MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

A new formulation of asset trading games in continuous time with essential forcing of variation exponent

Kei TAKEUCHI, Masayuki KUMON
and Akimichi TAKEMURA

METR 2007–46 August 2007

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.i.u-tokyo.ac.jp/mi/mi-e.htm

A new formulation of asset trading games in continuous time with essential forcing of variation exponent

Kei Takeuchi
Faculty of International Studies
Meiji Gakuin University
Masayuki Kumon
Risk Analysis Research Center
Institute of Statistical Mathematics
and
Akimichi Takemura
Graduate School of Information Science and Technology
University of Tokyo

August, 2007

Abstract

We introduce a new formulation of asset trading games in continuous time in the framework of the game-theoretic probability established by Shafer and Vovk [13]. In our formulation, the market moves continuously but an investor trades in discrete times, which can depend on the past path of the market. We prove that an investor can essentially force that the asset price path behaves with the variation exponent exactly equal to two. Our proof is based on embedding high-frequency discrete time games into the continuous time game and the use of the Bayesian strategy of Kumon, Takemura and Takeuchi [10] for discrete time coin-tossing games. We also clarify that the main growth part of the investor's capital processes is lucidly described by the information quantities, which are derived from the Kullback-Leibler information with respect to the empirical fluctuation of the asset price.

Keywords and phrases: Bayesian strategy, beta-binomial distribution, game-theoretic probability, Hölder exponent, Kullback-Leibler information, modulus of continuity, square root of dt effect.

1 Introduction

In this paper we present a new formulation of asset trading games in continuous time in the framework of the game-theoretic probability of Shafer and Vovk [13]. In the book by Shafer and Vovk, continuous-time games are formulated as limits of discrete time games by using techniques of nonstandard analysis. Although their approach is rigorously formulated in the framework of nonstandard analysis, it is desirable to give another formulation of continuous-time games in the game-theoretic probability within the conventional theory of analysis.

An asset trading game is a complete information game between an investor and the market. Following Chapter 9 of Shafer and Vovk [13] we denote these two players as Investor and Market. In our formulation Market moves continuously, but Investor moves in discrete times, depending on the past path of Market. The trading times of Investor need not be equally spaced. In this paper we mainly consider “limit order” strategy (rather than the “market order” strategy) of Investor. In the limit order strategy Investor trades a financial asset when the asset price or the increment of the asset price hits a certain level. We shall prove that by a high-frequency limit order type Bayesian strategy, Investor can essentially force the variation exponent of two in the price path of Market. The precise definition of essential forcing will be given in Section 2.

In an infinitely repeated series of fair betting games, a gambler cannot make gain for sure. This fact has been formulated and proved in the theory of martingales. But when the games are favorable to a gambler, for example if the results of the games are stochastically independent with positive expected value, to what extent can he exploit the situation and what would be a good strategy to be adopted? Several years after the advent of Shannon's celebrated work [14], this problem was first systematically studied by Kelly [7] in relation to the betting game interpretation of Shannon's mutual information quantity. In this spirit, betting games have been investigated among information theorists, which led to the notion of Cover's universal portfolios [2], [3]. One of the authors also wrote a note on it about forty years ago in Japanese, and presented the results in [16].

Recently Shafer and Vovk originated a new attractive field of game-theoretic probability and finance [13]. The most important point of their approach is that stochastic behavior of Market is not assumed a priori, but follows from the protocol of the game between Investor and Market. Shafer and Vovk established the general fact that in order to prevent Investor to make infinitely large gain, Market must behave as if she is stochastic and make the game fair in a stochastic sense. However the question remains what Inventor can make from Market's failure to do so. This issue was treated by Kumon and Takemura [8], where it is proved that when Market's moves are bounded, a simple single strategy forces the strong law of large numbers (SLLN) with the convergence rate of $O(\sqrt{\log n/n})$. Kumon, Takemura and Takeuchi [9] proved several versions of SLLN for the case that Market's moves are unbounded. For coin-tossing games, Kumon, Takemura and Takeuchi [10] considered a class of Bayesian strategies for Investor and established the important fact that if Market violates SLLN, then Investor can increase his capital exponentially fast and the the exponential growth rate is precisely described in terms of the

Kullback-Leibler information between the average of Market's moves when she violates SLLN and the average when she observes SLLN.

In this paper, we apply the results of [10] to asset trading games in continuous time. We consider implications of high-frequency limit order type Bayesian strategies and prove that Investor can make arbitrarily large gain if Market does not move jaggedly with the variation exponent exactly equal to two. In the literature on the mathematical finance, this phenomenon has been long recognized and understood as the fact that the fractional Brownian motion with the Hölder exponent $H \neq 1/2$ is not a semimartingale. See Rogers [12], Section 4.2 of Embrechts and Maejima [4] and Section 3 of Hobson [5]. Kunitomo [11] presented a similar result earlier. Vovk and Shafer [17] treated the $\sqrt{dt}$ effect using nonstandard analysis. The game-theoretic approach in the present paper and in [17] is advantageous, because no probabilistic model, such as the fractional Brownian motion, is imposed on the paths of Market. It can be an arbitrary continuous path in our formulation. Another fundamental strength of the game-theoretic approach is that we can give statements on an individual path of Market, whereas in measure-theoretic probability one can only make statements on measurable sets of the space of appropriate paths.

The organization of this paper is as follows. In Section 2 we formulate asset trading games and set up necessary notations and definitions. We also review the results on the Bayesian strategy of [10] for discrete time games embedded into the continuous-time game. We investigate the consequences of high-frequency Bayesian strategy in Section 3 and establish that Market is essentially forced to move with the variation exponent exactly equal to two. We end the paper with some concluding remarks in Section 4.

2 Asset trading games in continuous time

In this section we formulate asset trading games in continuous time and set up appropriate notations and definitions. We begin with an informal description of asset trading games in continuous time and its embedded discrete time game in Section 2.1. More precise definitions of the game and the move spaces of the players are presented in Section 2.2. In particular we will define the notion of essential forcing of an event by Investor. In Section 2.3 we review notions of the variation exponent and the Hölder exponent. In Section 2.4 we summarize results on Bayesian strategy for coin-tossing games in [10].

2.1 Formulation of asset trading games in continuous time

Suppose that there is a financial asset which is traded in a market in continuous time. Let $S(t)$ denote the price of the unit amount of the asset at time t. We assume that $S(t)$ is positive and a continuous function of t. We view that the price path $S(\cdot)$ is chosen by a player "Market". "Investor" enters the market at time $t = t_0 = 0$ (knowing the initial price $S(0)$) with the initial capital of $\mathcal{K}(0) = 1$ and he can buy or sell any amount of the asset at any time, provided that his capital always remains nonnegative. It is assumed that Investor can trade only at discrete time points $0 = t_0 < t_1 < t_2 < \cdots$, although he can decide the trading time t_i and the amount he trades at t_i, based on the path of $S(t)$ up to time t_i. Since $S(t)$ is continuous, when we say "up to time t_i", we do not need to distinguish whether Investor is allowed to use the value $S(t_i)$ or not. His repeated tradings up to time t_i also decide the amount M_i of the asset he holds for the interval $[t_i, t_{i+1})$. Again M_i can only depend on the path of $S(t)$ up to time t_i.

Let $\mathcal{K}(t)$ denote the capital of Investor at time t. It is written as

$$\begin{aligned} &\mathcal{K}(0) = 1, \\ &\mathcal{K}(t) = \mathcal{K}(t_i) + M_i(S(t) - S(t_i)) \quad \text{for} \quad t_i \le t < t_{i+1}. \end{aligned} \tag{1}$$

When M_i is negative, $\mathcal{K}(t)$ is the Investor's capital (expressed in cash), when he buys back $|M_i|$ units of the asset at the current price $S(t)$ at time t. As mentioned above Investor is required to keep $\mathcal{K}(t)$ nonnegative, whatever price path $S(\cdot)$ Market chooses. Also note that $\mathcal{K}(t)$ is continuous in t, since $S(t)$ is continuous in t.

By defining

$$\theta_i = \frac{M_i S(t_i)}{\mathcal{K}(t_i)},$$

we rewrite (1) as

$$\mathcal{K}(t) = \mathcal{K}(t_i)\left(1 + \theta_i \frac{S(t) - S(t_i)}{S(t_i)}\right) \quad \text{for} \quad t_i \le t < t_{i+1}$$

in terms of the return $(S(t) - S(t_i))/S(t_i)$ of the asset.

In this paper we mainly consider that Investor decides the trading times $t_1, t_2, \ldots$ by "limit order" strategy. Let $\delta_1, \delta_2 > 0$ be some constants and determine $t_1, t_2, \ldots$ as follows. After t_i are determined, let t_{i+1} be the first time after t_i when either

$$\frac{S(t_{i+1})}{S(t_i)} = 1 + \delta_1 \quad \text{or} \quad = \frac{1}{1 + \delta_2} \tag{2}$$

happens. In this scheme, although Investor enters the market at time $t_0 = 0$, he begins trading at time t_1. This process leads to a discrete time coin-tossing game embedded into the asset trading game as follows. Let

$$x_n = \frac{(1+\delta_2)S(t_{n+1}) - S(t_n)}{(\delta_1 + \delta_2 + \delta_1\delta_2)S(t_n)} = \begin{cases} 1, & \text{if } S(t_{n+1}) = S(t_n)(1+\delta_1), \\ 0, & \text{if } S(t_{n+1}) = S(t_n)/(1+\delta_2). \end{cases}$$

The risk neutral probability ρ of the coin-tossing game ([10],[15]) is given from

$$\rho\delta_1 - \frac{(1-\rho)\delta_2}{1+\delta_2} = 0,$$

which yields

$$\rho = \frac{\delta_2}{\delta_1 + \delta_2 + \delta_1\delta_2}.$$

Also write

$$\tilde{\mathcal{K}}_n = \mathcal{K}(t_{n+1}), \quad \nu_n = \frac{\delta_1 + \delta_2 + \delta_1\delta_2}{1+\delta_2}\theta_n.$$

Then we have the the following protocol of an embedded discrete time coin-tossing game.

EMBEDDED DISCRETE TIME COIN-TOSSING GAME
Protocol:

$\tilde{\mathcal{K}}_0 := 1$.
FOR $n = 1, 2, \dots$:
Investor announces $\nu_n \in \mathbb{R}$.
Market announces $x_n \in \{0, 1\}$.
$\tilde{\mathcal{K}}_n = \tilde{\mathcal{K}}_{n-1}(1 + \nu_n(x_n - \rho))$.
END FOR

This embedded discrete time game allows us to apply results on coin-tossing games to the asset trading game in continuous time. In particular we can apply the strong law of large numbers for coin-tossing games.

However it is to be noted that in the embedded game Market may decide to keep the variation of $S(t)$ small after t_n :

$$\frac{S(t_n)}{1+\delta_2} < S(t) < S(t_n)(1+\delta_1), \quad \forall t \geq t_n.$$

Then the embedded coin-tossing game is played only for n rounds and SLLN cannot be applied. Naturally we are tempted to make δ_1, δ_2 smaller, so that the total number of rounds increases, and we expect Investor's high-frequency tradings when δ_1 and δ_2 are small. But once δ_1, δ_2 are announced, Market can always make the variation even smaller. This suggests that we should formulate the asset trading game and the move spaces of the players more carefully.

2.2 Formal definition of asset trading games and the notion of essential forcing

Here we give definitions of asset trading games and the move spaces of the players. Also we define the notion of essential forcing of an event.

Market is required to choose a positive continuous function $S(\cdot)$ as her price path. Let

$$\Omega = C_{>0}[\mathbb{R}_+]$$

denote the set of positive continuous functions on $\mathbb{R}_+ = [0, \infty)$. This is the move space of Market, i.e. Market chooses an element $S(\cdot) \in \Omega$. We also call Ω the path space or the sample space. A subset E of Ω is called an *event*. A *variable* is a real-valued function $f : \Omega \to \mathbb{R}$ on the path space.

In order to define the move space of Investor, we need a game-theoretic definition of a stopping time (Section 5.3 of [13] and Section 1.1 of [6]) and a marked stopping time. A variable $\tau : \Omega \to [0, \infty) \cup \{\infty\}$ is called a *stopping time* if

$$\tau(S(\cdot)) < \infty \text{ and } S(u) = \tilde{S}(u),\ 0 \le u < \tau(S(\cdot)) \ \Rightarrow\ \tau(\tilde{S}(\cdot)) = \tau(S(\cdot)).$$

Investor's trading times are stopping times. When $\tau(S(\cdot)) = t < \infty$, we say that τ is realized at time point t. Investor also decides how many assets to hold at the time when τ is realized. A pair of variables

$$(\tau, m)\ :\ \Omega \to ([0, \infty) \cup \{\infty\}) \times \mathbb{R}$$

is a *marked stopping time* if τ is a stopping time and m depends only on the path up to the realized time of τ, i.e.

$$\tau(S(\cdot)) < \infty \text{ and } S(u) = \tilde{S}(u),\ 0 \le u < \tau(S(\cdot)) \ \Rightarrow\ m(\tilde{S}(\cdot)) = m(S(\cdot)).$$

We call m the mark associated with the stopping time τ. For definiteness, we define $m(S(\cdot)) = 0$ if $\tau(S(\cdot)) = \infty$.

A strategy $\mathcal{P}$ of Investor is a set of countably many marked stopping times

$$\mathcal{P} = \{(\tau_1, m_1), (\tau_2, m_2), \ldots\} \tag{3}$$

with the additional requirement that the stopping times are "discrete" in the following sense.

Definition 2.1. *A set of countably many stopping times $\{\tau_1, \tau_2, \ldots\}$ is discrete if for each $S(\cdot) \in \Omega$ there is no accumulation point of the set of realized stopping times.*

In the above definition, we are not requiring $\tau_1 \le \tau_2 \le \cdots$. For example, a strategy of Investor may consist of just two marked stopping times $\mathcal{P} = \{(\tau_1, m_1), (\tau_2, m_2)\}$, where τ_1 is the first time $S(t)$ hits a predetermined high value and τ_2 is the the first time $S(t)$ hits a predetermined low value. Then τ_1 may realize before τ_2 or vice versa. We use the notation $\tau_{(1)} \le \tau_{(2)} \le \cdots$ for the ordered realized stopping times.

By discreteness of the stopping times we require Investor to trade only finite number of times in every finite interval. The limit order type strategy in (2) clearly satisfies this requirement, because any continuous function on $[0, \infty)$ is uniformly continuous on the finite interval $[0, t]$. Under the above requirement, given a strategy $\mathcal{P}$ of Investor and a path $S(\cdot)$ of Market, the capital process $\mathcal{K}^{\mathcal{P}}(t) = \mathcal{K}^{\mathcal{P}}(t, S(\cdot))$ of Investor is defined as in (1) with $t_i = \tau_{(i)}(S(\cdot))$ and $M_i = m_{(i)}(S(\cdot))$, provided that the realized stopping times are all distinct. When realized time points of some stopping times coincide, for example when Investor employs nested strategies, we need to deal with obvious notational complications in adding up associated marks. But even when realized time points of some stopping times coincide, it is clear that discreteness requirement guarantees that the capital process $\mathcal{K}^{\mathcal{P}}(t)$ is written as a finite sum for each $t > 0$.

Furthermore we require that Investor observes his "collateral duty", i.e. starting with the initial capital of $\mathcal{K}^{\mathcal{P}}(0) = 1$, his strategy $\mathcal{P}$ has to satisfy

$$\mathcal{K}^{\mathcal{P}}(t, S(\cdot)) \geq 0, \quad \forall t > 0, \forall S(\cdot) \in \Omega.$$

In summary, the move space $\mathcal{F}_0 = \{\mathcal{P}\}$ of Investor is the set of strategies in (3) satisfying the discreteness of Definition 2.1 and the collateral duty.

We note that $\mathcal{F}_0$ is closed under finite static mixtures. Let $\mathcal{P}_j = \{(\tau_{ij}, m_{ij})\}_{i=1}^{\infty}$, $j = 1, 2$, be two strategies belonging to $\mathcal{F}_0$. For $0 < c_1, c_2 < 1$ with $c_1 + c_2 = 1$, Investor sets up two accounts with the initial capitals c_j, $j = 1, 2$. Then he employs $c_j\mathcal{P}_j = \{(\tau_{ij}, c_j m_{ij})\}_{i=1}^{\infty}$ to account j. This mixture is written as $c_1\mathcal{P}_2 + c_2\mathcal{P}_2 \in \mathcal{F}_0$ with the capital process $\mathcal{K}^{c_1\mathcal{P}_1+c_2\mathcal{P}_2}(t) = c_1\mathcal{K}^{\mathcal{P}_1}(t) + c_2\mathcal{K}^{\mathcal{P}_2}(t)$. By induction it is clear that $\mathcal{F}_0$ is closed with respect to any convex combination of finite number of strategies.

In the spirit of game-theoretic probability, we assume that Investor first announces his strategy $\mathcal{P}$ to Market and then Market decides her path $S(\cdot)$. Therefore the protocol of an asset trading game in continuous time is formulated as follows.

Asset Trading Game in Continuous Time
Protocol:

$\mathcal{K}(0) := 1$.

Investor announces $\mathcal{P} \in \mathcal{F}_0$.

Market announces $S(\cdot) \in \Omega$.

In game-theoretic probability, given some event $E \subset \Omega$, Investor is interpreted as the winner of the game if Market chooses a path $S(\cdot) \in E$ or else Investor's capital increases to infinity. In this case we say that Investor can force the event E. In order to prove forcing of an event E, as shown in Shafer and Vovk [13], it is essential to consider static mixture of countably many strategies of Investor. However in our formulation, countable mixing has a conceptual difficulty, because by a mixture of trading strategies with frequencies tending to infinity, we have to allow Investor to trade infinitely many times in a finite interval. Hence in this paper we use the following notion of "essential forcing" of an event E.

Definition 2.2. *In the asset trading game in continuous time, Investor can essentially force an event E, if for any $C > 0$ there exists a strategy $\mathcal{P}^C \in \mathcal{F}_0$ such that*

$$\sup_{0 \leq t < \infty} \mathcal{K}^{\mathcal{P}^C}(t, S(\cdot)) > C, \quad \forall S(\cdot) \in E^c.$$

We will discuss in Section 4 that essential forcing implies forcing in the sense of Shafer and Vovk [13] if we allow countable static mixtures. Therefore the notion of essential forcing is good enough for the development in the present paper. Also note that if Investor can essentially force a finite number of events $E_1, \dots, E_K$, he can essentially force the intersection $E_1 \cap \cdots \cap E_K$ by a finite mixture of appropriate strategies (cf. Lemma 3.2 of [13]).

We also give a somewhat stronger definition of essential forcing for a finite interval $[T_1, T_2] \subset [0, \infty)$.

Definition 2.3. *Investor can essentially force an event $E \subset \Omega$ in $[T_1, T_2]$ if for any $C > 0$ there exists a strategy $\mathcal{P}^C \in \mathcal{F}$ such that*

$$\sup_{T_1 \le t \le T_2} \mathcal{K}^{\mathcal{P}^C}(t, S(\cdot)) > C, \quad \forall S(\cdot) \in E^c.$$

2.3 Variation exponent and Hölder continuity

Here we summarize the notion of variation exponent and Hölder exponent (e.g. Section 4.1 of [4]). A continuous function f on the interval $[T_1, T_2]$ is called Hölder continuous (Lipschitz continuous) of order $\overline{H}$ on $[T_1, T_2]$ if for some $C > 0$

$$\frac{|f(y) - f(x)|}{|y - x|^{\overline{H}}} \le C, \qquad T_1 \le \forall x < \forall y \le T_2.$$

$\overline{H}$ is usually called the modulus of continuity or the Hölder exponent. In this paper we distinguish several closely related notions and we call $\overline{H}$ an *upper Hölder exponent.* In Section 3 we consider the set of functions

$$E_{\overline{H},C,T_1,T_2} = \Big\{ S \in \Omega \ \mid \ \frac{|\log S(y) - \log S(x)|}{|y - x|^{\overline{H}}} \le C, \ \ T_1 \le \forall x < \forall y \le T_2 \Big\}. \tag{4}$$

We also consider to bound the modulus of continuity (jaggedness of $S(\cdot)$) from below. Let $\mathbb{Q} \subset [0, \infty)$ be a given dense countable subset, such as the set of rational numbers. We define

$$\begin{aligned} \underline{E}_{\underline{H},C,T_1,T_2} = \{ S \in \Omega \ \mid \forall \epsilon > 0, \ \forall x \in [T_1, T_2 - \epsilon] \cap \mathbb{Q}, \exists y \in (x, T_2], \\ |\log S(y) - \log S(x)| \ge C\epsilon^{\underline{H}}, \ \ \frac{|\log S(y) - \log S(x)|}{|y - x|^{\underline{H}}} \ge C \}. \end{aligned} \tag{5}$$

This definition of bounding the jaggedness from below by a *lower Hölder exponent* $\underline{H}$ is convenient for our limit order type strategy.

Finally for $A > 0$ we write

$$E_{A,T_1,T_2} = \{ S \in \Omega \ \mid \ |\log S(y) - \log S(x)| \le A, \ T_1 \le \forall x < \forall y \le T_2 \}. \tag{6}$$

The modulus of continuity can also be understood from the viewpoint of total variation of a continuous function. Here we use the notion of strong p-variation from Section 11.6 of [13]. Let $\kappa : T_1 = t_0 < t_1 < \cdots < t_n = T_2$ be a division of the interval $[T_1, T_2]$. For $p \ge 1$ and a continuous function f define

$$\overline{\mathbf{var}}_f(p) = \sup_{\kappa} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p,$$

where sup is taken over all positive integers n and over all divisions κ. There exists a unique value $\overline{\mathrm{vex}}\, f \in [1,\infty]$ such that $\overline{\mathbf{var}}_f(p) < \infty$ for $p > \overline{\mathrm{vex}}\, f$ and $\overline{\mathbf{var}}_f(p) = \infty$ for for $p < \overline{\mathrm{vex}}\, f$. We call $\overline{\mathrm{vex}}\, f$ the *variation exponent* of f. It can be easily shown that $S \in E_{\overline{H},C,T_1,T_2}$ implies $\overline{\mathrm{vex}} \log S \le 1/\overline{H}$ and $S \in \underline{E}_{\underline{H},C,T_1,T_2}$ implies $\overline{\mathrm{vex}} \log S \ge 1/\underline{H}$. Note also that $\overline{\mathrm{vex}} \log S = \overline{\mathrm{vex}}\, S$ for $S \in \Omega$. From these relations we call $H = 1/\overline{\mathrm{vex}}\, S$ the *Hölder exponent* of S.

Results on the modulus of continuity of the paths of Brownian motion and fractional Brownian motion are summarized in Chapter IV of [1], Section 4.1 of [4] and Section 11.6 of [13].

2.4 Bayesian strategy for coin-tossing games

As discussed in Section 2.1 we mainly consider that Investor decides the trading times by the limit order type strategy in (2). In addition we consider that Investor specifies M_i by the Bayesian strategy in [10]. Here we briefly review the results of [10].

Suppose that Investor models Market's sequence of moves $x_1 x_2 \ldots$ $(x_i \in \{0,1\})$ in the embedded discrete time coin-tossing game of Section 2.1 by a probability distribution Q. Let $h_n = n\bar{x}_n = \sum_{i=1}^n x_i$ denote the number of heads and let $t_n = n - h_n$ denote the number of tails. The beta-binomial model is defined as

$$
\begin{aligned}
Q(x_1 \ldots x_n) &= \frac{1}{B(\alpha,\beta)} \int_0^1 p^{h_n+\alpha-1}(1-p)^{t_n+\beta-1} dp \\
&= \frac{(\Gamma(\alpha+h_n)/\Gamma(\alpha)) \times (\Gamma(\beta+t_n)/\Gamma(\beta))}{\Gamma(\alpha+\beta+n)/\Gamma(\alpha+\beta)},
\end{aligned}
$$

where $\alpha, \beta > 0$ are fixed and correspond to the prior numbers of heads and tails. We denote the conditional probability of $x_i = 1$ under Q given $x_1, \ldots, x_{i-1}$ by

$$
\hat{p}_i^Q = \hat{p}_i^Q(x_1, \ldots, x_{i-1}) = Q(x_i = 1 \mid x_1, \ldots, x_{i-1}).
$$

In this model

$$
\hat{p}_n^Q = \frac{B(\alpha+h_{n-1}+1, \beta+t_{n-1})}{B(\alpha+h_{n-1}, \beta+t_{n-1})} = \frac{\alpha+h_{n-1}}{\alpha+\beta+n-1},
$$

and the Investor's associated beta-binomial strategy is

$$
\nu_n^* = \frac{\hat{p}_n^Q - \rho}{\rho(1-\rho)}. \tag{7}
$$

The capital process $\tilde{\mathcal{K}}_n^*$ for this Bayesian strategy is explicitly written as

$$
\tilde{\mathcal{K}}_n^*(x_1 \ldots x_n) = \frac{Q(x_1 \ldots x_n)}{\rho^{h_n}(1-\rho)^{t_n}}. \tag{8}
$$

When both h_n and t_n are large, by using Stirling's formula

$$
\log \Gamma(x) = \left(x - \frac{1}{2}\right) \log x - x + \log \sqrt{2\pi} + O(x^{-1}),
$$

we can evaluate the log capital process $\log \tilde{\mathcal{K}}_n^*$ as

$$\log \tilde{\mathcal{K}}_n^* = nD\left(\frac{h_n}{n}\Big\|\rho\right) - \frac{1}{2}\log n + O(1),$$

where

$$D(p\|q) = p\log\frac{p}{q} + (1-p)\log\frac{1-p}{1-q}$$

denotes the Kullback-Leibler information between $0 < p < 1$ and $0 < q < 1$. This expression together with the Taylor expansion

$$D(\rho+\delta\|\rho) = \frac{\delta^2}{2\rho(1-\rho)} + O(\delta^3)$$

allows us to analyze the behavior of the capital process for a high-frequency Bayesian strategy of Investor in the next section.

3 Essential forcing of variation exponent in the asset trading game

Consider the asset trading game in continuous time in Section 2.2 and the events $E_{\overline{H},C,T_1,T_2}$ in (4), $E_{\underline{H},C,T_1,T_2}$ in (5) and E_{A,T_1,T_2} in (6). In this section we prove the following main result of this paper.

Theorem 3.1. *For every $\overline{H} > 0.5, A > 0, C > 0, 0 \le T_1 < T_2 \le T$, Investor can essentially force*

$$E_{\overline{H},C,T_1,T_2} \Rightarrow E_{A,T_1,T_2}.$$

For every $\underline{H} < 0.5, A > 0, C > 0, 0 \le T_1 < T_2 \le T$, Investor can essentially force

$$\underline{E}_{\underline{H},C,T_1,T_2} \Rightarrow E_{A,T_1,T_2}.$$

Here "$E_1 \Rightarrow E_2$" stands for $E_1^c \cup E_2$ (Section 4.1 of [13]) for two events $E_1, E_2 \subset \Omega$. Also from the proof of the theorem below, it will be clear that Investor can essentially force these events in the interval $[T_1, T_2]$. This theorem says that, within arbitrarily small constants, Market's path is essentially forced to have the variation exponent of two, unless she stays constant.

We give a proof of this theorem after some preliminary investigations of the limit order type strategy in Section 2.1 combined with the Bayesian strategy in Section 2.4 for the embedded discrete time game. Our proof is based on the limit order type strategy with sufficiently small $\delta = \delta_1 = \delta_2$ in (2). After the proof we also investigate the behavior of Investor's capital processes for the cases where δ_1 and δ_2 decrease with different rates.

First note that it suffices to consider the case $T_1 = 0$, because we can think that Investor enters the game at time $t = T_1$ instead of $t = 0$ and that he uses the the strategy described below from T_1 on. Write simply $T = T_2$, and thus we only consider the case $[T_1, T_2] = [0, T]$.

We take the limit order type strategy in Section 2.1. Write $\delta = (\delta_1, \delta_2)$, where $\delta_1, \delta_2 > 0$. Let $t_0 = 0 < t_1 < t_2 < \cdots$ be the sequence of Investor's trading time points determined by (2). Then we have the embedded discrete time coin-tossing game and the associated M_n's are determined by the Bayesian strategy in Section 2.4 in the form of ν_n^* in (7). The parameters $\alpha, \beta > 0$ for the Bayesian strategy are fixed throughout the rest of this section. It is clear that the resulting strategy $\mathcal{P} = \mathcal{P}^{\delta_1,\delta_2,\alpha,\beta}$ satisfies the collateral duty $\mathcal{K}^{\mathcal{P}}(t, S(\cdot)) \geq 0$, $\forall t > 0$, $\forall S(\cdot) \in \Omega$. We use the notation

$$\eta_i = \log(1 + \delta_i), \quad \delta_i = e^{\eta_i} - 1, \quad i = 1, 2,$$

and $\eta = (\eta_1, \eta_2)$. Define $n^* = n(T, \delta, S(\cdot))$ by $t_{n^*} < T \leq t_{n^*+1}$. Note that

$$n^*(T, \delta, S(\cdot)) \geq \frac{A}{\max(\eta_1, \eta_2)}$$

for every $S(\cdot) \in E^c_{A,0,T}$. Therefore n^* can be made arbitrarily large, uniformly in $S(\cdot) \in E^c_{A,0,T}$, by taking δ_1, δ_2 sufficiently small.

Now $\mathcal{K}(T) = \mathcal{K}^{\mathcal{P}^{\delta_1,\delta_2,\alpha,\beta}}(T, S(\cdot))$ is written as

$$\mathcal{K}(T) = \tilde{\mathcal{K}}^*_{n^*}\left(1 + \theta^*_n \frac{S(T) - S(t_{n^*})}{S(t_{n^*})}\right), \quad \theta^*_n = \frac{1 + \delta_2}{\delta_1 + \delta_2 + \delta_1\delta_2}\nu^*_n.$$

Since $\left|\frac{S(T)-S(t_{n^*})}{S(t_{n^*})}\right| < \max(\delta_1, \delta_2)$, we have

$$\log \mathcal{K}(T) = \log \tilde{\mathcal{K}}^*_{n^*} + O(1) = n^* D\left(\frac{h_{n^*}}{n^*}\Big\|\rho\right) - \frac{1}{2}\log n^* + O(1). \tag{9}$$

Define

$$TV(\eta, T) = \sum_{i=1}^{n^*} |\log S(t_i) - \log S(t_{i-1})| = h_{n^*}\eta_1 + t_{n^*}\eta_2, \tag{10}$$

$$L(\eta, T) = \log S(t_{n^*}) - \log S(0) = h_{n^*}\eta_1 - t_{n^*}\eta_2, \tag{11}$$

$$\sigma(\eta, T) = \frac{L(\eta, T)}{TV(\eta, T)} = \frac{h_{n^*}\eta_1 - t_{n^*}\eta_2}{h_{n^*}\eta_1 + t_{n^*}\eta_2}.$$

We call $TV(\eta, T)$ the total η-variation of $\log S(t)$ in the interval $[0, T]$. We also write

$$L(T) = \log S(T) - \log S(0) = L(\eta, T) + O(\max(\eta_1, \eta_2)).$$

Then we can express (9) as

$$\log \mathcal{K}(T) = n^* D\left(p(\eta, T)\|\rho\right) - \frac{1}{2}\log n^* + O(1), \tag{12}$$

where

$$p(\eta, T) = \frac{h_{n^*}}{n^*} = \frac{\eta_2(1+\sigma(\eta,T))}{\eta_1(1-\sigma(\eta,T)) + \eta_2(1+\sigma(\eta,T))}.$$

Also from (10) and (11), n^* can be written as

$$n^* = \left(\frac{\eta_1 + \eta_2 - \sigma(\eta,T)(\eta_1 - \eta_2)}{2\eta_1\eta_2}\right) TV(\eta, T). \tag{13}$$

Let $\eta_{1k} = a_1^{-k}$, $\eta_{2k} = a_2^{-k}$ for $a_1, a_2 > 1$, $k = 1, 2, \ldots$, and let $\log(1+\delta_{1k}) = \eta_{1k}$, $\log(1+\delta_{2k}) = \eta_{2k}$. We consider a sequence of the discretized games with $\delta_k = (\delta_{1k}, \delta_{2k})$ and let $\mathcal{K}_k(T)$ be the Investor's capital at $t = T$ for the beta-binomial strategy in each game. We denote the values of n^*, ρ by n_k^*, ρ_k corresponding to $\eta_k = (\eta_{1k}, \eta_{2k})$.

We are now ready to give a proof of Theorem 3.1.

Proof of Theorem 3.1. Take $a_1 = a_2 = a > 1$ and write $\eta_k = a^{-k}$, $\log(1 + \delta_k) = \eta_k$. Then we have

$$n_k^* = \frac{TV(\eta_k, T)}{\eta_k}, \quad p(\eta_k, T) = \frac{1 + \sigma(\eta_k, T)}{2}, \quad \rho_k = \frac{1}{2 + \delta_k}.$$

Note that $\rho_k \to 1/2$ as $k \to \infty$. More precisely

$$\rho_k = \frac{1}{2} - \frac{\delta_k}{4} + o(\delta_k).$$

Consider $n_k^* D(p(\eta_k, T)\|\rho_k)$ in (12). Since n_k^* can be made arbitrarily large uniformly in $S(\cdot) \in E_{A,0,T}^c$, we only need to consider k and $S(\cdot) \in E_{A,0,T}^c$ such that $p(\eta_k, T)$ is close to 1/2. Now use the Taylor expansion

$$D\left(\frac{1+d_1}{2}\middle\|\frac{1+d_2}{2}\right) = \frac{1}{2}(d_1 - d_2)^2 + o(|d_1 - d_2|^2),$$

with $d_1 = \sigma(\eta_k, T)$, $d_2 = -\delta_k/2$. Hence noting $\delta_k = e^{\eta_k} - 1 = a^{-k} + O(a^{-2k})$, we can evaluate $n_k^* D\left(p(\eta_k, T)\|\rho_k\right)$ as

$$\begin{aligned} n_k^* D\left(p(\eta_k, T)\|\rho_k\right) &\simeq a^k TV(\eta_k, T) \times \frac{1}{2}\left(\frac{L(T)}{TV(\eta_k, T)} + \frac{1}{2a^k}\right)^2 \\ &= \frac{1}{2}\left[\frac{a^k}{TV(\eta_k, T)} L^2(T) + L(T) + \frac{1}{4}\frac{TV(\eta_k, T)}{a^k}\right]. \end{aligned} \tag{14}$$

Let $\overline{H} > 0.5$ and consider $S(\cdot) \in E_{\overline{H},C,T_1,T_2}$. It is easily seen that there exists some c such that

$$TV(\eta_k, T) \le c a^{Bk}, \quad B = (1 - \overline{H})/\overline{H} < 1$$

for all k and for all $S(\cdot) \in E_{\overline{H},C,T_1,T_2}$. In this case $a^k/TV(\eta_k, T) \to \infty$ as $k \to \infty$ uniformly in $S(\cdot) \in E_{\overline{H},C,T_1,T_2}$. As seen from the argument below at the end of the proof, for $S(\cdot) \in E^c_{A,0,T}$ we only need to consider the case $|L(T)| \geq A/4$. Therefore $n_k^* D\left(p(\eta_k, T) \| \rho_k\right) \to \infty$ uniformly in $S(\cdot) \in E_{\overline{H},C,T_1,T_2}$. Also it is easily verified that $\log n_k^*$ in (9) is of smaller order than $n_k^* D\left(p(\eta_k, T) \| \rho_k\right)$.

Now let $\underline{H} < 0.5$ and consider $S(\cdot) \in \underline{E}_{\underline{H},C,T_1,T_2}$. Then there exist some c and k_0 such that

$$TV(\eta_k, T) \geq c a^{Bk}, \quad B = (1 - \underline{H})/\underline{H} > 1$$

for all $k \geq k_0$ and for all $S(\cdot) \in \underline{E}_{\underline{H},C,T_1,T_2}$. In this case $TV(\eta_k, T)/a^k \to \infty$ as $k \to \infty$ uniformly in $S(\cdot) \in \underline{E}_{\underline{H},C,T_1,T_2}$. Again $\log n_k^*$ can be ignored.

Thus we have the following behavior of $\mathcal{K}_k(T)$ according as the values of the upper and the lower Hölder exponents.

$$\begin{aligned}
&\text{If } \overline{H} > 0.5, S(\cdot) \in E_{\overline{H},C,T_1,T_2} \cap E^c_{A,0,T} \text{ and } |L(T)| \geq \frac{A}{4} && \text{then } \ \mathcal{K}_k(T) \to \infty. \\
&\text{If } \underline{H} < 0.5, S(\cdot) \in \underline{E}_{\underline{H},C,T_1,T_2} \cap E^c_{A,0,T} && \text{then } \ \mathcal{K}_k(T) \to \infty.
\end{aligned}$$

We can guarantee the condition $|L(T)| \geq A/4$ above in the following manner. Let Investor divide his initial capital $\mathcal{K}(0) = 1$ into two accounts with the initial capitals $\mathcal{K}_1(0) + \mathcal{K}_2(0) = 1$. At the first account, Investor follows the high-frequency trading strategy explained above. At the second account, Investor starts the game at the first time $t_A (< T)$ when $|\log S(t_A) - \log S(0)| \geq A/2$ and follows the same high-frequency trading strategy. We denote Investor's capitals of respective accounts at $t = T$ by $\mathcal{K}_{k1}(T),\ \mathcal{K}_{k2}(T)$. Then

$$\max(|\log S(T) - \log S(0)|, |\log S(T) - \log S(t_A)|) \geq \frac{A}{4}.$$

on $E^c_{A,0,T}$. Therefore at least one of $\mathcal{K}_{k1}(T),\ \mathcal{K}_{k2}(T)$ diverges to infinity. This proves the theorem. $\square$

For numerical comparison of capital processes it is useful to approximate the capital process for the simple case. If $TV(\eta_k, T) \simeq c a^{Bk}$, then (14) is rewritten as

$$n_k^* D\left(p(\eta_k, T) \| \rho_k\right) \simeq \frac{1}{2}\left[\frac{a^{(1-B)k}}{c} L^2(T) + L(T) + \frac{c a^{(B-1)k}}{4}\right]. \tag{15}$$

We also investigate the capital $\mathcal{K}_k(T)$ for other two cases: (ii) $a_1 < a_2$, (iii) $a_1 > a_2$. From (13) with $TV_k = TV(\eta_k, T),\ p_k = p(\eta_k, T)$ we have

$$n_k^* p_k \simeq \frac{1}{2} a_1^k \left(TV_k + L\right), \quad n_k^*(1 - p_k) \simeq \frac{1}{2} a_2^k \left(TV_k - L\right),$$

so that it follows

$$\begin{aligned}
n_k^* D\left(p_k \| \rho_k\right) &= n_k^* p_k \log \frac{p_k}{\rho_k} + n_k^*(1 - p_k) \log \frac{1 - p_k}{1 - \rho_k} \\
&\simeq \frac{1}{2}\left[a_1^k \left(TV_k + L\right) \log \frac{p_k}{\rho_k} + a_2^k \left(TV_k - L\right) \log \frac{1 - p_k}{1 - \rho_k}\right].
\end{aligned} \tag{16}$$

(ii) $a_1 < a_2$: In this case, $p_k,\ \rho_k \to 0$ as $k \to \infty$. However the expression (16) has the following approximation.

$$n_k^* D\left(p_k \| \rho_k\right) \simeq \frac{1}{2} a_1^k \left(TV_k + L\right) \left[\log \frac{TV_k + L}{TV_k - L} - \frac{2L}{TV_k + L}\right] \simeq \left(\frac{a_1^k}{TV(\eta_k, T)}\right) L^2(T). \quad (17)$$

Suppose that $TV(\eta_k, T) \simeq c a_1^{Bk}$. Then (17) is rewritten as

$$n_k^* D\left(p_k \| \rho_k\right) \simeq \frac{a_1^{(1-B)k}}{c} L^2(T), \tag{18}$$

and we can derive the behavior of $\mathcal{K}_k(T)$ as follows.

$$\text{If } \overline{H} > 0.5 \text{ and } |L(T)| \geq \frac{A}{4} \quad \text{then} \quad \mathcal{K}_k(T) \to \infty,$$

which is the only case such that $\mathcal{K}_k(T) \to \infty$.

(iii) $a_1 > a_2$: In this case, $p_k,\ \rho_k \to 1$ as $k \to \infty$. Again the expression (16) has the following approximation.

$$\begin{aligned} n_k^* D\left(p_k \| \rho_k\right) &\simeq \frac{1}{2} a_2^k \left(TV_k - L\right) \left[\log \frac{TV_k - L}{TV_k + L} + \frac{2L}{TV_k - L} + \frac{2L}{TV_k - L} a_2^{-k} + \frac{1}{2} a_2^{-2k}\right] \\ &\simeq \left(\frac{a_2^k}{TV(\eta_k, T)}\right) L^2(T) + L(T) + \frac{1}{4}\left(\frac{TV(\eta_k, T)}{a_2^k}\right). \end{aligned} \tag{19}$$

Suppose that $TV(\eta_k, T) \simeq c a_2^{Bk}$. Then (19) is rewritten as

$$n_k^* D\left(p_k \| \rho_k\right) \simeq \frac{a_2^{(1-B)k}}{c} L^2(T) + L(T) + \frac{c a_2^{(B-1)k}}{4}, \tag{20}$$

and as in the case of $a_1 = a_2$, the same behavior of $\mathcal{K}_k(T)$ are derived.

$$\begin{aligned} &\text{If } \overline{H} > 0.5 \text{ and } |L(T)| \geq \frac{A}{4} && \text{then} \quad \mathcal{K}_k(T) \to \infty. \\ &\text{If } \underline{H} < 0.5 && \text{then} \quad \mathcal{K}_k(T) \to \infty. \end{aligned}$$

We note that when $a = a_2$, the exponential growth part (20) is twice as large as (15).

4 Concluding remarks

In this paper we proposed a new formulation of continuous time games in the framework of the game-theoretic probability of Shafer and Vovk [13]. The present approach can be extended to prove that Investor can essentially force other properties of Market's path corresponding to various probability laws in continuous-time stochastic processes. Vovk [18] provided an approach to point processes and diffusion processes from prequential viewpoint, but it was not further developed from game-theoretic viewpoint.

In Section 2.2 we gave two definitions of essential forcing of an event, and there are other possibilities. For example we may allow $[T_1, T_2]$ to depend on C in Definition 2.3.

From theoretical perspective it is most important to consider taking the countable closure of the move space $\mathcal{F}_0$ of Investor. For the discrete time games, there is no conceptual difficulty in considering static mixtures of countably many strategies. As mentioned already in Section 2.2, the operation of countable mixture is needed to prove forcing an event in discrete time games. Even in continuous time games, conceptually there is no difficulty in dividing the initial capital into countably many accounts and apply separate strategies to each account. Suppose that Investor can essentially force an event E. Then he can divide his initial capital of one as

$$1 = \frac{1}{2} + \frac{1}{4} + \cdots$$

and put $1/2^i$ to the i-th account as the initial capital. He applies $\mathcal{P}^{2^i}$ to the i-th account until $\mathcal{K}^{\mathcal{P}^{2^i}} \geq 1$. Then he collects one (dollar) from each account and his capital diverges to infinity. This argument shows that if Investor can essentially force E, then he can force E, provided that static mixture of countable strategies are allowed.

Mathematically, however, we need to define the space of strategies allowed to Investor and show that the capital processes of strategies are well defined. These considerations are left to our future works.

Our main Theorem 3.1 is stated in terms of the essential forcing of events (4) and (5). There is some gap between these two sets of functions. In particular the set (5) may be too small. We used this definition for convenience in employing our simple limit order type strategy. A stronger statement should be stated in terms of the variation exponent $\overline{\mathrm{vex}}\, S$ defined at the end of Section 2.3 and this might require more sophisticated strategies of Investor.

Acknowledgment The authors are grateful to Norio Kono and Makoto Maejima for valuable suggestions.

References

[1] Andrei N. Borodin and Paavo Salminen. *Handbook of Brownian Motion – Facts and Formulae.* 2nd ed., Birkhäuser, Basel, 2002.

[2] Thomas M. Cover. Universal portfolios. *Mathematical Finance*, **1**, 1–29, 1991.

[3] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory.* 2nd ed., Wiley, New York, 2006.

[4] Paul Embrechts and Makoto Maejima. *Selfsimilar Processes.* Princeton University Press, New Jersey, 2002.

[5] David Hobson. A survey of mathematical finance. *Proc. R. Soc. London* A, **460**, 3369–3401, 2004.

[6] Yasunori Horikoshi and Akimichi Takemura. Implications of contrarian and one-sided strategies for the fair-coin game. `arXiv:math/0703743v1`, 2007.

[7] John L. Kelly Jr. A new interpretation of information rate. *Bell System Technical Journal*, **35**, 917–926, 1956.

[8] Masayuki Kumon and Akimichi Takemura. On a simple strategy weakly forcing the strong law of large numbers in the bounded forecasting game. *Annals of the Institute of Statistical Mathematics*, DOI 10.1007/s10463-007-0125-5, 2007.

[9] Masayuki Kumon, Akimichi Takemura and Kei Takeuchi. Game-theoretic versions of strong law of large numbers for unbounded variables. *Stochastics*, to appear, 2007.

[10] Masayuki Kumon, Akimichi Takemura and Kei Takeuchi. Capital process and optimality properties of a Bayesian Skeptic in coin-tossing games. `arXiv:math/0510662v1`. *Stochastic Analysis and Applications*, conditionally accepted, 2007.

[11] Naoto Kunitomo. Long-memory and geometric Brownian motion in security market models. Discussion Paper 92-F-12, Faculty of Economics, University of Tokyo, 1992.

[12] L. C. G. Rogers. Arbitrage with fractional Brownian motion. *Mathematical Finance*, **7**, 95–106, 1997.

[13] Glenn Shafer and Vladimir Vovk. *Probability and Finance: It's Only a Game!.* Wiley, New York, 2001.

[14] Claude E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, **27**, 379–423, 623–656, 1948.

[15] Akimichi Takemura and Taiji Suzuki. Game theoretic derivation of discrete distributions and discrete pricing formulas. *Journal of the Japan Statistical Society*, **37**, 87–104, 2007.

[16] Kei Takeuchi. *Kake no suuri to kinyu kogaku* (Mathematics of betting and financial engineering). Saiensusha, Tokyo, 2004. (in Japanese)

[17] Vladimir Vovk and Glenn Shafer. A game-theoretic explanation of the $\sqrt{dt}$ effect. Working Paper No.5. 2003. Available at `http://www.probabilityandfinance.com`

[18] Vladimir Vovk. Forecasting point and continuous processes. *Test*, **2**, 189-217, 1993.