial of q, which we call the characteristic quasi-polynomial of R. This paper gives the complete list of the characteristic quasi-polynomials of all irreducible root systems, and shows that the characteristic quasi-polynomial of an irreducible root system R is positive at $q \in \mathbb{Z}_{>0}$ if and only if q is greater than or equal to the Coxeter number of R.

Key words: characteristic quasi-polynomial, elementary divisor, hyperplane arrangement, root system.

1 Introduction

Let S be an arbitrary $m \times n$ integral matrix without zero columns. For each positive integer $q \in \mathbb{Z}_{>0}$, denote $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ and $\mathbb{Z}_q^\times = \mathbb{Z}_q \setminus \{0\}$. Consider the set

$$M_q(S) := \{\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{Z}_q^m : \mathbf{z}S \in (\mathbb{Z}_q^\times)^n\},$$

and its cardinality $|M_q(S)|$. In our recent paper [3], we showed that there exists a monic **quasi-polynomial** (periodic polynomial) $\chi_S(q)$ with integral coefficients of degree m such that

$$\chi_S(q) = |M_q(S)|, \quad q \in \mathbb{Z}_{>0}.$$

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Note that the set $M_q(S)$ is the complement of an arrangement of hyperplanes in the following sense: Let $S_1, S_2, \dots, S_n$ be the columns of S. Each set

$$H_{i,q} := \{\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{Z}_q^m : \mathbf{z}S_i = 0\}, \quad 1 \le i \le n,$$

can be called a “hyperplane” in $\mathbb{Z}_q^m$ by a slight abuse of terminology. Then

$$M_q(S) = \mathbb{Z}_q^m \setminus \bigcup_{i=1}^{n} H_{i,q}.$$

For a sufficiently large prime number q, $\chi_S(q)$ is known [1] to be equal to the **characteristic polynomial** [4, Def. 2.52] of the real arrangement consisting of the following hyperplanes (ignoring possible repetitions):

$$H_{i,\mathbb{R}} := \{\mathbf{z} = (z_1, \dots, z_m) \in \mathbb{R}^m : \mathbf{z}S_i = 0\}, \quad 1 \le i \le n.$$

It is thus natural to call the quasi-polynomial $\chi_S(q)$ the **characteristic quasi-polynomial** of S as in [3].

In this paper, we define and determine the characteristic quasi-polynomial $\chi_R(q)$ for **every irreducible root system** R. Let m be the rank of R and $n = |R|/2$. We assume that an $m \times n$ integral matrix $S = S(R) = [S_{ij}]$ satisfies

$$R_+ = \{\sum_{i=1}^{m} S_{ij}\alpha_i : j = 1, \dots, n\},$$

where R_+ is a set of positive roots and $B(R) = \{\alpha_1, \alpha_2, \dots, \alpha_m\}$ is the set of **simple roots** associated with R_+. In other words, S is a coefficient matrix of R_+ with respect to the basis $B(R)$. Define the **characteristic quasi-polynomial** $\chi_R(q) := \chi_S(q)$ for each irreducible root system R. Then $\chi_R(q)$ depends only upon R.

For example, for the root system $R = A_2 = \{\epsilon_i - \epsilon_j : 1 \le i \le 3,\ 1 \le j \le 3,\ i \ne j\}$, $B(A_2) = \{\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2 - \epsilon_3\}$ and $R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$, where $\epsilon_1, \epsilon_2, \epsilon_3$ are orthonormal, one has

$$S = S(A_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

It is easy to see that $\chi_{A_2}(q) = \chi_S(q) = (q-1)(q-2)$, which is equal to the ordinary characteristic polynomial of type A_2. In other words, the minimum period of the quasi-polynomial $\chi_{A_2}(q)$ is one. The minimum periods for all irreducible root systems are shown in the following table:

root system	minimum period	root system	minimum period
A_m	1	E_6	6
B_m	2	E_7	12
C_m	2	E_8	60
D_m	2	F_4	12
		G_2	6

The outline of this paper is as follows: In Section 2, we prove general results on $\chi_S(q)$ which are used in Sections 5 and 6. In Section 3, we study the case of root system A_m, which is the easiest case. We investigate the root systems B_m, C_m and D_m using the coset method in Section 4. The characteristic quasi-polynomials of these three root systems are closely related to each other. The cases of G_2 and F_4 are studied in Section 5. In Section 6, we study the remaining root systems E_m $(m = 6, 7, 8)$ which require the hardest calculations in this paper. We are aided by the computer package PARI/GP [5] and the theoretical results from Section 2. Lastly in Section 7, we state two results obtained from our calculations and the classification of irreducible root systems. Throughout this paper we use the table of irreducible root systems in [2] as our standard reference.

2 Results on the characteristic quasi-polynomial of an integral matrix

Let $\chi_S(t)$ be the characteristic quasi-polynomial of an $m \times n$ integral matrix S without zero columns. Fix a nonempty $J \subseteq [n] := \{1, 2, \ldots, n\}$ and define an $m \times |J|$ matrix S_J consisting of the columns of S corresponding to the set J. Let $e_{J,1}, \ldots, e_{J,\ell(J)} \in \mathbb{Z}_{>0}$ be the elementary divisors of S_J numbered so that $e_{J,1}|e_{J,2}|\cdots|e_{J,\ell(J)}$, where $\ell(J) := \operatorname{rank} S_J$. Write $e(J) := e_{J,\ell(J)}$, and define the **lcm period** $\rho_0(S)$ of S by

$$\begin{aligned}\rho_0 = \rho_0(S) &:= \operatorname{lcm}\{e(J) : J \subseteq [n],\ J \neq \emptyset\} \\ &= \operatorname{lcm}\{e(J) : J \subseteq [n],\ 1 \leq |J| \leq \min\{m, n\}\}.\end{aligned}$$

Then it is known ([3, Theorem 2.4]) that the lcm period ρ_0 is a period of $\chi_S(t)$.

It is further shown in [3] that the constituents of the quasi-polynomial $\chi_S(t)$ are the same for all q's with the same value of $\gcd\{\rho_0, q\}$. Let d be a positive integer which divides ρ_0, and define a monic polynomial $P_d(t) = P_{S,d}(t)$ with integral coefficients of degree m by

$$\chi_S(q) = P_d(q) \quad \text{for all} \quad q \in d + \rho_0\mathbb{Z}_{\geq 0}.$$

Put

$$e(J, d) := \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, d\}.$$

Then the following formula was essentially proved in our previous paper [3].

Theorem 2.1. *For each $d \in \mathbb{Z}_{>0}$ with $d|\rho_0$, the polynomial $P_d(t)$ is given by*

$$P_d(t) = \sum_{J \subseteq [n]} (-1)^{|J|} e(J, d) t^{m-\ell(J)},$$

where for $J = \emptyset$, we understand that $\ell(\emptyset) = 0$ and that $e(\emptyset, d) = 1$.

Proof. Obtained from [3, (10)] and the inclusion-exclusion principle. □

Theorem 2.2 ([3] Theorem 2.5). *The polynomial*

$$P_1(t) = \sum_{J \subseteq [n]} (-1)^{|J|} t^{m-\ell(J)}$$

is equal to the ordinary characteristic polynomial [4, Def. 2.52] of the real arrangement consisting of the hyperplanes (ignoring possible repetitions) $H_{1,\mathbb{R}}, H_{2,\mathbb{R}}, \ldots, H_{n,\mathbb{R}}$.

Corollary 2.3. *Suppose* $d, d' \in \mathbb{Z}_{>0}$ *both divide* ρ_0*, and assume the following condition holds true for some positive integer* s*:* $\gcd\{e(J), d\} = \gcd\{e(J), d'\}$ *for all* $J \subseteq [n]$ *with* $|J| \leq s$*. Then*

$$\deg\{P_d(t) - P_{d'}(t)\} < m - s.$$

In particular, we have $\deg\{P_d(t) - P_1(t)\} < m - s$ *if* $\gcd\{e(J), d\} = 1$ *for all* $J \subseteq [n]$ *with* $|J| \leq s$.

Proof. We apply Theorems 2.1 and 2.2. It is enough to show $e(J, d) = e(J, d')$ for $J \subseteq [n]$ with $\ell(J) \leq s$. We can choose a subset $J' \subseteq J$ such that $|J'| = \ell(J) \leq s$. Then $\gcd\{e(J'), d\} = \gcd\{e(J'), d'\}$. Since $e(J)|e(J')$ [3, Lemma 2.3], $\gcd\{e(J), d\} = \gcd\{e(J), d'\}$. This shows $e(J, d) = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, d\} = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, e(J), d\}$ $= \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, e(J), d'\} = \prod_{j=1}^{\ell(J)} \gcd\{e_{J,j}, d'\} = e(J, d')$. □

Corollary 2.4. *Suppose that* $d \in \mathbb{Z}_{>0}$ *and* $d' \in \mathbb{Z}_{>0}$ *both divide* ρ_0 *and that* $\gcd\{d, d'\} = 1$. *In addition, we assume the following condition holds true for some positive integer* s*:*

$$\gcd\{e(J), d\} = 1 \quad \textit{or} \quad \gcd\{e(J), d'\} = 1 \tag{1}$$

for all $J \subseteq [n]$ *with* $|J| \leq s$*. Then*

$$\deg\{P_1(t) + P_{dd'}(t) - P_d(t) - P_{d'}(t)\} < m - s.$$

Proof. Suppose $J \subseteq [n]$ with $\ell(J) \leq s$. It is enough to show

$$1 + e(J, dd') - e(J, d) - e(J, d') = 0.$$

We can choose a subset $J' \subseteq J$ such that $|J'| = \ell(J) \leq s$. Then either $\gcd\{e(J'), d\} = 1$ or $\gcd\{e(J'), d'\} = 1$ by (1). Since $e(J)|e(J')$,

$$\gcd\{e(J), d\} = 1 \quad \text{or} \quad \gcd\{e(J), d'\} = 1.$$

This shows that either $e(J, d) = 1$ or $e(J, d') = 1$. We finally have

$$\begin{aligned} 0 &= \{1 - e(J, d)\}\{1 - e(J, d')\} = 1 - e(J, d) - e(J, d') + e(J, d)e(J, d') \\ &= 1 - e(J, d) - e(J, d') + e(J, dd'). \end{aligned}$$

□

Corollary 2.5. *Suppose that* $d \in \mathbb{Z}_{>0}$ *and* $d' \in \mathbb{Z}_{>0}$ *both divide* ρ_0 *and that* $\gcd\{d, d'\} = 1$. *If* $e(J)$ *are prime powers or one for all* J*, we have* $P_{dd'}(t) = P_d(t) + P_{d'}(t) - P_1(t)$.

Proof. Easily follows from Corollary 2.4. □

The results in Corollaries 2.3, 2.4 and 2.5 will be used to find characteristic quasi-polynomials of root systems.

3 Characteristic quasi-polynomial of A_m

We follow PLATE I in [2]. Let $\{\epsilon_1, \ldots, \epsilon_{m+1}\}$ be an orthonormal basis for an $(m+1)$-dimensional Euclidean space W, and define

$$V := \left\{ \sum_{i=1}^{m+1} c_i \epsilon_i \in W : \sum_{i=1}^{m+1} c_i = 0 \right\}.$$

Then

$$R := \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq m+1\} \subset V, \quad |R| = m(m+1),$$

is an irreducible root system in V of type A_m. Then we may choose a set of positive roots

$$R_+ := \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq m+1\}.$$

Define $\alpha_i := \epsilon_i - \epsilon_{i+1},\ 1 \leq i \leq m$. Then $B := \{\alpha_1, \ldots, \alpha_m\}$ is the set of simple roots associated with R_+. We may express

$$R_+ = \left\{ \sum_{i \leq k \leq j} \alpha_k : 1 \leq i \leq j \leq m \right\}.$$

Let $n := |R_+| = m(m+1)/2$. Then the $m \times n$ matrix $S(A_m)$ consists of only 0's and 1's such that 1 appears consecutively in each column. For example

$$S(A_1) = \begin{bmatrix} 1 \end{bmatrix}, \quad S(A_2) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad S(A_3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

The characteristic quasi-polynomial $\chi_{A_m}(t)$ of the root system A_m is the characteristic quasi-polynomial of $S(A_m)$: $\chi_{A_m}(t) := \chi_{S(A_m)}(t)$. Let us enumerate the size of

$$M_q(S) = \{\mathbf{z} \in \mathbb{Z}_q^m : \mathbf{z}S \in (\mathbb{Z}_q^\times)^n\} = \{\mathbf{z} \in \mathbb{Z}_q^m : \sum_{i \leq k \leq j} z_k \neq 0 \quad (1 \leq i \leq j \leq m)\}.$$

First, there are $(q-1)$ ways to choose z_1. Next, there are $(q-2)$ ways to choose z_2, etc. Therefore we have

$$\chi_{A_m}(q) = |M_q(S)| = (q-1) \cdots (q-m).$$

Thus the characteristic quasi-polynomial $\chi_{A_m}(q)$ of A_m is equal to the ordinary characteristic polynomial.

4 Characteristic quasi-polynomials of B_m, C_m, D_m

4.1 The coset method

Let P be a non-singular $m \times m$ integral matrix. Consider a finite additive group $G := \mathbb{Z}_q^m$, and define a group homomorphism $\pi : G \to G$ by $\pi(\mathbf{z}) = \mathbf{z}P$, $\mathbf{z} \in G$. Consider the subgroup $H := \operatorname{im} \pi$ of G. Then it is not difficult to see that the index $(G : H)$ is equal to $b(q) := \gcd\{q, \det P\}$. Note that every fiber $\pi^{-1}(\mathbf{y})$ has the same cardinality $b(q)$ for any $\mathbf{y} \in H$. Let us express the set G/H of cosets as

$$G/H = \{\mathbf{g}_i + H : 1 \leq i \leq b(q)\}$$

for some complete set of representatives $\mathbf{g}_1, \ldots, \mathbf{g}_{b(q)} \in G,\ \mathbf{g}_1 \in H$.

Let S be an $m \times n$ integral matrix, and define an $m \times n$ integral matrix T by $T = PS$. Then we can write $f_S(q) := |\{\mathbf{y} \in G : \mathbf{y}S \in (\mathbb{Z}_q^\times)^n\}|$ as

$$\begin{aligned} f_S(q) = \left|\{\mathbf{y} \in G : \mathbf{y}S \in (\mathbb{Z}_q^\times)^n\}\right| &= \sum_{i=1}^{b(q)} \left|\{\mathbf{y} \in \mathbf{g}_i + H : \mathbf{y}S \in (\mathbb{Z}_q^\times)^n\}\right| \\ &= \frac{1}{b(q)} \sum_{i=1}^{b(q)} \left|\{\mathbf{z} \in G : (\mathbf{g}_i + \mathbf{z}P)S \in (\mathbb{Z}_q^\times)^n\}\right| \\ &= \frac{1}{b(q)} \sum_{i=1}^{b(q)} \left|\{\mathbf{z} \in G : \mathbf{z}T + \mathbf{g}_i S \in (\mathbb{Z}_q^\times)^n\}\right|. \end{aligned}$$

Define

$$f_i(q) := \left|\{\mathbf{z} \in G : \mathbf{z}T + \mathbf{g}_i S \in (\mathbb{Z}_q^\times)^n\}\right| \tag{2}$$

for $i = 1, \ldots, b(q)$. Then we have:

Theorem 4.1. *For an $m \times n$ integral matrix S, define $f_S(q) = |\{\mathbf{y} \in G : \mathbf{y}S \in (\mathbb{Z}_q^\times)^n\}|$. Then, for any non-singular $m \times m$ integral matrix P and the $m \times n$ integral matrix T defined by $T = PS$, we can write $f_S(q)$ as*

$$f_S(q) = \frac{1}{b(q)} \sum_{i=1}^{b(q)} f_i(q), \tag{3}$$

where $b(q) = \gcd\{q, \det P\}$, and $f_i(q)$, $1 \leq i \leq b(q)$, are defined in (2).

Note that when S has a zero column, (3) is trivially true because both sides are zero. Thus, we do not need the assumption that S has no zero column; when this assumption is satisfied, $f_S(q)$ is the characteristic quasi-polynomial $\chi_S(q)$.

We also note the following: $f_1(q) = |\{\mathbf{z} \in G : \mathbf{z}T \in (\mathbb{Z}_q^\times)^n\}| = f_T(q)$. Hence, when $b(q) = 1$ in particular (e.g., when P is unimodular), we have $f_S(q) = f_1(q) = f_T(q)$.

4.2 B_m

We follow PLATE II in [2]. Let $\{\epsilon_1, \dots, \epsilon_m\}$ be an orthonormal basis for an m-dimensional Euclidean space V. Let $m \geq 2$. Then

$$\begin{aligned} R \; &:= \; \{\pm\epsilon_i \ (1 \leq i \leq m), \ \pm(\epsilon_i - \epsilon_j) \ (1 \leq i < j \leq m), \\ &\qquad \pm(\epsilon_i + \epsilon_j) \ (1 \leq i < j \leq m)\} \subset V, \quad |R| = 2m^2, \end{aligned}$$

is an irreducible root system of type B_m. Then we may choose a set of positive roots

$$R_+ = \{\epsilon_i \ (1 \leq i \leq m), \ \epsilon_i - \epsilon_j \ (1 \leq i < j \leq m), \ \epsilon_i + \epsilon_j \ (1 \leq i < j \leq m)\}.$$

Define $\alpha_i := \epsilon_i - \epsilon_{i+1} \ (1 \leq i \leq m-1)$, $\alpha_m := \epsilon_m$. Then $B = \{\alpha_1, \dots, \alpha_m\}$ is the set of simple roots associated with R_+. We may express

$$R_+ \;=\; \left\{ \sum_{i \leq k \leq j} \alpha_k \ (1 \leq i \leq j \leq m), \ \sum_{i \leq k < j} \alpha_k + 2 \sum_{j \leq k \leq m} \alpha_k \ (1 \leq i < j \leq m) \right\}.$$

Let $n := |R_+| = m^2$. Then the $m \times n$ matrix $S := S(B_m)$ is the coefficient matrix of R_+ with respect to the set of simple roots B. For example,

$$S(B_2) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}, \ S(B_3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 2 & 2 & 2 \end{bmatrix}.$$

We want to find the characteristic quasi-polynomial $\chi_{B_m}(t) := \chi_{S(B_m)}(t)$ of B_m. Define an $m \times m$ matrix

$$P := \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & -1 & 1 \end{bmatrix}.$$

Then the $m \times n$ matrix $T = T(B_m) := PS$ is the coefficient matrix of R_+ with respect to the orthonormal basis $\epsilon_1, \dots, \epsilon_m$. Since P is unimodular, we have

$$\chi_{B_m}(q) = \chi_{S(B_m)}(q) = \chi_{T(B_m)}(q). \tag{4}$$

4.3 C_m

We follow PLATE III in [2]. Let $m \geq 3$.

$$\begin{aligned} R \; &:= \; \{\pm 2\epsilon_i \ (1 \leq i \leq m), \ \pm(\epsilon_i - \epsilon_j) \ (1 \leq i < j \leq m), \\ &\qquad \pm(\epsilon_i + \epsilon_j) \ (1 \leq i < j \leq m)\} \subset V, \quad |R| = 2m^2, \end{aligned}$$

is an irreducible root system in V of type C_m. Then we may choose a set of positive roots

$$R_+ = \{2\epsilon_i\ (1 \le i \le m),\ \epsilon_i - \epsilon_j\ (1 \le i < j \le m),\ \epsilon_i + \epsilon_j\ (1 \le i < j \le m)\}.$$

Define $\alpha_i := \epsilon_i - \epsilon_{i+1}\ (1 \le i \le m-1),\ \alpha_m := 2\epsilon_m$. Then $B = \{\alpha_1, \dots, \alpha_m\}$ is the set of simple roots associated with R_+. We may express

$$R_+ \;=\; \left\{ \sum_{i \le k \le j} \alpha_k\ (1 \le i \le j \le m),\ \sum_{i \le k < j} \alpha_k + 2 \sum_{j \le k < m} \alpha_k + \alpha_m\ (1 \le i \le j < m) \right\}.$$

Let $n := |R_+| = m^2$. Then the $m \times n$ matrix $S = S(C_m)$ is the coefficient matrix of R_+ with respect to the set of simple roots B. For example,

$$S(C_3) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We want to find the characteristic quasi-polynomial $\chi_{C_m}(t) := \chi_{S(C_m)}(t)$ of C_m. Define an $m \times m$ matrix

$$P := \begin{bmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ & -1 & & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & -1 & 2 \end{bmatrix}.$$

Then the $m \times n$ matrix $T = T(C_m) := PS$ is the coefficient matrix of R_+ with respect to $\epsilon_1, \dots, \epsilon_m$. Since $\det P = 2$, we have to consider two cases.

Case 1: When q is odd.

For odd q, we have $b(q) := \gcd\{q, \det P\} = 1$ and thus

$$\chi_S(q) = \chi_{T(C_m)}(q).$$

Case 2: When q is even.

For $\pi : G \to G$ defined by $\pi(\mathbf{z}) = \mathbf{z}P,\ \mathbf{z} \in G$, we have

$$H := \operatorname{im} \pi = \{(y_1, \dots, y_{m-1}, 2y_m) : y_1, \dots, y_m \in \mathbb{Z}_q\}.$$

Since $b(q) = (G : H) = \gcd\{q, \det P\} = 2$, we take $\mathbf{g}_1 = \mathbf{0} \in H$ and $\mathbf{g}_2 = (0, \dots, 0, 1) \in G \setminus H$. By Theorem 4.1

$$\chi_S(q) = \frac{1}{2}\{f_1(q) + f_2(q)\},$$

where $f_1(q) = \chi_{T(C_m)}(q)$ and

$$
\begin{aligned}
f_2(q) &= |\{\mathbf{z} \in G : \mathbf{z}T + \mathbf{g}_2 S \in (\mathbb{Z}_q^\times)^n\}| \\
&= |\{(z_1, \ldots, z_m) \in \mathbb{Z}_q^m : 2z_i + 1 \neq 0 \ (1 \leq i \leq m), \\
&\qquad z_i - z_j \neq 0 \ (1 \leq i < j \leq m), \\
&\qquad z_i + z_j + 1 \neq 0 \ (1 \leq i < j \leq m)\}| \\
&= |\{(z_1, \ldots, z_m) \in \mathbb{Z}_q^m : \\
&\qquad z_i - z_j \neq 0 \ (1 \leq i < j \leq m), \\
&\qquad z_i + z_j + 1 \neq 0 \ (1 \leq i < j \leq m)\}| \qquad (5) \\
&= m! \times |\{(c_1, \ldots, c_m) \in \mathbb{Z}^m : 0 \leq c_1 < \cdots < c_m < q, \\
&\qquad\quad c_i + c_j \neq q - 1 \ (1 \leq i < j \leq m)\}| \\
&= m! \times |\{(c_1, \ldots, c_m) \in \mathbb{Z}^m : 0 < c_1 < \cdots < c_m < q + 1, \\
&\qquad\quad c_i + c_j \neq q + 1 \ (1 \leq i < j \leq m)\}| \\
&= \chi_{T(B_m)}(q + 1).
\end{aligned}
$$

In the second equation of (5), we have used $\{\sum_{i \leq k \leq j} \alpha_k : 1 \leq i \leq j < m\} = \{\epsilon_i - \epsilon_j : 1 \leq i < j \leq m\}$, $\{\sum_{i \leq k \leq m} \alpha_k \ (1 \leq i \leq m),\ \sum_{i \leq k < j} \alpha_k + 2\sum_{j \leq k < m} \alpha_k + \alpha_m \ (1 \leq i \leq j < m)\} = \{2\epsilon_i \ (1 \leq i \leq m),\ \epsilon_i + \epsilon_j \ (1 \leq i < j \leq m)\}$ for R_+.

Therefore,

$$\chi_S(q) = \frac{1}{2}\{\chi_{T(C_m)}(q) + \chi_{T(B_m)}(q + 1)\}$$

for even q.

In summary,

$$\chi_{C_m}(q) = \chi_{S(C_m)}(q) = \begin{cases} \chi_{T(C_m)}(q) & \text{if } q \text{ is odd}, \\ \frac{1}{2}\{\chi_{T(C_m)}(q) + \chi_{T(B_m)}(q + 1)\} & \text{if } q \text{ is even}. \end{cases} \qquad (6)$$

4.4 D_m

We follow PLATE IV in [2]. Let $m \geq 4$.

$R := \{\pm(\epsilon_i - \epsilon_j) \ (1 \leq i < j \leq m),\ \pm(\epsilon_i + \epsilon_j) \ (1 \leq i < j \leq m)\} \subset V, \quad |R| = 2m(m - 1)$,

is an irreducible root system in V of type D_m. Then we may choose a set of positive roots

$$R_+ = \{\epsilon_i - \epsilon_j \ (1 \leq i < j \leq m),\ \epsilon_i + \epsilon_j \ (1 \leq i < j \leq m)\}.$$

Define $\alpha_i := \epsilon_i - \epsilon_{i+1} \ (1 \leq i \leq m - 1),\ \alpha_m := \epsilon_{m-1} + \epsilon_m$. Then $B = \{\alpha_1, \ldots, \alpha_m\}$ is the set of simple roots associated with R_+. We may express

$$R_+ = \left\{ \sum_{i \leq k \leq j} \alpha_k \ (1 \leq i \leq j < m), \right.$$

$$\sum_{i\le k<j}\alpha_k + 2\sum_{j\le k<m-1}\alpha_k + \alpha_{m-1} + \alpha_m \ (1\le i<j<m),$$
$$\left.\sum_{i\le k<m-1}\alpha_k + \alpha_m \ (1\le i<m)\right\}.$$

Let $n := |R_+| = m(m-1)$. Then the $m\times n$ matrix $S = S(D_m)$ is the coefficient matrix of R_+ with respect to the set of simple roots B. For example,

$$S(D_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

We want to find the characteristic quasi-polynomial $\chi_{D_m}(t) := \chi_{S(D_m)}(t)$ of D_m. Define an $m\times m$ matrix

$$P := \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & 1 \\ & & & -1 & 1 \end{bmatrix}.$$

Then the $m\times n$ matrix $T = T(D_m) := PS$ is the coefficient matrix of R_+ with respect to $\epsilon_1,\ldots,\epsilon_m$. Since $\det P = 2$, we have to consider two cases.

Case 1: When q is odd.

For odd q, we have $b(q) := \gcd\{q, \det P\} = 1$ and thus

$$\chi_S(q) = \chi_{T(D_m)}(q).$$

Case 2: When q is even.

We have

$$H = \operatorname{im}\pi = \{(y_1,\ldots,y_{m-1},y_{m-1}+2y_m) : y_1,\ldots,y_m\in\mathbb{Z}_q\}.$$

Since $b(q) = (G:H) = \gcd\{q,\det P\} = 2$, we take $\mathbf{g}_1 = \mathbf{0}\in H$ and $\mathbf{g}_2 = (0,\ldots,0,1)\in G\setminus H$. By Theorem 4.1

$$\chi_S(q) = \frac{1}{2}\left\{f_1(q)+f_2(q)\right\},$$

where $f_1(q) = \chi_{T(D_m)}(q)$ and

$$f_2(q) \ = \ |\{\mathbf{z}\in G : \mathbf{z}T + \mathbf{g}_2 S \in (\mathbb{Z}_q^\times)^n\}|$$

$$
\begin{aligned}
(7) \qquad\qquad &= |\{(z_1,\dots,z_m)\in\mathbb{Z}_q^m : z_i - z_j \neq 0\ (1\le i<j\le m),\\
&\qquad z_i + z_j + 1 \neq 0\ (1\le i<j\le m)\}|\\
&= \chi_{T(B_m)}(q+1)
\end{aligned}
$$

by (5). In the second equation of (7), we have used $\{\sum_{i\le k\le j}\alpha_k : 1\le i\le j<m\} = \{\epsilon_i-\epsilon_j : 1\le i<j\le m\}$, $\{\sum_{i\le k<j}\alpha_k + 2\sum_{j\le k<m-1}\alpha_k + \alpha_{m-1}+\alpha_m\ (1\le i<j<m),\ \sum_{i\le k<m-1}\alpha_k+\alpha_m\ (1\le i<m)\} = \{\epsilon_i+\epsilon_j : 1\le i<j\le m\}$ for R_+.

Therefore,

$$\chi_S(q) = \frac{1}{2}\{\chi_{T(D_m)}(q) + \chi_{T(B_m)}(q+1)\}$$

for even q.

In summary,

$$
(8) \qquad \chi_{D_m}(q) = \chi_{S(D_m)}(q) = \begin{cases} \chi_{T(D_m)}(q) & \text{if } q \text{ is odd,}\\ \frac{1}{2}\{\chi_{T(D_m)}(q)+\chi_{T(B_m)}(q+1)\} & \text{if } q \text{ is even.}\end{cases}
$$

4.5 Orthonormal basis

4.5.1 $\chi_{T(B_m)}(q)$ and $\chi_{T(D_m)}(q)$

We first prove the following lemma.

Lemma 4.2. *Assume that a matrix A satisfies the following three conditions:*
(1) each entry lies in $\{0,\pm1,\pm2\}$,
(2) each column contains at most two nonzero entries, and
(3) each column contains at most one entry from $\{\pm2\}$.
Then the elementary divisors of A lie in $\{1,2\}$.

Proof. Let us temporarily say that a matrix is of type (T) if it satisfies these three conditions. Denote the set of elementary divisors of A by $ED(A)$. Argue by an induction on the number of columns. When a matrix has only one column, the statement is obviously true. Suppose that a matrix A has more than one column.

Case 1. When $A=O$, $ED(A)=\emptyset$.

Case 2. When $A\neq O$ and each entry of A lies in $\{0,\pm2\}$, then $ED(A)=\{2\}$.

Case 3. If A has a column with only one nonzero entry $a\in\{\pm1\}$, then A is equivalent to

$$
\left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right]
$$

with B of type (T). Since $ED(B)\subseteq\{1,2\}$ by the induction assumption, $ED(A)\subseteq\{1,2\}$.

Case 4. If A has a column with exactly two nonzero entries, then A is equivalent to

$$A_1 = \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 1 & * & * & * \\ 0 & * & * & * \\ \vdots & * & * & * \\ 0 & * & * & * \end{array}\right].$$

By clearing the first row using the first column of A_1 we see that A is equivalent to

$$\left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 1 & & & \\ 0 & & & \\ \vdots & & C & \\ 0 & & & \end{array}\right].$$

Since

$$\begin{aligned} (1,1,0,\ldots,0)^{\mathrm{t}} - (1,-1,0,\ldots,0)^{\mathrm{t}} &= (0,2,0,\ldots,0)^{\mathrm{t}}, \\ (1,1,0,\ldots,0)^{\mathrm{t}} - (1,0,-1,\ldots,0)^{\mathrm{t}} &= (0,1,1,\ldots,0)^{\mathrm{t}}, \\ (1,1,0,\ldots,0)^{\mathrm{t}} - (1,0,1,\ldots,0)^{\mathrm{t}} &= (0,1,-1,\ldots,0)^{\mathrm{t}}, \end{aligned}$$

and so on, C is of type (T). Since $ED(C) \subseteq \{1,2\}$ by the induction assumption, $ED(A) \subseteq \{1,2\}$. $\square$

In the cases of $T(B_m)$ and $T(D_m)$, we have by Lemma 4.2 that $\gcd\{\rho_0, q\} = 1$ for odd $q \in \mathbb{Z}_{>0}$. Therefore, $\chi_{T(B_m)}(q)$ and $\chi_{T(D_m)}(q)$ for odd q are equal to the values of the characteristic polynomials of the real arrangements determined by the columns of $T(B_m)$ and $T(D_m)$, respectively ([3, Theorem 2.5]). Hence we have the following proposition.

Proposition 4.3. *For odd integers $q \in \mathbb{Z}_{>0}$, we have*

$$\chi_{T(B_m)}(q) = (q-1)(q-3)\cdots(q-2m+1), \tag{9}$$

$$\chi_{T(D_m)}(q) = (q-1)(q-3)\cdots(q-2m+3)(q-m+1). \tag{10}$$

Next, let us find $\chi_{T(B_m)}(q)$ and $\chi_{T(B_m)}(q)$ for even $q \in \mathbb{Z}_{>0}$.

Lemma 4.4. *We have the following equalities:*

$$\chi_{T(B_m)}(q) = \chi_{T(D_m)}(q-1) \quad \textit{for even } q \in \mathbb{Z}_{>0}, \tag{11}$$

$$\chi_{T(D_m)}(q) = \chi_{T(B_m)}(q) + m\chi_{T(B_{m-1})}(q) \quad \textit{for all } q \in \mathbb{Z}_{>0}. \tag{12}$$

Proof. For even $q \in \mathbb{Z}_{>0}$,

$$\chi_{T(B_m)}(q) = |\{(z_1,\ldots,z_m) \in \mathbb{Z}_q^m : z_i \neq 0 \ (1 \leq i \leq m),$$

$$\begin{aligned}
& \qquad\qquad z_i \neq \pm z_j \ (1 \leq i < j \leq m)\}| \\
= \ & m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 0 < c_1 < \cdots < c_m < q, \\
& \qquad\qquad c_i + c_j \neq q \ (1 \leq i < j \leq m)\}| \\
= \ & m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 1 - \frac{q}{2} \leq c_1 < \cdots < c_m < \frac{q}{2}, \\
& \qquad\qquad c_i + c_j \neq 0 \ (1 \leq i < j \leq m)\}| \\
= \ & |\{(z_1, \dots, z_m) \in \mathbb{Z}_{q-1}^m : z_i \neq \pm z_j \ (1 \leq i < j \leq m)\}| \\
= \ & \chi_{T(D_m)}(q-1),
\end{aligned}$$

where the third equality is seen by considering $c_i - (q/2)$ instead of $c_i, \ 1 \leq i \leq m$.

On the other hand, for all integers $q \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned}
\chi_{T(D_m)}(q) &= |\{(z_1, \dots, z_m) \in \mathbb{Z}_q^m : z_i \neq \pm z_j \ (1 \leq i < j \leq m)\}| \\
&= m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 0 \leq c_1 < \cdots < c_m < q, \\
&\qquad\qquad c_i + c_j \neq q \ (1 \leq i < j \leq m)\}| \\
&= m! \times |\{(c_1, \dots, c_m) \in \mathbb{Z}^m : 0 < c_1 < \cdots < c_m < q, \\
&\qquad\qquad c_i + c_j \neq q \ (1 \leq i < j \leq m)\}| \\
&\quad + m! \times |\{(0, c_2, \dots, c_m) \in \mathbb{Z}^m : 0 < c_2 < \cdots < c_m < q, \\
&\qquad\qquad c_i + c_j \neq q \ (2 \leq i < j \leq m)\}| \\
&= \chi_{T(B_m)}(q) + m\chi_{T(B_{m-1})}(q).
\end{aligned}$$

□

Theorem 4.5. *For even integers $q \in \mathbb{Z}_{>0}$, we have*

$$\chi_{T(B_m)}(q) = (q-2)(q-4)\cdots(q-2m+2)(q-m), \tag{13}$$

$$\chi_{T(D_m)}(q) = (q-2)(q-4)\cdots(q-2m+4) \times \{q^2 - 2(m-1)q + m(m-1)\}. \tag{14}$$

Proof. Equation (13) follows from (11) and (10); then (14) follows from (12) and (13). □

Proposition 4.3 and Theorem 4.5 imply in particular that each of the characteristic quasi-polynomials $\chi_{T(B_m)}(t)$ and $\chi_{T(D_m)}(t)$ has the minimum period two.

4.5.2 $\chi_{T(C_m)}(q)$

Lemma 4.2 implies that $\gcd\{\rho_0, q\} = 1$ for odd $q \in \mathbb{Z}_{>0}$ also in the case of $T(C_m)$. Since $T(B_m)$ and $T(C_m)$ define the same real arrangement, we have $\chi_{T(C_m)}(q) = \chi_{T(B_m)}(q)$ for odd $q \in \mathbb{Z}_{>0}$.

Proposition 4.6. *For odd integers $q \in \mathbb{Z}_{>0}$, we have*

$$\chi_{T(C_m)}(q) = (q-1)(q-3)\cdots(q-2m+1).$$

For even $q \in \mathbb{Z}_{>0}$, we can derive the following result:

Theorem 4.7. *For even integers $q \in \mathbb{Z}_{>0}$, we have*

$$\begin{aligned}\chi_{T(C_m)}(q) &= \chi_{T(C_m)}(q-1)\\ &= (q-2)(q-4)\cdots(q-2m).\end{aligned}$$

Proof. For even $q \in \mathbb{Z}_{>0}$,

$$\begin{aligned}\chi_{T(C_m)}(q) &= |\{(z_1,\dots,z_m) \in \mathbb{Z}_q^m : 2z_i \neq 0 \ (1 \le i \le m),\\ &\qquad z_i \neq \pm z_j \ (1 \le i < j \le m)\}|\\ &= m! \times |\{(c_1,\dots,c_m) \in \mathbb{Z}^m : 0 < c_1 < \cdots < c_m < q,\\ &\qquad c_i \neq \frac{q}{2} \ (1 \le i \le m),\ c_i + c_j \neq q \ (1 \le i < j \le m)\}|\\ &= m! \times |\{(c_1,\dots,c_m) \in \mathbb{Z}^m : 0 < c_1 < \cdots < c_m < q-1,\\ &\qquad c_i + c_j \neq q-1 \ (1 \le i < j \le m)\}|\\ &= \chi_{T(B_m)}(q-1) = \chi_{T(C_m)}(q-1),\end{aligned}$$

where the third equality is confirmed by transforming $c_i \mapsto c_i - 1$ for those c_i's with $c_i > q/2$. Thus we obtain the theorem by Proposition 4.6. □

We see that $\chi_{T(C_m)}(t)$ has also the minimum period two.

4.6 Conclusion on B_m, C_m and D_m

By equations (4), (6), (8) and the results in Section 4.5, we can obtain the characteristic quasi-polynomials of B_m, C_m and D_m:

Theorem 4.8. *The characteristic quasi-polynomials of B_m, C_m and D_m are*

$$\chi_{B_m}(q) = \chi_{C_m}(q) = \begin{cases}(q-1)(q-3)\cdots(q-2m+1) & \text{if } q \text{ is odd},\\ (q-2)(q-4)\cdots(q-2m+2)(q-m) & \text{if } q \text{ is even},\end{cases}$$

$$\chi_{D_m}(q) = \begin{cases}(q-1)(q-3)\cdots(q-2m+3)(q-m+1) & \text{if } q \text{ is odd},\\ (q-2)(q-4)\cdots(q-2m+4)\left\{q^2 - 2(m-1)q + \frac{m(m-1)}{2}\right\} & \text{if } q \text{ is even}.\end{cases}$$

Thus the minimum periods for B_m, C_m and D_m are equal to two.

5 Characteristic quasi-polynomials of G_2, F_4

In the rest of this paper we use the notation

$$\mathcal{E}_s := \{e(J) : J \subseteq [n],\ |J| \le s\}$$

for the $m \times n$ matrix $S = S(R)$ for a root system R and $s \in \mathbb{Z}_{>0}$.

5.1 Characteristic quasi-polynomial of G_2

We follow PLATE IX in [2]. Let $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ be an orthonormal basis for a 3-dimensional Euclidean space W, and define

$$V := \left\{ \sum_{i=1}^{3} c_i \epsilon_i \in W : \sum_{i=1}^{3} c_i = 0 \right\}.$$

Then

$R := \{\pm(\epsilon_i - \epsilon_j)\,(1 \le i < j \le 3), \pm(2\epsilon_1 - \epsilon_2 - \epsilon_3), \pm(2\epsilon_2 - \epsilon_1 - \epsilon_3), \pm(2\epsilon_3 - \epsilon_1 - \epsilon_2)\} \subset V,$
$|R| = 12,$

is an irreducible root system in V of type G_2. Then we may choose a set of positive roots

$$R_+ := \{\epsilon_1 - \epsilon_2, -2\epsilon_1 + \epsilon_2 + \epsilon_3, \epsilon_3 - \epsilon_1, \epsilon_3 - \epsilon_2, -2\epsilon_2 + \epsilon_1 + \epsilon_3, 2\epsilon_3 - \epsilon_1 - \epsilon_2\}.$$

Define $\alpha_1 := \epsilon_1 - \epsilon_2, \alpha_2 := -2\epsilon_1 + \epsilon_2 + \epsilon_3$. Then $B := \{\alpha_1, \alpha_2\}$ is the set of simple roots associated with R_+. We may express

$$R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Thus the 6×3 matrix $S(G_2)$ is given by

$$S(G_2) = \begin{bmatrix} 1 & 0 & 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

We find $\chi_{G_2}(q)$ as follows: First, the exponents of G_2 are $1, 5$. So we have

$$P_1(q) = q^2 - 6q + 5 = (q-1)(q-5),$$

which is the ordinary characteristic polynomial of type G_2. Next we compute:

$$\mathcal{E}_1 = \{1\}, \quad \mathcal{E}_2 = \{1, 2, 3\}.$$

Thus $\rho_0 = \operatorname{lcm} \mathcal{E}_2 = \operatorname{lcm}\{1, 2, 3\} = 6$. By Corollary 2.3, we have $P_d(q) = q^2 - 6q + \cdots$ for any $d|6$. Since

$$P_6 = P_2 + P_3 - P_1$$

by Corollary 2.5, it is enough to find P_2 and P_3. Therefore the special values

$$P_2(2) = |M_2(S)| = 0, \quad P_3(3) = |M_3(S)| = 0$$

are enough for us to obtain:

$$\chi_{G_2}(q) = \begin{cases} q^2 - 6q + 5 = (q-1)(q-5), & \gcd\{6, q\} = 1, \\ q^2 - 6q + 8 = (q-2)(q-4), & \gcd\{6, q\} = 2, \\ q^2 - 6q + 9 = (q-3)^2, & \gcd\{6, q\} = 3, \\ q^2 - 6q + 12, & \gcd\{6, q\} = 6. \end{cases}$$

Thus the minimum period for G_2 is 6.

5.2 Characteristic quasi-polynomial of F_4

We follow PLATE VIII in [2]. Let $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ be an orthonormal basis for a 4-dimensional Euclidean space V.

$$R := \{\pm\epsilon_i \ (1 \le i \le 4),\ \pm(\epsilon_i - \epsilon_j)\ (1 \le i < j \le 4),\ \pm(\epsilon_i + \epsilon_j)\ (1 \le i < j \le 4),$$
$$\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\ (16 \text{ of them})\} \subset V,$$

$|R| = 48$, is an irreducible root system in V of type F_4. Then we may choose a set of positive roots

$$R_+ = \{\epsilon_i \ (1 \le i \le 4),\ \epsilon_i - \epsilon_j\ (1 \le i < j \le 4),\ \epsilon_i + \epsilon_j\ (1 \le i < j \le 4),$$
$$\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\ (8 \text{ of them})\}.$$

Define $\alpha_1 := \epsilon_2 - \epsilon_3$, $\alpha_2 := \epsilon_3 - \epsilon_4$, $\alpha_3 := \epsilon_4$, and $\alpha_4 := \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$. Then $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is the set of simple roots associated with R_+. We may express

$$R_+ = \left\{ \sum_{i \le k \le j} \alpha_k \ (1 \le i \le j \le 4), \alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + \alpha_4, \right.$$
$$\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4,$$
$$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4, \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4,$$
$$\left. \alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \right\}.$$

Thus the 4×24 matrix $S = S(F_4)$, which is the coefficient matrix of R_+ with respect to the set of simple roots B, is:

$$S(F_4) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}.$$

We find $\chi_{F_4}(q)$ as follows: First, the exponents of F_4 are $1, 5, 7, 11$. So we have

$$P_1(q) = q^4 - 24q^3 + 190q^2 - 552q + 385 = (q-1)(q-5)(q-7)(q-11),$$

which is the ordinary characteristic polynomial of type F_4. Next we compute:

$$\mathcal{E}_1 = \{1\}, \quad \mathcal{E}_2 = \{1, 2\}, \quad \mathcal{E}_3 = \{1, 2, 4\}, \quad \mathcal{E}_4 = \{1, 2, 3, 4\}.$$

Thus $\rho_0 = \operatorname{lcm} \mathcal{E}_4 = \operatorname{lcm}\{1, 2, 3, 4\} = 12$. By Corollary 2.3, we have $P_d(q) = q^4 - 24q^3 + \cdots$ for any $d|12$. Since

$$P_6 = P_2 + P_3 - P_1, \quad P_{12} = P_3 + P_4 - P_1$$

by Corollary 2.5, it is enough to find P_2, P_3 and P_4. Also

$$\deg(P_2 - P_4) < 2, \quad \deg(P_3 - P_1) < 1$$

by Corollary 2.3. Therefore the following special values

$$P_2(2) = |M_2(S)| = 0,\ P_3(3) = |M_3(S)| = 0,\ P_4(4) = |M_4(S)| = 0,$$
$$P_4(8) = |M_8(S)| = 0,\ P_2(10) = |M_{10}(S)| = 0,\ P_2(14) = |M_{14}(S)| = 3456$$

are enough for us to obtain:

$$\chi_{F_4}(q) = \begin{cases} q^4 - 24q^3 + 190q^2 - 552q + 385 \\ \quad = (q-1)(q-5)(q-7)(q-11), \quad \gcd\{12, q\} = 1, \\ q^4 - 24q^3 + 208q^2 - 768q + 880 \\ \quad = (q-2)(q-10)(q^2 - 12q + 44), \quad \gcd\{12, q\} = 2, \\ q^4 - 24q^3 + 190q^2 - 552q + 513 \\ \quad = (q-3)(q-9)(q^2 - 12q + 19), \quad \gcd\{12, q\} = 3, \\ q^4 - 24q^3 + 208q^2 - 768q + 1024 \\ \quad = (q-4)^2(q-8)^2, \quad \gcd\{12, q\} = 4, \\ q^4 - 24q^3 + 208q^2 - 768q + 1008 \\ \quad = (q-6)^2(q^2 - 12q + 28), \quad \gcd\{12, q\} = 6, \\ q^4 - 24q^3 + 208q^2 - 768q + 1152, \quad \gcd\{12, q\} = 12. \end{cases}$$

Thus the minimum period for F_4 is 12.

6 Characteristic quasi-polynomials of E_6, E_7, E_8

For each of the root systems E_6, E_7 and E_8, we can find the characteristic quasi-polynomial by a similar method to the method in the previous section. First we compute $\mathcal{E}_s$ for each s. Then we have the lcm period ρ_0. For each constituent $P_d(t)$, $d|\rho_0$, apply Corollaries 2.3, 2.4 and 2.5 to get as much information as possible. Finally we actually count $|M_q(S)|$ for a large enough number of q's with $\gcd\{\rho_0, q\} = d$ and interpolate a polynomial. In this way, we obtain the characteristic quasi-polynomials for E_6, E_7 and E_8. For the evaluations of ρ_0's, we used PARI/GP [5].

6.1 Characteristic quasi-polynomial of E_6

We use PLATE V in [2] to get the 6×36 matrix $S = S(E_6)$:

$$S(E_6) = \left[\begin{array}{ccccccccccccccccccccc} 1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&1&0&0&0 \\ 0&1&0&0&0&0&0&1&0&0&0&0&1&1&0&0&1&0&1&1&0 \\ 0&0&1&0&0&0&1&0&1&0&0&1&1&0&1&0&1&1&1&0&1 \\ 0&0&0&1&0&0&0&1&1&1&0&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&0&0&0&0&1&1&0&0&1&1&1&0&1&1&1&1 \\ 0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&1&1 \end{array}\right.$$

$$\left.\begin{matrix}
1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{matrix}\right].$$

We find $\chi_{E_6}(q)$ as follows: First, the exponents of E_6 are $1, 4, 5, 7, 8, 11$. So we have

$$\begin{aligned}
P_1(q) &= q^6 - 36q^5 + 510q^4 - 3600q^3 + 13089q^2 - 22284q + 12320 \\
&= (q-1)(q-4)(q-5)(q-7)(q-8)(q-11),
\end{aligned}$$

which is the ordinary characteristic polynomial of type E_6. Next we compute:

$$\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = \{1\}, \quad \mathcal{E}_4 = \mathcal{E}_5 = \{1, 2\}, \quad \mathcal{E}_6 = \{1, 2, 3\}.$$

Thus $\rho_0 = \mathrm{lcm}\ \mathcal{E}_6 = \mathrm{lcm}\{1, 2, 3\} = 6$. By Corollary 2.3,

$$P_d(q) = q^6 - 36q^5 + 510q^4 - 3600q^3 + \cdots$$

for any $d|6$. Since

$$P_6 = P_2 + P_3 - P_1$$

by Corollary 2.5, it is enough to find P_2 and P_3. Also

$$\deg(P_3 - P_1) < 1$$

by Corollary 2.3. Therefore the following special values

$$P_2(2) = |M_2(S)| = 0,\ P_3(3) = |M_3(S)| = 0,\ P_2(4) = |M_4(S)| = 0,\ P_2(8) = |M_8(S)| = 0$$

are enough for us to obtain:

$$\chi_{E_6}(q) = \begin{cases}
q^6 - 36q^5 + 510q^4 - 3600q^3 + 13089q^2 - 22284q + 12320 \\
\quad = (q-1)(q-4)(q-5)(q-7)(q-8)(q-11), \\
\qquad\qquad \gcd\{6, q\} = 1, \\
q^6 - 36q^5 + 510q^4 - 3600q^3 + 13224q^2 - 23904q + 16640 \\
\quad = (q-2)(q-4)(q-8)(q-10)(q^2 - 12q + 26), \\
\qquad\qquad \gcd\{6, q\} = 2, \\
q^6 - 36q^5 + 510q^4 - 3600q^3 + 13089q^2 - 22284q + 12960 \\
\quad = (q-3)(q-9)(q^4 - 24q^3 + 195q^2 - 612q + 480), \\
\qquad\qquad \gcd\{6, q\} = 3, \\
q^6 - 36q^5 + 510q^4 - 3600q^3 + 13224q^2 - 23904q + 17280 \\
\quad = (q-6)^2(q^4 - 24q^3 + 186q^2 - 504q + 480), \\
\qquad\qquad \gcd\{6, q\} = 6.
\end{cases}$$

Thus the minimum period for E_6 is 6.

6.2 Characteristic quasi-polynomial of E_7

We use PLATE VI in [2] to get the 7×63 matrix $S = S(E_7)$:

$$S(E_7) = \left[\begin{array}{ccccccccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array} \right.$$

$$\begin{array}{cccccccccccccccccccccccc}
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 1 \\
1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 2 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1
\end{array}$$

$$\left. \begin{array}{cccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\
2 & 2 & 3 & 3 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 3 & 3 & 3 & 3 \\
2 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array} \right].$$

We find $\chi_{E_7}(q)$ as follows: First, the exponents of E_7 are $1, 5, 7, 9, 11, 13, 17$. So we have

$$\begin{aligned}
P_1(q) &= q^7 - 63q^6 + 1617q^5 - 21735q^4 + 162939q^3 - 663957q^2 + 1286963q - 765765 \\
&= (q-1)(q-5)(q-7)(q-9)(q-11)(q-13)(q-17),
\end{aligned}$$

which is the ordinary characteristic polynomial of type E_7. Next we compute:

$$\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = \{1\}, \quad \mathcal{E}_4 = \mathcal{E}_5 = \{1, 2\}, \quad \mathcal{E}_6 = \{1, 2, 3\}, \quad \mathcal{E}_7 = \{1, 2, 3, 4\}.$$

Thus $\rho_0 = \operatorname{lcm} \mathcal{E}_7 = \operatorname{lcm}\{1, 2, 3, 4\} = 12$. By Corollary 2.3,

$$P_d(q) = q^7 - 63q^6 + 1617q^5 - 21735q^4 + \cdots$$

for any $d|12$. Since

$$P_6 = P_2 + P_3 - P_1, \quad P_{12} = P_4 + P_3 - P_1$$

by Corollary 2.5, it is enough to find P_2, P_3 and P_4. Also

$$\deg(P_3 - P_1) < 2, \quad \deg(P_2 - P_4) < 1$$

by Corollary 2.3. Therefore the following special values

$$P_2(2) = |M_2(S)| = 0,\ P_3(3) = |M_3(S)| = 0,\ P_4(4) = |M_4(S)| = 0,\ P_3(9) = |M_9(S)| = 0,$$
$$P_2(10) = |M_{10}(S)| = 0,\ P_2(14) = |M_{14}(S)| = 0,\ P_2(22) = |M_{22}(S)| = 36288000$$

are enough for us to obtain:

$$\chi_{E_7}(q) = \begin{cases} q^7 - 63q^6 + 1617q^5 - 21735q^4 + 162939q^3 - 663957q^2 + 1286963q - 765765 \\ \quad = (q-1)(q-5)(q-7)(q-9)(q-11)(q-13)(q-17), \\ \qquad\qquad \gcd\{12, q\} = 1, \\ q^7 - 63q^6 + 1617q^5 - 21735q^4 + 163884q^3 - 689472q^2 + 1495808q - 1244880 \\ \quad = (q-2)(q-10)(q-13)(q-14)(q^3 - 24q^2 + 155q - 342), \\ \qquad\qquad \gcd\{12, q\} = 2, \\ q^7 - 63q^6 + 1617q^5 - 21735q^4 + 162939q^3 - 663957q^2 + 1304883q - 927045 \\ \quad = (q-3)(q-9)(q-15)(q^4 - 36q^3 + 438q^2 - 2052q + 2289), \\ \qquad\qquad \gcd\{12, q\} = 3, \\ q^7 - 63q^6 + 1617q^5 - 21735q^4 + 163884q^3 - 689472q^2 + 1495808q - 1290240 \\ \quad = (q-4)(q-5)(q-8)(q-16)(q^3 - 30q^2 + 263q - 504), \\ \qquad\qquad \gcd\{12, q\} = 4, \\ q^7 - 63q^6 + 1617q^5 - 21735q^4 + 163884q^3 - 689472q^2 + 1513728q - 1406160 \\ \quad = (q-6)(q^6 - 57q^5 + 1275q^4 - 14085q^3 + 79374q^2 - 213228q + 234360), \\ \qquad\qquad \gcd\{12, q\} = 6, \\ q^7 - 63q^6 + 1617q^5 - 21735q^4 + 163884q^3 - 689472q^2 + 1513728q - 1451520 \\ \quad = (q-12)(q^6 - 51q^5 + 1005q^4 - 9675q^3 + 47784q^2 - 116064q + 120960), \\ \qquad\qquad \gcd\{12, q\} = 12. \end{cases}$$

Thus the minimum period for E_7 is 12.

6.3 Characteristic quasi-polynomial of E_8

We use PLATE VII in [2] to get the 8×120 matrix $S = S(E_8)$:

$$S(E_8) = \left[\begin{array}{ccccccccccccccccccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \right.$$

$$
\begin{array}{ccccccccccccccccccccccccc}
0&0&1&1&0&0&0&0&0&1&1&0&0&1&0&0&1&1&1&0&0&1&0&1&1&1\\
0&0&1&0&1&1&1&0&0&1&1&1&1&0&1&0&1&1&1&1&1&0&1&1&1&1\\
0&0&1&1&1&1&0&1&0&1&1&1&1&1&0&1&1&1&2&1&1&1&1&1&2&1\\
1&0&1&1&1&2&1&1&1&2&1&2&1&1&1&1&1&2&2&2&2&1&1&2&2&2\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&2&1&1&1&2&1&1\\
1&1&0&1&1&0&1&1&1&0&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&0&0&0&0&1&1&1&0&0&0&1&1&1&1&1&0&0&0&1&1&1&0&0&1\\
0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&1&0&0&0
\end{array}
$$

$$
\begin{array}{ccccccccccccccccccccccccc}
0&1&0&1&1&1&0&0&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&2&1&1&1&1&1&1&2&1&1&1\\
1&1&1&2&2&1&1&1&1&2&2&1&1&2&1&2&2&2&2&1&1&2&2&2&2&1\\
2&1&2&2&2&2&2&2&2&3&2&2&2&2&2&3&2&3&2&2&2&3&3&3&2&2\\
2&1&1&2&1&2&2&2&1&2&2&2&2&1&2&2&2&2&2&2&2&2&2&2&2&2\\
1&1&1&1&1&1&2&1&1&1&1&2&2&1&1&1&2&1&1&2&2&2&1&1&2&2\\
1&1&1&0&1&1&1&1&1&0&1&1&1&1&1&0&1&1&1&1&2&1&1&1&1&2\\
0&1&1&0&0&0&0&1&1&0&0&0&1&1&1&0&0&0&1&1&1&0&0&1&1&1
\end{array}
$$

$$
\begin{array}{cccccccccccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&2&1&1&1&2&1&1\\
1&2&2&1&1&2&2&1&1&2&2&2&1&2&2&2&1&2&2&2&2&2&2&2\\
2&2&2&2&2&2&2&2&2&2&2&2&2&3&2&2&2&3&3&2&2&3&3&2\\
3&3&3&3&2&3&3&3&3&4&3&3&3&4&4&3&3&4&4&4&3&4&4&4\\
3&2&2&2&2&3&2&3&2&3&3&2&3&3&3&3&3&3&3&3&3&3&3&3\\
2&2&1&2&2&2&2&2&2&2&2&2&2&2&2&2&3&2&2&2&3&2&2&3\\
1&1&1&1&2&1&1&1&2&1&1&2&2&1&1&2&2&1&1&2&2&1&2&2\\
0&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&0&1&1&1&1&1&1
\end{array}
$$

$$
\left.
\begin{array}{ccccccccccccccccc}
2&1&1&2&1&1&2&1&2&2&2&2&2&2&2&2&2\\
2&2&2&2&2&2&2&3&2&3&2&3&3&3&3&3&3\\
3&3&2&3&3&3&3&3&3&3&4&4&4&4&4&4&4\\
4&4&4&4&4&5&4&5&5&5&5&5&6&6&6&6&6\\
3&3&4&3&4&4&4&4&4&4&4&4&4&5&5&5&5\\
2&3&3&3&3&3&3&3&3&3&3&3&3&3&4&4&4\\
2&2&2&2&2&2&2&2&2&2&2&2&2&2&2&3&3\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&2
\end{array}
\right] .
$$

We find $\chi_{E_8}(q)$ as follows: First, the exponents of E_8 are $1, 7, 11, 13, 17, 19, 23, 29$. So we have

$$
\begin{aligned}
P_1(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2616558q^4 \\
&\quad - 24693480q^3 + 130085780q^2 - 323507400q + 215656441 \\
&= (q-1)(q-7)(q-11)(q-13)(q-17)(q-19)(q-23)(q-29)
\end{aligned}
$$

which is the ordinary characteristic polynomial of type E_8. Next we compute:

$$
\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{E}_3 = \{1\}, \quad \mathcal{E}_4 = \mathcal{E}_5 = \{1, 2\},
$$

$$\mathcal{E}_6 = \{1,2,3\}, \quad \mathcal{E}_7 = \{1,2,3,4\}, \quad \mathcal{E}_8 = \{1,2,3,4,5,6\}$$

aided by PARI/GP [5]. Thus $\rho_0 = \operatorname{lcm} \mathcal{E}_8 = \operatorname{lcm}\{1,2,3,4,5,6\} = 60$. By Corollary 2.3,

$$P_d(q) = q^8 - 120q^7 + 6020q^6 - 163800q^5 + \cdots$$

for any $d|60$. Since

$$\begin{array}{lll} P_{10} = P_2 + P_5 - P_1, & P_{15} = P_3 + P_5 - P_1, & P_{20} = P_4 + P_5 - P_1, \\ P_{30} = P_6 + P_5 - P_1, & P_{60} = P_{12} + P_5 - P_1 & \end{array}$$

by Corollary 2.4, it is enough to find P_2, P_3, P_4, P_5, P_6 and P_{12}. Also

$$\deg(P_1 + P_6 - P_2 - P_3) < 1, \quad \deg(P_1 + P_{12} - P_3 - P_4) < 1$$

by Corollary 2.4 again. Moreover,

$$\deg(P_3 - P_1) < 3, \quad \deg(P_2 - P_4) < 2, \quad \deg(P_5 - P_1) < 1$$

by Corollary 2.3. Therefore the following special values

$P_2(2) = |M_2(S)| = 0$, $P_3(3) = |M_3(S)| = 0$, $P_4(4) = |M_4(S)| = 0$, $P_5(5) = |M_5(S)| = 0$,
$P_6(6) = |M_6(S)| = 0$, $P_4(8) = |M_8(S)| = 0$, $P_3(9) = |M_9(S)| = 0$, $P_{12}(12) = |M_{12}(S)| = 0$,
$P_2(14) = |M_{14}(S)| = 0$, $P_3(21) = |M_{21}(S)| = 0$, $P_2(22) = |M_{22}(S)| = 0$,
$P_2(26) = |M_{26}(S)| = 0$, $P_2(34) = |M_{34}(S)| = 6967296000$

are enough for us to obtain:

$$\begin{aligned}
P_1(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2616558q^4 \\
&\qquad -24693480q^3 + 130085780q^2 - 323507400q + 215656441 \\
&= (q-1)(q-7)(q-11)(q-13)(q-17)(q-19)(q-23)(q-29), \\
P_2(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2626008q^4 \\
&\qquad -25260480q^3 + 141860480q^2 - 418876800q + 435250816 \\
&= (q-2)(q-14)(q-22)(q-26)(q^4 - 56q^3 + 1068q^2 - 8344q + 27176), \\
P_3(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2616558q^4 \\
&\qquad -24693480q^3 + 130802580q^2 - 345011400q + 348264441 \\
&= (q-3)(q-9)(q-21)(q-27)(q^4 - 60q^3 + 1250q^2 - 10500q + 22749), \\
P_4(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2626008q^4 \\
&\qquad -25260480q^3 + 141860480q^2 - 424320000q + 516898816 \\
&= (q-4)(q-8)(q-16)(q-28)(q^4 - 64q^3 + 1428q^2 - 12536q + 36056), \\
P_5(q) &= q^8 - 120q^7 + 6020q^6 - 163800q^5 + 2616558q^4 \\
&\qquad -24693480q^3 + 130085780q^2 - 323507400q + 243525625
\end{aligned}$$

$$
\begin{aligned}
&= (q-5)(q-25)(q^6-90q^5+3195q^4-56700q^3+516183q^2-2120490q+1948205),\\
P_6(q) &= q^8-120q^7+6020q^6-163800q^5+2626008q^4\\
&\qquad -25260480q^3+142577280q^2-440380800q+587212416\\
&= (q-6)(q-18)(q^6-96q^5+3608q^4-66840q^3+632184q^2-2869344q+5437152),\\
P_{10}(q) &= q^8-120q^7+6020q^6-163800q^5+2626008q^4\\
&\qquad -25260480q^3+141860480q^2-418876800q+463120000\\
&= (q-10)(q^7-110q^6+4920q^5-114600q^4\\
&\qquad +1480008q^3-10460400q^2+37256480q-46312000),\\
P_{12}(q) &= q^8-120q^7+6020q^6-163800q^5+2626008q^4\\
&\qquad -25260480q^3+142577280q^2-445824000q+668860416\\
&= (q-12)(q-24)(q^6-84q^5+2708q^4-42120q^3+329784q^2-1257696q+2322432),\\
P_{15}(q) &= q^8-120q^7+6020q^6-163800q^5+2616558q^4\\
&\qquad -24693480q^3+130802580q^2-345011400q+376133625\\
&= (q-15)^2(q^6-90q^5+3095q^4-50700q^3+399183q^2-1310490q+1671705),\\
P_{20}(q) &= q^8-120q^7+6020q^6-163800q^5+2626008q^4\\
&\qquad -25260480q^3+141860480q^2-424320000q+544768000\\
&= (q-20)(q^7-100q^6+4020q^5-83400q^4\\
&\qquad +958008q^3-6100320q^2+19854080q-27238400),\\
P_{30}(q) &= q^8-120q^7+6020q^6-163800q^5+2626008q^4\\
&\qquad -25260480q^3+142577280q^2-440380800q+615081600,\\
P_{60}(q) &= q^8-120q^7+6020q^6-163800q^5+2626008q^4\\
&\qquad -25260480q^3+142577280q^2-445824000q+696729600.
\end{aligned}
$$

Thus the minimum period for E_8 is 60.

7 Two results

The following two results are obtained from our calculations and the classification of irreducible root systems.

Theorem 7.1. *For an irreducible root system R, the minimum period of the quasi-polynomial $\chi_R(q)$ is equal to the lcm period ρ_0.*

Theorem 7.2. *Let q be a positive integer. For an irreducible root system R with its Coxeter number h, $\chi_R(q) > 0$ if and only if $q \geq h$.*

Remark 7.3. *It is easy to see the point $(1, 1, \ldots, 1) \in \mathbb{Z}_q^m$ lies in $M_q(S)$ if $q \geq h$ because the sum of the elements of each column of S does not exceed the largest exponent. This shows $\chi_R(q) > 0$ if $q \geq h$. However, our proof of the "only if" part of Theorem 7.2 still requires the classification.*

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