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Computing the Degrees of All Cofactors in Mixed Polynomial Matrices

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Abstract

A mixed polynomial matrix is a polynomial matrix which has two kinds of nonzero coefficients: fixed constants that account for conservation laws and independent parameters that represent physical characteristics. This paper presents an algorithm for computing the degrees of all cofactors simultaneously in a regular mixed polynomial matrix. The algorithm is based on the valuated matroid intersection and all pair shortest paths. The technique is also used for improving the running time of the algorithm for minimizing the index of the differential-algebraic equation in the hybrid analysis for circuit simulation.

1 Introduction

This paper deals with the computation of the degrees of all cofactors in polynomial matrices, motivated by analysis of differential-algebraic equations (DAEs). Consider a linear DAE with constant coefficients

$$A_0\boldsymbol{x}(t) + A_1\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{f}(t), \tag{1}$$

where A_0 and A_1 are constant matrices. With the use of the Laplace transformation, the DAE is expressed as $A(s)\tilde{\boldsymbol{x}}(s) = \tilde{\boldsymbol{f}}(s) + A_1\boldsymbol{x}(0)$ by the polynomial matrix $A(s) = A_0 + sA_1$, where s is the variable for the Laplace transform that corresponds to $\mathrm{d}/\mathrm{d}t$, the differentiation with respect to time.

A polynomial matrix $A(s)$ is said to be *regular* if $A(s)$ is square and $\det A(s)$ is a nonvanishing polynomial. Since the solution $\boldsymbol{x}(t)$ is the inverse Laplace transform of $\tilde{\boldsymbol{x}}(s) = A(s)^{-1}(\tilde{\boldsymbol{f}}(s) + A_1\boldsymbol{x}(0))$, the positive powers of s in $A(s)^{-1}$ represent the number of differentiations of $\boldsymbol{f}(t)$ that appear in $\boldsymbol{x}(t)$. Thus the difficulty of the numerical solution of (1) depends on the degrees of entries of $A(s)^{-1}$, which can be determined from degrees of cofactors by Cramer's rule.

For regular polynomial matrices whose entries are of degree at most one, Bujakiewicz [1] proposed an efficient algorithm for finding the degrees of all cofactors under the assumption

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that coefficients of nonzero entries are independent parameters. Such a genericity assumption is supported by an argument that physical parameters like resistances in electric circuits are not precise in practice because of noises. However, there do exist exact numbers such as ± 1 that appear in the coefficients of Kirchhoff's conservation laws. This observation led Murota and Iri [12] to introduce the notion of a *mixed matrix*, which is a constant matrix that consists of two kinds of numbers as follows.

Accurate Numbers (Fixed Constants) Numbers that account for conservation laws are precise in values. These numbers should be treated numerically.

Inaccurate Numbers (Independent Parameters) Numbers that represent physical characteristics are not precise in values. These numbers should be treated combinatorially as nonzero parameters without reference to their nominal values. Since each such nonzero entry often comes from a single physical device, the parameters are assumed to be independent.

In order to deal with dynamical systems, it is natural to consider the polynomial matrix version, which is called a *mixed polynomial matrix* [11].

For a regular mixed polynomial matrix $A(s)$, we propose an algorithm for finding the degrees of all cofactors simultaneously, which is an extension of the result of Bujakiewicz [1]. The time complexity of the proposed algorithm is the same as that of the algorithm for the degree of $\det A(s)$ described by Murota [10]. The technique is also used to improve the complexity of the algorithm in [6] for finding an optimal hybrid analysis in which the index of the DAE to be solved attains the minimum.

The organization of this paper is as follows. Section 2 provides preliminaries on mixed polynomial matrices and valuated matroids. In Section 3, we describe the algorithm of Murota for computing the degree of the determinant of a regular mixed polynomial matrix. Section 4 gives a characterization of the degree of a cofactor. We present an algorithm for computing the degrees of all cofactors simultaneously in a regular mixed polynomial matrix and analyze its running time in Section 5. Finally, in Section 6, we discuss a similar problem which appears in the index minimization of the DAE in the hybrid analysis [6].

2 Preliminaries

This section is devoted to preliminaries on mixed polynomial matrices and valuated matroids. Valuated matroids are combinatorial abstractions of polynomial matrices.

A *generic matrix* is a matrix in which each nonzero entry is an independent parameter. A matrix $A(s)$ is called a *mixed polynomial matrix* if $A(s)$ is given by $A(s) = Q(s) + T(s)$ with a pair of polynomial matrices $Q(s) = \sum_{h=0}^{N} s^h Q_h$ and $T(s) = \sum_{h=0}^{N} s^h T_h$ that satisfy the following two conditions.

(MP-Q) The coefficients Q_h $(h = 0, \ldots, N)$ in $Q(s)$ are constant matrices.

(MP-T) The coefficients T_h $(h = 0, \ldots, N)$ in $T(s)$ are generic matrices.

A *layered mixed polynomial matrix* (or an *LM-polynomial matrix* for short) is defined to be a mixed polynomial matrix such that $Q(s)$ and $T(s)$ satisfying (MP-Q) and (MP-T) have disjoint nonzero rows. An LM-polynomial matrix $A(s)$ is expressed by $A(s) = \binom{Q(s)}{T(s)}$.

Dress and Wenzel [2] defined a *valuated matroid* to be a triple $\mathbf{M} = (V, \mathcal{B}, \omega)$ of a finite set V, a nonempty family $\mathcal{B} \subseteq 2^V$, and a function $\omega : \mathcal{B} \to \mathbf{R}$ that satisfy the following axiom (VM).

(VM) For any $B, B' \in \mathcal{B}$ and $u \in B \setminus B'$, there exists $v \in B' \setminus B$ such that $B \setminus \{u\} \cup \{v\} \in \mathcal{B}$, $B' \cup \{u\} \setminus \{v\} \in \mathcal{B}$, and $\omega(B) + \omega(B') \leq \omega(B \setminus \{u\} \cup \{v\}) + \omega(B' \cup \{u\} \setminus \{v\})$.

The function ω is called a *valuation.* For $B \in \mathcal{B}, u \in B$, and $v \in V \setminus B$, we define

$$\omega(B, u, v) = \omega(B \setminus \{u\} \cup \{v\}) - \omega(B).$$

By convention, we put $\omega(B, u, v) = -\infty$ if $B \setminus \{u\} \cup \{v\} \notin \mathcal{B}$.

The local optimality for the valuation implies the global optimality as follows.

Theorem 2.1 ([11, Theorem 5.2.7])**.** *A base $B \in \mathcal{B}$ satisfies $\omega(B) \geq \omega(B')$ for any $B' \in \mathcal{B}$ if and only if $\omega(B, u, v) \leq 0$ holds for any $u \in B$ and $v \in V \setminus B$.*

For $B \in \mathcal{B}$ and $B' \subseteq V$, we consider a bipartite graph, called the *exchangeability graph*, $G(B, B') = (B \setminus B', B' \setminus B; H)$ with

$$H = \{(u, v) \mid u \in B \setminus B', v \in B' \setminus B, B \setminus \{u\} \cup \{v\} \in \mathcal{B}\}.$$

We denote by $\hat{\omega}(B, B')$ the maximum weight of a perfect matching in $G(B, B')$, with respect to the edge weight $\omega(B, u, v)$, i.e.,

$$\hat{\omega}(B, B') = \max\{ \sum_{(u,v)\in M} \omega(B, u, v) \mid M \text{ is a perfect matching in } G(B, B')\}.$$

A necessary and sufficient condition for the unique existence of the maximum-weight perfect matching in $G(B, B')$ is given as follows.

Lemma 2.2 ([11, Lemma 5.2.32])**.** *Let $B \in \mathcal{B}$ and $B' \subseteq V$ with $|B' \setminus B| = |B \setminus B'| = h$. There exists exactly one maximum-weight perfect matching in $G(B, B')$ if and only if there exist $q : (B \setminus B') \cup (B' \setminus B) \to \mathbf{R}$ and indexings of elements of $B \setminus B'$ and $B' \setminus B$, say $B \setminus B' = \{u_1, \ldots, u_h\}$ and $B' \setminus B = \{v_1, \ldots, v_h\}$, such that*

$$\omega(B, u_j, v_i) + q(u_j) - q(v_i) \begin{cases} = 0 & (1 \leq i = j \leq h) \\ \leq 0 & (1 \leq i < j \leq h) \\ < 0 & (1 \leq j < i \leq h). \end{cases} \tag{2}$$

Then, $\hat{\omega}(B, B') = \sum_{i=1}^{h} q(v_i) - \sum_{i=1}^{h} q(u_i)$ holds.

The following lemma is called the "unique-max lemma."

Lemma 2.3 ([11, Lemma 5.2.35])**.** *Let $B \in \mathcal{B}$ and $B' \subseteq V$ with $|B'| = |B|$. If there exists exactly one maximum-weight perfect matching in $G(B, B')$, then $B' \in \mathcal{B}$ and $\omega(B') = \omega(B) + \hat{\omega}(B, B')$.*

Murota [8] introduced the *valuated independent assignment problem* as a generalization of the independent assignment problem [5]. The valuated independent assignment problem VIAP(r) parametrized by an integer r is as follows [11, p. 307].

[VIAP(r)] Given a bipartite graph $G = (V^+, V^-; E)$ with vertex sets V^+, V^- and edge set E, a pair of valuated matroids $\mathbf{M}^+ = (V^+, \mathcal{B}^+, \omega^+)$ and $\mathbf{M}^- = (V^-, \mathcal{B}^-, \omega^-)$, and a weight function $w : E \to \mathbf{R}$, find a triple (M, B^+, B^-) that maximizes

$$\Omega(M, B^+, B^-) := w(M) + \omega^+(B^+) + \omega^-(B^-),$$

where $w(M) = \sum\{w(a) \mid a \in M\}$, subject to the constraint that $M \subseteq E$ is a matching of size r and

$$\partial^+ M \subseteq B^+ \in \mathcal{B}^+, \quad \partial^- M \subseteq B^- \in \mathcal{B}^-, \tag{3}$$

where $\partial^+ M$ and $\partial^- M$ denote the set of vertices in V^+ and V^- incident to M, respectively.

An augmenting path algorithm for solving VIAP(r) has been developed in [9], where the unique-max lemma plays a key role.

3 Degree of Determinant

For a polynomial $a(s)$, we denote the degree of $a(s)$ by $\deg a$, where $\deg 0 = -\infty$ by convention. Let $\tilde{A}(s) = \tilde{Q}(s) + \tilde{T}(s)$ be an $n \times n$ regular mixed polynomial matrix with row set $\tilde{R}$ and column set $\tilde{C}$. We denote by $\tilde{A}[I, J]$ the submatrix of $\tilde{A}(s)$ with row set $I \subseteq \tilde{R}$ and column set $J \subseteq \tilde{C}$. In this section, we expound that the computation of

$$\delta_r(\tilde{A}) = \max_{I,J}\{\deg \det \tilde{A}[I, J] \mid |I| = |J| = r\},$$

the highest degree of a minor of order r, is reduced to solving VIAP(r) [10, 11].

Let us define

$$g_i = \max_{j \in \tilde{C}} \deg \tilde{Q}_{ij}(s) \quad (i \in \tilde{R}), \tag{4}$$

where $\tilde{Q}_{ij}(s)$ denotes the (i, j) entry of $\tilde{Q}(s)$. We now construct an associated $2n \times 2n$ LM-polynomial matrix

$$A(s) = \begin{pmatrix} Q(s) \\ T(s) \end{pmatrix} = \begin{array}{c} \\ R_Q \\ R_T \end{array} \!\!\begin{array}{c} \begin{array}{cc} \quad\tilde{R} & \quad\quad\tilde{C} \end{array} \\ \begin{pmatrix} D_Q(s) & \tilde{Q}(s) \\ -D_T(s) & \tilde{T}(s) \end{pmatrix} \end{array} \tag{5}$$

with column set $C = \tilde{R} \cup \tilde{C}$ and row set $R = R_Q \cup R_T$, where R_Q and R_T are disjoint copies of $\tilde{R}$. For each $i \in \tilde{R}$, we denote its copies by $i_Q \in R_Q$ and $i_T \in R_T$. Both $D_Q(s)$ and $D_T(s)$ are diagonal matrices. For each $i \in \tilde{R}$, the (i_Q, i) entry of $D_Q(s)$ is s^{g_i}, and the (i_T, i) entry of $D_T(s)$ is $t_i s^{g_i}$, where t_i is a new independent parameter.

For an LM-polynomial matrix $A(s) = \binom{Q(s)}{T(s)}$ in general, let R_Q and R_T denote the row sets of $Q(s)$ and $T(s)$. We also denote $|R_Q|$ and $|R_T|$ by m_Q and m_T, respectively. The degree of $\det A$ is expressed as follows.

Theorem 3.1 ([11, Theorem 6.2.5]). *For a regular LM-polynomial matrix $A(s) = \binom{Q(s)}{T(s)}$, we have*

$$\deg \det A = \max_{J \subseteq C, |J| = m_Q} \{\deg \det Q[R_Q, J] + \deg \det T[R_T, C \setminus J]\}.$$

The degrees of $\det Q[R_Q, J]$ and $\det T[R_T, C \setminus J]$ correspond to the valuation and the maximum weight of bipartite matchings, respectively. For $r = 0, 1, \ldots, m_T$, we define

$$\delta_r^{\mathrm{LM}}(A) = \max_{I,J} \{\deg \det A[R_Q \cup I, J] \mid I \subseteq R_T, J \subseteq C, |I| = r, |J| = m_Q + r\},$$

which designates the highest degree of a minor of order $m_Q + r$ with row set containing R_Q. Note that we have $\delta_{m_T}^{\mathrm{LM}}(A) = \deg \det A$ for a square LM-polynomial matrix $A(s)$.

For an associated LM-polynomial matrix $A(s)$ with an $n \times n$ mixed polynomial matrix $\tilde{A}(s)$, we have $m_Q = m_T = n$. The relation between $\delta_r(\tilde{A})$ and $\delta_r^{\mathrm{LM}}(A)$ is as follows.

Lemma 3.2 ([11, Lemma 6.2.6]). *Let $\tilde{A}(s) = \tilde{Q}(s) + \tilde{T}(s)$ be an $n \times n$ mixed polynomial matrix with row set $\tilde{R}$. We denote by $A(s)$ the associated LM-polynomial matrix defined by* (4) *and* (5)*. For an integer r with $0 \leq r \leq n$, we have*

$$\delta_r(\tilde{A}) = \delta_r^{\mathrm{LM}}(A) - \sum_{i \in \tilde{R}} g_i. \tag{6}$$

Remark 3.3. In fact, (6) holds for an associated LM-polynomial matrix defined by (5) if each g_i satisfies $g_i \geq \max_{j \in \tilde{C}} \deg \tilde{Q}_{ij}(s)$.

Example 3.4. Consider a mixed polynomial matrix

$$\tilde{A} = \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & t_1 s & 1 + t_2 s \end{pmatrix} = \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & t_1 s & t_2 s \end{pmatrix}$$

with row set $\tilde{R} = \{x_1, x_2, x_3\}$ and column set $\tilde{C} = \{y_1, y_2, y_3\}$. The associated LM-polynomial matrix defined by (4) and (5) is

$$A = \begin{array}{c@{\,}c} & \begin{array}{cccccc} x_1 & x_2 & x_3 & y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_{1Q} \\ x_{2Q} \\ x_{3Q} \\ x_{1T} \\ x_{2T} \\ x_{3T} \end{array} & \begin{pmatrix} s & 0 & 0 & 1 & 0 & s \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ -t_3 s & 0 & 0 & 0 & 0 & 0 \\ 0 & -t_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -t_5 & 0 & t_1 s & t_2 s \end{pmatrix} \end{array}.$$

Then we have $\delta_3(\tilde{A}) = 1$ and $\delta_3^{\mathrm{LM}}(A) = 2$, which satisfy (6).

By Lemma 3.2, $\delta_r(\tilde{A})$ is determined from $\delta_r^{\mathrm{LM}}(A)$. We now describe how to reduce the computation of $\delta_r^{\mathrm{LM}}(A)$ to VIAP(r).

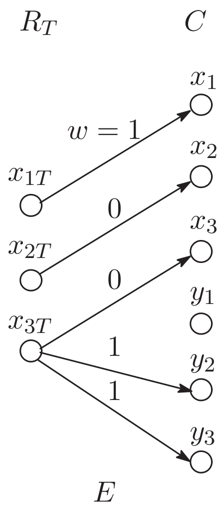

Figure 1: A bipartite graph G of Example 3.4.

Let $\mathbf{M}_Q = (C, \mathcal{B}_Q, \omega_Q)$ be a valuated matroid defined by

$$\mathcal{B}_Q = \{B \subseteq C \mid \det Q[R_Q, B] \neq 0\}, \quad \omega_Q(B) = \deg\det Q[R_Q, B] \quad (B \in \mathcal{B}_Q).$$

We denote the (i,j) entry of $T(s)$ by $T_{ij}(s)$. Consider a bipartite graph $G = (V^+, V^-; E)$ with $V^+ = R_T$, $V^- = C$, and $E = \{(i,j) \mid i \in R_T, j \in C, T_{ij}(s) \neq 0\}$. Let $\mathrm{VIAP}(A;r)$ denote $\mathrm{VIAP}(r)$ defined on G as follows. The valuated matroids $\mathbf{M}^+ = (V^+, \mathcal{B}^+, \omega^+)$ and $\mathbf{M}^- = (V^-, \mathcal{B}^-, \omega^-)$ attached to V^+ and V^- are defined by

$$\mathcal{B}^+ = \{R_T\}, \quad \omega^+(R_T) = 0,$$

and

$$\mathcal{B}^- = \{B \subseteq C \mid C \setminus B \in \mathcal{B}_Q\}, \quad \omega^-(B) = \omega_Q(C \setminus B) \quad (B \in \mathcal{B}^-).$$

The weight $w(a)$ of an arc $a = (i,j) \in E$ is given by $w(a) = \deg T_{ij}(s)$. Figure 1 illustrates G of Example 3.4.

A pair (M, B) of a matching $M \subseteq E$ and a base $B \in \mathcal{B}^-$ is called *feasible* for $\mathrm{VIAP}(A;r)$ if $|M| = r$ and $\partial^- M \subseteq B$. The value of a feasible pair (M,B) is given by

$$\begin{aligned}\Omega_r(M,B) &= w(M) + \omega^+(R_T) + \omega^-(B)\\ &= w(M) + \omega_Q(C \setminus B)\\ &= \deg\det Q[R_Q, C\setminus B] + \sum_{(i,j)\in M} \deg T_{ij}(s).\end{aligned}$$

A feasible pair that maximizes $\Omega_r(M,B)$ is called *optimal* for $\mathrm{VIAP}(A;r)$. The following theorem shows that the optimal value of $\mathrm{VIAP}(A;r)$ coincides with $\delta_r^{\mathrm{LM}}(A)$.

Theorem 3.5 ([11, Theorem 6.2.8]). *For a square LM-polynomial matrix $A(s)$ and an integer r with $0 \leq r \leq m_T$, we have*

$$\delta_r^{\mathrm{LM}}(A) = \max\{\Omega_r(M,B) \mid (M,B) \text{ is feasible for } \mathrm{VIAP}(A;r)\},$$

where the right-hand side is defined to be $-\infty$ if there exists no feasible pair (M,B).

We now describe the algorithm for computing $\delta_r^{\mathrm{LM}}(A)$, proposed by Murota [10, 11]. The algorithm solves $\mathrm{VIAP}(A;r)$ successively for $r = 0, 1, \ldots, m_T$. It maintains a feasible pair (M, B) that maximizes $\Omega_r(M, B)$.

Let us denote the reorientation of $a \in E$ by a°. With reference to G and (M, B), we construct an auxiliary graph $G^* = (R_T \cup C, E^*)$ with arc set $E^* = E \cup E^- \cup M^\circ$, where

$$E^- = \{(v, u) \mid u \in B, v \in C \setminus B, B \setminus \{u\} \cup \{v\} \in \mathcal{B}^-\}, \quad M^\circ = \{a^\circ \mid a \in M\}.$$

Note that the arcs in E^- have both ends in C and that the arcs in M° are directed from C to R_T. The arc length $\gamma : E^* \to \mathbf{Z}$ is defined by

$$\gamma(a) = \begin{cases} -w(a) & (a \in E) \\ w(a^\circ) & (a \in M^\circ) \\ -\omega^-(B, u, v) & (a = (v, u) \in E^-), \end{cases} \tag{7}$$

where $\omega^-(B, u, v) = \omega^-(B \setminus \{u\} \cup \{v\}) - \omega^-(B)$. We put $S^+ = R_T \setminus \partial^+ M$ and $S^- = B \setminus \partial^- M$. Let $\partial^+ a$ and $\partial^- a$ denote the initial and terminal vertices of a, respectively. Then the following fact holds.

Theorem 3.6 ([11, Theorem 5.2.62])**.** *Let (M, B) be an optimal pair for* $\mathrm{VIAP}(A;r)$ *and P be a shortest path from S^+ to S^- with respect to the arc length γ in G^* having the smallest number of arcs. Then $(\hat{M}, \hat{B})$ defined by*

$$\hat{M} = M \setminus \{a \in M \mid a^\circ \in P \cap M^\circ\} \cup (P \cap E), \tag{8}$$
$$\hat{B} = B \setminus \{\partial^- a \mid a \in P \cap E^-\} \cup \{\partial^+ a \mid a \in P \cap E^-\} \tag{9}$$

is optimal for $\mathrm{VIAP}(A; r+1)$.

Theorem 3.6 leads to the following algorithm for computing the degree of the determinant of a regular LM-polynomial matrix.

Algorithm for degree of determinant

Step 1: Find a maximum-weight base $B \in \mathcal{B}^-$ with respect to ω^-. Put $M := \emptyset$.

Step 2: Repeat (2-1)–(2-3) until $|M| = m_T$.

- **(2-1)** Construct an auxiliary graph G^* with respect to (M, B).
- **(2-2)** Find a shortest path P having the smallest number of arcs from S^+ to S^- with respect to the arc length γ in G^*.
- **(2-3)** Update (M, B) according to (8) and (9).

At each stage of this algorithm, it holds that $\delta_r^{\mathrm{LM}}(A) = \Omega_r(M, B)$ for $r = |M|$. At the end of the algorithm, we obtain an optimal pair (M, B) for $\mathrm{VIAP}(A; n)$.

4 Degree of Cofactor

Let $\tilde{A}(s)$ be an $n \times n$ regular mixed polynomial matrix and $A(s)$ be the associated LM-polynomial matrix defined by (4) and (5). In this section, we discuss the degree of a cofactor in $\tilde{A}(s)$. We first show that the degree of a cofactor in $\tilde{A}(s)$ is determined by that of the corresponding cofactor in $A(s)$.

Lemma 4.1. *Let $\tilde{A}(s)$ be an $n \times n$ mixed polynomial matrix and $A(s)$ be the associated LM-polynomial matrix defined by (4) and (5). For $k \in \tilde{R}$ and $l \in \tilde{C}$, we have*

$$\deg\det \tilde{A}[\tilde{R} \setminus \{k\}, \tilde{C} \setminus \{l\}] = \deg\det A[R \setminus \{k_T\}, C \setminus \{l\}] - \sum_{i \in \tilde{R}} g_i. \tag{10}$$

Proof. Applying Remark 3.3 to a mixed polynomial matrix $\tilde{A}[\tilde{R} \setminus \{k\}, \tilde{C} \setminus \{l\}]$ and an LM-polynomial matrix $A[R \setminus \{k_Q, k_T\}, C \setminus \{k, l\}]$, we have

$$\deg\det \tilde{A}[\tilde{R} \setminus \{k\}, \tilde{C} \setminus \{l\}] = \deg\det A[R \setminus \{k_Q, k_T\}, C \setminus \{k, l\}] - \sum_{i \in \tilde{R} \setminus \{k\}} g_i.$$

Since the degree of the (k_Q, k) entry of A is g_k and $A[R \setminus \{k_Q, k_T\}, \{k\}] = O$, it follows that

$$\deg\det A[R \setminus \{k_Q, k_T\}, C \setminus \{k, l\}] = \deg\det A[R \setminus \{k_T\}, C \setminus \{l\}] - g_k.$$

Thus we obtain (10). $\square$

By Lemma 4.1, it suffices to compute $\deg\det A[R \setminus \{k_T\}, C \setminus \{l\}]$ for $k \in \tilde{R}$ and $l \in \tilde{C}$. We now define the following problem.

[**DOC**$(A; k_T, l)$] Find a pair (M, B) of a matching $M \subseteq E$ and a base $B \in \mathcal{B}^-$ maximizing $w(M) + \omega^-(B)$ subject to

$$\partial^+ M = R_T \setminus \{k_T\}, \quad \partial^- M = B \setminus \{l\}, \quad l \in B. \tag{11}$$

A pair (M, B) that satisfies (11) is *feasible* for $\mathrm{DOC}(A; k_T, l)$. Similarly to Theorem 3.5, the degree of $\det A[R \setminus \{k_T\}, C \setminus \{l\}]$ coincides with the optimal value of $\mathrm{DOC}(A; k_T, l)$. The following proposition gives a sufficient condition for the optimality of $\mathrm{DOC}(A; k_T, l)$.

Proposition 4.2. *A feasible pair (M, B) for $\mathrm{DOC}(A; k_T, l)$ is optimal if there exists a pair of vectors $p : R_T \to \mathbf{R}$ and $q : C \to \mathbf{R}$ with $q(l) = 0$ such that*

(i) *$w(a) - p(\partial^+ a) + q(\partial^- a) \leq 0$ holds for $a \in E$,*

(ii) *$w(a) - p(\partial^+ a) + q(\partial^- a) = 0$ holds for $a \in M$,*

(iii) *B maximizes $\omega^-[-q]$, where $\omega^-[-q](B) \equiv \omega^-(B) - \sum_{u \in B} q(u)$.*

Proof. For any feasible pair (M', B') for $\mathrm{DOC}(A; k_T, l)$, we show that

$$w(M') + \omega^-(B') \leq w(M) + \omega^-(B). \tag{12}$$

By (i) and the feasibility of (M', B'), we have

$$w(M') + \omega^-(B') \leq p(\partial^+ M') - q(\partial^- M') + \omega^-(B') = p(R_T \setminus \{k_T\}) - q(B' \setminus \{l\}) + \omega^-(B'),$$

where $p(I) = \sum_{i \in I} p(i)$ and $q(J) = \sum_{j \in J} q(j)$. It follows from $q(l) = 0$ that

$$-q(B' \setminus \{l\}) + \omega^-(B') = -q(B') + \omega^-(B') = \omega^-[-q](B').$$

By (iii), we have $\omega^-[-q](B') \leq \omega^-[-q](B)$. Thus we obtain

$$w(M') + \omega^-(B') \leq p(R_T \setminus \{k_T\}) + \omega^-[-q](B) = p(R_T \setminus \{k_T\}) + \omega^-(B) - q(B),$$

which implies (12) by (ii) and $q(l) = 0$. □

With reference to an optimal pair (M, B) for $\mathrm{VIAP}(A; n)$, we construct the auxiliary graph G^*. For each pair of vertices u and v, let $d(u, v)$ denote the shortest path distance from u to v with respect to the arc length γ in G^*. If there exists no path from u to v, then we put $d(u, v) = \infty$. The degree of a cofactor is now characterized as follows.

Theorem 4.3. *Let (M, B) be an optimal pair for* $\mathrm{VIAP}(A; n)$*. Then we have*

$$\deg \det A[R \setminus \{k_T\}, C \setminus \{l\}] = \Omega_n(M, B) - d(l, k_T)$$

for any $k_T \in R_T$ and $l \in C$.

Let (M, B) be an optimal pair for $\mathrm{VIAP}(A; n)$ and P be a shortest path from l to k_T with respect to the arc length γ in G^* having the smallest number of arcs. We update (M, B) to $(\hat{M}, \hat{B})$ according to (8) and (9). Let $\{(v_i, u_i) \mid i = 1, \ldots, h\} = P \cap E^-$, where $h = |P \cap E^-|$, and the indices are chosen so that $v_h, u_h, \ldots, v_1, u_1$ appear on P in this order. In order to prove Theorem 4.3, we make use of the following lemma.

Lemma 4.4. *Let $G(B, \hat{B})$ be the exchangeability graph with respect to the valuated matroid $(V^-, \mathcal{B}^-, \omega^-)$. Then there exists exactly one maximum-weight perfect matching in $G(B, \hat{B})$. Moreover, we have*

$$\hat{\omega}^-(B, \hat{B}) = \sum_{i=1}^{h} d(l, v_i) - \sum_{i=1}^{h} d(l, u_i). \tag{13}$$

Proof. Consider $q(v) = d(l, v)$ for each $v \in V^-$. Then we have $q(v_i) - \omega^-(B, u_j, v_i) \geq q(u_j)$ for any $(v_i, u_j) \in E^-$. The equality holds if $i = j$ and the strict inequality does if $j < i$. Hence, by Lemma 2.2, there exists exactly one maximum-weight perfect matching in $G(B, \hat{B})$, and (13) holds. □

We are now ready to complete the proof of Theorem 4.3. Note that $(\hat{M}, \hat{B})$ is feasible for $\mathrm{DOC}(A; k_T, l)$. We claim that $(\hat{M}, \hat{B})$ is optimal for $\mathrm{DOC}(A; k_T, l)$.

Consider $p(u) = d(l, u)$ for $u \in V^+$ and $q(v) = d(l, v)$ for $v \in V^-$. We show that p, q, and $(\hat{M}, \hat{B})$ satisfy (i)–(iii) in Proposition 4.2. The definition of p and q implies that (i) and (ii) hold. By Lemmas 2.3 and 4.4, we have

$$\omega^-(\hat{B}) = \omega^-(B) + \hat{\omega}^-(B, \hat{B}). \tag{14}$$

It follows from (13) that

$$\hat{\omega}^-(B, \hat{B}) = \sum_{v \in \hat{B} \setminus B} q(v) - \sum_{u \in B \setminus \hat{B}} q(u) = q(\hat{B} \setminus B) - q(B \setminus \hat{B}) = q(\hat{B}) - q(B).$$

Thus we obtain $\omega^-(\hat{B}) - q(\hat{B}) = \omega^-(B) - q(B)$. This can be written as $\omega^-[-q](\hat{B}) = \omega^-[-q](B)$. By the definition of q, for any $u \in B$ and $v \in V^- \setminus B$, we have $q(v) - \omega^-(B, u, v) \geq q(u)$, which implies that $\omega^-(B) \geq \omega^-(B \setminus \{u\} \cup \{v\}) + q(u) - q(v)$. Hence

$$\begin{aligned}
\omega^-[-q](B \setminus \{u\} \cup \{v\}) &= \omega^-(B \setminus \{u\} \cup \{v\}) - q(B) + q(u) - q(v) \\
&\leq \omega^-(B) - q(B) = \omega^-[-q](B)
\end{aligned}$$

holds. Since the triple $(V^-, \mathcal{B}^-, \omega^-[-q])$ is a valuated matroid, it follows from Theorem 2.1 that $\omega^-[-q](B') \leq \omega^-[-q](B) = \omega^-[-q](\hat{B})$ holds for any $B' \in \mathcal{B}^-$, which implies (iii). Therefore, by Proposition 4.2, $(\hat{M}, \hat{B})$ is optimal for $\mathrm{DOC}(A; k_T, l)$.

Since the degree of $\det A[R \setminus \{k_T\}, C \setminus \{l\}]$ coincides with the optimal value of $\mathrm{DOC}(A; k_T, l)$, we have $\deg\det A[R \setminus \{k_T\}, C \setminus \{l\}] = w(\hat{M}) + \omega^-(\hat{B})$. It follows from (7) and (8) that

$$w(\hat{M}) = w(M) - \sum_{a^\circ \in P \cap M^\circ} w(a) + \sum_{a \in P \cap E} w(a) = w(M) - \sum_{a \in P \cap M^\circ} \gamma(a) - \sum_{a \in P \cap E} \gamma(a).$$

By (13) and (14), we obtain

$$\omega^-(\hat{B}) = \omega^-(B) + \hat{\omega}^-(B, \hat{B}) = \omega^-(B) - \sum_{a \in P \cap E^-} \gamma(a).$$

Therefore, we have $w(\hat{M}) + \omega^-(\hat{B}) = w(M) + \omega^-(B) - \sum_{a \in P} \gamma(a) = \Omega_n(M, B) - d(l, k_T)$. Thus $\deg\det A[R \setminus \{k_T\}, C \setminus \{l\}] = \Omega_n(M, B) - d(l, k_T)$ holds, which completes the proof of Theorem 4.3.

Example 4.5. For the LM-polynomial matrix of Example 3.4, Figure 2 exhibits an optimal pair (M, B) for $\mathrm{VIAP}(A; 3)$ and an auxiliary graph G^* with

$$M = \{(x_{1T}, x_1), (x_{2T}, x_2), (x_{3T}, y_3)\} \quad \text{and} \quad B = \{x_1, x_2, y_3\}.$$

Then we have $\Omega_3(M, B) = 2$. Consider the degree of $\det A[R \setminus \{x_{2T}\}, C \setminus \{y_1\}]$. A shortest path P from y_1 to x_{2T} in G^* is

$$P = \{(y_1, y_3), (y_3, x_{3T}), (x_{3T}, y_2), (y_2, x_2), (x_2, x_{2T})\}$$

and its shortest path distance is $d(y_1, x_{2T}) = \gamma(P) = -1$. It follows from Theorem 4.3 that

$$\deg\det A[R \setminus \{x_{2T}\}, C \setminus \{y_1\}] = \Omega_3(M, B) - d(y_1, x_{2T}) = 3.$$

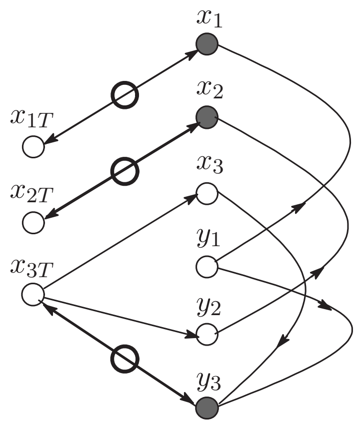

Figure 2: An auxiliary graph G^* of Example 3.4, where $\ominus$ and $\bullet$ denote arcs in M and vertices in B, respectively.

5 Degrees of All Cofactors

In this section, we present an algorithm for computing the degrees of all cofactors simultaneously and analyze its running time.

Theorem 4.3 suggests the following algorithm for computing the degrees of all cofactors in an $n \times n$ regular mixed polynomial matrix $\tilde{A}(s) = \tilde{Q}(s) + \tilde{T}(s)$. The output of this algorithm is a matrix Ψ whose (k, l) entry, denoted by ψ_{kl}, is the degree of the cofactor $\det \tilde{A}[\tilde{R} \setminus \{k\}, \tilde{C} \setminus \{l\}]$.

Algorithm for degrees of all cofactors

Step 1: Construct the $2n \times 2n$ associated LM-polynomial matrix $A(s)$ defined by (4) and (5).

Step 2: Find an optimal pair (M, B) for $\mathrm{VIAP}(A; n)$ by Algorithm for degree of determinant. Construct an auxiliary graph G^* with respect to (M, B).

Step 3: Compute the shortest path distances for all pairs of $k_T \in R_T$ and $l \in \tilde{C}$. For each $k \in \tilde{R}$ and $l \in \tilde{C}$, set $\psi_{kl} := \Omega_n(M, B) - d(l, k_T) - \sum_{i \in \tilde{R}} g_i$.

Step 4: Return Ψ.

We now discuss the running time of Algorithm for degrees of all cofactors. In Step 3, we can compute the shortest path distances for all pairs by the *Warshall-Floyd method* [3, 13] in $O(n^3)$ time. This is dominated by Algorithm for degree of determinant in Step 2. Thus the overall time complexity of Algorithm for degrees of all cofactors is the same as that of Algorithm for degree of determinant.

In order to reflect the dimensional consistency in conservation laws, Murota [7] introduced the following assumption.

(MP-Q2) Every nonvanishing minor of $\tilde{Q}(s)$ is a monomial in s.

For example, consider a linear time-invariant electric circuit. As for the coefficient matrix $\tilde{A}(s)$ of circuit equations, which consist of Kirchhoff's conservation laws (KCL and KVL) and constitutive equations, we assume that the physical parameters are independent. Then, $\tilde{A}(s)$ is an LM-polynomial matrix that satisfies (MP-Q2).

The assumption (MP-Q2) holds if and only if

$$\tilde{Q}(s) = D_R(s)\tilde{Q}(1)D_C(s) \tag{15}$$

for some diagonal matrices $D_R(s)$ and $D_C(s)$ with each diagonal entry being a monomial in s. Consequently, $\mathrm{VIAP}(A; r)$ reduces to an independent assignment problem [11, Remark 6.2.10], which allows us to state the time complexity of Algorithm for degree of determinant as follows.

Lemma 5.1. *Let $\tilde{A}(s)$ be an $n \times n$ regular mixed polynomial matrix. If $\tilde{A}(s)$ satisfies* (MP-Q2)*, we obtain an optimal pair for* $\mathrm{VIAP}(A; n)$ *in $O(n^4)$ time.*

Proof. Note that the associated LM-polynomial matrix $A(s)$ satisfies (MP-Q2). As an initial B in Step 1 of Algorithm for degree of determinant, we can set $B = \tilde{C}$. In Step 2, E^- can be constructed in $O(n^3)$ time. We can find the shortest path in Step 3 in $O(n^2)$ time. Thus the total complexity of Algorithm for degree of determinant is $O(n^4)$ time. □

Lemma 5.1 implies that the time complexity of Algorithm for degrees of all cofactors is $O(n^4)$ as follows.

Theorem 5.2. *Let $\tilde{A}(s)$ be an $n \times n$ regular mixed polynomial matrix that satisfies* (MP-Q2)*. Then the time complexity of* Algorithm for degrees of all cofactors *is $O(n^4)$.*

Proof. In Step 3, shortest path distances for all pairs of vertices are computed in $O(n^3)$ time by the Warshall-Floyd method. Hence Lemma 5.1 implies that the total complexity is $O(n^4)$. □

Gabow and Xu [4] devised an efficient scaling algorithm for an independent assignment problem. By using this algorithm, Algorithm for degrees of all cofactors can be implemented to run in $O(n^3 \log n \log(nN))$ time, where N denotes the highest degree of all the entries in $\tilde{A}(s)$.

6 Degree Matrix

This section presents an algorithm for computing a *degree matrix* defined as follows.

Definition 6.1 (degree matrix)**.** *Let $A(s) = \binom{Q(s)}{T(s)}$ be an $n \times n$ regular LM-polynomial matrix with row set $R = R_Q \cup R_T$ and column set C. Consider another LM-polynomial matrix $A'(s)$ defined by*

$$A'(s) = \begin{array}{c c} & \begin{array}{cc} C & \hat{C} \end{array} \\ \begin{array}{c} R_Q \\ R_T \end{array} & \begin{pmatrix} Q(s) & Q(s) \\ T(s) & O \end{pmatrix} \end{array},$$

where $\hat{C}$ is the copy of C. We denote the copy of $j \in C$ by $\hat{j} \in \hat{C}$. The degree matrix is the matrix $\Theta = (\theta_{kl})$ whose row and column sets are both identical with C such that each entry θ_{kl} is given by $\theta_{kl} = \deg\det A'[R, C \setminus \{l\} \cup \{\hat{k}\}]$.

We now explain the meaning of this degree matrix. Let us assume that $Q(s)$ is a constant matrix Q for simplicity. For an LM-polynomial matrix $A(s) = \binom{Q}{T(s)}$, consider the following transformation

$$\begin{pmatrix} S & O \\ O & I_{m_T} \end{pmatrix} \begin{pmatrix} Q \\ T(s) \end{pmatrix}, \tag{16}$$

where S is a nonsingular constant matrix and I_{m_T} is the identity matrix of order m_T. The transformation (16) does not change the entries in row set R_T and brings an LM-polynomial matrix into another LM-polynomial matrix. By a certain transformation of this type, we obtain an LM-polynomial matrix

$$\check{A}(s) = \begin{array}{c} R_Q \\ R_T \end{array} \begin{pmatrix} I_{m_Q} & Q' \\ & T(s) \end{pmatrix}.$$

We denote by X the column set of I_{m_Q}. Note that there exists a one-to-one correspondence between $k_Q \in R_Q$ and $l \in X$ with the (k_Q, l) entry of $\check{A}(s)$ being nonzero. The relation between the degree of a cofactor in $\check{A}(s)$ and an entry of the degree matrix Θ is as follows.

Lemma 6.2. *For any $k_Q \in R_Q$ and $l \in C$, we have $\theta_{kl} = \deg\det \check{A}[R \setminus \{k_Q\}, C \setminus \{l\}]$, where $k \in X$ is the column corresponding to row k_Q.*

Proof. Since we can transform $A(s)$ into $\check{A}(s)$ by row operations, we may assume that Θ is defined in terms of $\check{A}(s)$. Hence we have

$$\theta_{kl} = \deg\det \begin{pmatrix} \check{A}[R_Q, C \setminus \{l\}] & \check{A}[R_Q, \{k\}] \\ \check{A}[R_T, C \setminus \{l\}] & \mathbf{0} \end{pmatrix} = \deg\det \check{A}[R \setminus \{k_Q\}, C \setminus \{l\}],$$

because $\check{A}[R_Q, \{k\}]$ has only one nonzero entry in row k_Q. □

By Lemma 6.2, the entries in row k of Θ coincide with the degrees of cofactors obtained by deleting row k_Q from $\check{A}(s)$.

We now define the following problem.

[**DM**$(A; k, l)$] Find a pair (M, B) of a matching $M \subseteq E$ and a base $B \in \mathcal{B}^-$ maximizing $w(M) + \omega^-(B)$ subject to

$$\partial^+ M = R_T, \quad \partial^- M = B \setminus \{l\} \cup \{k\}, \quad l \in B, \quad k \notin B.$$

The value of θ_{kl} coincides with the optimal value of $\mathrm{DM}(A; k, l)$. The conditions (i)–(iii) in Proposition 4.2 also give a sufficient condition for the optimality of $\mathrm{DM}(A; k, l)$.

Let $A(s) = \binom{Q(s)}{T(s)}$ be an $n \times n$ regular LM-polynomial matrix. We can find an optimal pair (M, B) for $\mathrm{VIAP}(A; m_T)$ by using Algorithm for degree of determinant. We then construct the auxiliary graph G^* with respect to (M, B). The following theorem leads to an algorithm for computing the degree matrix. The proof is omitted as it is quite similar to that of Theorem 4.3.

Theorem 6.3. *Let (M, B) be an optimal pair for* $\mathrm{VIAP}(A; m_T)$*. For any $k \in C$ and $l \in C$, we have*

$$\theta_{kl} = \Omega_{m_T}(M, B) - d(l, k),$$

where $d(l, k)$ denotes the shortest path distance from l to k with respect to the arc length γ in G^*.

The algorithm for computing a degree matrix is summarized as follows. The output of this algorithm is a degree matrix $\Theta = (\theta_{kl})$.

Algorithm for degree matrix

Step 1: Find an optimal pair (M, B) for VIAP$(A; m_T)$ by Algorithm for degree of determinant.

Step 2: Construct an auxiliary graph G^* with respect to (M, B).

Step 3: Compute the shortest path distances for all pairs of $k \in C$ and $l \in C$. For each k and l, set $\theta_{kl} := \Omega_{m_T}(M, B) - d(l, k)$.

Step 4: Return Θ.

The time complexity of Algorithm for degree matrix is the same as that of Algorithm for degree of determinant, because the shortest path distances in Step 3 can be computed in $O(n^3)$ time by the Warshall-Floyd method [3, 13]. For example, if an LM-polynomial matrix $A(s)$ satisfies (MP-Q2), the total running time is $O(n^4)$. If $A(s)$ is a coefficient matrix of circuit equations, the complexity is improved under the genericity assumption that the physical parameters in the constitutive equations are algebraically independent.

Theorem 6.4. *For a linear time-invariant electric circuit with n elements, we denote by $A(s)$ a $2n \times 2n$ coefficient matrix of circuit equations. Then* Algorithm for degree matrix *can be implemented to run in $O(n^3)$ time, if the set of nonzero entries coming from the physical parameters are algebraically independent.*

Proof. Let us denote the row sets of $A(s)$ corresponding to KCL and KVL by R_I and R_V, respectively. We show that the time complexity of Algorithm for degree of determinant is $O(n^3)$. An initial B in Step 1 can be found in $O(n^3)$ time, because $A[R_I \cup R_V, C]$ is a constant matrix. In Step 2, the construction of E^- is as follows. Let B be a base, and Γ be a network graph of the circuit with vertex set W and edge set F. We split $C \setminus B$ into B_I and B_V such that $A[R_I, B_I]$ and $A[R_V, B_V]$ are nonsingular. Let us denote a spanning tree corresponding to B_I in Γ by T_I, and a cotree corresponding to B_V by $\overline{T}_V$. Consider subgraphs $\Gamma_I = (W, T_I)$ and $\Gamma_V = (W, F \setminus \overline{T}_V)$ of Γ. For each $e = (u, v) \in F \setminus T_I$, we find a path $P_I(e)$ from u to v in Γ_I in $O(n)$ time, because the number of edges is $O(n)$. Similarly, for each $e = (u, v) \in \overline{T}_V$, we find a path $P_V(e)$ from u to v in Γ_V in $O(n)$ time. Then, we obtain $E^- = \{(\bar{e}, e) \mid e \in F \setminus T_I, \bar{e} \in P_I(e)\} \cup \{(e, \bar{e}) \mid e \in \overline{T}_V, \bar{e} \in P_V(e)\}$. Thus E^- can be constructed in $O(n^2)$ time. A shortest path in Step 3 can be found in $O(n^2)$ time. Therefore, the time complexity of Algorithm for degree of determinant is $O(n^3)$, which implies that Step 1 of Algorithm for degree matrix requires $O(n^3)$ time.

In Step 3, the Warshall-Floyd method finds the shortest path distances in $O(n^3)$ time. Thus, the total time complexity of Algorithm for degree matrix is $O(n^3)$. □

The notion of the degree matrix plays a key role in the index reduction method for the DAE arising from the hybrid analysis in circuit simulation. Since the LM-polynomial matrix considered there is a coefficient matrix of the circuit equations, the degree matrix can be obtained in $O(n^3)$ time by Theorem 6.4. This improves the time complexity of finding the minimum index hybrid analysis in [6] by a factor of n^3.

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