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Characterization of a Complementary Sensitivity Property in Feedback Control – An Information Theoretic Approach –

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Abstract

This paper addresses a characterization of a complementary sensitivity property in feedback control using an information theoretic approach. We derive an integral-type constraint of the complementary sensitivity function with respect to the unstable zeros of the open-loop transfer function. It is an analogue of Bode's integral formula for the sensitivity gain. To show the constraint, we first show a conservation law of the entropy and mutual information of signals in the feedback system. Then, we clarify the relation between the mutual information of control signals and the unstable zeros of the open-loop transfer function.

1 Introduction

It has been known that control theory and information theory share a common background as both theories study signals and dynamical systems in general. One way to describe their difference is that the focal point of information theory is the signals involved in systems while control theory focuses more on systems which represent the relation between the input and output signals. Thus, in a certain sense, we may expect that they have a complementary relation. For this reason, studies on the interactions of the two theories have recently attracted a lot of attention. We briefly describe three research directions in the following.

In networked control systems, there certainly are issues related to both control and communication since communication channels with data losses,

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time delays, and quantization errors are employed between the plants and controllers (see, e.g., [1] and the references therein). To guarantee the overall control performance in such systems, it is important to evaluate the amount of information that the channels can transfer. Thus, for the analyses of networked control systems, information theoretic approaches are especially useful, and notions and results from this theory can be applied. For example, to characterize the properties of the channels, their capacity and rate of communication, which represent the number of bits that can be transfered at each time step, can be used. The results in [8] and [15] show the limitation in the communication rate for the existence of controllers, encoders, and decoders to stabilize discrete-time linear feedback systems.

On the other hand, by considering the interaction of control and communication, a certain problem in information theory can be dealt with as a control problem. The work of [4] shows an equivalence between feedback stabilization through an analog communication channel and a communication scheme based on feedback. As a consequence, the problem of finding optimal encoder and decoder in the communication system is reduced to the design of an optimal feedback controller.

While control theory, in many cases, considers systems that are linear time invariant, information theory imposes assumptions on the systems that are less stringent. This is because the focus there is more on the signals and not on their input-output relation. Thus, based on information theoretic approaches, we may expect to extend prior results in control theory. One such result can be found in [7], where a sensitivity property is analyzed and Bode's integral formula [2] is extended to a more general class of systems. A fundamental limitation of sensitivity functions is presented in relation to the poles of the plants.

In this paper, we follow the approach of [7] and characterize a complementary sensitivity property in a feedback system by measuring the entropies of the signals. In particular, we derive a limitation of the complementary sensitivity function with respect to the unstable zeros of the open-loop system. This limitation is shown in two steps as follows: We first show a conservation law of the entropy and mutual information of the signals in the feedback system. Then, we clarify the relation between the mutual information of a control signal and the unstable zeros of the open-loop transfer function. This result corresponds to the Bode's integral formula for the complementary sensitivity by [14]. Since this formula is derived from the viewpoint of information theory, in future research, we expect to generalize this result to the cases for nonlinear systems and networked control systems.

This paper is organized as follows: We first introduce Bode's integral formula and related works, and some notions and results in information theory in Section 2. In Section 3, we present the problem setting and some properties of the entropy and mutual information of the signals in the system. In Section 4, we show the main result of the paper. Finally, the conclusion is in Section 5.

2 Preliminaries

In this section, first, we introduce prior works related to the fundamental limitations on the sensitivity and complementary sensitivity functions. Then, we describe some notation and definitions used in the paper.

2.1 Bode's integral formula and related works

It is well known that the sensitivity and complementary sensitivity functions represent basic properties of feedback systems such as disturbance attenuation, sensor-noise reduction, and robustness against uncertainties in the plant model. One of the fundamental properties of the sensitivity functions is the water-bed effect for linear feedback systems. This was first shown in [2]. Although Bode's work deals with continuous-time systems, we present the corresponding result for discrete-time systems [13] as follows: Suppose that the open-loop system L is single-input single-output, linear time invariant, and strictly proper in Figure 1. If the open-loop system L and the feedback loop are stable, then the sensitivity function $S(z)$ given by

$$S(z) := \frac{1}{1+L(z)}$$

must satisfy

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|S(\mathrm{e}^{\mathrm{j}\omega})|\mathrm{d}\omega = 0.$$

This integral constraint on the sensitivity function is known as Bode's integral formula. Because of its importance, this formula has been generalized in many ways (e.g., [5, 11, 12, 6]).

In particular, the work by [14] gives an integral-type constraint of complementary sensitivity functions corresponding to Bode's integral formula. We briefly introduce this result next.

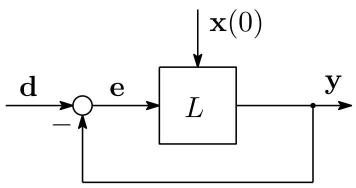

Figure 1: Discrete-time feedback control system.

Consider the system depicted in Figure 1. Let the state-space representation of L be given by

$$L : \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{y}(k) \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix}, \tag{1}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state, $\mathbf{y}(k) \in \mathbb{R}$ is the output, and $\mathbf{e}(k) \in \mathbb{R}$ is the error signal. Suppose that the relative degree of the open-loop transfer function $L(z)$ is $\nu \geq 1$. This implies that

$$\begin{aligned} &CA^{\nu-1}B \neq 0, \\ &CA^{j-1}B = 0, \quad j = 1, \cdots, \nu - 1. \end{aligned} \tag{2}$$

Here, let $D_0 := CA^{\nu-1}B$. This is the ratio of the leading coefficients of $L(z)$. Moreover, let $\mathcal{UZ}_L$ be the set of unstable zeros of $L(z)$:

$$\mathcal{UZ}_L := \{z \,|\, L(z) = 0, \, |z| \geq 1\}, \tag{3}$$

and let $T(z)$ be the complementary sensitivity function:

$$T(z) := \frac{L(z)}{1 + L(z)}. \tag{4}$$

Then, the following proposition holds.

Proposition 1 *[14] If the feedback system is stable, then the complementary sensitivity function $T(z)$ satisfies*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |T(\mathrm{e}^{\mathrm{j}\omega})| \mathrm{d}\omega = \sum_{\beta \in \mathcal{UZ}_L} \log |\beta| + \log |D_0|. \tag{5}$$

We write $\log_2(\cdot)$ simply as $\log(\cdot)$. This notation is also adopted in the following.

This relation has been shown by applying Jensen's formula, which is a well-known result in complex analysis. In this paper, we derive a limitation similar to (5) by evaluating the entropy and mutual information of signals in the feedback system.

2.2 Entropy and mutual information

In this section, we introduce some notation and basic results from information theory that we use in the paper (see [3]).

We adopt the following notation.

- We represent random variables using boldface letters, such as $\mathbf{x}$.

- Consider a discrete-time stochastic process $\{\mathbf{x}(k)\}_{k=0}^{\infty}$. We represent a sequence of random variables from $k = l$ to $k = m$ ($m \geq l$) as $\mathbf{x}_l^m := \{\mathbf{x}(k)\}_{k=l}^{m}$. In particular, when $l = 0$, we write $\mathbf{x}_l^m$ simply as $\mathbf{x}^m$.

- We use $\mathbf{x}$ instead of $\{\mathbf{x}(k)\}_{k=0}^{\infty}$ when it is clear from the context.

- The operation $\mathrm{E}[\cdot]$ denotes the expectation of a random variable.

Entropy is a notion widely used as a measure of uncertainty of a random variable. It is defined as follows.

Definition 1 *(Entropy and conditional entropy) The (differential) entropy $h(\mathbf{x})$ of a continuous random variable $\mathbf{x} \in \mathbb{R}$ with the probability density $p_{\mathbf{x}}$ is defined as*

$$h(\mathbf{x}) := -\int_{\mathbb{R}} p_{\mathbf{x}}(\xi) \log p_{\mathbf{x}}(\xi) \mathrm{d}\xi.$$

If $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ have a joint probability density function $p_{\mathbf{x},\mathbf{y}}$, we can also define the conditional entropy $h(\mathbf{x}|\mathbf{y})$ of $\mathbf{x}$ assuming $\mathbf{y}$ as

$$h(\mathbf{x}|\mathbf{y}) := -\int_{\mathbb{R}^2} p_{\mathbf{x},\mathbf{y}}(\xi, \eta) \log p_{\mathbf{x}|\mathbf{y}}(\xi, \eta) \mathrm{d}\xi \mathrm{d}\eta.$$

Next, we introduce mutual information, which is a measure of the amount of information that one random variable contains about another random variable.

Definition 2 *(Mutual information) The mutual information $I(\mathbf{x}; \mathbf{y})$ between $\mathbf{x} \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{R}$ with the joint probability density $p_{\mathbf{x},\mathbf{y}}$ is defined as*

$$I(\mathbf{x}; \mathbf{y}) := \int_{\mathbb{R}^2} p_{\mathbf{x},\mathbf{y}}(\xi, \eta) \log \frac{p_{\mathbf{x},\mathbf{y}}(\xi, \eta)}{p_{\mathbf{x}}(\xi) p_{\mathbf{y}}(\eta)} \mathrm{d}\xi \mathrm{d}\eta.$$

Note that we assume the existence of the probability density and the joint probability density functions in the above definitions.

The following is a list of basic properties of entropy and mutual information which are required in the paper. Their proofs can be found in, e.g., [3, 9, 10].

- *Symmetry and nonnegative property:*

$$\begin{aligned} I(\mathbf{x}; \mathbf{y}) &= I(\mathbf{y}; \mathbf{x}) \\ &= h(\mathbf{x}) - h(\mathbf{x}|\mathbf{y}) = h(\mathbf{y}) - h(\mathbf{y}|\mathbf{x}) \geq 0 \end{aligned} \tag{6}$$

- *Entropy and conditional entropy:* From the above property, the following holds:

$$h(\mathbf{x}|\mathbf{y}) \leq h(\mathbf{x}). \tag{7}$$

- *Chain rule:*

$$h(\mathbf{x},\mathbf{y}) = h(\mathbf{x}) + h(\mathbf{y}|\mathbf{x}) \tag{8}$$

- *Maximum entropy:* Consider a random vector $\mathbf{x} \in \mathbb{R}^m$ with variance $V_{\mathbf{x}} \in \mathbb{R}^{m\times m}$. The following holds:

$$h(\mathbf{x}) \leq \frac{1}{2}\log\left((2\pi \mathrm{e})^m \det V_{\mathbf{x}}\right). \tag{9}$$

  We have equality if $\mathbf{x}$ is Gaussian.

- *Data processing inequality:* Suppose that f is a measurable function on the appropriate space. Then the following holds:

$$h(\mathbf{x}|\mathbf{y}) \leq h(\mathbf{x}|f(\mathbf{y})). \tag{10}$$

  We have equality if f is invertible.

- *Transformations of random variables and their entropy:* Suppose that f is a piecewise C^1-class function and $\mathbf{x}$ and $\mathbf{y} = f(\mathbf{x})$ take continuous values. Then the following holds:

$$h(\mathbf{y}) = h(\mathbf{x}) + \mathrm{E}\left[\log|J_{\mathbf{x}}|\right], \tag{11}$$

  where $J_{\mathbf{x}}$ is the Jacobian of the transformation f.

- Suppose that f is any given function on the appropriate space. Then the following holds:

$$h(\mathbf{x} - f(\mathbf{y})|\mathbf{y}) = h(\mathbf{x}|\mathbf{y}). \tag{12}$$

Now we would like to introduce some notions for stochastic processes. The entropy rate is a time average of the entropy of a process and plays an important role in our analysis.

Definition 3 *(Entropy rate) The entropy rate $h_\infty(\mathbf{x})$ of a stochastic process $\mathbf{x}$ is defined as*

$$h_\infty(\mathbf{x}) := \limsup_{k\to\infty} \frac{h(\mathbf{x}^{k-1})}{k}.$$

Definition 4 *(Asymptotically stationary process) A zero mean stochastic process* $\mathbf{x}$ *(*$\mathbf{x}(k) \in \mathbb{R}$*) is asymptotically stationary if the following limit exists for every* $\gamma \in \mathbb{Z}$*:*

$$\overline{R}_{\mathbf{x}}(\gamma) := \lim_{k\to\infty} E\left[\mathbf{x}(k)\mathbf{x}(k+\gamma)\right].$$

For an asymptotically stationary process $\mathbf{x}$, we can define the asymptotic power spectral density $\overline{S}_{\mathbf{x}}$ using $\overline{R}_{\mathbf{x}}$ as

$$\overline{S}_{\mathbf{x}}(\omega) := \sum_{\gamma=-\infty}^{\infty} \overline{R}_{\mathbf{x}}(\gamma)\mathrm{e}^{-\mathrm{j}\gamma\omega}.$$

The following lemma, which is shown in [7], gives the relation between the entropy rate and the asymptotic power spectral density.

Lemma 1 *[7] If* $\mathbf{x}$ *is an asymptotically stationary process, then the following inequality holds:*

$$h_{\infty}(\mathbf{x}) \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} \log(2\pi\mathrm{e}\overline{S}_{\mathbf{x}}(\omega))\mathrm{d}\omega, \tag{13}$$

where the equality holds if, in addition, $\mathbf{x}$ *is a Gaussian process.*

3 Problem setting and some properties

In this section, we formulate our problem and present two key properties which are required to drive our main result. The first one shows a conservation law of the entropy, and the second one shows the relation between the mutual information and the zeros of the open-loop system.

3.1 Problem setting

Consider the system depicted in Figure 1. Suppose that the state-space representation of $L(z)$ is given by (1), and $\mathbf{x}(k) \in \mathbb{R}^n$, $\mathbf{d}(k) \in \mathbb{R}$, $\mathbf{e}(k) \in \mathbb{R}$, and $\mathbf{y}(k) \in \mathbb{R}$ are random variables. We characterize the complementary sensitivity function $T(z)$ in (4) by evaluating the entropy of signals. Here, it is assumed that the feedback system is stable in the mean-square sense, i.e.,

$$\sup_k \mathrm{E}[\mathbf{x}(k)^\top \mathbf{x}(k)] < \infty. \tag{14}$$

To deal with asymptotically stationary processes, we now define a complementary sensitivity-like function $\overline{T}$ by using the asymptotic power spectral densities of the input and output signals of T.

Definition 5 *(Complementary sensitivity-like function) If the stochastic processes* $\mathbf{d}$ *and* $\mathbf{y}$ *are asymptotically stationary, then the complementary sensitivity-like function is given by*

$$\bar{T}(\omega) := \sqrt{\frac{\bar{S}_{\mathbf{y}}(\omega)}{\bar{S}_{\mathbf{d}}(\omega)}}.$$

Remark 1 *If a stochastic process is stationary, its asymptotic power spectral density is equal to the ordinary power spectral density. Thus, when* $\mathbf{d}$ *and* $\mathbf{y}$ *are stationary, we have that*

$$\bar{T}(\omega) = \left|T(\mathrm{e}^{\mathrm{j}\omega})\right|.$$

This can be shown by the well-known relation between a linear time-invariant system with a stable transfer function and the power spectral densities of its input and output signals [9].

We consider the property of $\bar{T}$ instead of T, and derive a constraint similar to (5). We note that because of the relation given by Lemma 1, the ratio of the power spectral density in $\bar{T}$ can be expressed as the difference in the entropy rates of the input $\mathbf{d}$ and output $\mathbf{y}$ of T. Hence, in the following, we first analyze in difference of the entropy rates of $\mathbf{d}$ and $\mathbf{y}$ in Section 3.2. Next, we show the relation between the difference of the entropy rates and the unstable zeros of the open-loop transfer function $L(z)$ in Section 3.3. Finally, we show an integral-type constraint on the complementary sensitivity property with respect to the unstable zeros in Section 4.

We assume that $\mathbf{d}^k$ and $\mathbf{x}(0)$ are independent for every $k \in \mathbb{Z}_+$, and $|h(\mathbf{x}(0))| < \infty$[1].

3.2 The difference of the entropy rates

Here, we analyze the difference of the entropy rates $h_\infty(\mathbf{d})$ and $h_\infty(\mathbf{y})$.

The following proposition holds.

Proposition 2 *Consider the system depicted in Figure 1. The following inequality holds:*

$$h_\infty(\mathbf{y}) - h_\infty(\mathbf{d}) \geq \liminf_{k\to\infty} \frac{I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0))}{k} + \log|D_0|. \tag{15}$$

This relation is due to a conservation law of entropy between $\mathbf{d}$ and $\mathbf{y}$. We describe this in the following as a lemma.

[1]Actually, this assumption can be replaced with $|h(\mathbf{x}_u(0))| < \infty$ (see Section 3.3).

Lemma 2 *Consider the system depicted in Figure 1. The following relation holds:*

$$h(\mathbf{y}_\nu^{k+\nu}) = h(\mathbf{d}^k) + I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0)) + (k+1)\log|D_0|. \tag{16}$$

To derive this lemma, we have to consider how the entropy of $\mathbf{d}$ at time k, $h(\mathbf{d}(k))$, affects $h(\mathbf{y}(k))$. However, since the open-loop transfer function $L(z)$ is strictly proper, there is time delay of ν steps due to the relative degree of L, that is, $\mathbf{d}(k)$ has an influence on the output $\mathbf{y}$ only after time $k+\nu$.

To deal with this problem, we define the auxiliary system L_0 and the signal $\mathbf{y}^+$ as

$$L_0(z) := z^\nu L(z), \tag{17}$$
$$\mathbf{y}^+(k) := \mathbf{y}(k+\nu), \tag{18}$$

where ν is the relative degree of the open-loop transfer function $L(z)$. The state-space representation of $L_0(z)$ is given by:

$$\begin{aligned} L_0 : \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{y}^+(k) \end{bmatrix} &= \begin{bmatrix} A & B \\ CA^\nu & CA^{\nu-1}B \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix} \\ &=: \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{e}(k) \end{bmatrix}. \end{aligned}$$

It is clear that $D_0 \neq 0$ because of (2), and hence L_0 is a biproper system. The system in Figure 1 can be expressed as Figure 2 by using L_0 and $\mathbf{y}^+$.

We now consider a conservation law of the entropy between $\mathbf{d}$ and $\mathbf{y}^+$ instead of $\mathbf{y}$. The proof of Lemma 2 is provided in the following.

Proof. It follows that

$$\begin{aligned} &h(\mathbf{y}^+(i)|(\mathbf{y}^+)^{i-1}) \\ &= h(\mathbf{y}^+(i)|(\mathbf{y}^+)^{i-1}, \mathbf{x}(0)) + I(\mathbf{y}^+(i); \mathbf{x}(0)|(\mathbf{y}^+)^{i-1}) \\ &= h(\mathbf{y}^+(i)|\mathbf{d}^{i-1}, \mathbf{x}(0)) + I(\mathbf{y}^+(i); \mathbf{x}(0)|(\mathbf{y}^+)^{i-1}), \end{aligned}$$

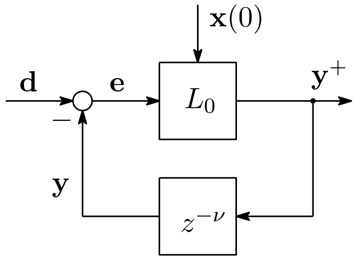

Figure 2: Equivalent system with the biproper system L_0.

where the first equality follows by (6), and the second one follows by (10). Moreover, using the property (11), we have that

$$\begin{aligned} h(\mathbf{y}^+(i)|(\mathbf{y}^+)^{i-1}) =& h(\mathbf{d}(i)|\mathbf{d}^{i-1}, \mathbf{x}(0)) + \log|D_0| \\ &+ I(\mathbf{y}^+(i); \mathbf{x}(0)|(\mathbf{y}^+)^{i-1}). \end{aligned}$$

Since $\mathbf{x}(0)$ and $\mathbf{d}(i)$ are independent, $\mathbf{x}(0)$ vanishes in the first term of the right-hand side of this equation. Thus, we have that

$$\begin{aligned} & h(\mathbf{y}^+(i)|(\mathbf{y}^+)^{i-1}) \\ & = h(\mathbf{d}(i)|\mathbf{d}^{i-1}) + \log|D_0| + I(\mathbf{y}^+(i); \mathbf{x}(0)|(\mathbf{y}^+)^{i-1}). \end{aligned} \tag{19}$$

Now, by summing both sides of (19) for $i = 0, 1, \cdots, k$, we obtain

$$h((\mathbf{y}^+)^k) = h(\mathbf{d}^k) + (k+1)\log|D_0| + I((\mathbf{y}^+)^k; \mathbf{x}(0)). \tag{20}$$

Here, we have used the chain rules:

$$\begin{aligned} h(\mathbf{a}^k) &= \sum_{i=0}^{k} h(\mathbf{a}(i)|\mathbf{a}^{i-1}), \\ I(\mathbf{a}^k; \mathbf{b}) &= \sum_{i=0}^{k} I(\mathbf{a}(i); \mathbf{b}|\mathbf{a}^{i-1}), \end{aligned}$$

which follow directly from (8). Finally, by the definition of $\mathbf{y}^+$, the relation (20) is equivalent to (16). □

Remark 2 *Lemma 2 shows that a conservation law of entropy holds between* $\mathbf{d}$ *and* $\mathbf{y}$. *Intuitively, one can understand that* $\log|D_0|$ *reflects the scaling caused by the system* L *(see (11)), and* $I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0))$ *shows the effect of the initial state* $\mathbf{x}(0)$, *which can be viewed as an external input between* $\mathbf{d}$ *and* $\mathbf{y}$, *on* $\mathbf{y}$.

In Proposition 2, the relation (15) can be shown by dividing (16) by k and taking the limsup as $k \to \infty$ on both sides.

3.3 Mutual information and unstable zeros

The relation between $h_\infty(\mathbf{d})$ and $h_\infty(\mathbf{y})$ has been clarified by Proposition 2. We next consider the relation between the mutual information term in (15) and the unstable zeros of the open-loop transfer function $L(z)$.

The mutual information is a quantity in the time domain. In general, however, it is difficult to deal with the zeros of transfer functions in this domain. Thus, we view the zeros of L as the poles of the inverse system of L. The poles are more convenient for our analysis because they can be expressed as the eigenvalues of the state matrix of the system. Moreover, this enables us to apply results in [7], where, for an unstable system, the mutual information between the initial state and the output of the system is related to its unstable poles.

One problem of this approach is that since L is strictly proper, the inverse system of L is improper. For this reason, we consider the inverse system of the biproper system L_0 defined by (17).

Let $\hat{L}_0$ denote the inverse system of L_0. The state-space representation of $\hat{L}_0$ is given by

$$\begin{aligned}\hat{L}_0 : \begin{bmatrix} \mathbf{x}(k+1) \\ \mathbf{e}(k) \end{bmatrix} \\ &= \begin{bmatrix} A_0 - B_0 D_0^{-1} C_0 & B_0 D_0^{-1} \\ -D_0^{-1} B_0 & D_0^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{y}^+(k) \end{bmatrix} \\ &=: \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \mathbf{y}^+(k) \end{bmatrix}. \end{aligned} \tag{21}$$

The system in Figure 2 can be equivalently expressed as Figure 3 by using $\hat{L}_0$.

Now, without loss of generality, $\hat{A}$ can be divided into the stable part $\hat{A}_s \in \mathbb{R}^{n_s \times n_s}$ and the unstable part $\hat{A}_s \in \mathbb{R}^{n_u \times n_u}$ such as

$$\hat{A} = \begin{bmatrix} \hat{A}_s & 0 \\ 0 & \hat{A}_u \end{bmatrix},$$

where all eigenvalues of $\hat{A}_s$ lie inside the unit circle, and those of $\hat{A}_u$ lie outside or on the unit circle. Let $\mathbf{x}_s(k) \in \mathbb{R}^{n_s}$ and $\mathbf{x}_u(k) \in \mathbb{R}^{n_u}$ be the parts of the state variable $\mathbf{x}(k)$ corresponding to $\hat{A}_s$ and $\hat{A}_u$, respectively. We similarly define $\hat{B}_s$ and $\hat{B}_u$ as the parts of $\hat{B}$.

We have the following proposition.

Proposition 3 *Consider the system depicted in Figure 3. If the system is stable in the mean-square sense (14), then the following inequality holds:*

$$\liminf_{k\to\infty} \frac{I(\mathbf{y}_\nu^{k+\nu}; \mathbf{x}(0))}{k} \geq \sum_{\beta \in \mathcal{UZ}_L} \log|\beta|, \tag{22}$$

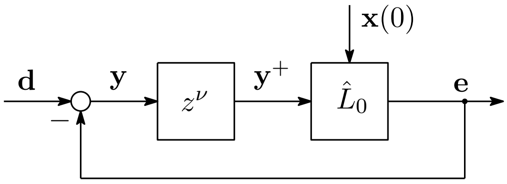

Figure 3: Equivalent system with the inverse system $\hat{L}_0$ of L_0.

where $\mathcal{UZ}_L$ is the set of unstable zeros of $L(z)$ given in (3).

Proof. From (21), we have that

$$\begin{aligned}\mathbf{x}_u(k) &= \hat{A}_u^k \mathbf{x}_u(0) + \sum_{i=0}^{k-1} \hat{A}_u^{k-1-i} \hat{B}_u \mathbf{y}^+(i) \\ &= \hat{A}_u^k \tilde{\mathbf{x}}(k),\end{aligned} \tag{23}$$

where $\tilde{\mathbf{x}}$ is given as

$$\tilde{\mathbf{x}}(k) := \mathbf{x}_u(0) + \sum_{i=0}^{k-1} \hat{A}_u^{-i-1} \hat{B}_u \mathbf{y}^+(i). \tag{24}$$

Let $V_{\mathbf{x}}(k)$ denote the variance of $\mathbf{x}(k)$. From the above equation, we have that

$$V_{\mathbf{x}_u}(k) = \left(\hat{A}_u^k\right) V_{\tilde{\mathbf{x}}}(k) \left(\hat{A}_u^k\right)^\top.$$

Thus, it follows that

$$\log\det\left(V_{\mathbf{x}_u}(k)\right) = 2k \log\left|\det \hat{A}_u\right| + \log\left(\det V_{\tilde{\mathbf{x}}}(k)\right).$$

From this and the property (9), we have that

$$\begin{aligned}&\frac{h(\tilde{\mathbf{x}}(k))}{k} \\ &\le \frac{\log\left\{(2\pi\mathrm{e})^{n_u}\det(V_{\tilde{\mathbf{x}}}(k))\right\}}{2k} \\ &= \frac{\log(2\pi\mathrm{e})^{n_u}}{2k} + \frac{\log\det(V_{\mathbf{x}_u}(k))}{2k} - \log\left|\det(\hat{A}_u)\right|.\end{aligned}$$

We have $\sup_k V_{\mathbf{x}_u}(k) < \infty$ because of the stability of the feedback system. Hence, we have

$$\limsup_{k\to\infty} \frac{h(\tilde{\mathbf{x}}(k))}{k} \le -\log\left|\det(\hat{A}_u)\right|. \tag{25}$$

Here, note the left-hand side of (22). It follows that

$$\begin{aligned}I((\mathbf{y}^+)^k; \mathbf{x}(0)) &\ge I((\mathbf{y}^+)^k; \mathbf{x}_u(0)) \\ &= h(\mathbf{x}_u(0)) - h(\mathbf{x}_u(0)|(\mathbf{y}^+)^k).\end{aligned}$$

The inequality is due to changing the variables from $\mathbf{x}(0)$ to $\mathbf{x}_u(0)$. Since, in (24), $\tilde{\mathbf{x}}(k)$ is denoted by $\mathbf{x}(0)$ and $(\mathbf{y}^+)^{k-1}$, we have

$$\begin{aligned}&h(\mathbf{x}_u(0)) - h(\mathbf{x}_u(0)|(\mathbf{y}^+)^k) \\ &= h(\mathbf{x}_u(0)) - h(\tilde{\mathbf{x}}(k)|(\mathbf{y}^+)^k) \\ &\ge h(\mathbf{x}_u(0)) - h(\tilde{\mathbf{x}}(k))\end{aligned}$$

by using (12). The inequality follows from (7). Then, we have

$$I((\mathbf{y}^+)^k; \mathbf{x}(0)) \geq h(\mathbf{x}_u(0)) - h(\tilde{\mathbf{x}}(k)). \tag{26}$$

Finally, from (25) and (26), we obtain

$$\begin{aligned}\liminf_{k\to\infty} \frac{I((\mathbf{y}^+)^k; \mathbf{x}(0))}{k} &\geq \log \left|\det\left(\hat{A}_u\right)\right| \\ &= \sum_{\lambda \in \mathcal{UP}_{\hat{L}_0}} \log|\lambda|,\end{aligned}$$

where $\mathcal{UP}_{\hat{L}_0}$ is the set of the unstable poles of $\hat{L}_0(z)$. We have (22) by expressing this equation in terms of $\mathbf{y}$, and using the fact that the set of the unstable poles of $\hat{L}_0(z)$ is equal to the set of unstable zeros of $L(z)$. □

Remark 3 *In general, from the viewpoint of the open-loop system, when the system is unstable, the system amplifies the initial state at a level depending on the size of the unstable poles (see, e.g., (23)). Hence, we can say that in systems having more unstable dynamics, the signals contain more information about the initial state. Therefore, in Figure 3, we can expect the mutual information between the input* $\mathbf{y}$ *and* $\mathbf{x}(0)$ *to be a function of the unstable poles. Proposition 3 corresponds to this observation.*

4 Main result

We are now in a position to present the main result of the paper. The following theorem provides an integral-type constraint on the complementary sensitivity-like function $\overline{T}$. This is obtained by the result of Proposition 2 and 3.

Theorem 1 *Consider the system depicted in Figure 1. If the system is stable in the mean-square sense (14), then the following holds:*

$$h_\infty(\mathbf{y}) - h_\infty(\mathbf{d}) \geq \sum_{\beta \in \mathcal{UZ}_L} \log|\beta| + \log|D_0|. \tag{27}$$

Additionally, if $\mathbf{d}$ *is an asymptotically stationary Gaussian process, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\overline{T}(\mathrm{j}\omega)\right| \mathrm{d}\omega \geq \sum_{\beta \in \mathcal{UZ}_L} \log|\beta| + \log|D_0|. \tag{28}$$

Proof. The relation (27) follows immediately by substituting (22) in Proposition 3 into (15) in Proposition 2.

Under the assumption that the input $\mathbf{d}$ is asymptotically stationary, the output process $\mathbf{y}$ is asymptotically stationary as well since the feedback system is stable and linear time-invariant. Thus, we have

$$h_\infty(\mathbf{d}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left(2\pi \mathrm{e}\overline{S}_{\mathbf{d}}(\omega)\right) \mathrm{d}\omega,$$
$$h_\infty(\mathbf{y}) \le \frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left(2\pi \mathrm{e}\overline{S}_{\mathbf{y}}(\omega)\right) \mathrm{d}\omega,$$

by using (13). Then, the following holds by (15):

$$\begin{aligned}\frac{1}{4\pi}\int_{-\pi}^{\pi} \log \frac{\overline{S}_{\mathbf{y}}(\omega)}{\overline{S}_{\mathbf{d}}(\omega)} \mathrm{d}\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|\overline{T}(\mathrm{j}\omega)\right| \mathrm{d}\omega \\ &\ge \liminf_{k\to\infty} \frac{I(\mathbf{y}_\nu^{k+\nu};\mathbf{x}(0))}{k} + \log|D_0|.\end{aligned}$$

We obtain (28) from this and (22). □

Remark 4 *The relation (28) is similar to (5) in Proposition 1, and has been shown independently of the result in [14]. We consider a complementary sensitivity property from the viewpoint of entropy and mutual information. We note that the entropy rate of a signal is a notion in the time domain and thus is well defined even for systems which do not have transfer function forms. This generalization is an important consequence of the information theoretic approach here. Moreover, this result will be useful for further extensions to networked control systems, nonlinear systems, and so on.*

Note that (28) is an inequality constraint. From our analysis, it is unclear when the equality holds here and moreover whether we can show a condition for the equality to hold by the information theoretic approach. However, as we described in Remark 1, when $\mathbf{d}$ *and* $\mathbf{y}$ *are stationary stochastic processes, we have* $\overline{T}(\omega) = |T(\mathrm{e}^{\mathrm{j}\omega})|$*. Thus, the equality in (28) holds from Proposition 1.*

5 Conclusion

This paper has addressed a characterization of a complementary sensitivity property by evaluating the entropy of signals in the feedback system. In particular, we have shown a constraint similar to Bode's integral formula (5). We would like to apply this result to networked control systems and nonlinear systems in future research.

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