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State Estimation under Information Constraints with Memory-less Encoders

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Abstract

Many sensor networks carry quantized data these days and it is interesting problem to know how much data rate is necessary and how we can encode efficiently for estimating state of a linear system. Preceding study shows that if full information is available at the encoder, we can estimate the state in arbitrary precision asymptotically with finite data rate which is determined by instability of the plant. In this paper, we propose a new quantization method that achieves bounded data rate even if the encoder is memory-less and does not know inputs to the plant. This result is related to so-called side information problem in information theory. We can consider inputs to the plant as side information available only at the decoder.

1 Introduction

Motivated by recent development of digital technology and communication, estimation and control in the presence of information constrains have received considerable attention. This problem introduces an information theoretical view point and a question how to choose the bits of information that would be most useful for estimation and control. In recent years, a number of researches have analyzed various version of this problem ([1], [2], [3]).

In this paper, we focus on state estimation of deterministic, discrete time, linear, time invariant system with a digital channel connecting a sensor to an estimator. The system may have bounded process noise. Our problem formulation follows that of Tatikonda and Mitter[2]. We consider the following linear time-invariant system:

$$
\begin{aligned}
x_{t+1} &= Ax_t + u_t, \\
y_t &= Cx_t,
\end{aligned}
$$

where x_t is the state, u_t is the input and y_t is the output at time $t \in \{0, 1, \dots\}$. We assume that (C, A) is observable but the estimator get the information of the output y_t only through digital link with a finite data rate.

We denote a tuple $(x_0, x_1, \dots, x_t)$ by x^t and so on. Encoders and decoders are maps defined as follows.

- Encoder: a encoder at time t is a map

$$\mathcal{E}_t : (x^t, \sigma^{t-1}, u^{t-1}) \to \sigma_t,$$

  where codeword σ_t takes its value in a finite set $\mathcal{I}_t$. The encoding rate R is defined by $R = \log |\mathcal{I}_t|$, where $|\mathcal{I}_t|$ denotes the cardinality of the set $\mathcal{I}_t$.

- Decoder: a decoder at time t is a map

$$\mathcal{D}_t : (\sigma^t, u^{t-1}) \to \tilde{x}_t,$$

  where $\tilde{x}_t$ is the estimator of x_t.

We can consider several configurations depending on which information is available at the encoder. They are called the *Information Pattern* of the encoder([2],[4]). Tatikonda and Mitter[2] discuss two classes of Information Pattern.

- Encoder Class One:

$$\mathcal{E}_t : (x^t, \sigma^{t-1}, u^{t-1}) \to \sigma_t.$$

  This encoder has full access to the information.

- Encoder Class Two (Figure 1):

$$\mathcal{E}_t : x_t \to \sigma_t.$$

  This encoder has no access to inputs, moreover, no access to past outputs or past encoded symbols, that is, it is memory-less. Note that we still allow memory for the decoder.

In both classes, we assume that the encoder and the decoder have knowledge of the dynamics of the plant and knowledge of the encoder and the decoder each other. Tatikonda and Mitter[2] show that the state can be estimated with zero-error asymptotically for class one encoders, but they state, in Proposition 6.1, that it is impossible to estimate the state with bounded error for class two encoders if the encoding rate is finite. In this paper, we introduce a new quantization method and demonstrate that we can achieve zero-error asymptotically even with class two encoders and a finite data rate. Moreover, we show that we can bound estimation error even if the bounded process noise is present.

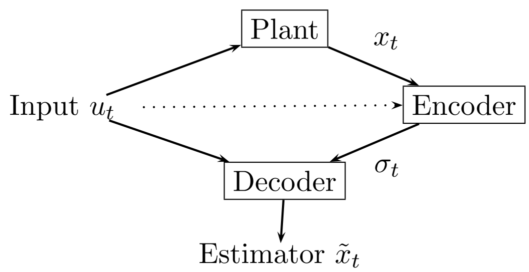

Figure 1: Information Pattern

2 Simple Example

We construct a simple one-dimensional plant and a encoder/decoder pair to illustrate that we can determine the system state with bounded error even if the encoder is memory-less. Let $x_t \in \mathbb{R}$ and define a plant, a encoder without memory and a decoder as the following.

- Plant:

$$x_{t+1} = 2x_t, \qquad x_0 \in [0, 2).$$

- Encoder:

$$\sigma_t = \lfloor x_t \rfloor \mod 2.$$

- Decoder:

$$\tilde{x}_{-1} = 0, \quad \tilde{x}_t = 2\tilde{x}_{t-1} + \sigma_t \quad (t \geq 0).$$

Here, $\lfloor x \rfloor$ denotes the maximum integer that is less than or equal to x. Apparently, $\{\sigma_t\}_0^\infty$ represent the binary expansion of x_0 and $\tilde{x}_t = \lfloor x_t \rfloor$. It follows that

$$\tilde{x} \leq x_t < \tilde{x}_t + 1.$$

This shows that the error $|\tilde{x}_t - x_t|$ is bounded even if the encoder is memory-less. This is a counter-example of Proposition 6.1 by Tatikonda and Mitter[2]. The point here is that, with conventional quantization methods, an encoded symbol corresponds to a region with finite volume in $\mathbb{R}$, on the other hand, the above encoder maps a region with infinite volume to an encoded symbol.

3 Main Result

In this section, we state a stronger result than the counter-example in the previous section. In the previous example, we assume that inputs to the plant is always zero, but here, we consider variable inputs, that are known only to the decoder, and bounded process disturbance. Moreover, we show that the estimation error can vanish asymptotically with sufficiently large encoding rate if disturbance is not present.

Consider the following n-dimensional linear system with inputs u_t and unknown bounded disturbance w_t.

$$x_{t+1} = Ax_t + u_t + w_t, \tag{1}$$
$$y_t = Cx_t, \tag{2}$$

where $t \geq 0$, $x_t, u_t, w_t \in \mathbb{R}^n$, $y_t \in \mathbb{R}^m$ and $|(w_t)_i| \leq (d)_i$ for a non-negative vector $d \in R^n$ and $i \in \{1, \dots, n\}$. We assume that (C, A) is observable and the initial value x_0 is also bounded:

$$(x_0)_i \in [(\tilde{x}_0 - L_0)_i, (\tilde{x}_0 + L_0)_i] \quad i \in \{1, \dots, n\}, \tag{3}$$

where $\tilde{x}_0, L_0 \in \mathbb{R}^n$ and $(L_0)_i > 0$. We denote a matrix with absolute values of elements of a matrix A by $\overline{A}$, that is,

$$(\overline{A})_{ij} = |(A)_{ij}|.$$

We introduce parameters $K_i, L_t, \delta_t \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times m}$, $N \in \mathbb{R}^{n\times n}$ and $F, S \in \mathbb{R}^{m\times m}$ to construct a encoder and a decoder as follows.

- Let $K_i \geq 2$ be an integer for $i \in \{1, \dots, m\}$ and

$$F = \begin{pmatrix} (K_1 - 1)^{-1} & & 0 \\ & \ddots & \\ 0 & & (K_m - 1)^{-1} \end{pmatrix}.$$

- Choose a matrix $M \in \mathbb{R}^{n\times m}$ such that $N \equiv A - MC$ is stable, and an invertible matrix $S \in \mathbb{R}^{m\times m}$. Because (C, A) is observable, there exists a such M.

- For $t \geq 0$, let

$$\delta_t = F\overline{S^{-1}C}L_t. \tag{4}$$

- For $t \geq 1$, let

$$L_{t+1} = (\overline{N} + \overline{MS}F\overline{S^{-1}C})L_t + d. \tag{5}$$

Note that these parameters does not depend on outputs y^t and can be computed without memory of past outputs and encoded symbols. Now we construct a encoder, a decoder and an observer.

- Encoder: $\mathcal{E}_t(y_t) = \sigma_t$, where σ_t is a n-dimensional integer vector and

$$(\sigma_t)_i = \left\lfloor \frac{(S^{-1}y_t)_i}{2(\delta_t)_i} \right\rfloor \mod K_i. \tag{6}$$

  Because $(\sigma_t)_i \in \{0, \cdots, K_i - 1\}$, the encoding rate R is given by $R = \sum_{i=1}^{n} \log K_i$.

- Observer: $O_t(\sigma^t, u^{t-1}) = \tilde{x}_t$, where

$$\tilde{x}_t = N\tilde{x}_{t-1} + u_{t-1} + MSr_{t-1}, \tag{7}$$

  and r_{t-1} is defined as follows.

- Decoder: $\mathcal{D}_t(\sigma^t, u^{t-1}) = r_t$, where

$$(r_t)_i = \left[\left\lfloor \frac{(\tilde{r}_t)_i}{2(\delta_t)_i} \right\rfloor + \left\{ \left((\sigma_t)_i - \left\lfloor \frac{(\tilde{r}_t)_i}{2(\delta_t)_i} \right\rfloor \right) \mod K_i \right\} \right] \cdot 2(\delta_t)_i + (\delta_t)_i, \tag{8}$$

  and

$$\tilde{r}_t = S^{-1}C\tilde{x}_t - \overline{S^{-1}C}L_t. \tag{9}$$

We should note that we do not allow memory for the encoder but allow for the decoder and the observer. The following theorem states that the estimation error of x_t by $\tilde{x}_t$ is bounded by L_t. The proof is given in the next section.

Theorem 3.1. *For $t \geq 0$ and $i = \{1, \cdots, n\}$,*

$$|(x_t - \tilde{x}_t)_i| \leq (L_t)_i. \tag{10}$$

If we can show that L_t is bounded by a constant or asymptotically vanishes with a finite encoding rate, we can conclude that estimation error is bounded even with class two encoders. For one-dimensional case, we can obtain a sufficient encoding rate explicitly.

Proposition 3.1 ([5]). *Suppose $n = 1$, $A = a \in \mathbb{R}$ and $C = c \in \mathbb{R}$ where $c \neq 0$. If $K_1 - 1 > |a|$, then*

$$L_t = \gamma^t L_0 + \frac{1 - \gamma^t}{1 - \gamma} d, \tag{11}$$

where $\gamma = |a|/(K_1 - 1) < 1$. It follows that

$$\lim_{t \to \infty} L_t = \frac{d}{K_1 - 1 - |a|}. \tag{12}$$

Therefore, if $d = 0$, the estimation error vanishes asymptotically.

Proof. Let $M = ac^{-1}$ and $S = c$, then $\overline{N} = 0$ and $\overline{S^{-1}C} = 1$. It follows from (5) that

$$L_{t+1} = \frac{|a|}{K_1 - 1} L_t + d = \gamma L_t + d.$$

Therefore, (11) and (12) follows. □

Tatikonda and Mitter[2] show that even if the encode has memory and knows inputs(class one encoders), the encoding rate R must be greater than $\log |a|$ so that the error is bounded. For the above memory-less encoder(class tow encoders), the sufficient condition is give by

$$R > \log(\lceil |a| \rceil + 1), \tag{13}$$

from the above proposition. For large $|a|$, the difference of these two rates is relatively small.

For multidimensional case($n \geq 2$), we have to choose a coordinate system, matrices M, S and coding rate carefully so that $(\overline{N} + \overline{MS}F\overline{S^{-1}C})$ in the equation (5) is stable. If $(\overline{N} + \overline{MS}F\overline{S^{-1}C})$ is stable, L_t is bounded, and asymptotically vanishes when $d = 0$. Generally, $\overline{N}$ may not be stable even if N is stable, however, if we choose appropriate coordinate system based on real Jordan canonical form of N as used by Tatikonda and Mitter[2], $\overline{N}$ is also stable. In that case, with sufficiently large K_i, $(\overline{N} + \overline{MS}F\overline{S^{-1}C})$ is also stable.

To obtain explicit sufficient encoding rate, the observable canonical form([6]) is more useful. If we choose appropriate coordinate system of x^n and an lower-triangular matrix S, we can have N and $S^{-1}C$ be matrices with 0,1 elements and MS be a matrices with coefficient of the canonical form. In this case,

$$\overline{N} + \overline{MS}F\overline{S^{-1}C} = N + \overline{MS}FS^{-1}C,$$

and the stability of above matrix is determined by $\overline{MS}F$ easily. Especially, if $m = 1$, $(-MS)$ is the vector of coefficients of the characteristic polynomial of A, and $S^{-1}C$ is $(0, \ldots, 0, 1)$. We have the following proposition.

Proposition 3.2. *Suppose $m = 1$, and (A, C) is observable. Let α_i be the coefficient of degree $(i-1)$ of the characteristic polynomial of A. If*

$$K_1 - 1 > \sum_{i=1}^{n} |\alpha_i| \gamma^{-n+i-1} \tag{14}$$

for some $0 < \gamma < 1$, then L_t is bounded.

To prove this proposition, we introduce some definitions and a lemma. Let $f_A(\lambda)$ be a characteristic polynomial of $n \times n$ matrix A and denote it as

$$f_A(\lambda) = \lambda^n + \sum_{i=1}^{n} \alpha_i \lambda^{i-1},$$

and define a polynomial $\bar{f}_A(\lambda)$ as

$$\bar{f}_A(\lambda) = \lambda^n - \sum_{i=1}^{n} |\alpha_i| \lambda^{i-1},$$

We denote the spectrum radius of A by $\rho(A)$, that is,

$$\rho(A) = \max_i |\lambda_i|,$$

where λ_i denotes i-th eigen value of the matrix A.

Lemma 3.1. *If $\bar{f}_A(\gamma) > 0$ for some $\gamma > 0$, then $\rho(A) < \gamma$.*

Proof. It is sufficient to show that $|f_A(\lambda)| > 0$ for all $\lambda \in \mathbb{C}$ which satisfies $|\lambda| \geq \gamma$. Since $|a + b| \geq |a| - |b|$ for any complex numbers a and b, we have

$$\begin{aligned} |f_A(\lambda)| &\geq |\lambda|^n - \left| \sum_{i=1}^{n} \alpha_i \lambda^{i-1} \right| \\ &\geq |\lambda|^n - \sum_{i=1}^{n} |\alpha_i| |\lambda|^{i-1} \\ &= |\lambda|^n (1 - \sum_{i=1}^{n} |\alpha_i| |\lambda|^{i-1-n}). \end{aligned}$$

Suppose $|\lambda| \geq \gamma$. Since $i - 1 - n < 0$,

$$\begin{aligned} |f_A(\lambda)| &\geq |\lambda|^n (1 - \sum_{i=1}^{n} |\alpha_i| \gamma^{i-1-n}) \\ &= |\lambda|^n \gamma^{-n} \bar{f}_A(\gamma). \end{aligned}$$

Therefore, $\bar{f}_A(\gamma) > 0$ implies $|f_A(\lambda)| > 0$. □

Now, we give the proof of Proposition 3.2.

Proof. Because $m = 1$ and (A, C) is observable, there is a coordinate transformation matrix T so that TAT^{-1} and CT^{-1} take the following observable canonical form:

$$\begin{aligned} A^* &\equiv TAT^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & -\alpha_1 \\ 1 & 0 & \dots & 0 & -\alpha_2 \\ 0 & 1 & \dots & 0 & -\alpha_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\alpha_n \end{pmatrix}, \\ C^* &\equiv CT^{-1} = (0, \dots, 0, 1), \end{aligned}$$

where α_i is the coefficient of degree $(i-1)$ of the characteristic polynomial of A. It is suffice to show that the state estimation error is bounded for new coordinate $x^* = Tx$ because T^{-1} is bounded. We choose $M = {}^t(-\alpha_1, -\alpha_2, \dots, -\alpha_n)$ and $S = 1$, then $N = A^* - MC^*$ is stable and have the following form:

$$N = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Now, we investigate the stability of the matrix $H \equiv \overline{N} + \overline{MS}F\overline{S^{-1}C}$.

$$H = N + \overline{M}FC = \begin{pmatrix} 0 & 0 & \dots & 0 & |\alpha_1|/(K_1-1) \\ 1 & 0 & \dots & 0 & |\alpha_2|/(K_1-1) \\ 0 & 1 & \dots & 0 & |\alpha_3|/(K_1-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & |\alpha_n|/(K_1-1) \end{pmatrix}.$$

Apparently, the characteristic polynomial of the matrix H is given by:

$$f_H(\lambda) = \lambda^n - \frac{1}{K_1 - 1}\sum_{i=1}^{n} |\alpha_i|\lambda^{i-1}.$$

and (14) implies $f_H(\gamma) > 0$. By lemma 3.1 and (5), we conclude that H is a stable matrix and L_t is bounded. □

We can also derive a weaker but intuitive sufficient condition.

Corollary 3.1. *Suppose $m = 1$, and (A, C) is observable. Let λ_i be the i-th eigen value of the matrix A. If*

$$\log K_1 > \sum_{i=1}^{n} \log(|\lambda_i|\gamma^{-1} + 1) \tag{15}$$

for some $0 < \gamma < 1$, then L_t is bounded.

Proof. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of the matrix A. For $\gamma > 0$, by Vieta's formulas, we have

$$\begin{aligned} \sum_{i=1}^{n} |\alpha_i|\gamma^{i-1} &= \left|\sum_i \lambda i\right| \gamma^{n-1} + \left|\sum_{i\neq j} \lambda_i\lambda_j\right| \gamma^{n-2} + \dots + |\lambda_1\lambda_2\dots\lambda_n| \\ &\leq \sum_i |\lambda i|\gamma^{n-1} + \sum_{i\neq j} |\lambda_i\lambda_j|\gamma^{n-2} + \dots + |\lambda_1\lambda_2\dots\lambda_n| \\ &= \prod_{i=1}^{n}(\gamma + |\lambda_i|) - \gamma^n \end{aligned}$$

Divided by γ^n, we have

$$\sum_{i=1}^{n} |\alpha_i| \gamma^{-n+i-1} \leq \prod_{i=1}^{n} (|\lambda_i| \gamma^{-1} + 1) - 1. \tag{16}$$

By (15) and (16), we have

$$K_1 > 1 + \sum_{i=1}^{n} |\alpha_i| \gamma^{-n+i-1}.$$

By Proposition 3.2, L_t is bounded. □

The constant γ appeared in Proposition 3.2 and Corollary 3.1 determines how fast the estimator approach to the true state. Smaller γ means better estimator but needs more information rate. If γ goes to 1, we obtain infimum rate for each sufficient condition. By continuity, we obtain sufficient conditions

$$\log K_1 > \log(1 + \sum_{i=1}^{n} |\alpha_i|) \tag{17}$$

and

$$\log K_1 > \sum_{i=1}^{n} \log(1 + |\lambda_i|) \tag{18}$$

from Proposition 3.2 and Corollary 3.1 respectively.

4 Proof of the Theorem

To prove the theorem, we prove the following key lemma first.

Lemma 4.1. *Let x, $\tilde{x}$ and $L > 0$ are real variables and $K \geq 2$ is a integer. If*

$$|x - \tilde{x}| \leq L,$$

then

$$\left(\left\lfloor \frac{x}{2\delta} \right\rfloor - \left\lfloor \frac{\tilde{x} - L}{2\delta} \right\rfloor \right) \bmod K = \left\lfloor \frac{x}{2\delta} \right\rfloor - \left\lfloor \frac{\tilde{x} - L}{2\delta} \right\rfloor, \tag{19}$$

where $\delta = \frac{L}{K-1} > 0$.

Proof. Because $\tilde{x} - L \leq x$, we have

$$0 \leq \left\lfloor \frac{x}{2\delta} \right\rfloor - \left\lfloor \frac{\tilde{x} - L}{2\delta} \right\rfloor. \tag{20}$$

Since $x \leq \tilde{x} + L = (\tilde{x} - L) + 2L$, it follows that

$$\begin{aligned} \left\lfloor \frac{x}{2\delta} \right\rfloor &\leq \left\lfloor \frac{\tilde{x} - L}{2\delta} + \frac{L}{\delta} \right\rfloor \\ &= \left\lfloor \frac{\tilde{x} - L}{2\delta} + K - 1 \right\rfloor \\ &= \left\lfloor \frac{\tilde{x} - L}{2\delta} \right\rfloor + K - 1. \end{aligned} \tag{21}$$

It follows from (20) and (21) that

$$0 \leq \left\lfloor \frac{x}{2\delta} \right\rfloor - \left\lfloor \frac{\tilde{x} - L}{2\delta} \right\rfloor \leq K - 1,$$

and

$$\left(\left\lfloor \frac{x}{2\delta} \right\rfloor - \left\lfloor \frac{\tilde{x} - L}{2\delta} \right\rfloor \right) \bmod K = \left\lfloor \frac{x}{2\delta} \right\rfloor - \left\lfloor \frac{\tilde{x} - L}{2\delta} \right\rfloor.$$

□

Now, we prove the Theorem 3.1.

Proof. By the definition (3) of $\tilde{x}_0$ and L_0, it is obvious that (10) is established for $t = 0$. We will show that (10) is established for time $t + 1$ under the assumption that (10) is established for some time t. Substituting (8) by (6),

$$\begin{aligned} \frac{(r_t)_i}{2(\delta_t)_i} &= \left\lfloor \frac{(\tilde{r}_t)_i}{2(\delta_t)_i} \right\rfloor + \left[\left\{ \left(\left\lfloor \frac{(S^{-1}y_t)_i}{(2\delta_t)_i} \right\rfloor \bmod K_i \right) - \left\lfloor \frac{(\tilde{r}_t)_i}{2(\delta_t)_i} \right\rfloor \right\} \bmod K_i \right] + \frac{1}{2} \\ &= \left\lfloor \frac{(\tilde{r}_t)_i}{2(\delta_t)_i} \right\rfloor + \left\{ \left(\left\lfloor \frac{(S^{-1}Cx_t)_i}{2(\delta_t)_i} \right\rfloor - \left\lfloor \frac{(\tilde{r}_t)_i}{2(\delta_t)_i} \right\rfloor \right) \bmod K_i \right\} + \frac{1}{2}. \end{aligned}$$

By the assumption,

$$\begin{aligned} |(S^{-1}Cx_t - S^{-1}C\tilde{x}_t)_i| &= |(S^{-1}C(x_t - \tilde{x}_t))_i| \\ &\leq \left(\overline{S^{-1}C} \cdot \overline{(x_t - \tilde{x}_t)} \right)_i \\ &\leq (\overline{S^{-1}C} L_t)_i. \end{aligned}$$

Substituting (19) with $x = (S^{-1}Cx_t)_i$, $\tilde{x} = (S^{-1}C\tilde{x}_t)_i$, $L = (\overline{S^{-1}C}L_t)_i$ and $\delta = (\delta_t)_i = (K_i - 1)^{-1}(\overline{S^{-1}C}L_t)_i$, it follows from lemma 4.1 and (9) that

$$\left(\left\lfloor \frac{(S^{-1}Cx_t)_i}{2(\delta_t)_i} \right\rfloor - \left\lfloor \frac{(\tilde{r}_t)_i}{2(\delta_t)_i} \right\rfloor \right) \bmod K_i = \left\lfloor \frac{(S^{-1}Cx_t)_i}{2(\delta_t)_i} \right\rfloor - \left\lfloor \frac{(\tilde{r}_t)_i}{2(\delta_t)_i} \right\rfloor.$$

Therefore,

$$\begin{aligned} \frac{(r_t)_i}{2(\delta_t)_i} &= \left\lfloor \frac{(\tilde{r}_t)_i}{2(\delta_t)_i} \right\rfloor + \left\lfloor \frac{(S^{-1}Cx_t)_i}{2(\delta_t)_i} \right\rfloor - \left\lfloor \frac{(\tilde{r}_t)_i}{2(\delta_t)_i} \right\rfloor + \frac{1}{2} \\ &= \left\lfloor \frac{(S^{-1}Cx_t)_i}{2(\delta_t)_i} \right\rfloor + \frac{1}{2}. \end{aligned}$$

Define $\varepsilon_t \in \mathbb{R}^n$ as

$$\begin{aligned}
(\varepsilon_t)_i &= (S^{-1}Cx_t)_i - (r_t)_i \\
&= (S^{-1}Cx_t)_i - \left\lfloor \frac{(S^{-1}Cx_t)_i}{2(\delta_t)_i} \right\rfloor \cdot 2(\delta_t)_i - (\delta_t)_i \\
&= \left\{ \frac{(S^{-1}Cx_t)_i}{2(\delta)_i} - \left\lfloor \frac{(S^{-1}Cx_t)_i}{2(\delta_t)_i} \right\rfloor \right\} \cdot 2(\delta_t)_i - (\delta_t)_i \\
&< (\delta_t)_i,
\end{aligned}$$

then we have $|(\varepsilon_t)_i| < (F\overline{S^{-1}C}L_t)_i$ by (4). It follows from (1) and (7) that

$$\begin{aligned}
|(x_{t+1} - \tilde{x}_{t+1})_i| &= |(N(x_t - \tilde{x}_t) + MCx_t - MSr_t + w_t)_i| \\
&= |(N(x_t - \tilde{x}_t) + MS\varepsilon_t + w_t)_i| \\
&\le |(N(x_t - \tilde{x}_t))_i| + |(MS\varepsilon_t)_i| + |(w_t)_i| \\
&\le (\overline{N}L_t)_i + (\overline{MS}F\overline{S^{-1}C}L_t)_i + d_i \\
&= \Big((\overline{N} + \overline{MS}F\overline{S^{-1}C})L_t \Big)_i + d_i \\
&= (L_{t+1})_i.
\end{aligned}$$

Therefore, (10) is also established for time $t+1$. $\square$

5 Summary

In this paper, we show that we can construct a encoder and a decoder that can estimate the state with bounded error even if the encoder is memory-less and does not have access to inputs of the plant. This complements the result by Tatikonda and Mitter[2] for encoders in class two. As far as we know, this type of quantization is new in this field and we believe that this method gives us new insight to understand which information bits are important for estimation and control.

Finally, we point out some relation between our result and studies in information theory. Encoders in class two has two limitations. One is memory-less and the other is unavailability of inputs. Our result suggests that we can construct a memory-less encoder at the expense of optimality of the encoding rate, that is, we need a slightly higher rate (13) than $\log|a|$ bits. This is the same situation in source coding that we can achieve optimum compression rate with the long block length that requires long memory. The latter limitation has a strong connection with problem of side information. We can consider inputs to the plant as side information available only at the decoder. In loss-less source coding, it is shown that the optimum encoding rate does not change whether side information is available at the encoder or not ([7],[8],[9]). Our result is consistent with this because we can achieve optimum rate asymptotically if the encoder is allowed to have memory. The quantizer we proposed here is inspired by bin-coding([10]). Though bin-coding is a random coding and ours is deterministic, it is the same idea that the source symbols should be spread over the codewords uniformly. We should emphasize that our result is not direct consequence of source coding with side information. It should be noted that for lossy source coding([11]), the optimum coding rate can be different depending on whether inputs is available at the encoder or not. Actually, our method can not be applied to the case when unbounded disturbance is present([3]).

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