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Semidefinite Programming for Uncertain Linear Equations in Static Analysis of Structures

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Abstract

This paper presents a method for computing a minimal bounding ellipsoid that contains the solution set to the uncertain linear equations. Particularly, we aim at finding a bounding ellipsoid for static response of structures, where both external forces and elastic moduli of members are assumed to be imprecisely known and bounded. By using the $\mathcal{S}$-lemma, we formulate a semidefinite programming (SDP) problem which provides an outer approximation of the minimal bounding ellipsoid. The minimum bounding ellipsoids are computed for nodal displacements of uncertain braced frames as the solutions of the presented SDP problems by using the primal-dual interior-point method.

Keywords

Semidefinite program; Data uncertainty; Uncertain linear equations; Interval analysis; Confidential bound; interior-point method

1 Introduction

This paper discusses a solution method for computing ellipsoidal bounds for the solution set to the uncertain linear equations, in which we suppose that the coefficient matrix as well as the right-hand-side vector of the system of linear equations possesses uncertainty. Particularly, in civil, mechanical and aerospace engineering, structural analyses considering the uncertainties have received fast-growing interests. This is because structures actually built always have various uncertainties caused by manufacturing errors, limitation of knowledge of input disturbances, observation errors, simplification for modeling, damage or deterioration of structural elements, etc.

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Probabilistic uncertainty modelings of structural analyses were studied extensively. Non-probabilistic uncertainty models have also been developed. In a non-probabilistic uncertainty model, a mechanical system is assumed to contain some unknown parameters which are supposed to be bounded. Such parameters are often referred to as *unknown-but-bounded* parameters. Ben-Haim and Elishakoff [3] developed the well-known *convex model* approach, with which Pantelides and Ganzerli [21] proposed a robust truss optimization method. The info-gap decision theory has been proposed by Ben-Haim [2]. Based on the info-gap uncertainty model, the authors proposed solution methods for robustness analysis of structures [14, 27].

The interval linear algebra has been well developed for *uncertain linear equations* (ULE) [1, 19], and has been employed in structural analyses considering various uncertainties [9, 16–18, 23]. In contrast to probabilistic modelings, non-probabilistic uncertainty modelings require only bounds on the uncertain parameters, and hence there exists no necessity to estimate the probabilistic distribution functions of uncertain parameters. Under the assumption of small variations of uncertain parameters, interval analyses of static response have been developed based on the first-order interval perturbation [9, 23]. Further refinements of the linear interval approach were proposed for static structural analyses including uncertainties [17, 18]. A comparison between the convex model analysis and interval analysis was given by Qiu [22].

For convex optimization problems, a unified methodology of *robust optimization* was developed by Ben-Tal and Nemirovski [6], in which the data in optimization problems are assumed to be unknown but bounded. Calafiore and El Ghaoui [8] proposed a method for finding ellipsoidal bounds of the solution set of ULE by using the semidefinite programming relaxation, where the uncertainty set of data matrix and vector is described by a linear fractional representation [10]. The authors formulated an SDP problem which provides a confidential ellipsoidal bound for static response of a truss including some bounded uncertain parameters [13].

In this paper, as a generalization of the results of Kanno and Takewaki [13], we aim at obtaining an ellipsoidal bound for static response of a general structure. We suppose that both the external forces and the member stiffnesses are known imprecisely. The conventional interval analysis of uncertain structures is included in this problem as a particular case. From the mathematical point of view, our problem corresponds to finding an ellipsoidal bound of the projection of the solution set of ULE, where the uncertainty set of data matrix has a particular structure. Indeed, our theory developed in this paper can be applied to a broader class of ULE. The data vector of ULE is assumed to be included in a bounded set described as a direct product of some ellipsoids.

By using quadratic embedding of the uncertain parameters and the $\mathcal{S}$-procedure [7, 13], we formulate a *semidefinite programming* (SDP) problem [11] that provides an outer approximation of bounding ellipsoid. This fundamental idea is similar to that used in [8], but our analysis is different because we take notice of a particular properties of stiffness matrix that cannot be described by the linear fractional representation. It should be emphasized that our uncertainty model of ULE is motivated by and suitable for the static analysis of structures with uncertainties.

It is known that SDP problems can be efficiently solved by using the primal-dual interior-point method [15], where the number of arithmetic operations required by the algorithm is bounded by a polynomial of problem size. Hence, our method finds a bounding ellipsoid within the polynomial time of problem size, on the contrary to the fact that most of methods based on the interval algebra have in general exponential complexity [4, section 6.5.3].

This paper is organized as follows. In section 2, in order to make this paper self-contained, we introduce SDP as well as some useful technical results. Section 3 defines the uncertainty model of a system of linear equations. In section 4, we formulate the minimization problem of a bounding ellipsoid. An approximation problem for finding the minimal bounding ellipsoid is formulated as an SDP problem in section 5. In section 6, we show how to embed the static analysis of structures with uncertainties into the general framework developed in sections 3–5. Numerical experiments are presented in section 7 for braced structures, while conclusions are drawn in section 8.

2 Preliminary results

In this paper, all vectors are assumed to be column vectors. The $(m+n)$-dimensional column vector $(\boldsymbol{u}^{\rm T}, \boldsymbol{v}^{\rm T})^{\rm T}$ consisting of $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^n$ is often written simply as $(\boldsymbol{u}, \boldsymbol{v})$. For a vector $\boldsymbol{p} = (p_i) \in \mathbb{R}^n$, let $\|\boldsymbol{p}\|_2$ and $\|\boldsymbol{p}\|_\infty$, respectively, denote the standard Euclidean norm and l_∞-norm of $\boldsymbol{p}$ defined as

$$\|\boldsymbol{p}\|_2 = (\boldsymbol{p}^{\rm T}\boldsymbol{p})^{1/2},$$
$$\|\boldsymbol{p}\|_\infty = \max_{i \in \{1,\dots,n\}} |p_i|.$$

For $\boldsymbol{p} = (p_i) \in \mathbb{R}^n$ and $\boldsymbol{q} = (q_i) \in \mathbb{R}^n$, we write $\boldsymbol{p} \geq \boldsymbol{0}$ and $\boldsymbol{p} \geq \boldsymbol{q}$, respectively, if $p_i \geq 0$ $(i = 1, \dots, n)$ and $\boldsymbol{p} - \boldsymbol{q} \geq \boldsymbol{0}$. We write $\mathrm{Diag}(\boldsymbol{p})$ for the $n \times n$ diagonal matrix with a vector $\boldsymbol{p} \in \mathbb{R}^n$ on its diagonal. For $\boldsymbol{p}_l \in \mathbb{R}^{n_l}$ $(l = 1, \dots, k)$, we simply write $\mathrm{Diag}(\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)$ instead of $\mathrm{Diag}((\boldsymbol{p}_1^{\rm T}, \dots, \boldsymbol{p}_k^{\rm T})^{\rm T})$. The jth column vector of the identity matrix $I \in \mathbb{R}^{n \times n}$ is denoted by $\boldsymbol{e}_j^{(n)}$.

For $p \in \mathbb{R}$ and $q \in \mathbb{R}$ satisfying $p \leq q$, we denote by $[p, q]$ the interval defined by

$$[p, q] = \{x \in \mathbb{R} | \, p \leq x \leq q\}.$$

For two sets $\mathcal{A} \subseteq \mathbb{R}^m$ and $\mathcal{B} \subseteq \mathbb{R}^n$, their Cartesian product is defined by $\mathcal{A} \times \mathcal{B} = \{(\boldsymbol{a}^{\rm T}, \boldsymbol{b}^{\rm T})^{\rm T} \in \mathbb{R}^{m+n} | \, \boldsymbol{a} \in \mathcal{A}, \ \boldsymbol{b} \in \mathcal{B}\}$. Particularly, we write $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. Define $\mathbb{R}^n_+ \subset \mathbb{R}^n$ by

$$\mathbb{R}^n_+ = \{\boldsymbol{p} \in \mathbb{R}^n | \, \boldsymbol{p} \geq \boldsymbol{0}\}.$$

The Moore–Penrose pseudo-inverse of $Q \in \mathbb{R}^{m \times n}$ is denoted by $Q^\dagger \in \mathbb{R}^{n \times m}$. The nullspace of Q is denoted by $\mathcal{N}(Q)$. Note that $\mathcal{N}(Q)$ is the set of all vectors $\boldsymbol{x} \in \mathbb{R}^n$ satisfying $Q\boldsymbol{x} = \boldsymbol{0}$. The basis of $\mathcal{N}(Q)$ is denoted by $Q^\perp$. The row rank of Q is denoted by $\mathrm{rank}(Q)$.

2.1 Semidefinite program

Let $\mathcal{S}^n \subset \mathbb{R}^{n \times n}$ denote the set of all $n \times n$ real symmetric matrices. We denote $\mathcal{S}^n_+ \subset \mathcal{S}^n$ the set of all symmetric positive semidefinite matrices. We write $P \succeq O$ and $P \succeq Q$, respectively, if $P \in \mathcal{S}^n_+$ and $P - Q \in \mathcal{S}^n_+$. For a matrix $P \in \mathbb{R}^{n \times n}$, $\mathrm{tr}(P)$ denotes the trace of P, i.e. the sum of the diagonal elements of P.

Let $A_i \in \mathcal{S}^n$ $(i = 1, \dots, m)$, $C \in \mathcal{S}^n$, and $\boldsymbol{b} = (b_i) \in \mathbb{R}^m$ be constant matrices and a constant vector. The *semidefinite programming* (SDP) problem refers to the optimization problem having the form of [11]

$$\min \{\mathrm{tr}(CX) : \mathrm{tr}(A_i X) = b_i \ (i = 1, \dots, m), \ \mathcal{S}^n \ni X \succeq O\}, \tag{1}$$

where X is a variable matrix. The dual of Problem (1) is formulated in the variables $\boldsymbol{y} \in \mathbb{R}^m$ as

$$\max \left\{ \boldsymbol{b}^{\mathrm{T}} \boldsymbol{y} : C - \sum_{i=1}^{m} A_i y_i \succeq O \right\}, \tag{2}$$

which is also an SDP problem.

Recently, SDP has received increasing attention for its various fields of application [5, 12, 13, 20]. It is well known that various convex optimization problems are included in SDP as particular cases [5]. The primal-dual interior-point method, which has been first developed for the linear program, has been naturally extended to SDP [11, 15]. It is theoretically guaranteed that the primal-dual interior-point method converges to optimal solutions of the primal-dual pair of SDP problems (1) and (2) within the number of arithmetic operations bounded by a polynomial of m and n.

2.2 Technical lemmas

The remainder of this section is devoted to introducing some technical results that will be used in the following sections.

Lemma 2.1 (Homogenization [8, Lemma A.3]). *Let $Q \in \mathcal{S}^n$, $\boldsymbol{p} \in \mathbb{R}^n$, and $r \in \mathbb{R}$. Then the following two conditions are equivalent:*

$$\text{(a)} : \quad \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} Q & \boldsymbol{p} \\ \boldsymbol{p}^{\mathrm{T}} & r \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \geq 0, \quad \forall \boldsymbol{x} \in \mathbb{R}^n;$$

$$\text{(b)} : \quad \begin{pmatrix} Q & \boldsymbol{p} \\ \boldsymbol{p}^{\mathrm{T}} & r \end{pmatrix} \succeq O.$$

Proof. The implication from (b) to (a) is trivial. We show that (a) implies (b) by the contradiction. Suppose that (b) does not holds, i.e. there exist $\boldsymbol{x}' \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$ satisfying

$$\begin{pmatrix} \boldsymbol{x}' \\ \eta \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} Q & \boldsymbol{p} \\ \boldsymbol{p}^{\mathrm{T}} & r \end{pmatrix} \begin{pmatrix} \boldsymbol{x}' \\ \eta \end{pmatrix} < 0. \tag{3}$$

If $\eta \neq 0$, then (3) is reduced to

$$\begin{pmatrix} \boldsymbol{x}'/\eta \\ 1 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} Q & \boldsymbol{p} \\ \boldsymbol{p}^{\mathrm{T}} & r \end{pmatrix} \begin{pmatrix} \boldsymbol{x}'/\eta \\ 1 \end{pmatrix} < 0,$$

which contradicts the assertion (a). Alternatively, if $\eta = 0$, then (3) is reduced to

$$\boldsymbol{x}'^{\mathrm{T}} Q \boldsymbol{x}' < 0. \tag{4}$$

Letting $\boldsymbol{x} = \zeta \boldsymbol{x}'$, the left-hand side of (a) is reduced to

$$(\boldsymbol{x}^{\mathrm{T}} Q \boldsymbol{x}')\zeta^2 + 2(\boldsymbol{p}^{\mathrm{T}} \boldsymbol{x}')\zeta + r, \tag{5}$$

which is regarded as a function of ζ. The condition (4) implies that (5) is not bounded below, from which it follows that there exists a ζ such that (5) becomes negative. Thus, we see the contradiction to (a). □

Let $f_0(\boldsymbol{x}),\ f_1(\boldsymbol{x}), \ldots, f_{m+k}(\boldsymbol{x})$ be quadratic functions in the variable $\boldsymbol{x} \in \mathbb{R}^n$ defined as

$$f_i(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}} Q_i \boldsymbol{x} + 2\boldsymbol{p}_i^{\mathrm{T}} \boldsymbol{x} + r_i, \quad i = 0, 1, \ldots, m+k,$$

where $Q_i \in \mathcal{S}^n$, $\boldsymbol{p}_i \in \mathbb{R}^n$, and $r_i \in \mathbb{R}$.

Lemma 2.2 ($\mathcal{S}$-lemma). *The implication*

$$f_1(\boldsymbol{x}) \geq 0, \ldots, f_m(\boldsymbol{x}) \geq 0 \quad \Longrightarrow \quad f_0(\boldsymbol{x}) \geq 0$$

holds if there exist $\tau_1, \ldots, \tau_m$ such that

$$\begin{aligned} &f_0(\boldsymbol{x}) \geq \sum_{i=1}^{m} \tau_i f_i(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \\ &\tau_1, \ldots, \tau_m \geq 0. \end{aligned}$$

Proof. See Boyd *et al.* [7, section 2.6.3] and the references therein. □

The following result immediately follows from Lemma 2.1 and Lemma 2.2:

Corollary 2.3. *The implication*

$$f_1(\boldsymbol{x}) \geq 0, \ldots, f_m(\boldsymbol{x}) \geq 0,\ f_{m+1}(\boldsymbol{x}) = 0, \ldots, f_{m+k}(\boldsymbol{x}) = 0 \quad \Longrightarrow \quad f_0(\boldsymbol{x}) \geq 0 \tag{6}$$

holds if there exist $\tau_1, \ldots, \tau_{m+k}$ satisfying

$$\begin{aligned} &\begin{pmatrix} Q_0 & \boldsymbol{p}_0 \\ \boldsymbol{p}_0^{\mathrm{T}} & r_0 \end{pmatrix} \succeq \sum_{i=1}^{m+k} \tau_i \begin{pmatrix} Q_i & \boldsymbol{p}_i \\ \boldsymbol{p}_i^{\mathrm{T}} & r_i \end{pmatrix}, \\ &\tau_1, \ldots, \tau_m \geq 0. \end{aligned}$$

Proof. Observe that the quadratic equation $f_i(\boldsymbol{x}) = 0$ is equivalent to the set of quadratic inequalities

$$f_i(\boldsymbol{x}) \geq 0, \quad -f_i(\boldsymbol{x}) \geq 0.$$

It follows from Lemma 2.2 that the implication (6) holds if there exist $\tau_1, \ldots, \tau_m,\ \rho_{m+1}^+, \ldots, \rho_{m+k}^+$, and $\rho_{m+1}^-, \ldots, \rho_{m+k}^-$ satisfying

$$\begin{aligned} &f_0(\boldsymbol{x}) \geq \sum_{i=1}^{m} \tau_i f_i(\boldsymbol{x}) + \sum_{i=m+1}^{m+k} \rho_i^+ f_i(\boldsymbol{x}) + \sum_{i=m+1}^{m+k} \rho_i^- (-f_i(\boldsymbol{x})), \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \\ &\tau_1, \ldots, \tau_m \geq 0,\ \rho_{m+1}^+, \ldots, \rho_{m+k}^+ \geq 0,\ \rho_{m+1}^-, \ldots, \rho_{m+k}^- \geq 0. \end{aligned}$$

The assertion of this corollary is obtained by putting $\tau_i = \rho_i^+ - \rho_i^-$ $(i = m+1, \ldots, m+k)$ and then applying Lemma 2.1. □

Lemma 2.4 (Lemma on the Schur complement). *Let*

$$X = \begin{pmatrix} P & A^{\mathrm{T}} \\ A & Q \end{pmatrix}$$

be a symmetric matrix with blocks $P \in \mathcal{S}^n$ and $Q \in \mathcal{S}^m$. Then

$$X \succeq O$$

if and only if

$$P \succeq O, \quad Q - AP^{\dagger}A^{\mathrm{T}} \succeq O, \quad (I - P^{\dagger}P)A^{\mathrm{T}} = O,$$

where $P^{\dagger}$ denotes the Moore–Penrose pseudo-inverse of P.

Proof. See Boyd *et al.* [7, pp.28]. □

3 Uncertain linear equations

Consider a system of linear equations

$$K\boldsymbol{u} = \boldsymbol{f}, \tag{7}$$

where $K \in \mathbb{R}^{m \times n}$ and $\boldsymbol{f} \in \mathbb{R}^m$. In (7), we assume that K and $\boldsymbol{f}$ have the bounded uncertainties, which shall be rigorously defined below.

In a typical example in structural mechanics of linear elasticity, the system of linear equations (7) is regarded as the equilibrium equations. In this case, $K \in \mathcal{S}_{+}^{n^{\mathrm{d}}}$ and $\boldsymbol{f} \in \mathbb{R}^{n^{\mathrm{d}}}$ denote the stiffness matrix and the external force vector, respectively, where n^{d} denotes the number of degrees of freedom of displacements. Section 6 is devoted to the details of formulations for the static analysis of structures under uncertainties. See, also, Example 3.3.

Suppose that K and $\boldsymbol{f}$ in (7) are known imprecisely. The nominal values, or the best estimates, of K and $\boldsymbol{f}$ are denoted by $\widetilde{K} \in \mathbb{R}^{m \times n}$ and $\widetilde{\boldsymbol{f}} \in \mathbb{R}^m$, respectively. Let $\boldsymbol{\zeta}_{\mathrm{a}} = (\zeta_{\mathrm{a}i}) \in \mathbb{R}^s$ and $\boldsymbol{\zeta}_{\mathrm{f}} = (\zeta_{\mathrm{f}j}) \in \mathbb{R}^{n^{\mathrm{f}}}$ denote the parameter vectors that are considered to be unknown, or uncertain. Here, s and n^{f} denote the numbers of the parameters $\zeta_{\mathrm{a}i}$ and $\zeta_{\mathrm{f}j}$, respectively. We describe the uncertainties of K and $\boldsymbol{f}$ in terms of the unknown $\boldsymbol{\zeta}_{\mathrm{a}}$ and $\boldsymbol{\zeta}_{\mathrm{f}}$, respectively. Suppose that K and $\boldsymbol{f}$ depend on $\boldsymbol{\zeta}_{\mathrm{a}}$ and $\boldsymbol{\zeta}_{\mathrm{f}}$ affinely as

$$K = \widetilde{K} + \sum_{i=1}^{s} a_i^0 \zeta_{\mathrm{a}i} K_i, \tag{8}$$

$$\boldsymbol{f} = \widetilde{\boldsymbol{f}} + F_0 \boldsymbol{\zeta}_{\mathrm{f}}, \tag{9}$$

where $\boldsymbol{a}^0 = (a_i^0) \in \mathbb{R}_{+}^s$, $K_i \in \mathbb{R}^{m \times n}$ $(i = 1, \dots, s)$, and $F_0 \in \mathbb{R}^{m \times n^{\mathrm{f}}}$ are constant vector and constant matrices.

Note that a_i^0 represents the magnitude of uncertainty of K_i. The matrix F_0 represents the magnitude of the uncertainty of f_j and the correlation of the uncertainties among $f_1, \dots, f_n$. Moreover, suppose that F_0 satisfies the following assumption:

Assumption 3.1. *The constant matrix $F_0 \in \mathbb{R}^{m \times n^{\mathrm{f}}}$ satisfies* $\mathrm{rank}(F_0) = n^{\mathrm{f}} \leq m$.

We utilize the fact that any matrix $K_i \in \mathbb{R}^{m \times n}$ in (8) is written as

$$K_i = \sum_{j=1}^{r} \boldsymbol{\beta}_{ij} \boldsymbol{b}_{ij}^{\mathrm{T}}, \tag{10}$$

where $\boldsymbol{\beta}_{ij} \in \mathbb{R}^m$ and $\boldsymbol{b}_{ij} \in \mathbb{R}^n$ are constant vector. It is usual that r depends on i. However, throughout the paper we write r instead of r_i for simplicity. Indeed, we can take $r = \max\{\mathrm{rank}(K_i) : i = 1, \dots, s\}$ without loss of generality. It is often that r is relatively small, especially in the static analysis of uncertain structures as discussed in section 6. If K_i is a symmetric matrix, then $\boldsymbol{\beta}_{ij}$ and $\boldsymbol{b}_{ij}$ in (10) are obtained through the eigenvalue decomposition as shown in Example 3.2 and Example 3.3 below.

Example 3.2 (decomposition of symmetric K_i). Suppose that $K_i \in \mathcal{S}^n$ is symmetric with $m = n$. Then the eigenvalues λ_j of K_i are defined as

$$K_i \boldsymbol{\phi}_j = \lambda_j \boldsymbol{\phi}_j, \quad j = 1, \dots, n. \tag{11}$$

Here, $\boldsymbol{\phi}_j \in \mathbb{R}^n$ $(j = 1, \dots, n)$ are the eigenvectors which are supposed to be orthonormal as

$$\boldsymbol{\phi}_j \boldsymbol{\phi}_k = \delta_{jk},$$

where δ_{jk} denotes Kronecker's delta. Since K_i is symmetric, the eigenvalues λ_j $(j = 1, \dots, n)$ are real values. From the eigenvalue decomposition it follows that K_i can be written as

$$K_i = \sum_{j=1}^{n} \lambda_j \boldsymbol{\phi}_j \boldsymbol{\phi}_j^{\mathrm{T}}. \tag{12}$$

Hence, by putting

$$\boldsymbol{\beta}_{ij} = \boldsymbol{\phi}_j, \quad \boldsymbol{b}_{ij} = \lambda_j \boldsymbol{\phi}_j, \quad j = 1, \dots, n,$$

we see that K_i in (12) is decomposed in the form of (10). ■

Example 3.3 (decomposition of positive semidefinite K_i). As a particular case of Example 3.2, suppose that $K_i \in \mathcal{S}^n$ is a symmetric and positive semidefinite matrix. For example, the member stiffness matrix of a structure satisfies this condition; see section 6. In the standard eigenvalue problem (11), suppose that λ_j $(j = 1, \dots, n)$ are arranged in non-ascending order and that $\mathrm{rank}(K_i) = r$, i.e.

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0, \quad \lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_n = 0.$$

Then, by putting

$$\boldsymbol{b}_{ij} = \sqrt{\lambda_j} \boldsymbol{\phi}_j, \quad j = 1, \dots, r,$$

the matrix K_i can be decomposed as

$$K_i = \sum_{j=1}^{r} \boldsymbol{b}_{ij} \boldsymbol{b}_{ij}^{\mathrm{T}},$$

which is a particular case of (10). ■

We next specify the uncertainty models of $\boldsymbol{\zeta}_{\mathrm{a}}$ and $\boldsymbol{\zeta}_{\mathrm{f}}$. Let $\Delta_l \in \mathbb{R}^{m_l \times n^{\mathrm{f}}}$ $(l = 1, \dots, n^{\Delta})$ denote constant matrices, where m_l $(l = 1, \dots, n^{\Delta})$ are natural numbers. Define the sets $\mathcal{Z}_{\mathrm{a}} \subset \mathbb{R}^s$ and $\mathcal{Z}_{\mathrm{f}} \subset \mathbb{R}^{n^{\mathrm{f}}}$ by

$$\mathcal{Z}_{\mathrm{a}} = \{ \boldsymbol{\zeta}_{\mathrm{a}} \in \mathbb{R}^s | \ 1 \geq \|\boldsymbol{\zeta}_{\mathrm{a}}\|_\infty \}, \tag{13}$$

$$\mathcal{Z}_{\mathrm{f}} = \left\{ \boldsymbol{\zeta}_{\mathrm{f}} \in \mathbb{R}^{n^{\mathrm{f}}} \middle| \ 1 \geq \|\Delta_l \boldsymbol{\zeta}_{\mathrm{f}}\|_2, \ l = 1, \dots, n^{\Delta} \right\}. \tag{14}$$

Assumption 3.4. *The set* $\mathcal{Z}_{\mathrm{f}}$ *defined by* (14) *is bounded.*

Assumption 3.4 is satisfied if the matrix Δ defined by

$$\Delta = \begin{pmatrix} \Delta_1 \\ \dots \\ \Delta_{n^{\Delta}} \end{pmatrix}$$

satisfies $\mathrm{rank}(\Delta) = n^{\mathrm{f}}$. Examples of Δ_l ($l = 1, \dots, n^{\Delta}$) will be shown in Example 3.6 and section 6.1. It is obvious that $\mathcal{Z}_{\mathrm{a}}$ defined by (13) is bounded. The uncertain parameters $\boldsymbol{\zeta}_{\mathrm{a}}$ and $\boldsymbol{\zeta}_{\mathrm{f}}$ are assumed to be running through the *uncertainty sets* $\mathcal{Z}_{\mathrm{a}}$ and $\mathcal{Z}_{\mathrm{f}}$ as

$$\boldsymbol{\zeta}_{\mathrm{a}} \in \mathcal{Z}_{\mathrm{a}}, \quad \boldsymbol{\zeta}_{\mathrm{f}} \in \mathcal{Z}_{\mathrm{f}}. \tag{15}$$

By using (8), (9), (10), and (15), the uncertainty sets $\mathcal{K}$ and $\mathcal{F}$ of K and $\boldsymbol{f}$, respectively, are obtained as

$$\mathcal{K} = \left\{ K \in \mathbb{R}^{m \times n} \;\middle|\; K = \widetilde{K} + \sum_{i=1}^{s} a_i^0 \zeta_{\mathrm{a}i} \sum_{j=1}^{r} \boldsymbol{\beta}_{ij} \boldsymbol{b}_{ij}^{\mathrm{T}}, \ \boldsymbol{\zeta}_{\mathrm{a}} \in \mathcal{Z}_{\mathrm{a}} \right\}, \tag{16}$$

$$\mathcal{F} = \left\{ \boldsymbol{f} \in \mathbb{R}^{m} \;\middle|\; \boldsymbol{f} = \widetilde{\boldsymbol{f}} + F_0 \boldsymbol{\zeta}_{\mathrm{f}}, \ \boldsymbol{\zeta}_{\mathrm{f}} \in \mathcal{Z}_{\mathrm{f}} \right\}, \tag{17}$$

where $\mathcal{Z}_{\mathrm{a}}$ and $\mathcal{Z}_{\mathrm{f}}$ have been defined in (13) and (14). As a consequence, the uncertainty model of the linear equations (7) is written as

$$K\boldsymbol{u} = \boldsymbol{f}, \quad (K, \boldsymbol{f}) \in \mathcal{K} \times \mathcal{F}, \tag{18}$$

which is the explicit formulation of the system of uncertain linear equations dealt with in this paper.

Define $\mathcal{U} \subset \mathbb{R}^n$ by

$$\mathcal{U} = \{ \boldsymbol{u} \in \mathbb{R}^n | \ \exists (K, \boldsymbol{f}) \in \mathcal{K} \times \mathcal{F}, \ K\boldsymbol{u} = \boldsymbol{f} \}, \tag{19}$$

which is the set of all possible solutions to the system (18) of uncertain linear equations. In what follows, we assume the following condition:

Assumption 3.5. *The set* $\mathcal{U}$ *is bounded.*

A sufficient condition for Assumption 3.5 will be given for the uncertain analysis of structures in section 6.2.

Example 3.6 (interval uncertainty of $\boldsymbol{f}$). The interval uncertainty model of the right-hand side vector $\boldsymbol{f}$ of the uncertain linear equations (18) is conventionally used in the interval linear algebra [1, 19], in which each component of $\boldsymbol{f}$ is assumed to perturb independently. For the static analysis of structures under uncertainties, this uncertainty model corresponds to the interval uncertainty of the external load and has been used extensively [9, 17, 18]. We show in this example that the uncertainty model of $\boldsymbol{f}$ defined by (14) and (17) includes the interval uncertainty model as a particular case. Let

$$m_1 = \cdots = m_n = 1, \quad n^{\Delta} = m.$$

Let $\boldsymbol{e}_l^{(m)} \in \mathbb{R}^m$ denote the lth column vector of the identity matrix $I \in \mathbb{R}^{m \times m}$. Note that $(\boldsymbol{e}_l^{(m)})^{\mathrm{T}} \boldsymbol{\zeta}_{\mathrm{f}}$ corresponds to the lth component $\zeta_{\mathrm{f}l}$ of $\boldsymbol{\zeta}_{\mathrm{f}}$. By putting

$$\Delta_l = (\boldsymbol{e}_l^{(m)})^{\mathrm{T}}, \quad l = 1, \dots, m,$$

the set $\mathcal{Z}_{\mathrm{f}}$ defined by (14) is explicitly written as

$$\mathcal{Z}_{\mathrm{f}} = \{ \boldsymbol{\zeta}_{\mathrm{f}} \in \mathbb{R}^m |\ 1 \geq \|\boldsymbol{\zeta}_{\mathrm{f}}\|_\infty \}. \tag{20}$$

Letting

$$F_0 = \mathrm{Diag}(\boldsymbol{f}^0),$$

and by using (20), the set $\mathcal{F}$ defined by (17) is written as

$$\mathcal{F} = \left\{ \boldsymbol{f} \in \mathbb{R}^m \;\middle|\; f_j \in \left[\tilde{f}_j - f_j^0, \tilde{f}_j + f_j^0 \right],\ j = 1, \dots, m \right\}.$$

Consequently, the uncertainty model $\boldsymbol{f} \in \mathcal{F}$ can be explicitly written as

$$\tilde{f}_j - f_j^0 \leq f_j \leq \tilde{f}_j + f_j^0, \quad j = 1, \dots, m,$$

which corresponds to the conventional interval uncertainty model. ■

Example 3.7 (interval uncertainty of K). In a manner similar to Example 3.6, we investigate the interval uncertainty model of the coefficient matrix K, which is often used in the interval linear algebra [1]. The interval uncertainty model of K is written as

$$K \in \{ K = (K_{pq}) \in \mathbb{R}^{m \times n} \,|\, \underline{K}_{pq} \leq K_{pq} \leq \overline{K}_{pq}\ (p = 1, \dots, m;\ q = 1, \dots, n) \}$$

i.e. each element K_{pq} of K is included in the interval $[\underline{K}_{pq}, \overline{K}_{pq}]$, and no correlation is assumed between any two elements. Define $\mathcal{K}^{\mathrm{int}}$ by

$$\mathcal{K}^{\mathrm{int}} = \left\{ K \in \mathbb{R}^{m \times n} \;\middle|\; K = \tilde{K} + \sum_{p=1}^{m} \sum_{q=1}^{n} a_{pq}^0 \zeta_{\mathrm{a}pq} \boldsymbol{e}_p^{(m)} (\boldsymbol{e}_q^{(n)})^{\mathrm{T}},\ \boldsymbol{\zeta}_{\mathrm{a}} \in \mathcal{Z}_{\mathrm{a}} \right\} \tag{21}$$

where

$$\mathcal{Z}_{\mathrm{a}} = \{ \boldsymbol{\zeta}_{\mathrm{a}} \in \mathbb{R}^{m \times n} \,|\, 1 \geq |\zeta_{\mathrm{a}pq}|\ (p = 1, \dots, m;\ q = 1, \dots, n) \} \tag{22}$$

By re-arranging the matrix $\boldsymbol{\zeta}_{\mathrm{a}} \in \mathbb{R}^{m \times n}$ in (21) and (22) as the vector in $\mathbb{R}^{mn}$, it is easy to see that (21) and (22) can be embedded into the forms of (16) and (13), respectively, where $s = mn$ and $r = 1$. The set $\mathcal{K}^{\mathrm{int}}$ is equivalently rewritten as

$$\mathcal{K}^{\mathrm{int}} = \left\{ K = (K_{pq}) \in \mathbb{R}^{m \times n} \;\middle|\; \tilde{K}_{pq} - a_{pq}^0 \leq K_{pq} \leq \tilde{K}_{pq} + a_{pq}^0\ (p = 1, \dots, m;\ q = 1, \dots, n) \right\},$$

which is nothing but an interval uncertainty model of K. ■

Example 3.6 and Example 3.7 imply that our uncertainty model defined by (16) and (17) includes the uncertainty model used in the conventional interval analysis as a particular case. However, we do not pay much attention to the interval uncertainty model from the view-point of application in structural analysis. It should be emphasized that the uncertainty model in (16) and (17) is motivated by a special property of the stiffness matrix of structures and the natural uncertainty model of external forces. This topic will be discussed in section 6 in detail, in which we will see that the uncertainty model in (16) and (17) is natural and suitable for static analysis of structures under uncertainties.

4 Minimum bounding ellipsoid

In this section, we formulate a problem for finding the minimal ellipsoid which bounds the solution set of the system (18) of uncertain linear equations. More precisely, we take notice of the projection of the set of all possible solutions to (18) when K and $\boldsymbol{f}$ are running through the uncertainty sets introduced in (16) and (17). Then we formulate the optimization problem for finding the minimal ellipsoid including the projection that we are interested in.

4.1 Ellipsoids

An ellipsoid in the μ-dimensional space can be described as

$$\mathcal{E} = \left\{ \boldsymbol{z} \in \mathbb{R}^{\mu} \,\middle|\, \boldsymbol{z} = \widehat{\boldsymbol{z}} + D\boldsymbol{y},\ 1 \geq \|\boldsymbol{y}\|_2,\ \boldsymbol{y} \in \mathbb{R}^{m^{\mathrm{y}}} \right\}, \tag{23}$$

where $\widehat{\boldsymbol{z}} \in \mathbb{R}^{\mu}$ is the *center* of the ellipsoid. The matrix $D \in \mathbb{R}^{\mu \times m^{\mathrm{y}}}$ is called the *shape matrix*, and satisfies $\mathrm{rank}(D) = m^{\mathrm{y}} \leq \mu$. By putting $P = DD^{\mathrm{T}} \in \mathcal{S}^{\mu}$, (23) is alternatively represented as

$$\mathcal{E}(P, \widehat{\boldsymbol{z}}) = \left\{ \boldsymbol{z} \in \mathbb{R}^{\mu} \,\middle|\, \begin{pmatrix} P & (\boldsymbol{z} - \widehat{\boldsymbol{z}}) \\ (\boldsymbol{z} - \widehat{\boldsymbol{z}})^{\mathrm{T}} & 1 \end{pmatrix} \succeq O \right\}, \tag{24}$$

where $P \succeq O$. Note that $\mathrm{tr}(P)$ corresponds to the sum of squares of the semi-axes lengths. We adopt $\mathrm{tr}(P)$ as the measure of size of an ellipsoid (24).

For $\mu = 1$, we see that (24) is reduced to

$$\mathcal{E}(P, \widehat{z}) = \left[\widehat{z} - P^{1/2}, \widehat{z} + P^{1/2} \right],$$

i.e. the ellipsoid $\mathcal{E}(P, \widehat{z})$ coincides with the interval. This observation implies that finding a bounding ellipsoid includes finding a confidence interval as a particular case.

4.2 Problem formulation

Let $G \in \mathbb{R}^{n \times \mu}$ be a constant matrix. A typical choice of G in structural analysis will be given in section 6.4 explicitly. Define $\mathcal{U}_G \subseteq \mathbb{R}^{\mu}$ by

$$\mathcal{U}_G = \left\{ G^{\mathrm{T}} \boldsymbol{u} \,\middle|\, \boldsymbol{u} \in \mathcal{U} \right\}, \tag{25}$$

where $\mathcal{U}$ has been introduced in (19). Note that $\mathcal{U}_G$ is the set of all possible realizations of $G^{\mathrm{T}}\boldsymbol{u}$ when $\boldsymbol{u}$ is a solution to the uncertain linear equations (18).

An ellipsoid $\mathcal{E}(P, \widehat{\boldsymbol{u}})$ in the μ-dimensional space is called a *bounding ellipsoid* of $\mathcal{U}_G$ if it satisfies

$$\mathcal{U}_G \subseteq \mathcal{E}(P, \widehat{\boldsymbol{u}}). \tag{26}$$

The condition (26) means that the ellipsoid $\mathcal{E}(P, \widehat{\boldsymbol{u}})$ includes all possible realization of $G^{\mathrm{T}}\boldsymbol{u}$ when K and $\boldsymbol{f}$ are running through the uncertainty sets introduced in (16) and (17). Hence, a bounding ellipsoid is regarded as an outer, or a conservative, approximation of $\mathcal{U}_G$.

Obviously, the bounding ellipsoid is desired to be as 'tight' as possible. Hence, we attempt to compute the minimum ellipsoid, in the sense of the measure $\mathrm{tr}(P)$, containing $\mathcal{U}_G$. This problem is formulated in the variables $P \in \mathcal{S}^{\mu}$ and $\widehat{\boldsymbol{u}} \in \mathbb{R}^{\mu}$ as

$$\min \left\{ \mathrm{tr}(P) : \mathcal{U}_G \subseteq \mathcal{E}(P, \widehat{\boldsymbol{u}}) \right\}. \tag{27}$$

If K and $\boldsymbol{f}$ are an interval matrix and vector, respectively, then finding an exact interval of x_i is known to be an NP-hard problem [24]. Thus, it is very difficult to find a global optimal solution of Problem (27). In the following section, we consider a tractable approximation problem of (27) which provides an outer approximation of $\mathcal{U}_G$. By an *outer approximation* we mean a set including all possible realization of $G^{\mathrm{T}}\boldsymbol{u}$.

5 Semidefinite programming approximation

The purpose of this section is to construct an efficiently computable problem approximating Problem (27).

Define the matrix Ψ_j and the vector $\boldsymbol{q}_j$ by

$$\Psi_j = \left(\boldsymbol{\beta}_{1j}, \dots, \boldsymbol{\beta}_{sj}\right) \in \mathbb{R}^{m \times s}, \quad j = 1, \dots, r. \tag{28}$$

$$\boldsymbol{q}_j = (q_{ij}) \in \mathbb{R}^s, \quad j = 1, \dots, r. \tag{29}$$

Let $\mathcal{N}(F_0^{\mathrm{T}})$ denote the left nullspace of F_0. The basis of $\mathcal{N}(F_0^{\mathrm{T}})$ is denoted by $(F_0^{\mathrm{T}})^{\perp}$. Let $F_0^{\dagger} \in \mathbb{R}^{n^{\mathrm{f}} \times n}$ denote the Moore–Penrose pseudo-inverse of F_0.

Proposition 5.1. *$\boldsymbol{u} \in \mathcal{U}$ if and only if there exists a set of vectors $\boldsymbol{q}_j \in \mathbb{R}^s$ $(j = 1, \dots, r)$ satisfying*

$$\left[(F_0^{\mathrm{T}})^{\perp}\right]^{\mathrm{T}} \left[\widetilde{K}\boldsymbol{u} + \sum_{j=1}^{r} \Psi_j \boldsymbol{q}_j - \widetilde{\boldsymbol{f}}\right] = \boldsymbol{0}, \tag{30}$$

$$F_0^{\dagger}\left[\widetilde{K}\boldsymbol{u} + \sum_{j=1}^{r} \Psi_j \boldsymbol{q}_j - \widetilde{\boldsymbol{f}}\right] = \boldsymbol{\zeta}_{\mathrm{f}}, \tag{31}$$

$$q_{ij} = a_i^0 \zeta_{\mathrm{a}i} \boldsymbol{b}_{ij}^{\mathrm{T}} \boldsymbol{u}, \quad j = 1, \dots, r;\ i = 1, \dots, m, \tag{32}$$

$$\boldsymbol{\zeta}_{\mathrm{f}} \in \mathcal{Z}_{\mathrm{f}}, \tag{33}$$

$$\boldsymbol{\zeta}_{\mathrm{a}} \in \mathcal{Z}_{\mathrm{a}}. \tag{34}$$

Proof. Observe that the relation

$$\sum_{i=1}^{s} a_i^0 \zeta_{\mathrm{a}i} \boldsymbol{\beta}_{ij} \boldsymbol{b}_{ij}^{\mathrm{T}} \boldsymbol{u} = \left(\boldsymbol{\beta}_{1j}, \dots, \boldsymbol{\beta}_{sj}\right) \begin{pmatrix} a_1^0 \zeta_{\mathrm{a}1} \boldsymbol{b}_{1j}^{\mathrm{T}} \boldsymbol{u} \\ \dots \\ a_s^0 \zeta_{\mathrm{a}s} \boldsymbol{b}_{sj}^{\mathrm{T}} \boldsymbol{u} \end{pmatrix} = \Psi_j \boldsymbol{q}_j \tag{35}$$

holds, where $\boldsymbol{q}_j$ is defined by (29) and (32). From (16) and (35), we see that $K \in \mathcal{K}$ satisfies

$$K\boldsymbol{u} = \widetilde{K}\boldsymbol{u} + \sum_{j=1}^{r} \left(\sum_{i=1}^{s} a_i^0 \zeta_{\mathrm{a}i} \boldsymbol{\beta}_{ij} \boldsymbol{b}_{ij}^{\mathrm{T}} \boldsymbol{u}\right) = \widetilde{K}\boldsymbol{u} + \sum_{j=1}^{r} \Psi_j \boldsymbol{q}_j \tag{36}$$

with (32) and (34). Accordingly, by using the definition (17) of $\mathcal{F}$, the uncertain linear equations (18) are equivalently rewritten as

$$\widetilde{K}\boldsymbol{u} + \sum_{j=1}^{r} \Psi_j \boldsymbol{q}_j = \widetilde{\boldsymbol{f}} + F_0 \boldsymbol{\zeta}_{\mathrm{f}} \tag{37}$$

with (32), (33), and (34). For simplicity, we introduce $\boldsymbol{p} \in \mathbb{R}^m$ by

$$\boldsymbol{p} = \widetilde{K}\boldsymbol{u} + \sum_{j=1}^{r} \Psi_j \boldsymbol{q}_j - \widetilde{\boldsymbol{f}}. \tag{38}$$

Then the condition (37) is equivalently rewritten as

$$F_0 \boldsymbol{\zeta}_{\mathrm{f}} = \boldsymbol{p}. \tag{39}$$

Note that (39) is regarded as a system of linear equations in terms of $\boldsymbol{\zeta}_{\mathrm{f}}$. Observe that there exists $\boldsymbol{\zeta}_{\mathrm{f}}$ satisfying (33) and (39) if and only if (i) (39) has a solution $\boldsymbol{\zeta}_{\mathrm{f}}$ and (ii) the solution $\boldsymbol{\zeta}_{\mathrm{f}}$ satisfies (33). The condition (i) holds if and only if $\boldsymbol{p}$ is orthogonal to $\mathcal{N}(F_0^{\mathrm{T}})$ [25], i.e.

$$\left[(F_0^{\mathrm{T}})^{\perp}\right]^{\mathrm{T}} \boldsymbol{p} = \boldsymbol{0}. \tag{40}$$

Substitution (38) into (40) yields (30). Recall that F_0 has the full row rank from Assumption 3.1, which implies $\mathcal{N}(F_0) = \emptyset$. Hence, any solution to (39) is written as

$$\boldsymbol{\zeta}_{\mathrm{f}} = F_0^{\dagger} \boldsymbol{p}. \tag{41}$$

By substituting (38) into (41), we see that the condition (ii) is equivalent to (31) and (33). □

Remark 5.2. We investigate the formulations (30)–(34) in some degenerate cases. If $\mathrm{rank}(F_0) = n$, then we see that $F_0^{\dagger} = F_0^{-1}$ and $\mathcal{N}(F_0^{\mathrm{T}}) = \emptyset$. Hence, in this case, (30) is omitted from the set of conditions (30)–(34). On the other hand, if $\boldsymbol{f}$ is certain, which is modeled by letting $n^{\mathrm{f}} = 0$, then it is easy to see that $\boldsymbol{u} \in \mathcal{U}$ if and only if there exist $\boldsymbol{q}_j \in \mathbb{R}^s$ $(j = 1, \dots, r)$ satisfying

$$\widetilde{K}\boldsymbol{u} + \sum_{j=1}^{r} \Psi_j \boldsymbol{q}_j - \widetilde{\boldsymbol{f}} = \boldsymbol{0},$$

(32), and (34). These formulations are valid even if $a_i^0 = 0$ for some $i \in \{1, \dots, s\}$. ■

Proposition 5.1 implies that the uncertain equilibrium equations (18) are equivalent to the system (30)–(34). In comparison with the original system (18), it is of interest to note that the unknown parameters $\boldsymbol{\zeta}_{\mathrm{f}}$ and $\boldsymbol{\zeta}_{\mathrm{a}}$ appear only on the right-hand side of the equations (30)–(32). We next eliminate these unknown parameters $\boldsymbol{\zeta}_{\mathrm{f}}$ and $\boldsymbol{\zeta}_{\mathrm{a}}$ by using the quadratic embedding technique [8, 13]. The following proposition shows the key idea of the elimination of the unknown parameters:

Proposition 5.3. *Let $\boldsymbol{y} = (y_i) \in \mathbb{R}^r$ and $\boldsymbol{z} = (z_i) \in \mathbb{R}^r$. There exists $\zeta \in \mathbb{R}$ satisfying*

$$\boldsymbol{y} = \zeta \boldsymbol{z},$$
$$1 \geq |\zeta|$$

if and only if $\boldsymbol{y}$ and $\boldsymbol{z}$ satisfy

$$\|\boldsymbol{y}\|^2 \leq \|\boldsymbol{z}\|^2,$$
$$y_i z_{i+1} = y_{i+1} z_i, \quad i = 1, \dots, r-1.$$

It is easy to prove Proposition 5.3, and hence the proof is omitted.

Define the matrix Ψ and the vector $\boldsymbol{q}$ by

$$\Psi = \begin{pmatrix}\Psi_1, \ldots, \Psi_r\end{pmatrix} \in \mathbb{R}^{m\times sr},$$

$$\boldsymbol{q} = \begin{pmatrix}\boldsymbol{q}_1\\ \ldots\\ \boldsymbol{q}_r\end{pmatrix} \in \mathbb{R}^{sr},$$

in order to simplify the notation. Let $\boldsymbol{e}_i^{(s)} \in \mathbb{R}^s$ denote the ith column vector of the identity matrix $I \in \mathbb{R}^{s\times s}$. Define the vector $\widehat{\boldsymbol{e}}_i$ by

$$\widehat{\boldsymbol{e}}_i = \begin{pmatrix}\boldsymbol{e}_i^{(s)}\\ \ldots\\ \boldsymbol{e}_i^{(s)}\end{pmatrix} \in \mathbb{R}^{sr}.$$

Let $E_i^{j,k} \in \mathbb{R}^{sr\times sr}$ denote a matrix, the (j,k)-block of which is equal to $\mathrm{Diag}(\boldsymbol{e}_i^{(s)})$ and the remaining blocks are equal to $s\times s$ zero matrices. For example, if $r = 3$, then $E_i^{2,3}$ denotes

$$E_i^{2,3} = \begin{pmatrix}O & O & O\\ O & O & \mathrm{Diag}(\boldsymbol{e}_i^{(s)})\\ O & O & O\end{pmatrix} \in \mathbb{R}^{3s\times 3s}.$$

Let $\widehat{n}$ be

$$\widehat{n} = sr + n \tag{42}$$

for simplicity. Define the constant symmetric matrices $\Omega_0 \in \mathcal{S}^{\widehat{n}+1}$, $\Omega_{\mathrm{f}l} \in \mathcal{S}^{\widehat{n}+1}$, $\Omega_{\mathrm{a}i} \in \mathcal{S}^{\widehat{n}+1}$, and $\Theta_{ij} \in \mathcal{S}^{\widehat{n}+1}$ by

$$\Omega_0 = -\begin{pmatrix}\Psi^{\mathrm{T}}\\ \widetilde{K}^{\mathrm{T}}\\ -\widetilde{\boldsymbol{f}}^{\mathrm{T}}\end{pmatrix} (F_0^{\mathrm{T}})^{\perp}\left[(F_0^{\mathrm{T}})^{\perp}\right]^{\mathrm{T}} \begin{pmatrix}\Psi & \widetilde{K} & -\widetilde{\boldsymbol{f}}\end{pmatrix}, \tag{43}$$

$$\Omega_{\mathrm{f}l} = \mathrm{Diag}(\mathbf{0},\mathbf{0},1) - \begin{pmatrix}\Psi^{\mathrm{T}}\\ \widetilde{K}^{\mathrm{T}}\\ -\widetilde{\boldsymbol{f}}^{\mathrm{T}}\end{pmatrix} (F_0^{\dagger})^{\mathrm{T}}\Delta_l^{\mathrm{T}}\Delta_l F_0^{\dagger} \begin{pmatrix}\Psi & \widetilde{K} & -\widetilde{\boldsymbol{f}}\end{pmatrix}, \quad l = 1,\ldots,n^{\Delta}, \tag{44}$$

$$\Omega_{\mathrm{a}i} = (a_i^0)^2 \begin{pmatrix}O\\ \Psi\\ \mathbf{0}^{\mathrm{T}}\end{pmatrix} \mathrm{Diag}(\widehat{\boldsymbol{e}}_i) \begin{pmatrix}O & \Psi^{\mathrm{T}} & \mathbf{0}\end{pmatrix} - \mathrm{Diag}(\widehat{\boldsymbol{e}}_i,\mathbf{0},0), \quad i = 1,\ldots,s, \tag{45}$$

$$\Theta_{ij} = \begin{pmatrix}I\\ O\\ \mathbf{0}^{\mathrm{T}}\end{pmatrix} \left[E_i^{j,j+1} - E_i^{j+1,j}\right] \begin{pmatrix}O & \Psi^{\mathrm{T}} & \mathbf{0}\end{pmatrix} + \begin{pmatrix}O\\ \Psi\\ \mathbf{0}^{\mathrm{T}}\end{pmatrix} \left[E_i^{j,j+1} - E_i^{j+1,j}\right] \begin{pmatrix}I & O & \mathbf{0}\end{pmatrix},$$
$$j = 1,\ldots,r-1, \quad i = 1,\ldots,s. \tag{46}$$

In the following proposition, the uncertain parameters $\boldsymbol{\zeta}_{\mathrm{a}}$ and $\boldsymbol{\zeta}_{\mathrm{f}}$ in (30)–(34) are eliminated based on the idea of the quadratic embedding (Proposition 5.3):

Proposition 5.4. *There exist* $\boldsymbol{\zeta}_{\rm f}$ *and* $\boldsymbol{\zeta}_{\rm a}$ *satisfying* (30)–(34) *if and only if* $\boldsymbol{\xi} \in \mathbb{R}^{\hat{n}+1}$ *defined by*

$$\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{u} \\ 1 \end{pmatrix}$$

satisfies

$$\boldsymbol{\xi}^{\rm T}\Omega_0\boldsymbol{\xi} \geq 0, \tag{47}$$

$$\boldsymbol{\xi}^{\rm T}\Omega_{{\rm f}l}\boldsymbol{\xi} \geq 0, \quad l = 1,\dots,n^{\Delta}, \tag{48}$$

$$\boldsymbol{\xi}^{\rm T}\Omega_{{\rm a}i}\boldsymbol{\xi} \geq 0, \quad i = 1,\dots,s, \tag{49}$$

$$\boldsymbol{\xi}^{\rm T}\Theta_{ij}\boldsymbol{\xi} = 0, \quad j = 1,\dots,r-1;\ i = 1,\dots,s. \tag{50}$$

Proof. Observe that the equation (30) is equivalently rewritten as the quadratic inequality

$$\left\| \left[(F_0^{\rm T})^{\perp}\right]^{\rm T} \left[\widetilde{K}\boldsymbol{u} + \Psi\boldsymbol{q} - \widetilde{\boldsymbol{f}}\right] \right\|_2^2 \leq 0, \tag{51}$$

because the left-hand side of (51) is nonnegative. From the definition (43) of Ω_0, we see that (51) is equivalent to (47). For each $l = 1,\dots,n^{\Delta}$, observe that the condition

$$1 \geq \|\Delta_l \boldsymbol{\zeta}_{\rm f}\|_2$$

is satisfied if and only if

$$1 - \boldsymbol{\zeta}_{\rm f}^{\rm T}\Delta_l^{\rm T}\Delta_l\boldsymbol{\zeta}_{\rm f} \geq 0 \tag{52}$$

is satisfied. By substituting (31) into (52) and using the definition (44) of $\Omega_{{\rm f}j}$, we see that the condition (31) and (33) is equivalently rewritten as (48). For each $i = 1,\dots,s$, it follows from Proposition 5.3 that there exists a $\zeta_{{\rm a}i} \in \mathbb{R}$ satisfying

$$q_{ij} = a_i^0 \zeta_{{\rm a}i}\boldsymbol{b}_{ij}^{\rm T}\boldsymbol{u}, \quad 1 \geq |\zeta_{{\rm a}i}|, \quad j = 1,\dots,r$$

if and only if q_{ij} and $\boldsymbol{u}$ satisfy

$$(a_i^0)^2 \sum_{j=1}^{r} (\boldsymbol{b}_{ij}^{\rm T}\boldsymbol{u})^2 - \sum_{j=1}^{r} q_{ij}^2 \geq 0, \tag{53}$$

$$q_{ij}(\boldsymbol{b}_{i,j+1}^{\rm T}\boldsymbol{u}) - q_{i,j+1}(\boldsymbol{b}_i^{\rm T}\boldsymbol{u}) = 0, \quad j = 1,\dots,r-1. \tag{54}$$

From the definition (45) of $\Omega_{{\rm a}i}$, we see that the condition (53) is equivalent to

$$\boldsymbol{\xi}\Omega_{{\rm a}i}\boldsymbol{\xi} \geq 0.$$

Similarly, from the definition (46) of Θ_{ij}, the condition (54) is equivalently rewritten as

$$\boldsymbol{\xi}\Theta_{ij}\boldsymbol{\xi} = 0, \quad j = 1,\dots,r-1.$$

From this observation it follows that there exists $\boldsymbol{\zeta}_{\rm a}$ satisfying (32) and (34) if and only if $\boldsymbol{\xi}$ satisfies (49) and (50), which completes the proof. □

Remark 5.5. If $r = 1$ in (10), i.e. if $K_1, \ldots, K_s$ in (8) are rank-one matrices, then the quadratic equations (50) are omitted from the assertion of Proposition 5.4. ■

Proposition 5.4 implies that the uncertain linear equations (30)–(34) are equivalent to a finite number of quadratic inequalities (47)–(49) and quadratic equations (50). It should be emphasized that the unknown parameters $\boldsymbol{\zeta}_{\mathrm{f}}$ and $\boldsymbol{\zeta}_{\mathrm{a}}$ have been eliminated as a result of this quadratic embedding.

Let $w_0 \in \mathbb{R}$, $\boldsymbol{w}_{\mathrm{f}} = (w_{\mathrm{f}l}) \in \mathbb{R}^{n^{\Delta}}$, $\boldsymbol{w}_{\mathrm{a}} = (w_{\mathrm{a}i}) \in \mathbb{R}^{s}$, and $S = (S_{ij}) \in \mathbb{R}^{s \times (r-1)}$. Define the matrix-valued function $Y : \mathbb{R} \times \mathbb{R}^{n^{\Delta}} \times \mathbb{R}^{s} \times \mathbb{R}^{s \times (r-1)} \to \mathcal{S}^{\widehat{n}+1}$ by

$$Y(w_0, \boldsymbol{w}_{\mathrm{f}}, \boldsymbol{w}_{\mathrm{a}}, S) = w_0 \Omega_0 + \sum_{l=1}^{n^{\Delta}} w_{\mathrm{f}l} \Omega_{\mathrm{f}l} + \sum_{i=1}^{s} w_{\mathrm{a}i} \Omega_{\mathrm{a}i} + \sum_{i=1}^{s} \sum_{j=1}^{r-1} S_{ij} \Theta_{ij}. \tag{55}$$

Proposition 5.6. *The condition*

$$\mathcal{U}_G \subseteq \mathcal{E}(P, \widehat{\boldsymbol{u}}) \tag{56}$$

is satisfied if there exist w_0, $\boldsymbol{w}_{\mathrm{f}}$, $\boldsymbol{w}_{\mathrm{a}}$, *and* S *satisfying*

$$\begin{pmatrix} P & \begin{pmatrix} O & G^{\mathrm{T}} & -\widehat{\boldsymbol{u}} \end{pmatrix} \\ \begin{pmatrix} O \\ G \\ -\widehat{\boldsymbol{u}}^{\mathrm{T}} \end{pmatrix} & \mathrm{Diag}(\boldsymbol{0}, \boldsymbol{0}, 1) - Y(w_0, \boldsymbol{w}_{\mathrm{f}}, \boldsymbol{w}_{\mathrm{a}}, S) \end{pmatrix} \succeq O, \tag{57}$$

$$w_0 \geq 0, \quad \boldsymbol{w}_{\mathrm{f}} \geq \boldsymbol{0}, \quad \boldsymbol{w}_{\mathrm{a}} \geq \boldsymbol{0}. \tag{58}$$

Proof. Letting $\widehat{G} \in \mathbb{R}^{(\widehat{n}+1) \times \mu}$ be

$$\widehat{G}^{\mathrm{T}} = \begin{pmatrix} O & G^{\mathrm{T}} & -\widehat{\boldsymbol{u}} \end{pmatrix},$$

we see that

$$\widehat{G}^{\mathrm{T}} \boldsymbol{\xi} = G^{\mathrm{T}} \boldsymbol{u} - \widehat{\boldsymbol{u}}. \tag{59}$$

From (24) and (59), we see that the ellipsoid $\mathcal{E}(P, \widehat{\boldsymbol{u}})$ contains the point $G^{\mathrm{T}} \boldsymbol{u}$ if and only if the condition

$$\begin{pmatrix} P & \widehat{G}^{\mathrm{T}} \boldsymbol{\xi} \\ \boldsymbol{\xi}^{\mathrm{T}} \widehat{G} & 1 \end{pmatrix} \succeq O \tag{60}$$

is satisfied. By using Lemma 2.4, (60) is equivalently rewritten as

$$1 - (\widehat{G}^{\mathrm{T}} \boldsymbol{\xi}) P^{\dagger} (\widehat{G}^{\mathrm{T}} \boldsymbol{\xi}) \geq 0, \tag{61}$$

$$P \succeq O, \tag{62}$$

$$(I - P^{\dagger} P) \widehat{G}^{\mathrm{T}} \boldsymbol{\xi} = \boldsymbol{0}. \tag{63}$$

The ellipsoid $\mathcal{E}(P, \widehat{\boldsymbol{u}})$ that lies in the space $\{ G^{\mathrm{T}} \boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{R}^n \}$ satisfies the flatness condition (see Proposition 4.2 in [13])

$$(I - P^{\dagger} P) \widehat{G}^{\mathrm{T}} = O. \tag{64}$$

Note that (64) implies (63). Moreover, (62) should be satisfied from the definition (24) of the ellipsoid $\mathcal{E}(P, \widehat{\boldsymbol{u}})$. It follows from Proposition 5.4 that the ellipsoid $\mathcal{E}(P, \widehat{\boldsymbol{u}})$ satisfying (62) satisfies the condition

$$\mathcal{U}_G \subseteq \mathcal{E}(P, \widehat{\boldsymbol{u}}) \subseteq \{ G^{\mathrm{T}} \boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{R}^n \}$$

if and only if (64) holds, and (61) is satisfied for any $\boldsymbol{\xi}$ satisfying (47)–(50). Observe that (61) is equivalently rewritten as

$$\boldsymbol{\xi}^{\mathrm{T}} \left[\mathrm{Diag}(\mathbf{0}, \mathbf{0}, 1) - \widehat{G} P^{\dagger} \widehat{G}^{\mathrm{T}} \right] \boldsymbol{\xi} \geq 0.$$

Hence, by applying Corollary 2.3, (61) is satisfied for any $\boldsymbol{\xi}$ satisfying (47)–(50) if there exist w_0, $\boldsymbol{w}_{\mathrm{f}}$, $\boldsymbol{w}_{\mathrm{a}}$, and S satisfying

$$\left[\mathrm{Diag}(\mathbf{0}, \mathbf{0}, 1) - \widehat{G} P^{\dagger} \widehat{G}^{\mathrm{T}} \right] \succeq Y(w_0, \boldsymbol{w}_{\mathrm{f}}, \boldsymbol{w}_{\mathrm{a}}, S), \tag{65}$$

$$w_0 \geq 0, \quad \boldsymbol{w}_{\mathrm{f}} \geq \mathbf{0}, \quad \boldsymbol{w}_{\mathrm{a}} \geq \mathbf{0}. \tag{66}$$

where Y has been defined in (55). From Lemma 2.4 it follows that the conditions (62), (64), and (65) are equivalent to (57), which concludes the proof. □

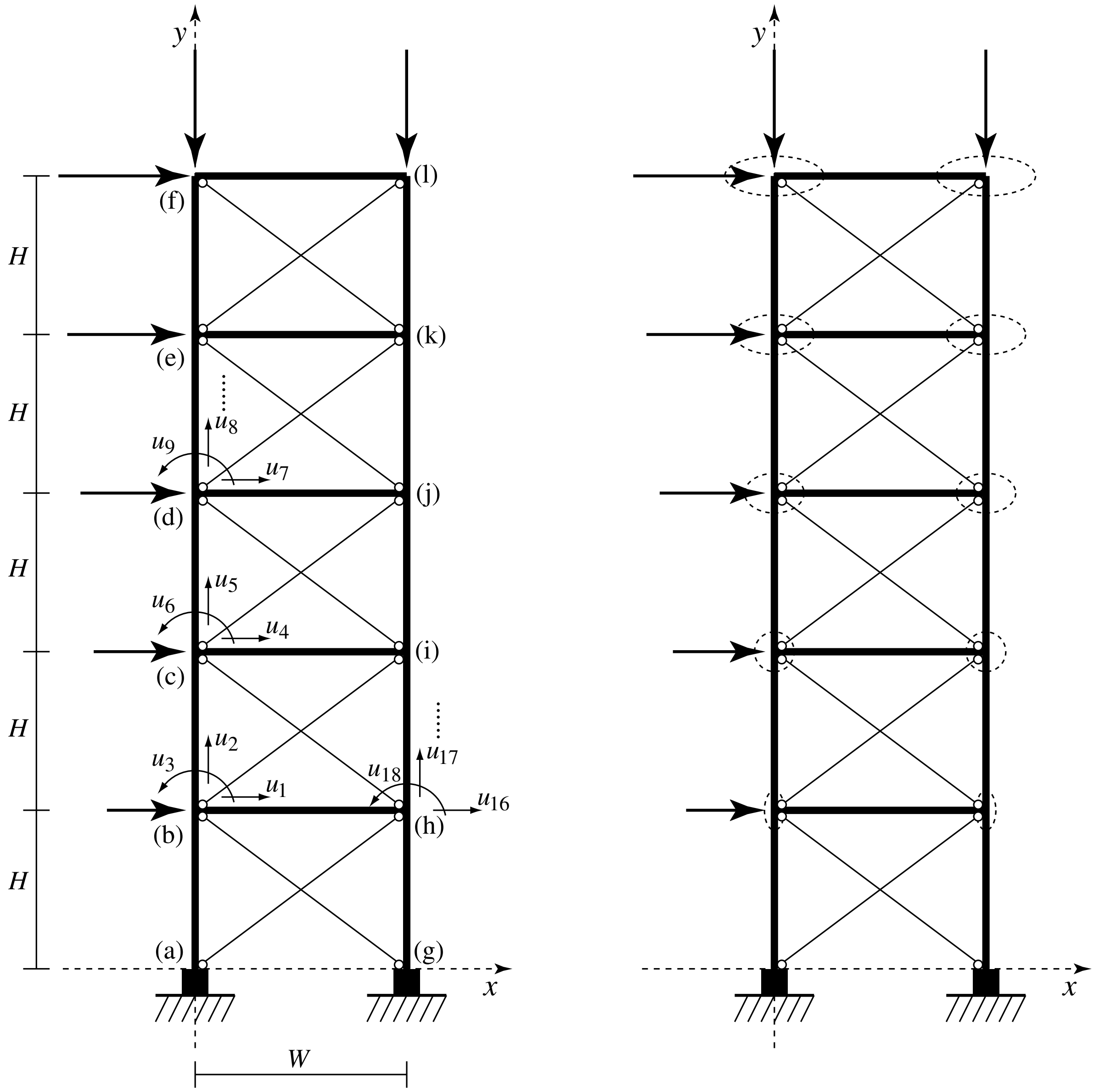
Proposition 5.6 implies that $\mathcal{E}(P, \widehat{\boldsymbol{u}})$ is guaranteed to be a confidence ellipsoid of $\mathcal{U}_G$ if (57) and (58) are satisfied, i.e. Proposition 5.6 presents a sufficient condition of the constraint condition of Problem (27). This naturally motivates us to solve the following problem in the variables $P \in \mathcal{S}^{\mu}$, $\widehat{\boldsymbol{u}} \in \mathbb{R}^{\mu}$, $w_0 \in \mathbb{R}$, $\boldsymbol{w}_{\mathrm{f}} \in \mathbb{R}^{n^{\Delta}}$, $\boldsymbol{w}_{\mathrm{a}} \in \mathbb{R}^{s}$, and $S \in \mathbb{R}^{s \times (r-1)}$:

$$\left. \begin{array}{ll} \min & \mathrm{tr}(P) \\ \mathrm{s.t.} & \begin{pmatrix} P & \begin{pmatrix} O & G^{\mathrm{T}} & -\widehat{\boldsymbol{u}} \end{pmatrix} \\ \begin{pmatrix} O \\ G \\ -\widehat{\boldsymbol{u}}^{\mathrm{T}} \end{pmatrix} & \mathrm{Diag}(\mathbf{0}, \mathbf{0}, 1) - Y(w_0, \boldsymbol{w}_{\mathrm{f}}, \boldsymbol{w}_{\mathrm{a}}, S) \end{pmatrix} \succeq O, \\ & w_0 \geq 0, \quad \boldsymbol{w}_{\mathrm{f}} \geq \mathbf{0}, \quad \boldsymbol{w}_{\mathrm{a}} \geq \mathbf{0}. \end{array} \right\} \tag{67}$$

Note that (67) is an SDP problem, because Y defined by (55) is a (matrix-valued) linear function of w_0, $\boldsymbol{w}_{\mathrm{f}}$, $\boldsymbol{w}_{\mathrm{a}}$, and S. Hence, (67) can be solved by using the primal-dual interior-point method efficiently. The optimal solution of (67) yields an outer ellipsoidal approximation of $\mathcal{U}_G$, that is optimal in the sense of the sufficient condition provided by Proposition 5.6.

6 Static response of braced frames

Consider a linearly elastic, rigidly-jointed frame with some pin-jointed braces in the two- or three-dimensional space. Small rotations and small strains are assumed. An example of a five-story braced frame is illustrated in Figure 1 (i). Each brace is modeled as a truss element. Let n^{b} and n^{t} denote the numbers of beam elements and truss elements. The total number of elements is denoted by $n^{\mathrm{m}} = n^{\mathrm{b}} + n^{\mathrm{t}}$. In this and the following section, let a_i ($i = 1, \dots, n^{\mathrm{m}}$) denote the elastic modulus of the ith member.



(i) model and coordinates

(ii) uncertainty of $\boldsymbol{f}$

Figure 1: 5-story framed structure.

Let $\boldsymbol{u} \in \mathbb{R}^{n^{\rm d}}$ and $\boldsymbol{f} \in \mathbb{R}^{n^{\rm d}}$ denote the vectors of nodal displacements and external forces, respectively, where $n^{\rm d}$ denotes the number of degrees of freedom of nodal displacements. The system of equilibrium equations is written as

$$K\boldsymbol{u} = \boldsymbol{f}, \tag{68}$$

where $K \in \mathcal{S}^{n^{\rm d}}$ denotes the stiffness matrix. In the case of the example of Figure 1 (i), we see that $n^{\rm d} = 30$, $n^{\rm b} = 15$, and $n^{\rm t} = 10$. We assume that both the stiffness matrix K and the external load $\boldsymbol{f}$ have uncertainties. Thus, the uncertainty analysis of structures fall into a particular case of the uncertain linear equations (18) with $m = n = n^{\rm d}$.

Suppose that the uncertainty of K is caused by the uncertainty of stiffness of each member. The locations of nodes are assumed to be certain. We describe the uncertainty of stiffness of each

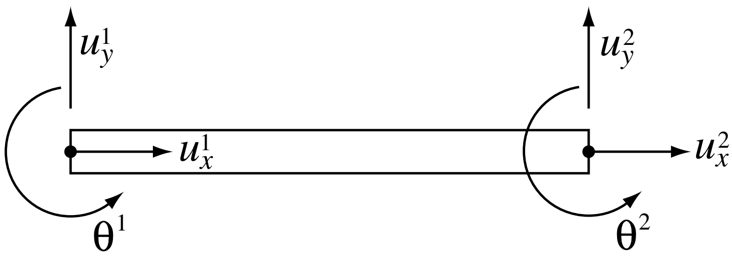


Figure 2: Local coordinate of the ith member.

member in terms of the uncertainty of the elastic modulus a_i. The nominal values of the elastic moduli and the external load, respectively, are denoted by $\tilde{\boldsymbol{a}} = (\tilde{a}_i) \in \mathbb{R}^{n^{\mathrm{b}}+n^{\mathrm{t}}}$ and $\tilde{\boldsymbol{f}} = (\tilde{f}_j) \in \mathbb{R}^{n^{\mathrm{d}}}$.

Consider the local coordinate system for the ith member illustrated in Figure 2. The element displacement vector $\boldsymbol{u}_i^{\mathrm{e}}$ with respect to the local coordinate is written as

$$\boldsymbol{u}_i^{\mathrm{e}} = \begin{pmatrix} u_x^1 & u_y^1 & \theta^1 & u_x^2 & u_y^2 & \theta^2 \end{pmatrix}^{\mathrm{T}}. \tag{69}$$

The global coordinate system which defines $\boldsymbol{u}$ and $\boldsymbol{f}$ are assigned as illustrated in Figure 1 (i). Let $T_i \in \mathbb{R}^{6 \times n^{\mathrm{d}}}$ denote the constant transformation matrix from the global coordinate system of the displacements to the local coordinate system, i.e.

$$\boldsymbol{u}_i^{\mathrm{e}} = T_i \boldsymbol{u}. \tag{70}$$

6.1 Uncertainty of nodal load

We define the global coordinate system for the nodal load vector $\boldsymbol{f} \in \mathbb{R}^{n^{\mathrm{d}}}$ in a manner similar to $\boldsymbol{u}$ illustrated in Figure 1 (i), where $n^{\mathrm{d}} = 30$. Suppose that the uncertain external forces are applied to all free nodes (b)–(f) and (h)–(l). Note that no external moments are applied.

Let $\boldsymbol{f}^{(l)} \in \mathbb{R}^2$ denotes the external forces applied at the lth free node, where the indices of nodes are labeled in the order of (b), …,(f), (h), …,(l). A component of $\boldsymbol{f}^{(l)}$ coincides with an appropriate component of $\boldsymbol{f}$. For example, $\boldsymbol{f}^{(1)} = (f_1, f_2)^{\mathrm{T}}$ denotes the external force vector applied at the node (b). It may be reasonable to assume that the uncertainties of external forces applied to two different nodes have no correlation, while $\boldsymbol{f}^{(l)}$ is included in one ellipsoid. This implies that the external load $\boldsymbol{f}$ is running through the ellipsoids depicted with the dashed lines in Figure 1 (ii). In this uncertainty model, two parameters are required to represent the uncertainty of $\boldsymbol{f}^{(l)}$ for each $l = 1, \dots, 10$. Hence, we see that the total number of uncertain parameters $\boldsymbol{\zeta}_{\mathrm{f}}$ in (14) and (17) is $n^{\mathrm{f}} = 20$.

Assume that, for each $l = 1, \dots, 10$, the two semi-axes of the ellipsoid including $\boldsymbol{f}^{(l)}$ are parallel to the x- and y-directions in Figure 1. For example, the ellipsoidal uncertainty model of $\boldsymbol{f}^{(1)}$ is written as

$$\boldsymbol{f}^{(1)} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} + \begin{pmatrix} f_1^0 \zeta_{\mathrm{f}1} \\ f_2^0 \zeta_{\mathrm{f}2} \end{pmatrix}, \quad 1 \geq \|\Delta_1 \boldsymbol{\zeta}_{\mathrm{f}}\|_2,$$

where $\Delta_1 \in \mathbb{R}^{2\times 20}$ is defined by

$$\Delta_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Here, f_1^0 and f_2^0 represent the 'magnitudes' of uncertainties of $\boldsymbol{f}^{(1)}$ in the x- and y-directions, respectively.

Accordingly, by defining F_0 and Δ_l as

$$F_0 = \mathrm{Diag}(\boldsymbol{f}^0), \tag{71}$$

$$\Delta_l = \begin{pmatrix} (\boldsymbol{e}_{2l-1}^{(20)})^{\mathrm{T}} \\ (\boldsymbol{e}_{2l}^{(20)})^{\mathrm{T}} \end{pmatrix}, \quad l = 1, \dots, 10 \tag{72}$$

with $\boldsymbol{f}^0 = (f_j^0) \in \mathbb{R}^{20}$ and $n^{\Delta} = 10$, we see that the uncertainty model of $\boldsymbol{f}$ is embedded into (14) and (17). In this model, the external force vector $\boldsymbol{f}^{(l)}$ of the lth node is running through the ellipsoid, and hence the components of $\boldsymbol{f}^{(l)}$ have some correlation with each other. Moreover, for $l \neq l'$, $\boldsymbol{f}^{(l)}$ and $\boldsymbol{f}^{(l')}$ are included in two different ellipsoids and have no correlation. Thus, we see that the uncertainty model of $\boldsymbol{f}$ introduced in section 3 is naturally required in the structural analysis considering uncertainties.

6.2 Uncertainty of member stiffnesses

Let $K_i \in \mathcal{S}_+^{n^{\mathrm{d}}}$ denote the member stiffness matrix divided by the elastic modulus of the ith member, the explicit form of which shall be given in section 6.3. The stiffness matrix K in (68) is written in terms of the elastic moduli as

$$K = \sum_{i=1}^{n^{\mathrm{m}}} a_i K_i. \tag{73}$$

Note that K_i $(i = 1, \dots, n^{\mathrm{m}})$ are positive semidefinite and assumed to be certain, because we have supposed that the locations of nodes are certain.

We next specify the uncertainty model of $\boldsymbol{a}$. Since a braced-frame is an assemblage of nodes connected by some independent members, the uncertainty of elastic modulus of a member may not affect those of the other members. Let $\widetilde{\boldsymbol{a}} = (\widetilde{a}_i) \in \mathbb{R}^{n^{\mathrm{m}}}$ denote the nominal value of $\boldsymbol{a}$, which satisfies $\widetilde{a}_i > 0$ $(i = 1, \dots, n^{\mathrm{m}})$. We describe the uncertainty of $\boldsymbol{a}$ in terms of the uncertain parameters $\boldsymbol{\zeta}_{\mathrm{a}} \in \mathcal{Z}_{\mathrm{a}}$, where $\mathcal{Z}_{\mathrm{a}}$ has been defined in (13) and we put $s = n^{\mathrm{b}} + n^{\mathrm{t}}$. Suppose that $\boldsymbol{a}$ depends on $\boldsymbol{\zeta}_{\mathrm{a}}$ affinely as

$$a_i = \widetilde{a}_i + a_i^0 \zeta_{\mathrm{a}i}, \quad i = 1, \dots, n^{\mathrm{m}}, \tag{74}$$

where $a_i^0 \in \mathbb{R}_+$ is a constant which represents the magnitude of uncertainty of a_i.

Define $\widetilde{K} \in \mathcal{S}^{n^{\mathrm{d}}}$ by

$$\widetilde{K} = \sum_{i=1}^{n^{\mathrm{m}}} \widetilde{a}_i K_i. \tag{75}$$

It follows from (73), (74), and (75) that the stiffness matrix K is written in terms of $\boldsymbol{\zeta}_{\rm a}$ as

$$K = \widetilde{K} + \sum_{i=1}^{n^{\rm m}} a_i^0 \zeta_{{\rm a}i} K_i.$$

Hence, from (16), the uncertainty set of K is obtained as

$$\mathcal{K} = \left\{ K \in \mathcal{S}^n \;\middle|\; K = \widetilde{K} + \sum_{i=1}^{n^{\rm m}} a_i^0 \zeta_{{\rm a}i} K_i,\ \boldsymbol{\zeta}_{\rm a} \in \mathcal{Z}_{\rm a} \right\}, \tag{76}$$

where $\mathcal{Z}_{\rm a}$ has been defined in (13).

Assume that $\boldsymbol{a}^0$ satisfies

$$\widetilde{a}_i > a_i^0, \quad i = 1, \dots, n^{\rm m}, \tag{77}$$

which implies

$$a_i > 0,\ i = 1, \dots, n^{\rm m}, \quad \forall \boldsymbol{\zeta}_{\rm a} \in \mathcal{Z}_{\rm a}. \tag{78}$$

From the mechanical point of view, it is natural to assume the condition (78), because a_i denotes the elastic modulus of the ith member. Recall that the member stiffness matrices $a_i K_i$ $(i = 1, \dots, n^{\rm m})$ are positive semidefinite. Hence, any $K \in \mathcal{K}$ is positive semidefinite if (77) is satisfied. Thus, we see that (77) is a sufficient condition for Assumption 3.5.

6.3 Decomposition of stiffness matrix

In this section, the uncertainty set (76) of K is embedded into the form of (16). We assume without loss of generality that the beam elements are labeled by the indices $i = 1, \dots, n^{\rm b}$, while the truss elements are labeled as $i = n^{\rm b} + 1, \dots, n^{\rm b} + n^{\rm t}$.

Firstly, we consider the contributions of the beam elements to the stiffness matrix. Let $\boldsymbol{c} = (c_i) \in \mathbb{R}^{n^{\rm b}}$ denote the vector of cross-sectional areas of beam elements. The vector of second-moments of areas, or moments of inertia, is denoted by $\boldsymbol{t} = (t_i) \in \mathbb{R}^{n^{\rm b}}$. Recall that a_i denotes the elastic modulus of the ith member. The member stiffness matrix $a_i K_i^{\rm b} \in \mathcal{S}^6$ with respect to the local coordinate in Figure 2 is written as

$$a_i K_i^{\rm b} = \frac{a_i}{l_i^3} \begin{pmatrix} c_i l_i^2 & 0 & 0 & -c_i l_i^2 & 0 & 0 \\ 0 & 12 t_i & 6 t_i l_i & 0 & -12 t_i & 6 t_i l_i \\ 0 & 6 t_i l_i & 4 t_i l_i^2 & 0 & -6 t_i l_i & 2 t_i l_i^2 \\ -c_i l_i^2 & 0 & 0 & c_i l_i^2 & 0 & 0 \\ 0 & -12 t_i & -6 t_i l_i & 0 & 12 t_i & -6 t_i l_i \\ 0 & 6 t_i l_i & 2 t_i l_i^2 & 0 & -6 t_i l_i & 4 t_i l_i^2 \end{pmatrix},$$

where l_i denote the undeformed length of the ith member. Observe that $K_i^{\rm b}$ can be decomposed as

$$K_i^{\rm b} = \sum_{j=1}^{3} \widehat{\boldsymbol{b}}_{ij} \widehat{\boldsymbol{b}}_{ij}^{\rm T}, \tag{79}$$

with

$$\widehat{\boldsymbol{b}}_{i1} = \sqrt{\frac{c_i}{l_i}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \widehat{\boldsymbol{b}}_{i2} = \sqrt{\frac{t_i}{l_i^3}} \begin{pmatrix} 0 \\ 2\sqrt{3} \\ \sqrt{3}l_i \\ 0 \\ -2\sqrt{3} \\ \sqrt{3}l_i \end{pmatrix}, \quad \widehat{\boldsymbol{b}}_{i3} = \sqrt{\frac{t_i}{l_i}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Note that $K_i^{\rm b}$ is positive semidefinite and $\mathrm{rank}(K_i^{\rm b}) = 3$. We can see that $\widehat{\boldsymbol{b}}_{i1}$, $\widehat{\boldsymbol{b}}_{i2}$, and $\widehat{\boldsymbol{b}}_{i3}$ are the eigenvectors of $K_i^{\rm b}$ corresponding to the non-zero eigenvalues of $K_i^{\rm b}$. Thus, the decomposition (79) is regarded as the Cholesky factorization.

Secondly, we consider the truss elements, which are labeled as $i = n^{\rm b} + 1, \dots, n^{\rm b} + n^{\rm t}$. Let a_i and c_i denote the elastic modulus and cross-sectional area of the ith member, respectively. Since the truss member can transmit the axial force only, the member stiffness matrix $a_i K_i^{\rm t} \in \mathcal{S}^6$ with respect to the local coordinate in Figure 2 is written as

$$a_i K_i^{\rm t} = \frac{a_i c_i}{l_i} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{80}$$

It is easy to see that $K_i^{\rm t}$ is positive semidefinite and $\mathrm{rank}(K_i^{\rm t}) = 1$. The Cholesky factorization yields the decomposition

$$K_i^{\rm t} = \widehat{\boldsymbol{b}}_{i1} \widehat{\boldsymbol{b}}_{i1}^{\rm T}.$$

Finally, we construct the stiffness matrix K of the whole structure by condensing the member stiffness matrices. Let $K(\boldsymbol{a}) \in \mathcal{S}^{n^{\rm d}}$ denote the stiffness matrix of the structure with respect to the global coordinate system, which is regarded as the (matrix-valued) function of $\boldsymbol{a}$. Recall that the transformation from the local coordinate to the global coordinate is given by (70). Hence, the stiffness matrix K in (73) is obtained as

$$K = \sum_{i=1}^{n^{\rm b}} a_i T_i^{\rm T} K_i^{\rm b} T_i + \sum_{i=n^{\rm b}+1}^{n^{\rm b}+n^{\rm t}} a_i T_i^{\rm T} K_i^{\rm t} T_i. \tag{81}$$

For each $i = 1, \dots, n^{\rm b} + n^{\rm t}$, define the vectors $\boldsymbol{b}_{ij} \in \mathbb{R}^{n^{\rm d}}$ $(j = 1, 2, 3)$ as

$$\boldsymbol{b}_{ij} = T_i^{\rm T} \widehat{\boldsymbol{b}}_{ij}, \quad j = 1, \dots, 3.$$

Then (81) is equivalently rewritten as

$$K = \sum_{i=1}^{n^{\rm b}} a_i \sum_{j=1}^{3} \boldsymbol{b}_{ij} \boldsymbol{b}_{ij}^{\rm T} + \sum_{i=n^{\rm b}+1}^{n^{\rm b}+n^{\rm t}} a_i \boldsymbol{b}_{i1} \boldsymbol{b}_{i1}^{\rm T}. \tag{82}$$

Thus, we see in (10) that $r = 3$ for beam elements and $r = 1$ for truss elements. From (76) and (82), the uncertainty set $\mathcal{K}$ of K is explicitly written as

$$\mathcal{K} = \left\{ K \in \mathcal{S}^n \;\middle|\; K = \widetilde{K} + \sum_{i=1}^{n^{\rm b}} a_i^0 \zeta_{{\rm a}i} \sum_{j=1}^{3} \boldsymbol{b}_{ij} \boldsymbol{b}_{ij}^{\rm T} + \sum_{i=n^{\rm b}+1}^{n^{\rm b}+n^{\rm t}} a_i^0 \zeta_{{\rm a}i} \boldsymbol{b}_{i1} \boldsymbol{b}_{i1}^{\rm T},\ \boldsymbol{\zeta}_{\rm a} \in \mathcal{Z}_{\rm a} \right\},$$

which is in the form of (16).

6.4 Bound for nodal displacement

Recall that the global coordinate system defining the displacement vector $\boldsymbol{u} \in \mathbb{R}^{n^{\rm d}}$ is assigned as shown in Figure 1 (i). In section 6.1, we have investigated an uncertainty model of $\boldsymbol{f}$ such that the external force applied at each node is included in the independently defined ellipsoid. In accordance with this uncertainty model, it is natural to seek for a bounding ellipsoid for each nodal displacement vector. Note that bounds for rotation angles of the nodes are not considered.

Let $\boldsymbol{u}^{(l)} \in \mathbb{R}^2$ denotes the nodal displacement vector of the lth free node. For example, $\boldsymbol{u}^{(1)} = (u_1, u_2)^{\rm T}$ denotes the displacement of the node (b). What we attempt to do is finding a minimum ellipsoid satisfying

$$\boldsymbol{u}^l \in \mathcal{E}(\boldsymbol{P}, \widehat{\boldsymbol{u}}), \quad \forall \boldsymbol{u} \in \mathcal{U},$$

where

$$\boldsymbol{P} \in \mathcal{S}^2, \quad \widehat{\boldsymbol{u}} \in \mathbb{R}^2, \quad \mu = 2.$$

In the case of the node (b), define $G \in \mathbb{R}^{n^{\rm d} \times 2}$ in (25) by

$$G = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \end{pmatrix}^{\rm T}.$$

Then the minimum bounding ellipsoid including all possible $\boldsymbol{u}^{(1)}$ is obtained by solving Problem (67).

Accordingly, we now obtain the details of explicit formulation of Problem (67) in the case of the braced frame in Figure 1. It should be emphasized that the procedures in section 6 are not restricted to frame structures. It is easy to apply the proposed method to a general finitely-discretized structure. Particularly, we can decompose element stiffness matrices by using the numerical Cholesky factorization, although we have shown the analytical decomposition in section 6.3. It is often that r in (10) is small for an element stiffness matrix.

7 Numerical experiments

The minimum bounding ellipsoids are computed for structures by solving Problem (67). We solve the SDP problem (67) by using SeDuMi Ver. 1.05 [26], which implements the primal-dual interior-point method for the linear programming problems over symmetric cones. Computation has been carried out on Pentium M (1.5 GHz with 1.0 GB memory) with MATLAB Ver. 6.5.1 [28].

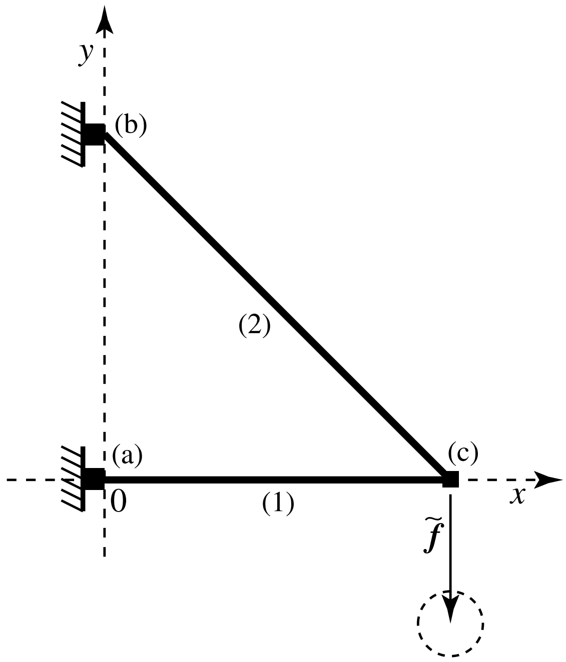


Figure 3: 2-bar frame.

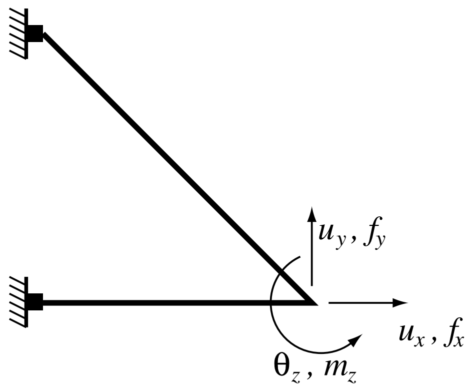


Figure 4: Coordinate system of the 2-bar frame.

7.1 2-bar plane frame

Consider a two-bar plane frame illustrated in Figure 3. The nodes (a) and (b) are fixed-supports located at $(x, y) = (0, 0)$ and $(0, 200.0)$ in cm, respectively, while the node (c) is free. The members (1) and (2) are modeled as the Euler–Bernoulli beam elements, i.e. $n^{\mathrm{d}} = 3$ and $n^{\mathrm{b}} = 2$. The lengths of members (1) and (2) are 200.0 cm and $200\sqrt{2}$ cm, respectively. We assume that the member stiffnesses and the external forces are uncertain. The uncertainty of stiffness of each member is represented in terms of the uncertainty of elastic modulus.

The cross-sectional areas and the moment of inertia are given as

$$c_i = 24.0 \text{ cm}^2, \quad t_i = 72.0 \text{ cm}^4, \quad i = 1, 2.$$

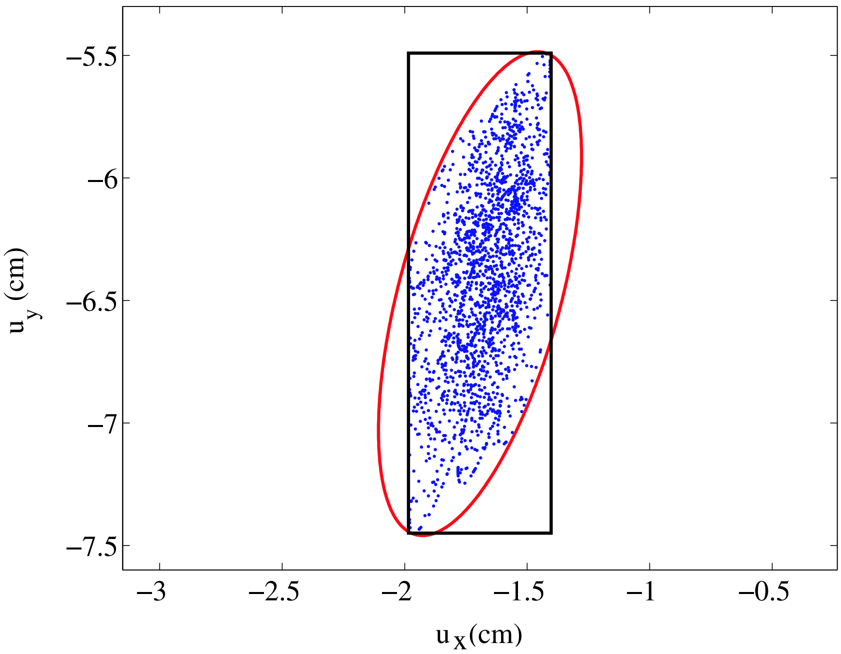


Figure 5: Bounding ellipsoid and bounding box for (u_x, u_y) at the node (c) of the 2-bar frame.

Let a_i denote the elastic modulus of the ith member. The nominal elastic moduli are given as

$$\widetilde{a}_i = 200.0 \text{ GPa}, \quad i = 1, 2.$$

The nominal external load is given as

$$\widetilde{\boldsymbol{f}} = \begin{pmatrix} f_x \\ f_y \\ m_z \end{pmatrix} = \begin{pmatrix} 0 \\ -4000.0 \\ 0 \end{pmatrix} \text{ kN},$$

where the definition of coordinate system is illustrated in Figure 4.

Consider the ellipsoidal uncertainty model for the external forces investigated in section 6.1. Note that the uncertain external moment is not considered. The uncertainties of $\boldsymbol{a}$ and $\boldsymbol{f}$ are defined as

$$a_i = \widetilde{a}_i + a_i^0 \zeta_{\mathrm{a}i}, \quad 1 \geq |\zeta_{\mathrm{a}i}|, \quad i = 1, 2, \tag{83}$$

$$\boldsymbol{f} = \widetilde{\boldsymbol{f}} + \begin{pmatrix} f_1^0 \zeta_{\mathrm{f}1} \\ f_2^0 \zeta_{\mathrm{f}2} \\ 0 \end{pmatrix}, \quad 1 \geq \left\| \begin{pmatrix} \zeta_{\mathrm{f}1} \\ \zeta_{\mathrm{f}2} \end{pmatrix} \right\|. \tag{84}$$

The uncertainty model (84) is embedded in the form of (17) by putting

$$F_0 = \begin{pmatrix} f_1^0 & 0 \\ 0 & f_1^0 \\ 0 & 0 \end{pmatrix}, \quad \Delta_l = I \in \mathbb{R}^{2\times 2}, \quad n^{\Delta} = 1,$$

where $n^{\mathrm{f}} = 2$. In (83) and (84), the coefficients of uncertainty are given as

$$a_i^0 = 20.0 \text{ GPa}, \quad i = 1, 2;$$
$$f_j^0 = 200.0 \text{ kN}, \quad j = 1, 2.$$

Table 1: Definition of the nominal load $\widetilde{\boldsymbol{f}}$.

node	force (kN)	
	x-direction	y-direction
(b)	150.0	0
(c)	200.0	0
(d)	250.0	0
(e)	300.0	0
(f)	350.0	-1000.0
(l)	0	-1000.0

Consequently, the external forces are running through the circle depicted with the dotted line in Figure 3. Note again that no external moments are applied even in uncertain cases.

Let $\boldsymbol{u} = (u_x, u_y, \theta_z) \in \mathbb{R}^3$ denote the displacement vector of the node (c), where the definition of coordinate system is shown in Figure 4. We randomly generate a number of $\boldsymbol{\zeta}_{\rm a}$ and $\boldsymbol{\zeta}_{\rm f}$ satisfying (83) and (84), and solve the corresponding equilibrium equations (7). The solutions (u_x, u_y) obtained are shown in Figure 5 as many points. Note that we do not aim at finding the bound for the rotation angle θ_z in this example.

The minimal confidence bounds on (u_x, u_y) are found by solving the SDP problem (67). The bounding ellipsoid for (u_x, u_y) is computed in a manner similar to section 6.4. The obtained parameters of a bounding ellipsoid are

$$P^* = \begin{pmatrix} 0.1716 & 0.2317 \\ 0.2317 & 0.9744 \end{pmatrix}, \quad \widehat{\boldsymbol{u}}^* = \begin{pmatrix} -1.6926 \\ -6.4722 \end{pmatrix}.$$

Figure 5 depicts the bounding ellipsoid $\mathcal{E}(P^*, \widehat{\boldsymbol{u}}^*)$ obtained.

Optimal bounding intervals for u_x and u_y are also computed by putting $\mu = 1$ in (67). The obtained optimal bounding box is also shown in Figure 5, whose parameters are

$$\begin{pmatrix} -1.9845 \\ -7.4496 \end{pmatrix} \le \begin{pmatrix} u_x \\ u_y \end{pmatrix} \le \begin{pmatrix} -1.4023 \\ -5.4907 \end{pmatrix}.$$

It is observed from Figure 5 that all generated (u_x, u_y) are included in the computed bounding ellipsoid and bounding box. We can also see that these bounds are sufficiently tight. Moreover, the bounding ellipsoid seems to represent the characteristics of distribution of (u_x, u_y) more clearly compared with the bounding box.

7.2 Braced plane frame

Consider the plane frame shown in Figure 1, which consists of 10 columns, 5 beams, and 10 braces. The columns and beams are modeled as the Euler–Bernoulli beam-column elements, while the braces are modeled as the truss elements, i.e. $n^{\rm b} = 15$ and $n^{\rm t} = 10$. The nodes (a) and (g) are the fixed-supports, and hence $n^{\rm d} = 30$. We set $W = 400.0$ cm and $H = 300.0$ cm. The equilibrium equations

Table 2: Definition of the uncertainty coefficients $\boldsymbol{f}^0$ of the external force.

node	f_j^0 (kN)	
	x-direction	y-direction
(b), (h)	16.0	32.0
(c), (i)	32.0	32.0
(d), (j)	48.0	32.0
(e), (k)	64.0	32.0
(f), (l)	80.0	32.0

with uncertainties are embedded into the standard form (18) of uncertain linear equations with

$$m = n = n^{\rm d}, \quad s = n^{\rm b} + n^{\rm t}.$$

The cross-sectional area and the moment inertia of each beam are 60.0 cm^2 and 500.0 cm^4, respectively. For each column, the cross-sectional area and the moment inertia are set as 40.0 cm^2 and 213.3 cm^4, respectively. The cross-sectional area of each brace is 8.0 cm^2.

Suppose that the stiffnesses of all members are uncertain, and the uncertain external forces are applied at all free nodes (b)–(f) and (h)–(l). The uncertainty of stiffness of each member is represented in terms of the uncertainty of elastic modulus. Let a_i denote the elastic modulus of the ith member. The nominal elastic moduli are given as

$$\tilde{a}_i = 200.0 \text{ GPa}, \quad i = 1, \dots, n^{\rm b} + n^{\rm t}$$

for both the beam and truss elements. In (74), the coefficients of uncertainty are given as

$$a_i^0 = 20.0 \text{ GPa}, \quad i = 1, \dots, n^{\rm b} + n^{\rm t}.$$

As the nominal external force $\tilde{\boldsymbol{f}}$, the nodal loads are applied to the nodes (b)–(f) and (l) as listed in Table 1. As discussed in section 6.1, suppose that uncertainties of external forces applied to two different nodes have no correlation, and each external nodal load is included in one ellipsoid. Consequently, the external load $\boldsymbol{f}$ is running through the 10 ellipsoids depicted with the dashed lines in Figure 1 (ii). The semi-axes of each ellipsoid are parallel to the x- and y-directions in Figure 1 (ii). The uncertainty model $\mathcal{F}$ in (17) is defined by (71) and (72), where $n^{\Delta} = 10$. The coefficients $\boldsymbol{f}^0$ of the uncertainty in (71) are listed in Table 2.

Based on the formulation investigated in section 6, we compute the minimal bounding ellipsoid of the nodal displacement of each free node by solving the SDP problem (67). Note that we have solved 10 SDP problems in total. Each SDP problem has 71 variables, 36 linear inequalities, and the constraint that the symmetric matrix in $\mathcal{S}^{88}$ should be positive semidefinite. The average and the standard deviation of CPU time, respectively, required for solving one SDP problem are 7.89 sec and 1.42 sec. The ellipsoids obtained are shown in Figure 6. We randomly generate a number of $\boldsymbol{\zeta}_{\rm a}$ and $\boldsymbol{\zeta}_{\rm f}$, and compute the corresponding displacements. The obtained displacements are also shown in Figure 6 as many dots.

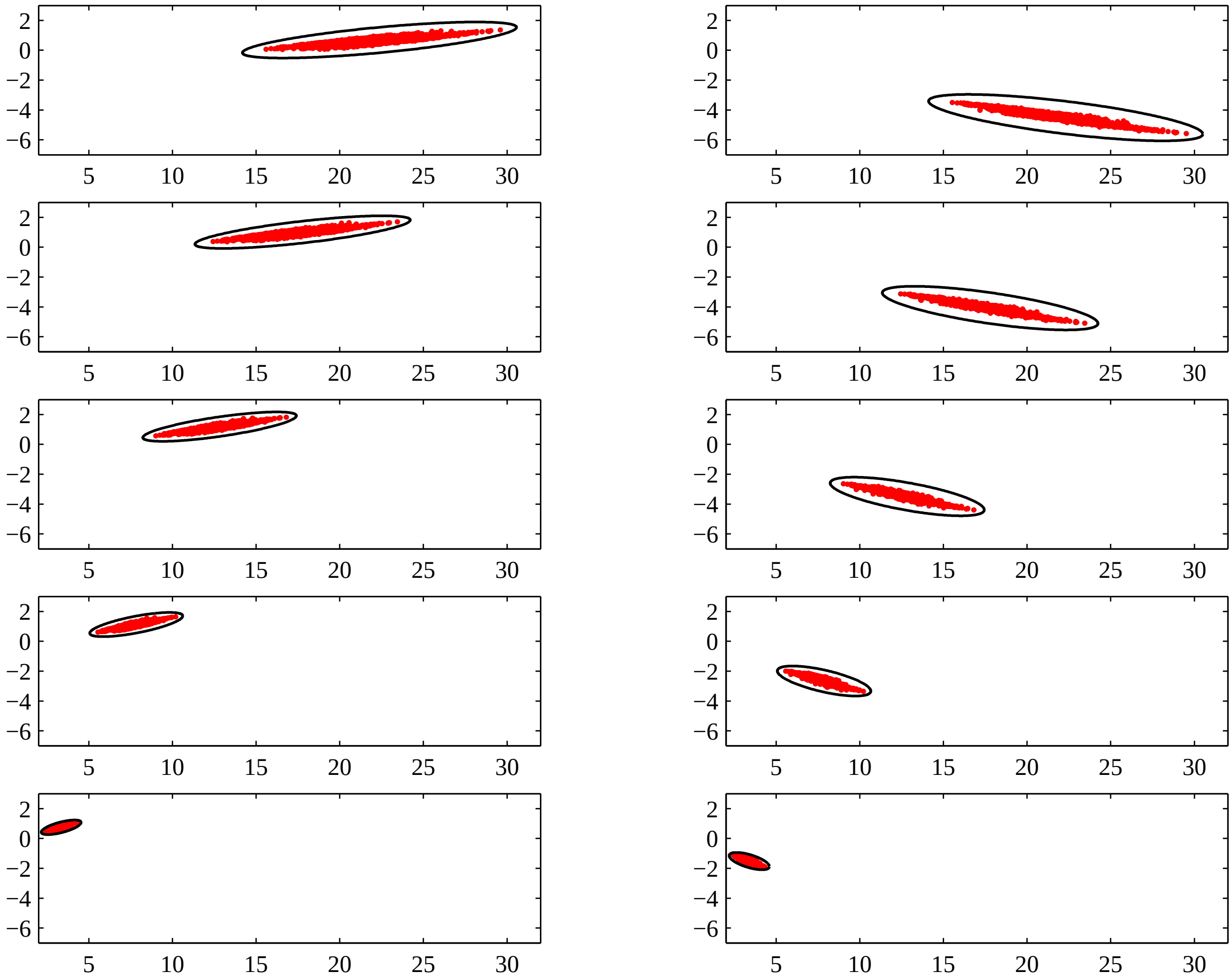


Figure 6: Bounding ellipsoids and the nodal displacements (in cm) for randomly generated $\zeta_{\rm a}$ and $\zeta_{\rm f}$.

Similarly, we compute the minimal bounding interval for each component of the displacements by solving the SDP problem (67) with $\mu = 1$. Note that bounds for the rotation angles of nodes are not computed, and hence we have solved 20 SDP problems in total. Each SDP problem has 68 variables, 36 linear inequalities, and the constraint that the symmetric matrix in $\mathcal{S}^{87}$ should be positive semidefinite. The average and the standard deviation of CPU time, respectively, required for solving one SDP problem are 6.06 sec and 0.71 sec. The obtained intervals (boxes) are illustrated in Figure 7. The displacements corresponding to randomly generated $\zeta_{\rm a}$ and $\zeta_{\rm f}$ are also depicted in Figure 7. It is observed from Figure 6 and Figure 7 that the bounds obtained are sufficiently tight.

8 Conclusions

In this paper, we have proposed a technique for computing confidential bounds on the solution set of the uncertain linear equations. We have proposed the general framework of uncertainty model for the data matrix and right-hand-side vector of the linear equations, which is applicable to the static analysis of structures affected by uncertainties. Both external forces and elastic moduli of structural

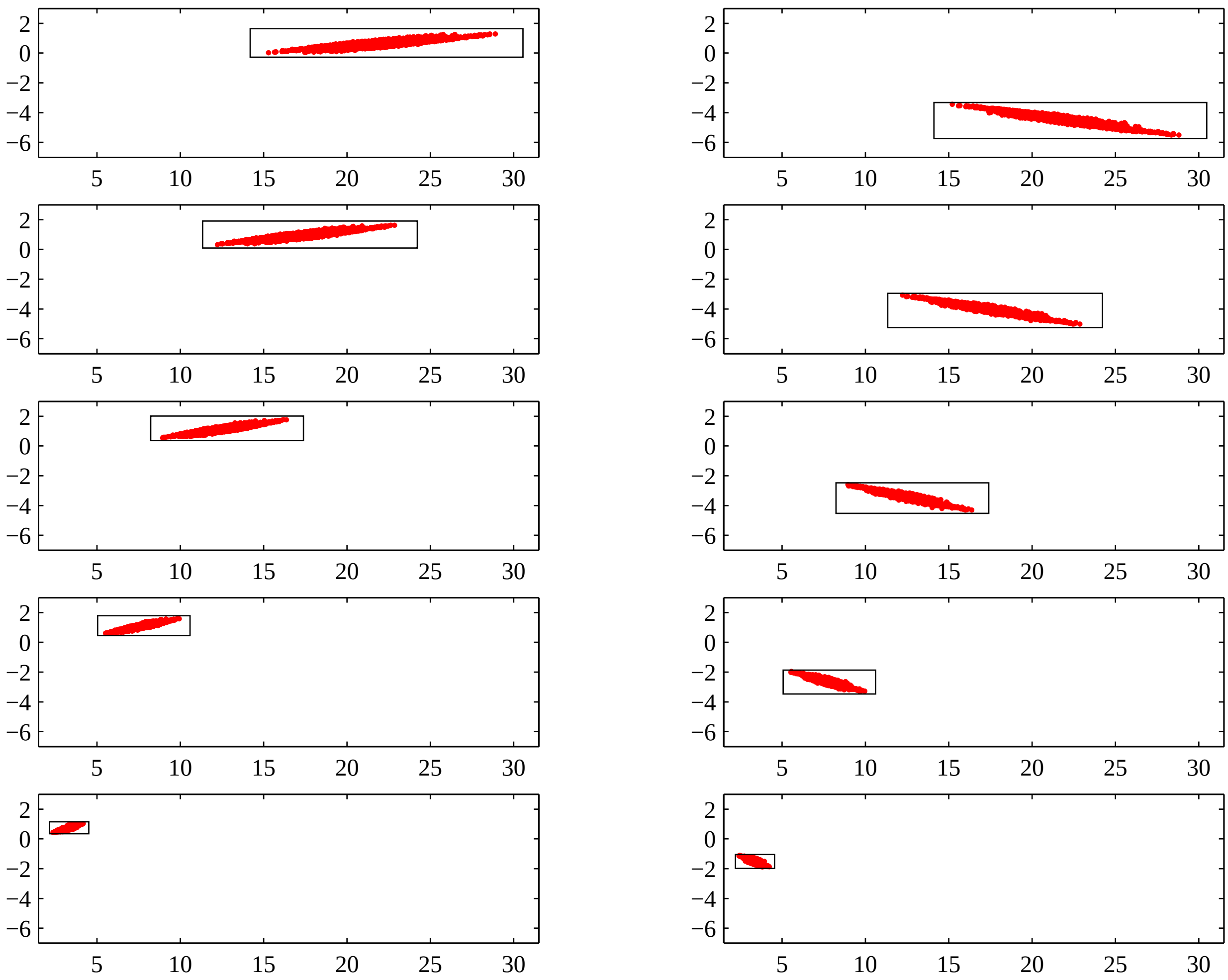


Figure 7: Bounding boxes and the nodal displacements (in cm) for randomly generated $\zeta_{\rm a}$ and $\zeta_{\rm f}$.

elements are assumed to be imprecisely known and bounded.

It has been shown that an ellipsoidal bound for all realizations of static response of a structure can be obtained efficiently by solving a semidefinite programming (SDP) problem, which is a convex optimization problem. By using the quadratic embedding of uncertainty parameters and the $\mathcal{S}$-lemma, we formulated an approximation problem for finding the minimal bounding ellipsoid as an SDP problem. Since finding a confidence ellipsoid is a generalization of finding a confidence interval, our problem dealt with in this paper includes the conventional interval analysis of structures (or the uncertain linear equations) as a particular case. Our method has polynomial-time complexity of problem size, while interval calculus approaches have in general exponential complexity.

It should be emphasized that the so-called convex model approaches have been developed based on the first-order perturbation with respect to the uncertain parameters, whereas the proposed method uses a semidefinite relaxation technique without the first-order approximation. Hence, the proposed method can be applied to cases in which the magnitudes of uncertainties are relatively large. Compared with confidence intervals, confidence ellipsoids may help intuitive understanding of characteristics of mechanical response, e.g. distribution of nodal displacements. Besides these advantages, SDP problems can be solved by using the well-developed software. Hence, our major task is limited to input the constant matrices and vectors defining the SDP problems, and no effort is required to develop any software for the special purpose.

In the numerical examples, the SDP problems presented have been solved by using the primal-dual interior-point method. It has been shown that confidence ellipsoids of nodal displacements of frame structures can be obtained effectively. We have also illustrated through numerical examples that the obtained ellipsoidal or interval bounds are sufficiently tight even for moderately large magnitudes of perturbations, although no theoretical result is to date available for sharpness of approximation.

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