MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

Discrete L-/M-Convex Function Minimization Based on Continuous Relaxation

Satoko MORIGUCHI and Nobuyuki TSUCHIMURA

METR 2007–59 November 2007

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html

Discrete L-/M-Convex Function Minimization Based on Continuous Relaxation

Satoko MORIGUCHI and Nobuyuki TSUCHIMURA

Department of Mechanical Engineering
Faculty of Science and Technology
Sophia University
s-morigu@sophia.ac.jp
and
Department of Mathematical Informatics
Graduate School of Information Science and Technology
University of Tokyo
tutimura@mist.i.u-tokyo.ac.jp

November, 2007

Abstract

We consider the problem of minimizing a nonlinear discrete function with L$^\natural$-/M$^\natural$-convexity proposed in the theory of discrete convex analysis. For this problem, steepest descent algorithms and steepest descent scaling algorithms are known. In this paper, we use continuous relaxation approach which minimizes the continuous variable version first in order to find a good initial solution of a steepest descent algorithm. For discrete L$^\natural$-/M$^\natural$-convex functions, we give proximity theorems showing that a discrete global minimizer exists in the neighborhood of a continuous global minimizer. These proximity theorems afford theoretical guarantees for the efficiency of the proposed algorithms.

1 Introduction

In recent research towards a unified framework of discrete convex analysis [11], the concept of M$^\natural$-convex functions was proposed as an extension of that of valuations on matroids invented by Dress and Wenzel [2]. The concept of L$^\natural$-convex functions, which generalize the Lovász extension of submodular set functions [8], was also proposed in the theory of discrete convex analysis. These two concepts of discrete convexity are conjugate to each other, and a Fenchel-type duality theorem holds for L$^\natural$- and M$^\natural$-convex/concave functions [11]. Applications of L$^\natural$-/M$^\natural$-convexity can be found in mathematical economics with indivisible commodities [1, 12, 13], system analysis by mixed polynomial matrices [10], etc. These two discrete convexities play central roles in the theory of discrete convex analysis [11] and provide a nice framework of nonlinear combinatorial optimization; global optimality is guaranteed by local optimality and descent algorithms work for minimization. Steepest descent algorithms, which terminate in pseudo-polynomial time, and steepest descent scaling algorithms, which terminate in polynomial time with the aid of a scaling technique, are also known. The proximity theorems on a scaled local optimum for L$^\natural$-convexity and M$^\natural$-convexity guarantee the efficiency of scaling algorithms.

The objective of this paper is to show that we can minimize an L$^\natural$-/M$^\natural$-convex function more efficiently by using continuous relaxation approach which minimizes the continuous variable version first in order to find a good initial solution of a steepest descent algorithm. In general, for discrete function minimization, we can say that the rounded continuous relaxation solution is almost certainly nonoptimal and may be very far away from the optimal integer solution. For separable convex optimization problems, proximity results between the continuous and integral optimal solutions were obtained [4, 5]. In this paper, for the discrete L$^\natural$-/M$^\natural$-convex function minimization problems, which are nonseparable optimization problems, we give proximity theorems showing that a discrete global minimizer exists in the neighborhood of a continuous global minimizer. Based on our new proximity, we can minimize a discrete L$^\natural$-/M$^\natural$-convex function efficiently by using continuous relaxation. In order to compare the performance of our new continuous relaxation approach with those of the previously proposed algorithms, we make numerical experiments with randomly generated test problems. It is observed from numerical results that our new approach is much faster than the previously proposed algorithms.

2 Preliminaries

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a function. The effective domain and the epigraph of f are given by

$$\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid f(x) < +\infty\}, \quad \mathrm{epi} f = \{(x, \alpha) \in \mathbf{R}^n \times \mathbf{R} \mid \alpha \geq f(x)\}.$$

For a function $f : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$, we use the notation $\mathrm{dom}_{\mathbf{Z}}\, f = \{x \in \mathbf{Z}^n \mid f(x) < +\infty\}$ for the effective domain of f.

A convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be proper if $\mathrm{dom}\, f \neq \emptyset$, and closed if $\mathrm{epi} f$ is a closed set. For a closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, $\arg\min f \neq \emptyset$ if $\mathrm{dom}\, f$ is bounded.

2.1 L-Convex Functions

For vectors $p, q \in \mathbf{Z}^n$, we write $p \vee q$ and $p \wedge q$ for their componentwise maximum and minimum. We write $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{Z}^n$. A function

$g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is called L-convex [11] if it satisfies

(SBF[Z]) $g(p) + g(q) \geq g(p \vee q) + g(p \wedge q) \quad (p, q \in \mathbf{Z}^n)$,
(TRF[Z]) $\exists r \in \mathbf{R}$ such that $g(p+\mathbf{1}) = g(p)+r \quad (p \in \mathbf{Z}^n)$,

where it is understood that the inequality (SBF) is satisfied if $g(p)$ or $g(q)$ is equal to $+\infty$.

A function $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is called L$^\natural$-convex [3, 11] if it is obtained from an L-convex function $\tilde{g}(p_0, p_1, \ldots, p_n)$ by restriction, i.e.,

$$g(p_1, \ldots, p_n) = \tilde{g}(0, p_1, \ldots, p_n). \tag{1}$$

It turns out that L$^\natural$-convexity can be characterized by a kind of generalized submodularity:

(SBF$^\natural$[Z]) $g(p) + g(q) \geq g((p - \alpha\mathbf{1}) \vee q) + g(p \wedge (q + \alpha\mathbf{1}))$
$(0 \leq \alpha \in \mathbf{Z}, p, q \in \mathbf{Z}^n)$,

which is called translation submodularity.

The concepts of L-/L$^\natural$-convexity can also be defined for functions in real variables through an appropriate adaptation of the conditions (SBF[**Z**]) and (TRF[**Z**]). Namely, we call a function $\bar{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ L-convex [11] if $\bar{g}$ is convex and satisfies

(SBF[R]) $\bar{g}(p) + \bar{g}(q) \geq \bar{g}(p \vee q) + \bar{g}(p \wedge q) \quad (p, q \in \mathbf{R}^n)$,
(TRF[R]) $\exists r \in \mathbf{R}$ such that $\bar{g}(p+\mathbf{1}) = \bar{g}(p)+r \quad (p \in \mathbf{R}^n)$.

L$^\natural$-convex functions are defined as the restriction of L-convex functions, as in (1), and are characterized by

(SBF$^\natural$[R]) $\bar{g}(p) + \bar{g}(q) \geq \bar{g}((p - \alpha\mathbf{1}) \vee q) + \bar{g}(p \wedge (q + \alpha\mathbf{1}))$
$(0 \leq \alpha \in \mathbf{R}, p, q \in \mathbf{R}^n)$.

Throughout the paper, we assume that a continuous L$^\natural$-convex function is a closed proper convex function.

Minimization of a continuous L$^\natural$-convex function is tractable with a firm theoretical basis provided by convex analysis. For minimization of a discrete L$^\natural$-convex function, we have the following optimality criterion, which shows that global minimality is characterized by local minimality. The characteristic vector of $X \subseteq \{1, 2, \ldots, n\}$ is denoted by $\chi_X \in \{0, 1\}^n$.

Theorem 2.1 (Theorem 7.14 in [11])**.** *Let $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be a discrete L$^\natural$-convex function. For $p \in \mathrm{dom}_{\mathbf{Z}}\, g$, $g(p) \leq g(q)$ $(q \in \mathbf{Z}^n)$ if and only if*

$$g(p) \leq g(p \pm \chi_X) \quad (X \subseteq \{1, 2, \ldots, n\}). \tag{2}$$

2.2 M-Convex Functions

Let $\chi_i \in \{0,1\}^n$ denote the characteristic vector of $i \in \{1,2,\ldots,n\}$. For a vector $x \in \mathbf{Z}^n$ and an element $i \in \{1,2,\ldots,n\}$, $x(i)$ means the component of x with index i. For a vector $x \in \mathbf{Z}^n$ and a set $X \subseteq \{1,2,\ldots,n\}$, we write $x(X) = \sum_{i \in X} x(i)$. We write the positive and negative supports of a vector x by

$$\mathrm{supp}^+(x) = \{i \in \{1,2,\ldots,n\} \mid x(i) > 0\},$$
$$\mathrm{supp}^-(x) = \{i \in \{1,2,\ldots,n\} \mid x(i) < 0\}.$$

A function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be M-convex if it satisfies (M-EXC[$\mathbf{Z}$]):

(M-EXC[Z]) $\forall x, y \in \mathrm{dom}_{\mathbf{Z}} f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

We call a function $\bar{f} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ M-convex if it is convex and satisfies (M-EXC[$\mathbf{R}$]):

(M-EXC[R]) $\forall x, y \in \mathrm{dom}\, \bar{f}$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$ satisfying

$$\bar{f}(x)+\bar{f}(y) \geq \bar{f}(x-\alpha(\chi_i-\chi_j))+\bar{f}(y+\alpha(\chi_i-\chi_j)) \quad (\alpha \in [0,\alpha_0]). \tag{3}$$

A continuous M-convex function is said to be closed proper M-convex if it is closed proper convex, in addition. The effective domain of a closed proper M-convex function is contained in a hyperplane $\{x \in \mathbf{R}^n \mid \sum_{i=1}^n x(i) = r\}$ for some $r \in \mathbf{R}$. In view of this, we say that a function $\bar{f} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is M$^\natural$-convex if the function $\tilde{f} : \mathbf{R}^{n+1} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\tilde{f}(x_0, x) := \begin{cases} \bar{f}(x) & ((x_0, x) \in \mathbf{R}^{n+1}, x_0 = r - \sum_{i=1}^n x(i)), \\ +\infty & \text{(otherwise)} \end{cases}$$

is M-convex. We say that $\bar{f}$ is closed proper M$^\natural$-convex if it is closed proper convex, in addition. M$^\natural$-convexity of $\bar{f}$ is characterized by the following exchange property (M$^\natural$-EXC[$\mathbf{R}$]):

(M$^\natural$-EXC[R]) $\forall x, y \in \mathrm{dom}\, \bar{f}$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}$, $\exists \alpha_0 > 0$ satisfying (3),

where $\chi_0 = \mathbf{0}$.

Throughout the paper, we assume that a continuous M$^\natural$-convex function $\bar{f}$ is a closed proper convex function.

Minimization of a continuous M$^\natural$-convex function is tractable with a firm theoretical basis provided by convex analysis. For minimization of a discrete M$^\natural$-convex function, we have the following optimality criterion which shows that global minimality is characterized by local minimality.

Theorem 2.2 (Theorem 6.26 in [11]). *Let $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be a discrete $M^\natural$-convex function. For $x \in \mathrm{dom}_{\mathbf{Z}} f$, $f(x) \leq f(y)$ $(y \in \mathbf{Z}^n)$ if and only if*

$$f(x) \leq f(x - \chi_i + \chi_j) \quad (i, j \in \{1, 2, \ldots, n\} \cup \{0\}). \tag{4}$$

3 Proposed Algorithms

For discrete $\mathrm{L}^\natural$-convex and $\mathrm{M}^\natural$-convex function minimization, our continuous relaxation approach and proximity theorems between the discrete minimizer and the relaxation solution are given in Sections 3.1 and 3.2, respectively. Section 3.3 is devoted to the proofs of proximity theorems.

3.1 Algorithm for L-Convex Functions

The local characterization of global minimality for $\mathrm{L}^\natural$-convex functions (Theorem 2.1) naturally leads to the following steepest descent algorithm [11, Sec. 10.3.1].

Steepest descent algorithm for an $\mathbf{L}^\natural$-convex function g

S0: Find a vector $p \in \mathrm{dom}\, g$.
S1: Find $\varepsilon \in \{1, -1\}$ and $X \subseteq \{1, 2, \ldots, n\}$ that minimizes $g(p + \varepsilon\chi_X)$.
S2: If $g(p) \leq g(p + \varepsilon\chi_X)$, then stop ($p$ is a minimizer of g).
S3: Set $p := p + \varepsilon\chi_X$ and go to S1.

Step S1, i.e., the verification of (2), amounts to minimizing a pair of submodular set functions which can be done in polynomial time [6, 14, 15]. Furthermore, the steepest descent algorithm, which is a pseudo-polynomial time algorithm, can be made more efficient with the aid of a scaling technique. The resulting steepest descent scaling algorithm [11, Sec. 10.3.2] terminates in polynomial time. This is guaranteed by the proximity theorem (Theorem 7.18 in [11]) on a scaled local optimum for L-convexity.

The following is another "proximity theorem," showing that a continuous relaxation solution of a discrete $\mathrm{L}^\natural$-convex function minimization problem exists in a neighborhood of the integer minimizer.

Theorem 3.1. *Let $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be a discrete $L^\natural$-convex function and $\bar{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a continuous $L^\natural$-convex function with* $\arg\min \bar{g} \neq \emptyset$. *We assume that*

$$g(p) = \bar{g}(p) \quad (p \in \mathbf{Z}^n).$$

Then, for any $p^* \in \arg\min g$, there exists some $\bar{p} \in \arg\min \bar{g}$ such that

$$p^* - n\mathbf{1} < \bar{p} < p^* + n\mathbf{1}.$$

The proof of Theorem 3.1 is given later in Section 3.3.

In the reverse direction, we obtain the following proximity theorem, a main theorem of this paper, which shows that a minimizer of a discrete L$^\natural$-convex function g exists in a neighborhood of the continuous relaxation solution. We assume now the boundedness of the effective domain.

Theorem 3.2. *Let $g : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be a discrete L$^\natural$-convex function and $\bar{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a continuous L$^\natural$-convex function with* $\arg\min \bar{g} \neq \emptyset$*. We assume that*

$$g(p) = \bar{g}(p) \quad (p \in \mathbf{Z}^n).$$

For any $\bar{p} \in \arg\min \bar{g}$, there exists some $p^* \in \arg\min g$ such that

$$\bar{p} - n\mathbf{1} < p^* < \bar{p} + n\mathbf{1}.$$

The proof of Theorem 3.2 is also given later in Section 3.3.

Based on Theorem 3.2, to minimize a discrete L$^\natural$-convex function g, we propose a continuous relaxation approach which is the steepest descent algorithm starting with a continuous relaxation solution as the initial solution. Theorem 3.2 guarantees that our continuous relaxation approach is efficient if the relaxation solution can be found fast. In order to find the relaxation solution, we can utilize continuous convex minimization algorithms for $\bar{g}$ since a continuous L$^\natural$-convex function is convex by the definition.

3.2 Algorithm for M-Convex Functions

The local characterization of global minimality for M$^\natural$-convex functions (Theorem 2.2) naturally leads to the following steepest descent algorithm [11, Sec. 10.1.1].

Steepest descent algorithm for an M$^\natural$-convex function f

S0: Find a vector $x \in \mathrm{dom}\, f$.
S1: Find $i, j \in \{1, 2, \ldots, n\} \cup \{0\}$ $(i \neq j)$ that minimizes $f(x - \chi_i + \chi_j)$.
S2: If $f(x) \leq f(x - \chi_i + \chi_j)$, then stop ($x$ is a minimizer of f).
S3: Set $x := x - \chi_i + \chi_j$ and go to S1.

The steepest descent algorithm terminates in pseudo-polynomial time. Furthermore, this can be made more efficient with the aid of a scaling technique. The resulting steepest descent scaling algorithm [9], see also [11, Sec. 10.1.2], terminates in polynomial time. This is guaranteed by the proximity theorem (Theorem 6.37 in [11]) on a scaled local optimum for M-convexity.

The following is another "proximity theorem," showing that a continuous relaxation solution of a discrete M$^\natural$-convex function minimization problem exists in a neighborhood of the integer minimizer.

Theorem 3.3. *Let $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be a discrete M$^\natural$-convex function and $\bar{f} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a continuous M$^\natural$-convex function with* $\arg\min \bar{f} \neq \emptyset$*. We assume that*

$$f(x) = \bar{f}(x) \quad (x \in \mathbf{Z}^n).$$

Then, for any $x^* \in \arg\min f$, there exists some $\bar{x} \in \arg\min \bar{f}$ such that

$$x^* - n\mathbf{1} < \bar{x} < x^* + n\mathbf{1}.$$

The proof of Theorem 3.3 is given later in Section 3.3.

In the reverse direction, we obtain the following proximity theorem, another main theorem of this paper, which shows that a minimizer of a discrete M$^\natural$-convex function g exists in a neighborhood of the continuous relaxation solution. We assume now the boundedness of the effective domain.

Theorem 3.4. *Let $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ be a discrete M$^\natural$-convex function and $\bar{f} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a continuous M$^\natural$-convex function with* $\arg\min \bar{f} \neq \emptyset$*. We assume that*

$$f(x) = \bar{f}(x) \quad (x \in \mathbf{Z}^n).$$

For any $\bar{x} \in \arg\min \bar{f}$, there exists some $x^* \in \arg\min f$ such that

$$\bar{x} - n\mathbf{1} < x^* < \bar{x} + n\mathbf{1}.$$

The proof of Theorem 3.4 is also given later in Section 3.3.

Based on Theorem 3.4, to minimize a discrete M$^\natural$-convex function f, we propose a continuous relaxation approach which is the steepest descent algorithm starting with a continuous relaxation solution as the initial solution. Theorem 3.4 guarantees that our continuous relaxation approach is efficient if the relaxation solution can be found fast. In order to find the relaxation solution, we can utilize continuous convex minimization algorithms for $\bar{f}$ since a continuous M$^\natural$-convex function is convex by the definition.

3.3 Proofs

We give proofs of Theorems 3.1, 3.2, 3.3 and 3.4.

Proof for Theorem 3.1. For an integer $s \geq 2$, we define $g_s : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ as

$$g_s(p) := \bar{g}\left(\frac{p}{s}\right) \quad (p \in \mathbf{Z}^n).$$

We have

$$g(p) = g_s(sp) \quad (p \in \mathbf{Z}^n). \tag{5}$$

For all $p, q \in \mathbf{Z}^n$ and $0 \leq \alpha \in \mathbf{Z}$, we have

$$\begin{aligned}
g_s(p) + g_s(q) &= \bar{g}(\frac{p}{s}) + \bar{g}(\frac{q}{s}) \\
&\geq \bar{g}((\frac{p}{s} - \frac{\alpha}{s}\mathbf{1}) \vee \frac{q}{s}) + \bar{g}(\frac{p}{s} \wedge (\frac{q}{s} + \frac{\alpha}{s}\mathbf{1})) \\
&= \bar{g}(\frac{(p - \alpha\mathbf{1}) \vee q}{s}) + \bar{g}(\frac{p \wedge (q + \alpha\mathbf{1})}{s}) \\
&= g_s((p - \alpha\mathbf{1}) \vee q) + g_s(p \wedge (q + \alpha\mathbf{1})),
\end{aligned}$$

where the inequality is by translation submodularity (SBF$^\natural$[$\mathbf{R}$]). This means discrete L$^\natural$-convexity of g_s.

Optimality criterion for g, i.e., (2), yields

$$g_s(sp^*) \leq g_s(sp^* \pm s\chi_X) \ (X \subseteq \{1, 2, \ldots, n\})$$

from (5). By applying L-proximity theorem on a scaled local optimum (Theorem 7.18 (2) in [11]) to g_s and sp^*, there exists $p_s \in \arg\min g_s$ with

$$sp^* - (s-1)n\mathbf{1} \leq p_s \leq sp^* + (s-1)n\mathbf{1}. \tag{6}$$

Dividing all parts of (6) by s shows

$$p^* - n\mathbf{1} < p^* - \frac{s-1}{s}n\mathbf{1} \leq \frac{p_s}{s} \leq p^* + \frac{s-1}{s}n\mathbf{1} < p^* + n\mathbf{1}.$$

Put $K := \{p \in \mathbf{R}^n \mid p^* - n\mathbf{1} \leq p \leq p^* + n\mathbf{1}\}$. Since K is compact, every sequence in K has a convergent subsequence, the limit point of which belongs to K. For $k \in \mathbf{N}$, we suppose $s_k = 2^k$, $p_{s_k} \in \arg\min g_{s_k}$ and $\frac{p_{s_k}}{s_k} \in K$. From a sequence $\{\frac{p_{s_k}}{s_k}\}$, we take a convergent subsequence $\{\frac{p_{s_{k_i}}}{s_{k_i}}\}$. We put $\lim_{i\to\infty} \frac{p_{s_{k_i}}}{s_{k_i}} = p' \in K$. Continuity of $\bar{g}$ implies $\lim_{i\to\infty} \bar{g}(\frac{p_{s_{k_i}}}{s_{k_i}}) = \bar{g}(\lim_{i\to\infty} \frac{p_{s_{k_i}}}{s_{k_i}}) = \bar{g}(p')$. Note that $\{\bar{g}(\frac{p_{s_{k_i}}}{s_{k_i}})\}$ is monotonically decreasing sequence ($\bar{g}(\frac{p_{s_{k_1}}}{s_{k_1}}) \geq \bar{g}(\frac{p_{s_{k_2}}}{s_{k_2}}) \geq \cdots \geq \bar{g}(\frac{p_{s_{k_i}}}{s_{k_i}}) \geq \cdots$) and

$$\bar{g}(p') \leq \bar{g}(\frac{p_{2^{k_i}}}{2^{k_i}}) = \min g_{2^{k_i}} \quad (i \in \mathbf{N}). \tag{7}$$

Now, we prove $\bar{g}(p') = \min \bar{g}$, i.e., $p' \in \arg\min \bar{g}$ by contradiction. Assume $\bar{g}(p') > \min \bar{g}$ and put $\varepsilon_0 := \bar{g}(p') - \min \bar{g} > 0$. We fix an arbitrary $\bar{p} \in \arg\min \bar{g}$. For any number $\varepsilon > 0$, there exist $N \in \{s_{k_i} \mid i = 1, 2, \ldots\}$ and $q := \sum_{k=0}^{N} \frac{b_k}{2^k}$ with $b_0 := \lfloor \bar{p} \rfloor$ and $b_k \in \{0, 1\}^n$ such that $2^N q \in \mathbf{Z}$, $|\bar{p} - q| < \varepsilon$. Continuity of $\bar{g}$ gives

$$\forall \varepsilon' > 0, \exists \delta_{\varepsilon'} > 0, : |x - y| < \delta_{\varepsilon'}, \Rightarrow |\bar{g}(x) - \bar{g}(y)| < \varepsilon'$$

for $x = \bar{p}$ and $y = q$. Now, considering the number $\varepsilon' = \frac{\varepsilon_0}{2}$, we have

$$\min g_{2^N} \leq \bar{g}(q) < \min \bar{g} + \frac{\varepsilon_0}{2} < \min \bar{g} + \varepsilon_0 = \bar{g}(p'),$$

contradicting (7). This proves $\bar{g}(p') = \min \bar{g}$. $\square$

Proof for Theorem 3.2. To use Theorem 3.1 in the reverse direction, we consider the case where $\bar{g} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ has a unique minimizer $\bar{p}$. Then, the fact that the condition

$$p^* - n\mathbf{1} < \bar{p} < p^* + n\mathbf{1}$$

holds for all $p^* \in \arg\min g$ is immediate from Theorem 3.1. In particular, there exists $p^* \in \arg\min g$ satisfying this condition.

We consider a perturbation of $\bar{g}$ so that we can use this fact. We arbitrarily fix a minimizer $\bar{p} \in \arg\min \bar{g}$. For any number $\varepsilon > 0$, we define functions $\bar{g}_\varepsilon : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ and $g_\varepsilon : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ as

$$\bar{g}_\varepsilon(p) := \bar{g}(p) + \sum_{i=1}^{n} \varepsilon(p(i) - \bar{p}(i))^2 \quad (p \in \mathbf{R}^n)$$

and

$$g_\varepsilon(p) := \bar{g}_\varepsilon(p) \quad (p \in \mathbf{Z}^n).$$

The functions $\bar{g}_\varepsilon$ and g_ε are L$^\natural$-convex by Theorem 7.11(1) in [11] and $\bar{g}_\varepsilon$ has a unique minimizer $\bar{p}$. Recall that we assume now the boundedness of the effective domain. We fix sufficiently small ε such that $p^*_\varepsilon \in \arg\min g_\varepsilon$ is also a minimizer of g. Then we apply the fact in the case of a unique continuous minimizer to show that there exists $p^*_\varepsilon \in \arg\min g$ such that $p^*_\varepsilon - n\mathbf{1} < \bar{p} < p^*_\varepsilon + n\mathbf{1}$. □

Proof for Theorem 3.3. For an integer $s \geq 2$, we define $f_s : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ as

$$f_s(x) := \bar{f}\left(\frac{x}{s}\right) \quad (x \in \mathbf{Z}^n).$$

We have

$$f(x) = f_s(sx) \quad (x \in \mathbf{Z}^n). \tag{8}$$

Since $\bar{f}$ is continuous M$^\natural$-convex, for all $x, y \in \mathbf{Z}^n$ and $i \in \mathrm{supp}^+(x-y)$, there exist $j \in \mathrm{supp}^-(x-y) \cup \{0\}$ and $\alpha_0 > 0$, satisfying

$$\begin{aligned} f_s(x) + f_s(y) &= \bar{f}(\frac{x}{s}) + \bar{f}(\frac{y}{s}) \\ &\geq \bar{f}(\frac{x}{s} - \alpha(\chi_i - \chi_j)) + \bar{f}(\frac{y}{s} + \alpha(\chi_i - \chi_j)) \\ &= \bar{f}(\frac{x - s\alpha(\chi_i - \chi_j)}{s}) + \bar{f}(\frac{y + s\alpha(\chi_i - \chi_j)}{s}) \end{aligned}$$

for all $\alpha \in [0, \alpha_0]$. Here, when $s \geq \lceil \frac{1}{\alpha_0} \rceil$, we have $s\alpha_0 \geq 1$ and we can choose α so that $s\alpha = 1$. This means

$$f_s(x) + f_s(y) \geq f_s(x - \chi_i + \chi_j) + f_s(y + \chi_i - \chi_j),$$

that is to say, f_s is discrete M$^\natural$-convex.

Optimality criterion for f, i.e., (4), yields

$$f_s(sx^*) \leq f_s(sx^* - s(\chi_i - \chi_j)) \ (i, j \in \{1, 2, \ldots, n\} \cup \{0\})$$

from (8). By applying M-proximity theorem on a scaled local optimum (Theorem 6.37 (2) in [11]) to f_s and sx^*, there exists $x_s \in \arg\min f_s$ with

$$sx^* - (s-1)n\mathbf{1} \leq x_s \leq sx^* + (s-1)n\mathbf{1}. \tag{9}$$

Dividing all parts of (9) by s shows

$$x^* - n\mathbf{1} < x^* - \frac{s-1}{s}n\mathbf{1} \leq \frac{x_s}{s} \leq x^* + \frac{s-1}{s}n\mathbf{1} < x^* + n\mathbf{1}.$$

We can prove, for $k \in \mathbf{N}$, $s_k = 2^k$ and $x_{s_k} \in \arg\min f_{s_k}$, in the same way to the proof of Theorem 3.1, that there exists a convergent subsequence $\{\frac{x_{s_{k_i}}}{s_{k_i}}\}$, the limit point of which is $x' \in \{x \in \mathbf{R}^n \mid x^* - n\mathbf{1} < x < x^* + n\mathbf{1}\}$ and $x' \in \arg\min \bar{f}$. □

Proof for Theorem 3.4. For any number $\varepsilon > 0$, we define functions $\bar{f}_\varepsilon : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ and $f_\varepsilon : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ as

$$\bar{f}_\varepsilon(x) := \bar{f}(x) + \sum_{i=1}^{n} \varepsilon(x(i) - \bar{x}(i))^2 \quad (x \in \mathbf{R}^n)$$

and

$$f_\varepsilon(x) := \bar{f}_\varepsilon(x) \quad (x \in \mathbf{Z}^n).$$

Since the functions $\bar{f}_\varepsilon$ and f_ε are M$^\natural$-convex by Theorem 6.15(1) in [11], we can complete the proof in the similar way to the proof of Theorem 3.2. □

4 Numerical Experiments

We here mainly compare the performance of our new continuous relaxation approach with those of the previously proposed algorithms. We observe from numerical experiments that our approach is much faster than the previous algorithms.

We implemented three algorithms for minimization of a discrete L$^\natural$-convex function shown in Table 1 and four algorithms for minimization of a discrete M$^\natural$-convex function shown in Table 2 in the C language to compare the performance of these algorithms.

We use the following libraries:

- 'L-BFGS' by J. Nocedal[1] with its C++ wrapper by T. Kudo[2], which is an implementation of quasi-Newton method for continuous function

[1] http://www.ece.northwestern.edu/~nocedal/lbfgs.html
[2] http://chasen.org/~taku/software/misc/lbfgs/

Table 1: Algorithms we implemented for L$^\natural$-convex function minimization.

symbol	algorithm
SD	steepest descent algorithm [11, Sec. 10.3.1]
SCALING	steepest descent scaling algorithm [11, Sec. 10.3.2]
RELAX	our new continuous relaxation approach

Table 2: Algorithms we implemented for M$^\natural$-convex function minimization.

symbol	algorithm
SD	steepest descent algorithm [11, Sec. 10.1.1]
SD2	modified steepest descent algorithm [9]
SCALING	steepest descent scaling algorithm [9], [11, Sec. 10.1.2]
RELAX	our new continuous relaxation approach

optimization [7]. As the routine requires the gradient of the objective function, we calculate a finite-difference approximation by calling the function evaluation oracle $n+1$ times. We use this only in RELAX (our new continuous relaxation approach).

- 'SFM8' by S. Iwata, which is an implementation of Iwata–Fleischer–Fujishige [6]. This minimizes a submodular function with $\mathrm{O}(n^5 \log_2 M)$ function evaluations, where M is the maximum absolute value of the submodular function. We use this for the L$^\natural$-convex case.
- 'SIMD-oriented Fast Mersenne Twister' developed by M. Saito and M. Matsumoto[3], which generates pseudorandom numbers. We make use of this to generate test problems.

As test problems for discrete L$^\natural$-convex function minimization, we consider the following function:

$$g(p) = \sum_{i=1}^{n} h_i(p(i)) + \sum_{1 \le i < j \le n} h_{ij}(p(i) - p(j)) \quad (p \in \mathbf{Z}^n),$$

where $h_i(z) = a_i(z - c_i)^2 + b_i(z - c_i)$ and $h_{ij}(z) = a_{ij}z^2 + b_{ij}z$ are univariate functions. In our continuous relaxation approach, we use

$$\bar{g}(p) = \sum_{i=1}^{n} h_i(p(i)) + \sum_{1 \le i < j \le n} h_{ij}(p(i) - p(j)) \quad (p \in \mathbf{R}^n).$$

For each n, we generate ten test problems with randomly chosen integer variables $1 \le a_i, a_{ij} \le n$, $-n^2 \le b_i, c_i, b_{ij} \le n^2$. For each problem, we

[3] http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/SFMT/

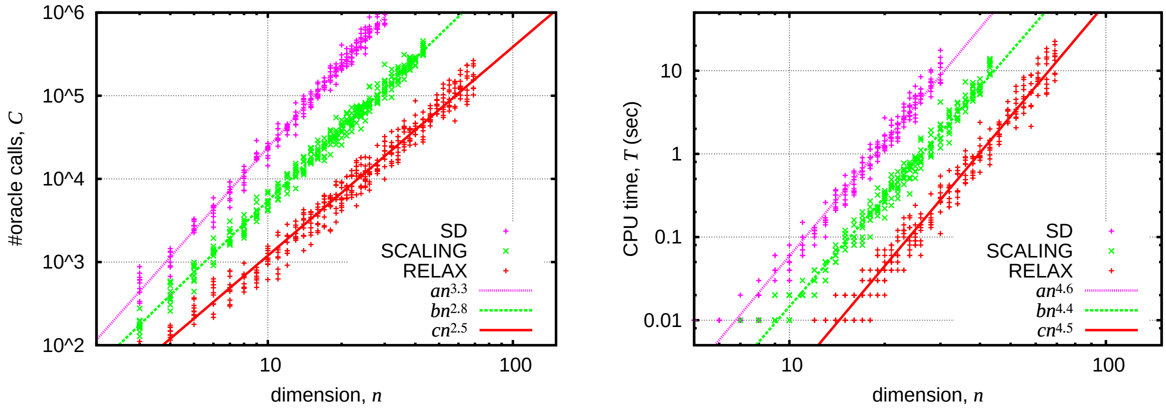

Figure 1: The number of oracle calls and CPU time for L$^\natural$-convex function minimization.

randomly choose an initial discrete solution p_0 satisfying $-10n \leq p_0(i) \leq 10n$.

As test problems for discrete M$^\natural$-convex function minimization, we consider the following function:

$$f(x) = \sum_{X \in \mathcal{T}} \{a_X x(X)^2 + b_X x(X) + c_X\} \quad (x \in \mathbf{Z}^n),$$

where $\mathcal{T}$ is a laminar family. In our continuous relaxation approach, we use

$$\bar{f}(x) = \sum_{X \in \mathcal{T}} \{a_X x(X)^2 + b_X x(X) + c_X\} \quad (x \in \mathbf{R}^n).$$

For each n, we generate ten test problems with randomly chosen real variables $0 < a_X \leq 1000$, $-1000 \leq b_X, c_X \leq 1000$ for $X \in \mathcal{T}$.

Our computational environment is the following: HP dx5150 SF/CT, AMD Athlon 64 3200+ processor (2.0GHz, 512KB L2 cache), 4GB memory, Vine Linux 4.1 (kernel 2.6.16), gcc 3.3.6.

All the algorithms implemented here provide an optimal solution under the assumption that an oracle for computing L$^\natural$-/M$^\natural$-convex function values is available. We measure the number of oracle calls and CPU time for each problem. Our numerical results are summarized in Figures 1 and 2. The left of Figure 1 shows the relationship between the number of oracle calls C and dimension n for L$^\natural$-convex function minimization, and the right shows the relationship between CPU time T and n. Figure 2 shows the case of M$^\natural$-convex function minimization. In all the algorithms the relationship is linear in $\log C$ and $\log n$, which implies $C = \mathrm{O}(n^l)$ for some l. Also, the relationship is linear in $\log T$ and $\log n$, which implies $T = \mathrm{O}(n^l)$ for some l. These results are displayed in Tables 3 and 4.

By numerical experiments with randomly generated test problems, we can conclude that our continuous relaxation approach is faster than the previously proposed algorithms.

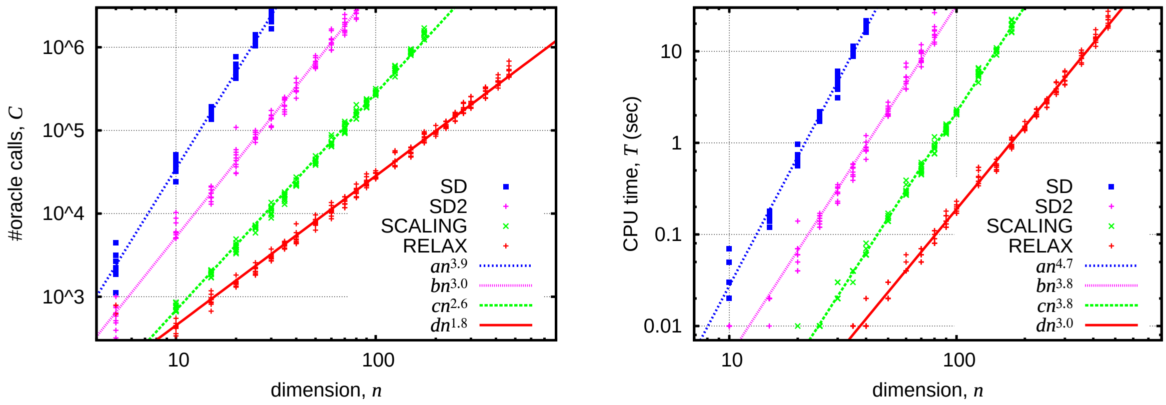

Figure 2: The number of oracle calls and CPU time for M$^\natural$-convex function minimization.

Table 3: Observed computational complexity for L$^\natural$-convex function minimization.

algorithm	SD	SCALING	RELAX
oracle calls C	$n^{3.3}$	$n^{2.8}$	$n^{2.5}$
CPU time T	$n^{4.6}$	$n^{4.4}$	$n^{4.5}$

Table 4: Observed computational complexity for M$^\natural$-convex function minimization.

algorithm	SD	SD2	SCALING	RELAX
oracle calls C	$n^{3.8}$	$n^{3.0}$	$n^{2.6}$	$n^{1.8}$
CPU time T	$n^{4.7}$	$n^{3.8}$	$n^{3.8}$	$n^{3.0}$

Acknowledgement

The authors thank Hiroshi Hirai, Satoru Iwata, and Kazuo Murota for stimulating comments. This work is supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

[1] V. Danilov, G. Koshevoy, and K. Murota, "Discrete convexity and equilibria in economies with indivisible goods and money," Math. Social Sci., **41**, pp. 251–273, 2001.

[2] A.W.M. Dress and W. Wenzel, "Valuated matroids," Adv. Math., **93**, pp. 214–250, 1992.

[3] S. Fujishige and K. Murota, “Notes on L-/M-convex functions and the separation theorems,” Math. Program., **88**, pp. 129–146, 2000.

[4] D.S. Hochbaum and J.G. Shanthikumar, “Convex Separable Optimization is Not Much Harder than Linear Optimization,” J. ACM, **37**, pp. 843–862, 1990.

[5] D.S. Hochbaum, “Lower and Upper Bounds for the Allocation Problem and Other Nonlinear Optimization Problems,” Mathematics of Operations Research, **19**, pp. 390–409, 1994.

[6] S. Iwata, L. Fleischer, and S. Fujishige, “A combinatorial strongly polynomial algorithm for minimizing submodular functions,” J. ACM, **48**, pp. 761–777, 2001.

[7] D.C. Liu and J. Nocedal, “On the limited memory method for large scale optimization,” Math. Prog. B, **45**, pp. 503–528, 1989.

[8] L. Lovász, “Submodular functions and convexity,” in Mathematical Programming—The State of the Art, eds. A. Bachem, M. Grötschel and B. Korte, pp.235–257, Springer-Verlag, 1983.

[9] S. Moriguchi, K. Murota, and A. Shioura, “Scaling Algorithms for M-convex Function Minimization,” IEICE Transactions on Fundamentals, **E85-A**, pp. 922–929, 2002.

[10] K. Murota, Matrices and Matroids for Systems Analysis, Springer, Berlin, 2000.

[11] K. Murota, Discrete Convex Analysis, Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[12] K. Murota and A. Tamura, “New characterizations of M-convex functions and their applications to economic equilibrium models with indivisibilities,” Discrete Appl. Math., **131**, pp. 495-512, 2003.

[13] K. Murota and A. Tamura, “Application of M-convex submodular flow problem to mathematical economics,” Japan J. Indust. Appl. Math., **20**, pp. 257–277, 2003.

[14] J.B. Orlin, “A faster strongly polynomial algorithm for submodular function minimization,” Proceedings of IPCO 12, M. Fischetti and D. Williamson, eds., Ithaca, NY, pp. 240–251, 2007.

[15] A. Schrijver, “A combinatorial algorithm minimizing submodular functions in strongly polynomial time,” J. Comb. Theory, Ser., **B 80**, pp. 346–355, 2000.