MATHEMATICAL ENGINEERING TECHNICAL REPORTS

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(Communicated by Kazuo MUROTA)

METR 2007–51 September 2007

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September 21, 2007

Abstract

It has been pointed out that the notion of the sum of roots can be utilized for solving (parametric) polynomial spectral factorization. This paper reveals another aspect of the sum of roots as an indicator of achievable performance limitations. It is shown that performance limitations for some $\mathcal{H}_2$ control problems can be expressed as the difference of two sums of roots. An optimization approach combining the two aspects of the sum of roots is also shown possible that minimizes the achievable performance limitation over plant parameters.

1 Introduction

There has been an increased interest in the search for fundamental performance limitations achievable by feedback control [1, 2, 3]. Achievable performance levels, in terms of various criteria, are expressed by various plant characteristics such as unstable poles, non-minimum phase zeros, time delays, plant gain, and pole/zero directions. There is however a restriction in such results in that it is in general difficult to relate plant physical parameters (length, mass, etc.) to those characteristics and thus observation of the effect of physical parameters on the achievable performance level is not straightforward.

In this paper it is attempted to characterize some $\mathcal{H}_2$ performance limitations using what is called the *sum of roots*. It is still not simple to relate physical parameters to the sum of roots. Nevertheless the advantage of the characterization

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in terms of the sum of roots, rather than usual plant characteristics, is that it does allow parametric optimization to be executed in case where the plant has some tuning parameters. The optimization approach is accomplished by the exploitation of the sum of roots as a means to solve polynomial spectral factorization.

The notion of the *sum of roots* is initially introduced as an indicator of 'average stability' [4]. It is later discovered that the sum of roots allows observation of an intriguing connection between polynomial spectral factorization and an algebraic approach of Gröbner bases and that polynomial spectral factorization is solvable by an algebraic approach in an efficient manner via the sum of roots [5]. Furthermore it is shown possible to extend the approach to the parametric case [6]. Thus the notion of the sum of roots is proven useful for carrying out polynomial spectral factorization. Another aspect of the sum of roots is revealed in this paper and an interesting relationship between the sum of roots and some $\mathcal{H}_2$ performance limitations is presented. More specifically it is shown that the performance limitation for the $\mathcal{H}_2$ regulation problem is expressed in a simple manner in terms of two sums of roots obtained from the plant and polynomial spectral factorization. By way of a new analysis tool named the *reciprocal transform* [7, 8], an expression for the $\mathcal{H}_2$ tracking performance limitation is also derived.

The efficacy of the sum of roots is demonstrated by combining the two aspects of the sum of roots. Given a plant with parameters, the best achievable performance level cannot in general be related directly to parameters. Instead parametric optimization is performed in such a way that the performance level is expressed as a root of a polynomial whose coefficients contains parameters. Another algebraic tool called quantifier elimination can then be employed for optimization over plant parameters.

The paper is organized as follows. An illustrative example is firstly presented in Section 2 in order to give readers some ideas on the sum of roots and the results to be derived later in the paper. Section 3 is devoted to the exposition of the computation tool aspect of the sum of roots, and solution of polynomial spectral factorization via the sum of roots is reviewed. Then, in Section 4, the performance limitation aspect of the sum of roots is shown and it is proven that the best achievable performance levels for the $\mathcal{H}_2$ regulation and tracking problems can be expressed simply as the difference of two sums of roots. Section 5 discusses the formulation of the optimization problem over parameters as a quantifier elimination problem. In Section 6, the numerical example presented in Section 2 is revisited and a further explanation is provided. Some concluding remarks are made in Section 7.

2 An Illustrative Example

This section indicates what is shown in this paper by way of an example plant with tuning parameters. The $\mathcal{H}_2$ regulation problem [9], which is formally formulated in Subsection 4.1, is considered. However, unlike the ordinary optimal control problem setting, it is assumed here that, given a fixed plant, the optimal controller can always be designed and also used to construct the closed-loop system. More precisely, given a plant with tuning parameters, the task is to choose the values of the parameters from an admissible region so that the best optimal (i.e., 'best of the best') performance may be achieved. The idea is that the parameters correspond to some physical quantities (such as mass) that a designer can decide under practical restrictions and that those design parameters are to be determined for the best final result. It is emphasized that parameters are tuning/design parameters, rather than uncertain ones.

In [9], an expression for the best $\mathcal{H}_2$ regulation performance (achievable by an optimal controller), which is denoted by $E^\star$, is derived in terms of some characteristics of single-input-single-output (SISO) $P(s)$:

$$E^\star(P) = 2\sum_k p_k^{\mathrm{a}} + \frac{1}{\pi}\int_0^\infty \log\bigl(1 + |P(j\omega)|^2\bigr)d\omega \ , \tag{1}$$

where p_k^{a}'s are unstable poles of P. When a fixed plant (i.e., a plant without parameters) is given, the above formula may be used to compute the achievable performance level. Nevertheless, in the case of a plant with parameters, it is in general impossible to get exact expressions for p_k^{a}'s (or one for $\sum_k p_k^{\mathrm{a}}$) and also to evaluate the integral. Parametric optimization, i.e., getting an expression for $E^\star(P)$ in the presence of parameters, based on (1) is thus impractical. An alternative characterization of the best performance level that can be exploited for optimization over parameters is desired.

Now the following plant is employed as a numerical example:

$$P(s) = \frac{(3-2q_1)(1+q_2^2)}{s(s-2q_1^2+q_2)} =: \frac{P_N(s)}{P_D(s)} \ ,$$

where $\mathbf{q} = (q_1, q_2)$ are parameters which have to be chosen from the admissible range

$$\mathcal{Q} = \bigl\{\mathbf{q} = (q_1, q_2) \bigm| q_1 \in [0,1], q_2 \in [0,1]\bigr\} \ .$$

The task is first to find an expression for $E^\star(P)$ with parameters and further to optimize over $\mathcal{Q}$ the best performance level:

$$\inf_{\mathbf{q}\in\mathcal{Q}} E^\star(P) \ .$$

Whilst it is in general impossible, exact computation can be carried out for this particular low order system, yielding

$$E^\star(P) = \sigma_{\mathrm{M}} - \sigma_{\mathrm{P}} \ , \tag{2}$$

where

$$\sigma_{\mathrm{M}} = \sqrt{4q_1^4 - 4q_1^2q_2 - 4q_1q_2^2 + 7q_2^2 - 4q_1 + 6} \ , \tag{3}$$

$$\sigma_{\mathrm{P}} = -2q_1^2 + q_2 \ . \tag{4}$$

The characterization of σ_{M} and σ_{P} is revealed in the sequel. Firstly the poles of P (that is, the roots of P_D) are 0 and $2q_1^2 - q_2$. It can be seen that $-\sigma_{\mathrm{P}}$ is the sum of them:

$$-\sigma_{\mathrm{P}} = 0 + (2q_1^2 - q_2) \; .$$

Similarly, $-\sigma_{\mathrm{M}}$ is shown to be the sum of roots of the spectral factor of an even polynomial constructed from the plant. More specifically construct an even polynomial in s from P_N and P_D:

$$\begin{aligned} &P_N(s)P_N(-s) + P_D(s)P_D(-s) \\ &\qquad = s^4 - (2q_1^2 - q_2)^2 s^2 + (3 - 2q_1)^2(1 + q_2^2)^2 \; . \end{aligned} \tag{5}$$

Let $M_D(s)$ be a polynomial with a positive leading coefficient such that $M_D(s)M_D(-s)$ is equal to (5) and M_D has roots in the open left half plane only. For this particular example, M_D can be written in closed form in terms of parameters:

$$M_D(s) = s^2 + \sqrt{4q_1^4 - 4q_1^2 q_2 - 4q_1 q_2^2 + 7q_2^2 - 4q_1 + 6}\, s + (3 - 2q_1)(1 + q_2^2) \; .$$

It is observed that the coefficient of s is identical to σ_{M} in (3). This implies that $-\sigma_{\mathrm{M}}$ is the sum of roots of M_D since the negative of the coefficient of the second highest term is the sum of roots of a monic polynomial. Even though it is not always possible to get explicit expressions for $E^\star(P)$ or the sum of roots of M_D, it is always the case (under some assumptions) that $E^\star(P)$ can simply be expressed as in (2), that is as the difference of the two sums of roots. In the $\mathcal{H}_2$ *tracking* problem considered in Subsection 4.2, the achievable performance level can also be related to two sums of roots. These facts indicate that the quantity of the sum of roots has a direct link to the control performance limitation.

Interesting as it may be, readers would wonder how the relationship like (2) can be of any use. Another aspect of the sum of roots is the key to its effectiveness. It can be confirmed that (3) is the largest real root of the following polynomial in σ:

$$\begin{aligned} &\sigma^4 - 2(2q_1^2 - q_2)^2\sigma^2 \\ &\qquad + (4q_1^4 - 4q_1^2 q_2 - 4q_1 q_2^2 + 7q_2^2 - 4q_1 + 6) \\ &\qquad\qquad \times (4q_1^4 - 4q_1^2 q_2 + 4q_1 q_2^2 - 5q_2^2 + 4q_1 - 6) \; . \end{aligned} \tag{6}$$

As is already stated, it is in general impossible to express the sum of roots of M_D (equivalently, σ_{M}) in closed form. Nevertheless it is shown possible to derive a polynomial which has the quantity σ_{M} as its largest real root by making use of a useful structural property of polynomial spectral factorization [5, 6]. (Finding M_D is in fact polynomial spectral factorization.) In [5], an intriguing connection between polynomial spectral factorization and the Gröbner basis theory is pointed out through the notion of the sum of roots. Based on the relationship an algebraic

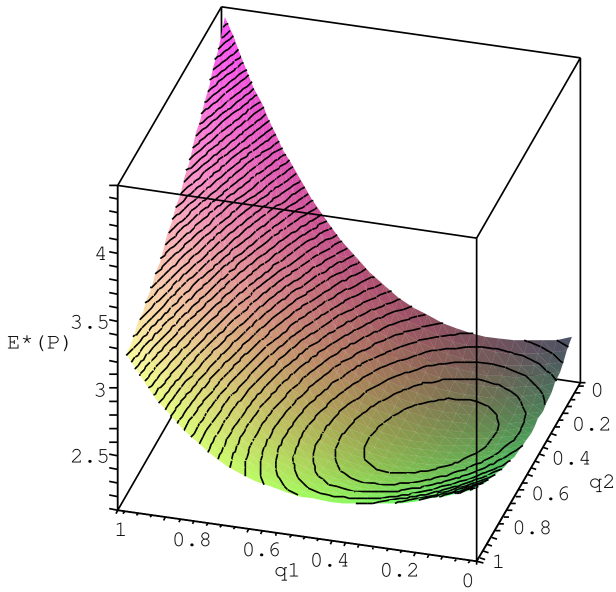

Figure 1: $E^{\star}(P)$ drawn from exact expression (2).

method for polynomial spectral factorization is devised that can deal with parameters as they are (i.e., without substituting parameters with numbers) [6].

Now two aspects of the sum of roots have been stated: the sum of roots for expressing performance limitations and the sum of roots for computation of polynomial spectral factorization. It is in general impossible to get a expression of the performance limitation explicitly in terms of parameters, but it is possible to express the performance limitation *in terms of the sum of roots* and parameters, and moreover to compute a polynomial which has the *sum of roots as one of its roots*. Therefore, by way of the two aspects of the sum of roots, the performance limitation can be related with parameters. This useful link can be exploited and an algebraic method called quantifier elimination (QE) can be employed for optimization over parameters. A QE package called QEPCAD B [10] shows that, when $\mathbf{q} \in \mathcal{Q}$,

$$\eta_1 \leq E^{\star}(P) \leq 2 + \sqrt{6}\,, \tag{7}$$

where η_1 is the third largest real root of

$$16\eta^5 - 531\eta^4 + 1488\eta^3 + 2824\eta^2 - 8640\eta + 1360\,,$$

i.e., $\eta_1 \simeq 2.1468$. It is noted here that these lower and upper bounds are exact. QEPCAD B also shows that the lower bound is achieved when $\mathbf{q} \simeq (0.27782, 0.43913)$. The result agrees with the graph of $E^{\star}$ in Fig. 1 plotted from the exact expression (2), which shows the correctness of (7).

3 Solution of Polynomial Spectral Factorization via the Sum of Roots

3.1 Polynomial Spectral Factorization: Problem Formulation

Consider an even polynomial in x of degree $2n$

$$f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0 \ , \tag{8}$$

where a_{2i}, $i = 0, \ldots, n$, are assumed to be real (i.e., not functions in parameters) here for simplicity. It is assumed without loss of generality that the leading coefficient is positive: $a_{2n} > 0$. Assume that $f(x)$ does not have roots on the imaginary axis. Polynomials derived from most control problems satisfy this assumption and it is by no means a severe restriction. Because of the assumptions that $f(x)$ is an even polynomial and that $f(x)$ has no pure imaginary roots, there are exactly n roots in the open left half plane and n roots in the open right half plane. The task is then to express $f(x)$ as a product of two polynomials: one that captures the open left half plane (LHP) roots and the other the open right half plane (RHP) roots.

Definition 1 *The* spectral factorization *of $f(x)$ in* (8) *is a decomposition of $f(x)$ of the following form:*

$$a_{2n}f(x) = (-1)^n g(x)g(-x) \ , \tag{9}$$

where

$$g(x) = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0 \ , \ \ b_n = a_{2n} \ ,$$

and $g(x)$ has roots in the open left *half plane only. The polynomial $g(x)$ is called the* spectral factor *of $f(x)$.*

3.2 Sum of Roots

In this subsection the notion of the sum of roots is formally reviewed. Let $\alpha_1, \ldots, \alpha_n$ be the n roots of $f(x)$ in the open *left* half plane. The n roots in the open *right* half plane can then be written as $-\alpha_1, \ldots, -\alpha_n$. By using α_i's, $f(x)$ and $g(x)$ can be written as

$$\begin{aligned} f(x) &= a_{2n}\prod_{i=1}^{n}(x-\alpha_i)(x+\alpha_i) = a_{2n}\prod_{i=1}^{n}(x^2-\alpha_i^2) \ , \\ g(x) &= a_{2n}\prod_{i=1}^{n}(x-\alpha_i) \ , \end{aligned} \tag{10}$$

respectively.

Now the *sum of roots (SoR)* is defined as the following quantity:

$$\sigma = -(\alpha_1 + \alpha_2 + \cdots + \alpha_n) \ . \tag{11}$$

The name is derived from the fact that $-\sigma$ is the sum of roots of the spectral factor $g(x)$. Note that $\mathrm{Re}\,(-\alpha_i) > 0$. Also, for each non-real root of $f(x)$, its complex conjugate has the same real part, which leads to the following fact.

Fact 2 ([4]) *The quantity σ is real and positive.*

Remark 1 *It may seem natural to define the quantity $\sum \alpha_i$ (rather than $\sum(-\alpha_i)$) as the SoR. However the definition* (11) *is used to keep consistency with the notation in previous work [4, 5]. Moreover the SoR so defined yields an expression like $s^n + \sigma s^{n-1} + \cdots$, and is considered to make engineers 'feel stable'. Some may argue that this is something to do with the fact that the authors are all 'positive' thinkers!*

3.3 Solution via the Sum of Roots

This subsection expounds the solution of the polynomial spectral factorization problem via the SoR developed in [5, 6], to which readers are referred for further detail. The solution approach is based on an interesting relationship between polynomial spectral factorization and the theory of Gröbner bases that the SoR makes conspicuous.

Firstly, by expanding the right hand side of (10) and comparing it with the right hand side of (1), it is straightforward to find the relationship

$$b_{n-1} = a_{2n}\sigma \; . \tag{12}$$

The SoR σ and b_{n-1} can thus be viewed interchangeably. In particular, in the case where $a_{2n} = 1$, these two quantities are identical.

A set of algebraic equations in terms of b_i's can be obtained by considering b_i's as variables and equating the coefficients of the both sides of (9). This set has a useful property, which is the seminal point of the development that follows.

Lemma 3 *Given $f(x)$ and $g(x)$ as in* (8) *and* (1)*, respectively, consider b_i, $i = 0, \ldots, n-1$, as variables. A set of algebraic equations in terms of b_i's is obtained by comparing the coefficients of* (9)*. Then the set $\mathcal{G}$ of the polynomials obtained from the polynomial parts of the equations forms the reduced Gröbner basis of the ideal generated by itself with respect to the graded reverse lexicographic order $b_{n-1} \succ \cdots \succ b_0$.*

For Gröbner bases and associated ideas such as the graded reverse lexicographic order, readers are referred to, e.g., [11].

The ideal $\langle \mathcal{G} \rangle$ is called the *ideal of spectral factorization*. The following lemma can immediately be deduced from Lemma 3 and the Gröbner basis theory.

Lemma 4 *The ideal of spectral factorization is 0 dimensional and the number of its zeros with multiplicities counted is 2^n.*

Let $\mathcal{P} = \big\{(\epsilon_1, \ldots, \epsilon_n) \mid \epsilon_i \in \{1, -1\}\big\}$. In the *generic case*, distinct combinations of $(\epsilon_1, \ldots, \epsilon_n) \in \mathcal{P}$ give distinct values of $\epsilon_1\alpha_1 + \cdots + \epsilon_n\alpha_n$. The generic case has a different basis which is easy to deal with.

Theorem 5 *In the generic case the ideal of spectral factorization has a Gröbner basis of so-called* shape form *with respect to any elimination ordering* $\{b_0, \ldots, b_{n-2}\} \succ\succ b_{n-1}$*:*

$$\left\{\hat{S}_f(b_{n-1}), b_{n-2} - \hat{h}_{n-2}(b_{n-1}), \ldots, b_0 - \hat{h}_0(b_{n-1})\right\} ,$$

where $\hat{S}_f$ *is a polynomial of degree exactly* 2^n *and* $\hat{h}_i$*'s are polynomials of degree strictly less than* 2^n*.*

Given a set of multivariate polynomials, the set does not in general have a shape basis, and, even if it does, computation of the shape basis may be prohibitively expensive. Nevertheless the properties of the set $\mathcal{G}$ of polynomials stated in Lemma 3 and Theorem 5 allow efficient computation. Due to the fact that $\mathcal{G}$ is a Gröbner basis and also by knowing that the ideal $\langle\mathcal{G}\rangle$ has a shape basis, computation of the shape basis from $\mathcal{G}$ can effectively be performed by means of the basis conversion (change-of-order) technique [11, Appendix D, §2], [12].

Theorem 5 along with the relationship (12) implies that there is a polynomial of degree 2^n defining the SoR σ and that each coefficient of $g(x)$ is described as a polynomial in σ:

$$S_f(\sigma) = 0 \; , \;\; b_{n-1} = a_{2n}\sigma \; , \;\; b_{n-2} = h_{n-2}(\sigma) \; , \;\; \ldots \; , \;\; b_0 = h_0(\sigma) \; ,$$

where $S_f(\sigma) := \hat{S}_f(a_{2n}\sigma)$ and $h_i(\sigma) := \hat{h}_i(a_{2n}\sigma)$. The problem of polynomial spectral factorization thus boils down to finding a root of $S_f(\sigma)$. In fact the SoR has a preferable property and not all roots are to be pursued. Of 2^n roots of $S_f(\sigma)$, the true σ is always the largest real root. Furthermore, under the assumption that there is no imaginary axis roots in $f(x)$, the SoR σ is always a simple root of S_f. That is, even though S_f may have multiple roots, the SoR is always a simple root of S_f.

The *singular case* happens for instance when $\alpha_i = \alpha_j$ for some pair of (i, j), $1 \le i, j \le n$, $i \ne j$, or when $\alpha_i + \alpha_j - \alpha_k = 0$ for some 3-tuple (i, j, k), $1 \le i, j, k \le n$, $i \ne j$, $j \ne k$, $k \ne i$. Even in this case it is possible to compute a shape basis; the degree of $\hat{S}_f$ (or equivalently, that of S_f) is smaller in that case. Also it is still the case that the SoR is the largest real root of S_f and that it is a simple root under the assumption that there is no imaginary axis roots in $f(x)$.

In the case where the coefficients a_{2k} of $f(x)$ are polynomials in parameters $\mathbf{q} = (q_1, q_2, \ldots, q_m)$, it is still possible to carry out polynomial spectral factorization by means of the SoR. The crucial point in this case is to identify the area $\mathtt{C} \subset \mathbb{R}^m$ such that, for any $\mathbf{c} \in \mathtt{C}$, $a_{2n}(\mathbf{c}) \ne 0$ and the number of roots of $f(x, \mathbf{c})$ in the left half plane is n. As is seen in Section 4, imaginary axis roots of $f(x, \mathbf{q})$ result from pole-zero cancellation in the plant. In practice, a possibility of pole-zero cancellation is to be examined beforehand, and conditions of parameters under

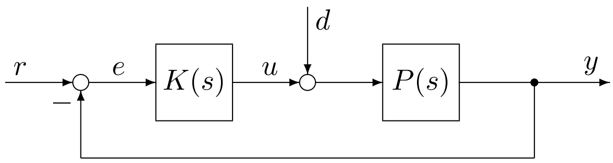

Figure 2: Unity feedback system configuration.

which such cancellation occurs are to be identified so that the admissible region $\mathcal{Q}$ belongs to $\mathtt{C}$.

For such $\mathtt{C}$, the polynomial set $\mathcal{G}$ can be computed from the equations (9) and (1), and $\mathcal{G}$ is still a Gröbner basis for ideals generated by itself with respect to the graded reverse lexicographic order $b_{n-1} \succ \cdots \succ b_0$. By using methods for *comprehensive Gröbner systems* of parametric ideals [13, 14, 15], the Gröbner basis of the ideal of spectral factorization with respect to a fixed elimination ordering $\{b_0, \ldots, b_{n-2}\} \succ\succ b_{n-1}$ can always be computed.

4 Characterization of Performance Limitations via the Sum of Roots

4.1 $\mathcal{H}_2$ Regulation Problem

This subsection considers the $\mathcal{H}_2$ regulation problem and attempts to get an expression of the achievable performance level in terms of the SoR. In Fig. 2, $P(s)$ is the transfer function of the SISO continuous-time, linear, time-invariant plant to be controlled, and $K(s)$ is the transfer function of the controller. The signals $r(t)$, $u(t)$, $y(t)$, $d(t)$, and $e(t) := r(t) - u(t)$ are the reference input, the control input, the control output, the disturbance input, and the error signal, respectively. In the $\mathcal{H}_2$ regulation problem, the disturbance input is taken to be the impulse signal, and it is assumed that there is no reference signal, i.e., $d(t) = \delta(t)$, $r(t) \equiv 0$. The task is to regulate the plant output to zero under an input penalty. The quadratic cost function

$$E(P,\, K) := \int_0^\infty \big(|y(t)|^2 + |u(t)|^2\big)\,dt$$

is employed to measure the performance level, and the best (minimal) performance level is written as

$$E^\star(P) := \inf_{K \text{ stabilizing}} E(P,\, K)\ .$$

Some technical assumptions are made on the plant [9]:

- P is strictly proper;
- P is minimum phase (but can be unstable).

Under these assumptions, $E^\star$ can be expressed as in (1). A similar expression can also be derived in the case of weighted control output penalty.

The aim here is to show that the best $\mathcal{H}_2$ regulation performance $E^\star$ can be expressed explicitly in terms of *two sums of roots*. As is pointed out in Section 2, one sum of roots is directly obtained from the transfer function P. The other sum of roots comes from the spectral factor of an even polynomial (as in Section 3) constructed from the plant P.

Write the plant as

$$P(s) = \frac{P_N(s)}{P_D(s)} , \tag{13}$$

where P_N and P_D are coprime polynomials. Without loss of generality, P_D is assumed to be monic. If the order of P is n, then P_N and P_D can be written as

$$P_N(s) = c^{\mathrm{n}}_{n-1}s^{n-1} + c^{\mathrm{n}}_{n-2}s^{n-2} + \cdots + c^{\mathrm{n}}_1 s + c^{\mathrm{n}}_0 , \tag{14}$$

$$P_D(s) = s^n + c^{\mathrm{d}}_{n-1}s^{n-1} + \cdots + c^{\mathrm{d}}_1 s + c^{\mathrm{d}}_0 . \tag{15}$$

(Note that P is strictly proper.) Let $M_D(s)$ be the spectral factor of

$$P_N(s)P_N(-s) + P_D(s)P_D(-s) \ \Big(= M_D(s)M_D(-s)\Big) . \tag{16}$$

Notice that (16) does not have imaginary axis roots due to the coprimeness of P_N and P_D. The assumption that there is no imaginary axis root in the even polynomial to be factorized is therefore a natural one, rather than a restrictive one. Since the degree of P_D is strictly larger than that of P_N, M_D is monic in this case and thus can be written as

$$M_D(s) = s^n + b_{n-1}s^{n-1} + \cdots + b_0 .$$

Define

$$\sigma_{\mathrm{M}} := b_{n-1} , \ \ \sigma_{\mathrm{P}} := c^{\mathrm{d}}_{n-1} .$$

Notice that $-\sigma_{\mathrm{M}}$ is the sum of roots of M_D and $-\sigma_{\mathrm{P}}$ is the sum of roots of P_D, as in Section 2. Furthermore, σ_{P} can be immediately read off from P_D (i.e., from the plant). The quantity σ_{M} arises from polynomial spectral factorization and thus corresponds to the SoR discussed in Section 3. It can be confirmed that M_D is in fact the characteristic polynomial of the closed-loop system constructed with the optimal controller. Therefore, σ_{P} and σ_{M} indicate the degrees of 'average stability' of the plant and of the achieved closed-loop system, respectively.

Now the following theorem is stated that shows that $E^\star$ is the difference between the two sums of roots.

Theorem 6 *The performance limitation $E^\star$ of the $\mathcal{H}_2$ regulation problem can be written as*

$$E^\star(P) = \sigma_M - \sigma_P . \tag{17}$$

The proof is given in Appendix A.

Theorem 6 indicates an intriguing fact: the best achievable performance level is the difference between the degrees of 'average stability' of the plant to be controlled and of the closed-loop system constructed with the optimal controller. Roughly speaking, the more unstable the plant is, the worse performance one can get. It is pointed out however that the closed-loop poles are implicit functions in the poles/zeros of the plant and even the plant gain. Therefore the performance improvement/deterioration with respect to plant pole locations is not so direct as it looks.

4.2 $\mathcal{H}_2$ Tracking Problem

This subsection deals with the $\mathcal{H}_2$ tracking problem and derives an expression for the best achievable performance level similar to the one for the $\mathcal{H}_2$ regulation case. Whilst it may be possible to derive it directly, the expression for the optimal $\mathcal{H}_2$ *tracking* performance level is deduced from the $\mathcal{H}_2$ *regulation* problem counterpart. A new tool named the reciprocal transform [7, 8] that 'preserves' performance limitations is employed for the derivation.

Firstly the $\mathcal{H}_2$ tracking problem is formulated by using the feedback system configuration in Fig. 2. Let the reference input be the step input, and suppose that there is no disturbance signal: $r(t) = 0\,(t < 0),\ 1\,(t \geq 0)$, $d(t) \equiv 0$. The task is to make the plant output follow the reference signal whilst penalizing the control input. The performance level is measured by the quadratic cost function

$$J(P,\, K) := \int_0^\infty \big(|e(t)|^2 + |u(t)|^2\big)dt\ .$$

The best (minimal) performance level is denoted by

$$J^\star(P) := \inf_{K \text{ stabilizing}} J(P,\, K)\ .$$

It is assumed that [9]

- P is written as $P(s) = \frac{P_0(s)}{s}$ where P_0 is some (marginally) stable transfer function;
- P does not have a zero at $s = 0$.

Notice that P should be marginally stable, but can be non-minimum phase. Under these assumptions a closed form expression for $J^\star$ in terms of the plant characteristics is derived [9]:

$$J^\star(P) = 2\sum_k \frac{1}{z_k^{\mathrm{a}}} + \frac{1}{\pi}\int_0^\infty \frac{1}{\omega^2}\log\!\left(1 + \frac{1}{|P(j\omega)|^2}\right)d\omega\ ,$$

where z_k^{a}'s are non-minimum phase zeros of P.

Now a new effective tool for unifying analysis on control performance limitations achievable by feedback control is reviewed. Consider a class of systems

$$\mathbf{\Gamma} := \left\{ G(s) \in \mathbf{R}_{\mathrm{p}}^{m\times m} \,\middle|\, \det\bigl(G(0)\bigr) \neq 0 \right\} ,$$

where $\mathbf{R}_{\mathrm{p}}$ is the set of proper rational functions in s. The *reciprocal transform* of a system in $\mathbf{\Gamma}$ is defined as follows.

Definition 7 ([7, 8]) *For $G(s) \in \mathbf{\Gamma}$, its reciprocal transform is defined as*

$$\mathsf{RT}\bigl(G(s)\bigr) := G^{-1}\Bigl(\frac{1}{s}\Bigr) .$$

By this transform, a proper plant with an integrator is transformed into a strictly proper plant. Also it can be confirmed that, when $P(s)$ satisfies the assumptions made for the $\mathcal{H}_2$ tracking problem, $\hat{P}(s) := \mathsf{RT}\bigl(P(s)\bigr)$ satisfies those for the $\mathcal{H}_2$ regulation problem. The best achievable performance levels for P and $\hat{P}$ exhibit an alluring connection.

Lemma 8 ([7, 8]) *Consider $P \in \mathbf{\Gamma}$ and its reciprocal transform $\hat{P} := \mathsf{RT}\bigl(P\bigr)$. The best achievable performance level for P in the $\mathcal{H}_2$* tracking *problem, $J^\star(P)$, is the same as the best achievable performance level for $\hat{P}$ in the $\mathcal{H}_2$* regulation *problem, $E^\star(\hat{P})$:*

$$J^\star(P) = E^\star\bigl(\hat{P}\bigr) .$$

The above lemma clearly indicates that $J^\star$ can again be expressed as the difference between two sums of roots, but those related to the reciprocal plant this time. That is the sum of roots has a direct link to the performance limitation in the $\mathcal{H}_2$ tracking problem as well.

An investigation into the effect of reciprocal transform on poles and zeros leads to an expression of $J^\star$ in terms of the characteristics of the original plant P.

Theorem 9 *The performance limitation $J^\star$ of the $\mathcal{H}_2$ tracking problem can be written as*

$$J^\star(P) = \sum_i \Bigl(-\frac{1}{\alpha_i}\Bigr) - \sum_\ell \Bigl(-\frac{1}{z_\ell}\Bigr) , \tag{18}$$

where α_i's are the roots of the spectral factor M_D of (16) *and z_ℓ's are the zeros of the plant P.*

See Appendix B for the proof.

It is pointed out that M_D is again the characteristic polynomial of the closed-loop system constructed with the optimal controller. Contrary to the $\mathcal{H}_2$ regulation case where the expression for $E^\star$ is in terms of pole locations, the best $\mathcal{H}_2$ tracking performance level is expressed in terms of *time constants*, or more specifically, as the difference between the 'average response speed' of the closed-loop system and that of the plant *zeros*. Under the assumption that the achieved closed-loop poles are the same, it is observed that having zeros of the plant far away from the origin does not affect the performance level whether they are minimum phase or non-minimum phase. As the time constant of a minimum phase zero increases, the plant bandwidth widens and the controller can 'see' the plant more. As a result a better performance can be accomplished. Conversely a non-minimum phase zero near the origin brings about an inverse response and degrades the achievable performance. Again it is emphasized that the closed-loop poles are determined by poles/zeros of the plant and the plant gain and that the relationship between the plant zeros and $J^\star$ is not straightforward. However, in the case of the $\mathcal{H}_2$ tracking problem, it is the time constants of the plant zeros (both minimum and non-minimum phase) that have a direct impact on the best achievable performance level.

5 Parametric Optimization via Quantifier Elimination

Section 4 has shown that the best achievable $\mathcal{H}_2$ regulation and tracking performance levels can be expressed in a very simple manner in terms of the sums of roots. The results reviewed in Section 3 implies that, when a plant with parameters is given as in the numerical example demonstrated in Section 2, a polynomial with parametric coefficients can be obtained that has one of the sums of roots as a root. These results can be exploited for parametric optimization based on quantifier elimination (QE) [16].

A QE-based optimization approach has been proposed that can solve possibly non-convex optimization problems under polynomial constraints [17]. Such an approach is applicable to the parametric optimization problem considered in Section 2 owing to the derived facts that $E^\star$ may be expressed as a linear function in σ_{M} and σ_{P} and that a polynomial that has σ_{M} as one of its roots can be computed. In fact, σ_{M} can be removed from (17) and S_f, and a polynomial one of whose roots is $E^\star$ can be obtained, which is denoted by S_E. Due to the linear relationship between $E^\star$ and σ_{M}, the true $E^\star$ is the largest real root of S_E, as well.

Now it is shown how the optimization problem is formulated as a QE problem. Suppose that the set of constraints on parameters (i.e., $\mathbf{q} \in \mathcal{Q}$) can be written as $\varphi(\mathbf{q})$ where $\varphi(\mathbf{q})$ is assumed to consist of a set of algebraic expressions (equalities/inequalities) in parameters. The optimization problem may then be cast as

$$\exists \mathbf{q} \; \big(\, [\, E^\star \text{ is the largest real root of } S_E(E^\star, \mathbf{q}) \,] \; \wedge \; \varphi(\mathbf{q}) \, \big) \; .$$

In order to apply a QE algorithm, the condition '$[\, E^\star$ is the largest real root of $S_E(E^\star, \mathbf{q}) \,]$' needs to be expressed as a set of algebraic expressions. Given a polynomial, the fact that a particular value is its largest real root can be described as the condition that the value is a root of the polynomial and also that there is no real root between that value and $+\infty$. The Sturm-Habicht sequence [18] yields an algebraic condition for the number of polynomial roots in an interval on the real axis when a polynomial with real parametric coefficients is provided. More specifically the condition is written as a set of polynomial inequalities in terms of the coefficients of the original polynomial ($S_E(E^\star, \mathbf{q})$, in this case).

It is thus shown that a QE package such as QEPCAD B [10] can be employed for the parametric optimization problem. A QE algorithm will eliminate quantified variables $\mathbf{q}$ (i.e., all the parameters) and give a set of polynomial inequalities defining the set of values that $E^\star$ can take when $\mathbf{q}$ varies inside $\mathcal{Q}$. Moreover, by tracing down the intermediate results during a phase called the 'lifting phase', it can be found which sets of values of parameters achieve its minimum and maximum.

6 Numerical Example Revisited

This section revisits the numerical example considered in Section 2 and a further detail of how the example is solved is provided. Firstly expressions for σ_{M} and σ_{P} are obtained. It is immediate from the denominator of P that σ_{P} is expressed as in (4). In order to find a polynomial that has σ_{M} as one of its roots, polynomial spectral factorization with parameters is performed for the even polynomial (5). Write its spectral factor as

$$M_D(s) = s^2 + b_1 s + b_0 \; .$$

By comparing the coefficients of (5) and those of $M_D(s)M_D(-s)$, the following polynomial equations are obtained:

$$\begin{cases} b_1^2 - 2b_0 - 4q_1^4 + 4q_1^2 q_2 - q_2^2 = 0 \; , \\ b_0^2 - (1+q_2^2)^2(3-2q_1)^2 = 0 \; . \end{cases}$$

As is stated in Lemma 3, the polynomial parts form the reduced Gröbner basis with respect to the graded reverse lexicographic order $b_1 \succ b_0$. By means of 'parametric' basis conversion, a shape basis is obtained, and the polynomial (6) is obtained that has σ_{M} as its largest real root.

Theorem 6 along with (4) leads to

$$E^\star = \sigma_{\mathrm{M}} - \sigma_{\mathrm{P}} = \sigma_{\mathrm{M}} + 2q_1^2 - q_2 \; .$$

Write

$$\eta = \sigma_{\mathrm{M}} + 2q_1^2 - q_2 \; ,$$

and solve it for σ_{M} to substitute σ_{M} in (6). Then,

$$\begin{aligned} S_E(\eta) &:= S_f(\eta - 2q_1^2 + q_2) \\ &= \eta^4 + (-8q_1^2 + 4q_2)\eta^3 + (16q_1^4 - 16q_1^2 q_2 + 4q_2^2)\eta^2 - 36 + 96q_1 q_2^2 \\ &\qquad + 48q_1 - 72q_2^2 - 16q_1^2 - 36q_2^4 + 48q_1 q_2^4 - 32q_1^2 q_2^2 - 16q_1^2 q_2^4 \end{aligned}$$

is obtained. Notice that the largest real root of $S_E(\eta)$ is the best achievable performance $E^\star$.

By computing the Sturm-Habicht sequence for S_E and simplifying the obtained condition, the condition that η is the largest real root of S_E is shown to be

$$S_E(\eta) = 0 \ \wedge \ \eta - 2q_1^2 + q_2 > 0 \ \wedge \ \eta(\eta - 4q_1^2 + 2q_2) > 0 \ . \tag{19}$$

The first equation $S_E(\eta) = 0$ obviously requires that η should be a root of S_E, and the rest of the condition specifies that η should be the largest real one. Optimization of $E^\star$ over parameters q_1, q_2 thus boils down to the following QE problem:

$$\exists q_1 \, \exists q_2 \ (\text{ Condition (19) } \wedge \ 0 \leq q_1 \leq 1 \ \wedge \ 0 \leq q_2 \leq 1 \ ) \ .$$

QEPCAD B is then applied to the above problem and (7) is obtained. By tracing down the CAD tree during the lifting phase, it is found that the minimum value is attained when $\mathbf{q} \simeq (0.27782, 0.43913)$ and that the maximum value is achieved when $(q_1, q_2) = (1, 0)$. Fig. 1 confirms that the obtained result is correct.

7 Concluding Remarks

This paper has shown that the quantity named the sum of roots is closely related to the best achievable performance levels of the $\mathcal{H}_2$ regulation and tracking problems. Also shown is an algebraic optimization approach that exploits the combination of the derived result and another aspect of the sum of roots as a computation means for polynomial spectral factorization. The property that the sum of roots is the largest real root of a polynomial is further to be exploited for efficient optimization.

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A Proof of Theorem 6

To prove Theorem 6, the following formulae are required:

$$\frac{1}{\pi}\int_0^\infty \log\left(\frac{\omega^2+a^2}{\omega^2}\right)d\omega = a\ ,\quad a\in\mathbb{C}\ ,\quad \operatorname{Re}a>0\ , \tag{20}$$

$$\frac{1}{\pi}\int_0^\infty \log\left\{\left(\frac{\omega^2+(ja)^2}{\omega^2}\right)^2\right\}d\omega = 0\ ,\quad a\in\mathbb{R}\ . \tag{21}$$

Theorem 6 can be proven as follows. The right hand side of (1) is to be proven identical to the right hand side of (17). Firstly it is noted that imaginary axis poles do not contribute to the value of the summation on the right hand side of (1). Thus it can be assumed without loss of generality that unstable poles p_k^{a} are in the *open* right half plane, i.e., $\operatorname{Re}p_k^{\mathrm{a}} > 0$. Further write the roots of P_D in the closed left half plane (i.e., (marginally) stable poles of P) as p_j^{s}. Then, P_D can be written as

$$P_D(s) = \prod_j (s-p_j^{\mathrm{s}})\prod_k (s-p_k^{\mathrm{a}})\ . \tag{22}$$

Write the roots of (16) in the open left half plane as α_i. (Remember that (16) does not have imaginary axis roots.) The spectral factor M_D of (16) can be written as

$$M_D(s) = \prod_i (s-\alpha_i)\ . \tag{23}$$

By expanding (22) and (23), it can be seen that

$$\sigma_{\mathrm{P}} = \sum_j(-p_j^{\mathrm{s}}) + \sum_k(-p_k^{\mathrm{a}})\ ,\quad \sigma_{\mathrm{M}} = \sum_i(-\alpha_i)\ .$$

Now another expression for the integrand is derived:

$$
\begin{aligned}
1 + |P(j\omega)|^2 &= 1 + \frac{|P_N(j\omega)|^2}{|P_D(j\omega)|^2} = \frac{|P_N(j\omega)|^2 + |P_D(j\omega)|^2}{|P_D(j\omega)|^2} = \frac{|M_D(j\omega)|^2}{|P_D(j\omega)|^2} \\
&= \frac{M_D(j\omega)M_D(-j\omega)}{P_D(j\omega)P_D(-j\omega)} = \frac{\prod_i \big(\omega^2 + (-\alpha_i)^2\big)}{\prod_j \big(\omega^2 + (-p_j^{\mathrm{s}})^2\big) \prod_k \big(\omega^2 + (p_k^{\mathrm{a}})^2\big)} \ .
\end{aligned} \tag{24}
$$

Notice that the degrees of the numerator and the denominator of (24) in ω are identical and that the real parts of $-\alpha_i$, $-p_j^{\mathrm{s}}$ and p_k^{a} are all non-negative. Formulae (20), (21) can then be used to derive

$$
\frac{1}{\pi}\int_0^\infty \log\big(1 + |P(j\omega)|^2\big) d\omega = \sum_i (-\alpha_i) - \Big(\sum_j (-p_j^{\mathrm{s}}) + \sum_k p_k^{\mathrm{a}}\Big) \ .
$$

From this and (1),

$$
\begin{aligned}
E^\star &= 2\sum_k p_k^{\mathrm{a}} + \Big\{\sum_i (-\alpha_i) - \Big(\sum_j (-p_j^{\mathrm{s}}) + \sum_k p_k^{\mathrm{a}}\Big)\Big\} \\
&= \sum_i (-\alpha_i) - \Big(\sum_j (-p_j^{\mathrm{s}}) + \sum_k (-p_k^{\mathrm{a}})\Big) = \sigma_{\mathrm{M}} - \sigma_{\mathrm{P}} \ .
\end{aligned}
$$

This concludes the proof.

B Proof of Theorem 9

In this proof, symbols without the hat ($\hat{}$) mark are used to denote those related to the original plant P, whilst symbols with it denote those related to the transformed plant $\hat{P}$. Denote by $-\hat{\sigma}_{\mathrm{M}}$ and $-\hat{\sigma}_{\mathrm{P}}$ the sum of roots of $\hat{M}_D$ and that of $\hat{P}_D$, respectively, as is defined in Subsection 4.1. Theorem 6 and Lemma 8 together imply that

$$
J^\star(P) = E^\star\big(\hat{P}\big) = \hat{\sigma}_{\mathrm{M}} - \hat{\sigma}_{\mathrm{P}} \ . \tag{25}
$$

So expressions of $\hat{\sigma}_{\mathrm{M}}$ and $\hat{\sigma}_{\mathrm{P}}$ in terms of the characteristics of P are sought.

Let the degree of P be n. Write P, P_N, and P_D as in (13), (14), and (15), respectively. Since P has an integrator, the constant term of P_D is zero, i.e., $c_0^{\mathrm{d}} = 0$. Also, P_N and P_D are coprime, and therefore the constant term of P_N is non-zero, i.e., $c_0^{\mathrm{n}} \neq 0$. Then, $\hat{P}$ can be written as

$$
\hat{P}(s) = \mathsf{RT}\big(P(s)\big) = \frac{P_D(\frac{1}{s})}{P_N(\frac{1}{s})} = \frac{\hat{P}_N(s)}{\hat{P}_D(s)} \ ,
$$

where

$$\hat{P}_N(s) = \frac{1}{c_0^{\mathrm{n}}}\left(c_1^{\mathrm{d}} s^{n-1} + c_2^{\mathrm{d}} s^{n-2} + \cdots + c_{n-1}^{\mathrm{d}} s + 1\right),$$
$$\hat{P}_D(s) = \frac{1}{c_0^{\mathrm{n}}}\left(c_0^{\mathrm{n}} s^{n} + c_1^{\mathrm{n}} s^{n-1} + \cdots + c_{n-2}^{\mathrm{n}} s^2 + c_{n-1}^{\mathrm{n}} s\right) .$$

If the degree of P_N is n_{z}, i.e., the first non-zero coefficient of P_N is $c_{n_{\mathrm{z}}}^{\mathrm{n}}$, there are n_{z} (finite) zeros in P. Since

$$\hat{P}_D(s) = \frac{s^n}{c_0^{\mathrm{n}}} P_N\left(\frac{1}{s}\right) ,$$

n_{z} zeros of P are transformed into n_{z} poles of $\hat{P}$ and, if ζ is a root of $P_N(s)$, then $\frac{1}{\zeta}$ is a root of $\hat{P}_D(s)$. The remaining $(n - n_{\mathrm{z}})$ poles of $\hat{P}$ are located at $s = 0$. Therefore,

$$\hat{\sigma}_{\mathrm{P}} = \sum_{\ell}\left(-\frac{1}{z_\ell}\right) , \tag{26}$$

where z_ℓ's are the zeros of P.

Now consider $\hat{\sigma}_{\mathrm{M}}$. Observe that

$$\begin{aligned}\hat{P}_N(s)&\hat{P}_N(-s) + \hat{P}_D(s)\hat{P}_D(-s) \\ &= \frac{(-1)^n s^{2n}}{\left(c_0^{n}\right)^2}\left\{P_N\left(\frac{1}{s}\right)P_N\left(-\frac{1}{s}\right) + P_D\left(\frac{1}{s}\right)P_D\left(-\frac{1}{s}\right)\right\} .\end{aligned} \tag{27}$$

Again, if ζ is a root of $P_N(s)P_N(-s) + P_D(s)P_D(-s)$, then $\frac{1}{\zeta}$ is a root of (27). Moreover a root ζ in the left (resp., right) half plane is transformed into a root $\frac{1}{\zeta}$ in the left (resp., right) half plane. Due to the coprimeness of P_N and P_D, there is no imaginary axis root in $P_N(s)P_N(-s) + P_D(s)P_D(-s)$, which suggests that (27) has neither imaginary axis roots nor roots at infinity. It can thus be concluded that

$$\hat{\sigma}_{\mathrm{M}} = \sum_{i}\left(-\frac{1}{\alpha_i}\right) , \tag{28}$$

where α_i's are the (open) left half plane roots of $P_N(s)P_N(-s) + P_D(s)P_D(-s)$.

Equations (25), (26) and (28) together lead to (18).