# MATHEMATICAL ENGINEERING TECHNICAL REPORTS

# Optimization-Based Stability Analysis of Structures under Unilateral Constraints

Yoshihiro KANNO and Makoto OHSAKI

METR 2008–09 March 2008

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

**WWW page: http://www.keisu.t.u-tokyo.ac.jp/research/techrep/index.html**

# Optimization-Based Stability Analysis of Structures under Unilateral Constraints

Yoshihiro Kanno [a,1] , Makoto Ohsaki [b,2]

[a] *Department of Mathematical Informatics,*
*University of Tokyo, Tokyo 113-8656, Japan*
E-mail: `kanno@mist.i.u-tokyo.ac.jp`

[b] *Department of Architecture and Architectural Engineering,*
*Kyoto University, Kyoto 615-8540, Japan*
E-mail: `ohsaki@archi.kyoto-u.ac.jp`

**Abstract**

This paper discusses an optimization-based technique for determining the stability of a given equilibrium point of the unilaterally constrained structural system, which is subjected to the static load. We deal with the three problems in mechanics sharing the common mathematical properties: (i) structures containing no-compression cables; (ii) frictionless contacts; (iii) elastic-plastic trusses with nonnegative hardening. It is shown that the stability of a given equilibrium point of these structures can be determined by solving a maximization problem of a convex function over a convex set. Based on the difference of convex functions (DC) optimization, we propose an algorithm to solve the stability determination problem, at each iteration of which a second-order cone programming (SOCP) problem is to be solved. The problems presented are solved for various structures to determine the stability of given equilibrium points.

**Keywords**

Frictionless contact; Unilateral constraint; Nonsmooth mechanics; Second-order cone program; Stability analysis.

[1] Corresponding author. Address: Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Bunkyo, Tokyo 113-8656, Japan. E-mail: `kanno@mist.i.u-tokyo.ac.jp`. Phone: +81-3-5841-6906, Fax: +81-3-5841-6886.

[2] Address: Department of Architecture and Architectural Engineering, Graduate School of Engineering, Kyoto University, Nishikyo, Kyoto 615-8540, Japan. E-mail: `ohsaki@archi.kyoto-u.ac.jp`. Phone: +81-75-383-2901, Fax: +81-75-383-2972.

# 1 Introduction

This paper discusses a numerical technique for stability determination of a given equilibrium point of a statically-loaded finite-dimensional mechanical system, when the displacements and/or stresses are subjected to unilateral constraints. The stability of the static equilibrium point is a radically important issue of nonlinear mechanics in civil, mechanical and aeronautical engineering [11]. Therefore, various criteria on the stability of equilibrium points have been proposed for elastic-inelastic structures [11, 25, 32].

It is known that various classes of structural systems are governed by the unilateral constraints on displacements and/or stresses; see, e.g. Duvaut and Lions [9]. In this paper, we propose a numerical method for the stability determination of the following three problems in nonsmooth mechanics that share the common mathematical formulations:

(i) Structures containing no-compression cables;

(ii) Frictionless contacts;

(iii) elastic-plastic trusses with nonnegative hardening.

A cable member cannot transmit compression force. This property is referred to as the *stress unilateral behavior*, which was studied by Panagiotopoulos [24] by means of the variational inequality. Later, the existence and uniqueness of static equilibrium points of cable networks were investigated in [5, 15, 36]. In this paper, we deal with the structures containing some cable members, and investigate the stability determination problem of such a structure.

For frictionless contacts, finding the equilibrium paths of an elastic beam which possibly contact with rigid obstacles is classical but still receives much attention [14, 35, 37]. A unilaterally constrained elastic plate have also been studied widely [8]. Mathematically, these problems are regarded as a tracing problem of a continuation of solutions of the family of variational inequalities [10, 20]. Besides these numerical path-tracing methods, the stability in frictionless contacts was investigated theoretically by Klarbring [17] and the references therein. Tschöpe *et al.* [33] proposed a numerical method for finding limit points in large-deformation frictionless contacts. We attempt in this paper to determine the stability of an equilibrium point which is given a priori. Note that our aim is neither to find bifurcation points nor to compute equilibrium paths.

Elastic-plastic analysis can be regarded as a problem in which stresses are subjected to the unilateral constraints [29], if infinitesimal displacements are considered. Hill [13] derived a sufficient condition for stability of a given equilibrium point of elastic-plastic structures. In this paper, for elastic-plastic problems, we restrict ourselves to truss structures for simplicity.

In this paper, we aim at proposing a numerical method to determine the stability of the given equilibrium point of a structures subjected to unilateral constraints. It is emphasized that the method should be applicable to large-scale problems, particularly, problems with a large number of unilateral constraints. We achieve this aim by using numerical optimization. For frictional contact problems, a mathematical programming approach was proposed to find the directionally unstable points by Pinto da Costa *et al.* [26], which is based on the enumeration of solutions to the complementarity problem. Casciaro and Mancusi [7] performed the imperfection sensitivity analysis numerically by solving a nonconvex quadratic programming problem.

We give some explanations regarding the perspective of a combinatorial property of stability analysis that we are interested in, by using an example of (i). Consider an elastic beam illustrated in Figure 1 (which is assumed to be discretized into some finite elements, if necessary). The axial force is applied to the beam so that the buckling may possibly occur. The beam is supported by 11 cables, which can transmit only tensile forces and are in their natural (unstressed) lengths at the pre-buckling state. Observe that not all cables are in tensile states at the post-buckling state: some cables (referred to as *active cables*) stretch with nonzero tensile forces, whereas the other cables (referred to as *inactive cables*) slacken without any forces. The inactive cables can be neglected for the sake of the stability analysis. If we know the 'correct' set of all active cables, then the tangent stiffness matrix can be evaluated immediately by neglecting inactive cables. The stability of an equilibrium point is determined as usual from the sign of the minimal eigenvalue of the tangent stiffness matrix.

It should be emphasized that the difficulty of our problem arises from the fact that *we cannot know the set of active cables a priori*. It is essentially a combinatorial problem to choose the 'correct' sets of active and inactive cables which are compatible to the minimal incremental potential energy. Similarly, in the case of (ii), we cannot know the set of nodes which remain in contact a priori; in the case of (iii), the set of loading members cannot be known a priori. The details of these specific problems are discussed in sections 3–5. To the authors' knowledge, no efficient methods have ever been proposed to resolve this combinatorial property in stability analyses of unilaterally constrained mechanical systems.

We present a unified perspective, as well as general formulations, for the stability determination of the structural systems (i)–(iii). Provided that the equilibrium point is given, we shall show that the directional stability of structural systems belonging to these three classes can be determined by solving an optimization problem, specifically, the maximization problem of a convex quadratic function over a convex homogeneous quadratic inequality and some linear inequalities. This is regarded as the first contribution of the paper. This claim is also regarded as an expository introduction of stability analyses to the community working on mathematical programs. Indeed, we recognize that stability analyses can be a natural application of some classes of nonconvex quadratic programming problems.

As the second contribution, we next propose a solution technique for the stability determination problem presented. Since the problem is nonconvex, the conventional local method may converge to a local solution, which implies that the stability cannot be determined correctly. On the other hand, any global optimization method seems to be too expensive for mechanical engineers from the view point of computational cost. It should be noted that, when the equilibrium-path following method is carried out, one may wish to employ the stability determination at each equilibrium point obtained. Moreover, in designing process of structures, stability under a given load should be checked repeatedly for different designs of a structure. Consequently, from a practical point of view, it seems that a global optimization approach is not suitable for stability analysis. Hence, in this paper, we choose a local method, which quite often converges to the global optimal solution.

We show that the stability determination problem presented can be embedded into the form of the *DC* (difference of convex functions) *program* [4]. The DC programming problem is the minimization of the difference of two convex functions. The DC algorithm was proposed as a local

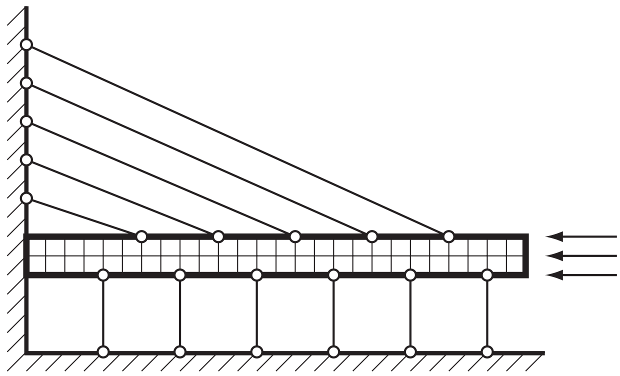


Figure 1: Finitely discretized elastic structure containing 11 cable members.

method solving the DC programming problem, and has been examined by various kinds of DC programming problems; see the review paper [4] and the references therein. It has been observed that the DC algorithm quite often converges to global optimal solution, in spite of the fact that the convergence to the global solution is not guaranteed theoretically.

With the aid of DC algorithm, we propose an algorithm for the stability determination problem, which is not limited to toy-size problems. Indeed, it is speculated through our numerical experiments that the algorithm finds global solutions successfully. At each iteration of the algorithm, we solve a *second-order cone programming* (SOCP) problem [1], which can be solved efficiently by using the primal-dial interior-point method.

This paper is organized as follows. The notation used in this paper is prepared in section 1.1. In section 2, we introduce the optimization problem that we are interested in, and an algorithm is proposed. Three specific problems in mechanics (i)–(iii) are investigated in sections 3–5, respectively. In section 3, we consider the structures including some cable members. The frictionless contact problem, which might be the most typical example of the mechanical systems subjected to unilateral constraints, is investigated in section 4. In section 5, we consider elastic-plastic trusses. Numerical experiments are presented in section 6 for various structures investigated in the three sections precedent. Our results are summarized in section 7. All proofs appear in section 8.

### 1.1 Notation

All vectors are assumed to be column vectors in this paper. The $(m+n)$-dimensional column vector $(\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{v}^{\mathrm{T}})^{\mathrm{T}}$ consisting of $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^n$ is often written simply as $(\boldsymbol{u}, \boldsymbol{v})$. The $L_p$-norm of the vector $\boldsymbol{v} = (v_i) \in \mathbb{R}^n$ is defined by $\|\boldsymbol{v}\|_p = \sum_{i=1}^n (|v_i|^p)^{1/p}$. Particularly, the $L_2$-norm (or the standard Euclidean norm) is written simply as $\|\boldsymbol{v}\| = (\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v})^{1/2}$.

We denote by $\mathbb{R}^n_+ \subset \mathbb{R}^n$ and $\mathbb{R}^n_{++} \subset \mathbb{R}^n$ the nonnegative and positive orthants, respectively, i.e.

$$\mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \boldsymbol{0}\},$$
$$\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_i) \in \mathbb{R}^n \mid x_i > 0 \ (i = 1, \dots, n)\}.$$

We write $\boldsymbol{v} \geq \boldsymbol{0}$ and $\boldsymbol{v} \geq \boldsymbol{w}$, respectively, if $\boldsymbol{v} \in \mathbb{R}^n_+$ and $\boldsymbol{v} - \boldsymbol{w} \geq \boldsymbol{0}$. The set of all $n \times n$ real symmetric matrices is denoted by $\mathcal{S}^n \subset \mathbb{R}^{n \times n}$. Let $\mathcal{S}^n_+ \subset \mathcal{S}^n$ and $\mathcal{S}^n_{++} \subset \mathcal{S}^n$ denote the sets of all positive semidefinite matrices and all positive definite matrices, respectively. We write $\boldsymbol{P} \succeq \boldsymbol{O}$ and $\boldsymbol{P} \succeq \boldsymbol{Q}$, respectively, if $\boldsymbol{P} \in \mathcal{S}^n_+$ and $\boldsymbol{P} - \boldsymbol{Q} \succeq \boldsymbol{O}$.

The cardinality of the set $\mathcal{A}$ is denoted by $|\mathcal{A}|$. The inner product $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w}$ of the vectors $\boldsymbol{v} \in \mathbb{R}^n$ and $\boldsymbol{w} \in \mathbb{R}^n$ is also written as $\langle \boldsymbol{v}, \boldsymbol{w} \rangle$. The identity matrix with an appropriate size is denoted by $\boldsymbol{I}$.

# 2 General framework for stability analysis

## 2.1 Stability criterion

Consider a finite-dimensional structure in two- or three-dimensional space. The structure is subjected to static nodal loads. Let $\boldsymbol{\xi}^0 \in \mathbb{R}^k$ denote the vector of state variables describing the static equilibrium point, which consists of the total nodal displacement vector and the generalized stress vector. Suppose that we are given the equilibrium point $\boldsymbol{\xi}^0$ under the specified external load. The vector of infinitesimal incremental displacements from $\boldsymbol{\xi}^0$ is denoted by $\boldsymbol{u} \in \mathbb{R}^{n^{\mathrm{d}}}$, where $n^{\mathrm{d}}$ is the number of degrees of freedom. We denote by $\mathcal{A}(\boldsymbol{\xi}^0) \subseteq \mathbb{R}^{n^{\mathrm{d}}}$ the set of all admissible incremental displacements $\boldsymbol{u}$ satisfying the boundary conditions.

### 2.1.1 Stability condition for elastic structures

For elastic structures subjected to unilateral stress and/or displacement constraints, the total potential energy $\Pi(\boldsymbol{u})$ is defined for admissible incremental displacements vector $\boldsymbol{u} \in \mathcal{A}(\boldsymbol{\xi}^0)$. By application of Liapunov's direct method [18], a sufficient condition of stability is given as follows.

**Definition 2.1 (stability of an elastic system).** The equilibrium point $\boldsymbol{\xi}^0$ is said to be stable if $\Pi : \boldsymbol{u} \mapsto \Pi(\boldsymbol{u})$ is continuously differentiable at any $\boldsymbol{u} \in \mathcal{A}(\boldsymbol{\xi}^0)$ and if $\Pi$ has an isolated minimum at $\boldsymbol{u} = \boldsymbol{0}$.

We denote the tangential stiffness matrix by $\boldsymbol{K}(\boldsymbol{u}; \boldsymbol{\xi}^0) \in \mathcal{S}^{n^{\mathrm{d}}}$, which, in general, is a matrix-valued function $\boldsymbol{K}(\cdot; \boldsymbol{\xi}^0) : \boldsymbol{u} \mapsto \boldsymbol{K}(\boldsymbol{u}; \boldsymbol{\xi}^0)$. Let $v(\boldsymbol{u})$ denote the (twice of) second-order term of the increment of the potential energy corresponding to $\boldsymbol{u}$ at $\boldsymbol{\xi}^0$, which can be written as

$$v(\boldsymbol{u}) = \boldsymbol{u}^{\mathrm{T}} \boldsymbol{K}(\boldsymbol{u}, \boldsymbol{\xi}^0) \boldsymbol{u}. \tag{1}$$

Define $v^* \in \mathbb{R}$ by

$$v^* := \min_{\boldsymbol{u}} \left\{ v(\boldsymbol{u}) \mid \boldsymbol{u} \in \mathcal{A}(\boldsymbol{\xi}^0),\ \|\boldsymbol{u}\|_p = 1 \right\}. \tag{2}$$

Note that we shall put $p = 2$ in our numerical method for stability analysis. However, the discussions in the remainder of this section are valid for any $p \in [1, \infty]$.

The sufficient conditions for stability and instability are then written as follows.

**Definition 2.2 (sufficient conditions for stability and instability of an elastic system).** The equilibrium point $\boldsymbol{\xi}^0$ is said to be stable (resp. unstable) if $v^* > 0$ (resp. $v^* < 0$).

Roughly speaking, the equilibrium point $\boldsymbol{\xi}^0$ is stable, if $v(\boldsymbol{u}) > 0$ for any $\boldsymbol{u} \in \mathbb{R}^{n^{\mathrm{d}}} \setminus \{\boldsymbol{0}\}$ which is kinematically admissible.

### 2.1.2 Stability condition for elastic-plastic structures

For elastic-plastic structures, the total potential energy cannot be defined, because the incremental work is path-dependent. Therefore, we employ *directional instability* [6, 13, 25] which is defined by using $v^*$, introduced in (2), as follows.

**Definition 2.3 (sufficient conditions for directional stability of an elastic-plastic system).** The equilibrium point $\boldsymbol{\xi}^0$ is said to be directionally stable (resp. directionally unstable) if $v^* > 0$ (resp. $v^* < 0$). The equilibrium point is unstable if it is directionally unstable.

Note that the directional stability is a necessary condition for the stability of an elastic-plastic structure. In fact, a directionally stable structure may be unstable if indirect paths are considered.

However, we restrict ourselves to the directional stability of elastic-plastic structure. In the remainder of the paper, we omit the term *directional* as far as no confusion is possible.

### 2.1.3 Stability determination problem

It follows from sections 2.1.1 and 2.1.2 that the stability of both elastic and elastic-plastic systems are determined by finding a global optimal solution of the problem (2). Unfortunately, (2) is a nonconvex optimization problem, since the constraint $\|\boldsymbol{u}\|_p = 1$ in (2) is nonconvex. Furthermore, we focus on the cases in which the matrix $\boldsymbol{K}(\boldsymbol{u};\boldsymbol{\xi}^0)$ is indefinite (even for the fixed $\boldsymbol{u}$).

*Remark 2.4.* As the simplest particular case of (2) with $p = 2$, suppose that $\boldsymbol{K}(\boldsymbol{u};\boldsymbol{\xi}^0)$ does not depend on $\boldsymbol{u}$ and the constraint $\boldsymbol{u} \in \mathcal{A}(\boldsymbol{\xi}^0)$ of (2) does not exist. Then the global optimal solution of (2) can be obtained easily. More precisely, suppose that

$$\mathcal{A}(\boldsymbol{\xi}^0) = \mathbb{R}^{n^{\mathrm{d}}}, \tag{3}$$

$$\boldsymbol{K}(\boldsymbol{u};\boldsymbol{\xi}^0) = \boldsymbol{K}^0(\boldsymbol{\xi}^0), \quad \forall \boldsymbol{u} \in \mathcal{A}(\boldsymbol{\xi}^0), \tag{4}$$

where $\boldsymbol{K}^0(\boldsymbol{\xi}^0) \in \mathcal{S}^{n^{\mathrm{d}}}$ is a constant (but generically indefinite) matrix. Then $v^*$ defined by (2) coincides with the minimum eigenvalue of $\boldsymbol{K}^0(\boldsymbol{\xi}^0)$, although (2) is still nonconvex. Certainly, we are interested in the cases in which at least one of (3) and (4) does not hold. Practical situations of such cases are discussed specifically in sections 3.1, 4.1, and 5.1. Gander *et al.* [12] investigated a particular case of the problem (2), referred to as the *constrained eigenvalue problem*, in which the condition (4) holds and $\mathcal{A}(\boldsymbol{\xi}^0)$ is represented by some linear equalities. In order to solve this problem, a DC algorithm was proposed by Tao and An [31]. Note that we are interested in the case in which $\mathcal{A}(\boldsymbol{\xi}^0)$ is expressed by some linear inequalities, and thus our problem is different from the constrained eigenvalue problem proposed in [12]. ■

*Remark 2.5.* An interesting problem related to the constrained eigenvalue problem discussed in Remark 2.4 arises for the case in which $\boldsymbol{K}^0(\boldsymbol{\xi}^0)$ is indefinite and $p = \infty$. In this case, the problem (2) is NP-hard even if the conditions (3) and (4) are satisfied. Thus, in contrast to the case of $p = 2$, it is difficult to obtain the global optimal solution of (2) with $p = \infty$. Nesterov [21] proposed an approximation algorithm for this problem with the theoretically guaranteed approximation ratio. An important application of the problem (2) is the trust-region subproblem [22, section 18.5]. Thus, in spite of its simplicity in formulation, the problem (2) still receives much attention from the both view points of algorithms and applications, while in this paper we concentrate on the stability analysis, based on Definition 2.3, arising from various structural engineering applications. ■

### 2.2 Reduction to feasibility problem

Hereinafter, we simply put $p = 2$ in the stability determination problem (2). We write $\|\boldsymbol{p}\|$ instead of $\|\boldsymbol{p}\|_2$ for simplicity.

In sections 3–5, we show that the second-order term of the incremental potential energy is written as

$$v(\boldsymbol{u}, \boldsymbol{z}) = \boldsymbol{u}^{\mathrm{T}} \boldsymbol{Q}_0 \boldsymbol{u} + \boldsymbol{z}^{\mathrm{T}} \boldsymbol{Q}_1 \boldsymbol{z}$$

by introducing some auxiliary variables $\boldsymbol{z} \in \mathbb{R}^m$. We also show that $v^*$ defined by (2) can be obtained as the optimal value of an optimization problem in the form of

$$v^* = \min_{\boldsymbol{u}, \boldsymbol{z}} \left\{ \boldsymbol{u}^{\mathrm{T}} \boldsymbol{Q}_0 \boldsymbol{u} + \boldsymbol{z}^{\mathrm{T}} \boldsymbol{Q}_1 \boldsymbol{z} \mid (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{F}, \ \|\boldsymbol{u}\|^2 = 1 \right\}. \tag{5}$$

See Proposition 3.2, the problem (43), and Proposition 5.2 for the detail of each specific problem. Note that $\boldsymbol{Q}_0 \in \mathcal{S}^{n^{\mathrm{d}}}$ and $\boldsymbol{Q}_1 \in \mathcal{S}^m_{++}$ are constant in (5). Furthermore, $\mathcal{F} \subseteq \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m$ is a convex set that can be represented in the form of

$$\mathcal{F} = \left\{ (\boldsymbol{u}, \boldsymbol{z}) \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \mid \boldsymbol{A}_u \boldsymbol{u} + \boldsymbol{A}_z \boldsymbol{z} \geq \boldsymbol{0} \right\}, \tag{6}$$

where $\boldsymbol{A}_u$ and $\boldsymbol{A}_z$ are constant matrices with appropriate sizes. We assume throughout the paper that the problem (5) is feasible. The stability of $\boldsymbol{\xi}^0$ is determined by solving (5) instead of (2). Notice here that $v^*$ is positive if $\boldsymbol{Q}_0$ is positive definite. Therefore, we focus on the case in which $\boldsymbol{Q}_0$ is indefinite.

Choose a constant, $\widetilde{\lambda} \in \mathbb{R}_{++}$, so that the condition

$$\boldsymbol{Q}_0 + \widetilde{\lambda} \boldsymbol{I} \in \mathcal{S}^n_{++} \tag{7}$$

is satisfied, i.e. $\widetilde{\lambda}$ is greater than the absolute value of the smallest eigenvalue of $\boldsymbol{Q}_0$. Define $\widetilde{\boldsymbol{Q}}_0 \in \mathcal{S}^{n^{\mathrm{d}}}_{++}$ by

$$\widetilde{\boldsymbol{Q}}_0 = \boldsymbol{Q}_0 + \widetilde{\lambda} \boldsymbol{I}. \tag{8}$$

For simplicity, we use the following notations:

$$\boldsymbol{x} := \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{z} \end{pmatrix}, \quad \widetilde{f}(\boldsymbol{u}, \boldsymbol{z}) = \boldsymbol{u}^{\mathrm{T}} \widetilde{\boldsymbol{Q}}_0 \boldsymbol{u} + \boldsymbol{z}^{\mathrm{T}} \boldsymbol{Q}_1 \boldsymbol{z}, \quad g(\boldsymbol{u}, \boldsymbol{z}) = \|\boldsymbol{u}\|^2 - 1. \tag{9}$$

Consider the following problem, which defines $\widetilde{v}$:

$$\begin{aligned} \widetilde{v} &= \min_{\boldsymbol{x}} \left\{ \widetilde{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{F}, \ g(\boldsymbol{x}) \geq 0 \right\} \\ &= \min_{\boldsymbol{u}, \boldsymbol{z}} \left\{ \boldsymbol{u}^{\mathrm{T}} \widetilde{\boldsymbol{Q}}_0 \boldsymbol{u} + \boldsymbol{z}^{\mathrm{T}} \boldsymbol{Q}_1 \boldsymbol{z} \mid (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{F}, \ \|\boldsymbol{u}\|^2 - 1 \geq 0 \right\}. \end{aligned} \tag{10}$$

The following proposition implies that any optimal solution of (5) is also optimal for (10), and vice versa.

**Proposition 2.6 (relation between** (5) **and** (10)**).** *The problems* (5) *and* (10) *share the same set of optimal solutions. Moreover,* $\widetilde{v} = v^* + \widetilde{\lambda}$ *holds.*

Note that all proofs for this section appear in section 8.1.

The following is an immediate corollary of Definition 2.3 and Proposition 2.6:

**Corollary 2.7 (stability determination based on (10)).** *The equilibrium point $\boldsymbol{\xi}^0$ is stable (resp. unstable) if $\widetilde{v} > \widetilde{\lambda}$ (resp. $\widetilde{v} < \widetilde{\lambda}$).*

Proposition 2.6 implies that we can obtain an optimal solution of (5) by solving the problem (10). From Corollary 2.7 we see that it is important for structural engineering whether the optimal value of (10) satisfies $\widetilde{v} > \widetilde{\lambda}$ or not. This motivates us to propose an algorithm for (10) which for most cases converges to the global optimal solution within the computational time acceptable for real engineering applications. We need further reformulation of (10) to a tractable form.

By exchanging the objective function and the nonconvex constraint of the problem (10), we consider the following family of problems with respect to a parameter $\lambda$:

$$g^*(\lambda) := \max_{\boldsymbol{x}} \left\{ g(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{F},\ \widetilde{f}(\boldsymbol{x}) \leq \lambda \right\}, \tag{11}$$

which defines the function $g^* : \mathbb{R}_{++} \to \mathbb{R}$. Note that (11) is a maximization of the convex function over the convex set.

**Proposition 2.8 (relation between (10) and (11)).** *For any $\lambda \in \mathbb{R}_{++}$,*

(i) *$\widetilde{v} \geq \lambda$ implies $g^*(\lambda) \leq 0$.*

(ii) *$g^*(\lambda) \leq 0$ implies $\widetilde{v} \geq \lambda$.*

The relation investigated in Proposition 2.8 is sometimes called *duality* between the objective and constraint functions [34]. The following is the key result of this paper, which implies that the stability can be determined by solving the problem (11) instead of the problem (10).

**Theorem 2.9 (stability determination based on (11)).** *The equilibrium point $\boldsymbol{\xi}^0$ is*

(i) *stable if $g^*(\widetilde{\lambda}) < 0$;*

(ii) *unstable if $g^*(\widetilde{\lambda}) > 0$.*

*Furthermore,*

(iii) *$\widetilde{v} = \lambda$ if and only if $g^*(\lambda) = 0$.*

**Corollary 2.10 (optimality condition of (10)).** *A feasible solution $\overline{\boldsymbol{x}}$ of the problem (10) is optimal if and only if $g^*(\widetilde{f}(\overline{\boldsymbol{x}})) = 0$.*

It is clear that a feasible solution $\overline{\boldsymbol{x}}$ of (10) is optimal if and only if $\widetilde{f}(\overline{\boldsymbol{x}}) = \widetilde{v}$. Consequently, Corollary 2.10 follows Theorem 2.9 (iii) immediately.

## 2.3 Sequential convex optimization algorithm

We have seen in Theorem 2.9 that the stability of the given equilibrium point is determined by solving the problem (11). In this section we first propose an algorithm for (11), at each iteration of which a convex programming problem is solved.

We first show that the problem (11) can be reformulated as a *DC* (difference of convex functions) *programming* problem [4]. We denote by $I_{\widetilde{\mathcal{F}}}(\cdot;\lambda) : \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \to (-\infty, +\infty]$ the indicator function of the feasible set of (11), i.e.,

$$I_{\widetilde{\mathcal{F}}}(\boldsymbol{x};\lambda) = \begin{cases} 0, & \text{if } \boldsymbol{x} \in \mathcal{F},\ \widetilde{f}(\boldsymbol{x}) \le \lambda, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let $\rho \in \mathbb{R}_{++}$ be a constant. Define $h_1 : \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \to \mathbb{R}$ and $h_2 : \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \to (-\infty, +\infty]$ by

$$h_1(\boldsymbol{x}) = \frac{\rho}{2}\|\boldsymbol{x}\|^2 + g(\boldsymbol{x}), \tag{12}$$

$$h_2(\boldsymbol{x};\lambda) = I_{\widetilde{\mathcal{F}}}(\boldsymbol{x};\lambda) + \frac{\rho}{2}\|\boldsymbol{x}\|^2. \tag{13}$$

Note that $h_1$ and $h_2$ are strictly convex. Then the problem (11) is equivalently rewritten as

$$\max_{\boldsymbol{x}} \left\{ h_1(\boldsymbol{x}) - h_2(\boldsymbol{x};\lambda) \mid \boldsymbol{x} \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \right\}, \tag{14}$$

which is a DC programming problem. The dual problem of (14) is written as

$$\max_{\boldsymbol{y}} \left\{ h_2^*(\boldsymbol{y};\lambda) - h_1^*(\boldsymbol{y}) \mid \boldsymbol{y} \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m \right\}, \tag{15}$$

where $h_1^*$ and $h_2^*$ denote the conjugate functions of $h_1$ and $h_2$, respectively.

*Remark 2.11.* The choice of the pair of functions $h_1$ and $h_2$ in (12) and (13) is not unique. Indeed, it is known that there exist infinitely many pairs of strictly convex functions $h_1$ and $h_2$ such that (14) becomes equivalent to (11). See [4] for more details. Certainly, for example, we may choose any $\rho \in \mathbb{R}_{++}$ in (12) and (13), while the convergence property of the algorithm presented below may depend on the choice of $\rho$. In our numerical examples, we choose $\rho = 0.1$; see section 6. ■

The DC algorithm generates two sequences $\{\boldsymbol{x}^k\}$ and $\{\boldsymbol{y}^k\}$ by defining $\boldsymbol{x}^{k+1}$ and $\boldsymbol{y}^k$, respectively, as the solutions to the convex problems [4]

$$\max_{\boldsymbol{x} \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m} \left\{ \left[ (\boldsymbol{y}^k)^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}^k) + h_1(\boldsymbol{x}^k) \right] - h_2(\boldsymbol{x};\lambda) \right\}, \tag{16}$$

$$\max_{\boldsymbol{y} \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m} \left\{ \left[ (\boldsymbol{x}^k)^{\mathrm{T}}(\boldsymbol{y} - \boldsymbol{y}^{k-1}) + h_2^*(\boldsymbol{y}^{k-1};\lambda) \right] - h_1^*(\boldsymbol{y}) \right\}. \tag{17}$$

Observe that (16) (resp. (17)) is defined from (14) (resp. (15)) by replacing $h_1$ (resp. $h_2(\cdot;\lambda)$) by its affine approximation. The DC algorithm yields the update scheme

$$\boldsymbol{y}^k \in \partial h_1(\boldsymbol{x}^k), \tag{18}$$

$$\boldsymbol{x}^{k+1} \in \partial h_2^*(\boldsymbol{y}^k;\lambda). \tag{19}$$

Then it is known that the sequences $\{h_1(\boldsymbol{x}^k) - h_2(\boldsymbol{x}^k;\lambda)\}$ and $\{h_2^*(\boldsymbol{y}^k;\lambda) - h_1^*(\boldsymbol{y}^k)\}$ of the objective functions of (14) and (15) generated by the DC algorithm increase monotonically [4].

From the definition (12) of $h_1$, the update scheme (18) of $\boldsymbol{y}$ is explicitly written as

$$\boldsymbol{y}^k := \rho\boldsymbol{x}^k + \nabla g(\boldsymbol{x}^k), \tag{20}$$

because $h_1$ is continuously differentiable. Substitution of (20) into (16) yields

$$\max_{\boldsymbol{x}} \left\{ (\rho \boldsymbol{x}^k + \nabla g(\boldsymbol{x}^k))^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}^k) + \left( \frac{\rho}{2}\|\boldsymbol{x}^k\|^2 + g(\boldsymbol{x}^k) \right) - \frac{\rho}{2}\|\boldsymbol{x}\|^2 \mid \boldsymbol{x} \in \mathcal{F},\ \widetilde{f}(\boldsymbol{x}) \le \lambda \right\}. \tag{21}$$

By multiplying a constant $2/\rho$ and eliminating the constant terms, the objective function of (21) is simplified without changing the optimal solution as

$$\max_{\boldsymbol{u},\boldsymbol{z}} \left\{ -\left\| \boldsymbol{u} - (1 + (2/\rho))\boldsymbol{u}^k \right\|^2 - \left\| \boldsymbol{z} - \boldsymbol{z}^k \right\|^2 \mid (\boldsymbol{u}, \boldsymbol{z}) \in \mathcal{F},\ \widetilde{f}(\boldsymbol{u}, \boldsymbol{z}) \le \lambda \right\}. \tag{22}$$

The following algorithm solves a convex problem (22) sequentially to obtain a solution of (11).

**Algorithm 2.12 (sequential convex optimization method for** (11)**).**

**Step 0:** Choose $(\boldsymbol{u}^0, \boldsymbol{z}^0) \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^m$, $\rho > 0$, and the tolerance $\epsilon > 0$. Set $k := 0$.

**Step 1:** Find the optimal solution $(\boldsymbol{u}^{k+1}, \boldsymbol{z}^{k+1})$ of (22).

**Step 2:** If $\|(\boldsymbol{u}^{k+1}, \boldsymbol{z}^{k+1}) - (\boldsymbol{u}^k, \boldsymbol{z}^k)\| \le \epsilon$, then stop. Otherwise, set $k \leftarrow k + 1$, and go to Step 1.

Algorithm 2.12 is guaranteed to be well-defined by the following proposition in the sense that the subproblem (22) solved at each iteration has the unique solution.

**Proposition 2.13 (property of the subproblem).** *The convex problem* (22) *has the unique optimal solution.*

The proof is easy, and hence is omitted. The following corollary follows Theorem 2.9.

**Corollary 2.14 (sufficient condition for instability).** *Put $\lambda := \widetilde{\lambda}$ in the problem* (11)*, and let $\boldsymbol{x}^* = (\boldsymbol{u}^*, \boldsymbol{z}^*)$ be an accumulation point of a sequence $\{\boldsymbol{x}^k\}$ generated by Algorithm 2.12. If $g(\boldsymbol{x}^*) > 0$, then the equilibrium point $\boldsymbol{\xi}^0$ is unstable.*

Since Algorithm 2.12 is based on a local optimality condition of the problem (11), it cannot guarantee the global optimality of a solution obtained. However, it has been observed (see, e.g. [4]) that the DC algorithm very often converges to global optimal solutions of various nonconvex optimization problems in practice. Therefore, from Theorem 2.9 and the fact that Algorithm 2.12 with $\lambda := \widetilde{\lambda}$ provides a lower bound of $g^*(\widetilde{\lambda})$, the equilibrium point is stable for most cases if $g(\boldsymbol{x}^*) < 0$.

Note that, from Theorem 2.9, it is sufficient to compute $g^*(\widetilde{\lambda})$ in order to determine the stability. On the contrary, when we want to know the incremental displacement corresponding to the minimum increment of potential energy, the problem (10) is to be solved. It follows from Corollary 2.10 that (10) can be solved by using a bi-section method, in which we solve (11) several times with various values of $\lambda$. Provided that Algorithm 2.12 converges to the global optimal solution of (11), the following algorithm computes the global optimal solution of (10):

**Algorithm 2.15 (bisection method for** (10)**).**

**Step 0:** Choose $\underline{\lambda}^0$ and $\overline{\lambda}^0$ satisfying $0 < \underline{\lambda}^0 \le \lambda^* \le \overline{\lambda}^0$, and the tolerance $\epsilon > 0$. Set $k := 0$.

**Step 1:** If $\overline{\lambda}^k - \underline{\lambda}^k \le \epsilon$, then stop. Otherwise, set $\lambda := (\underline{\lambda}^k + \overline{\lambda}^k)/2$.

**Step 2:** Find an optimal solution $(\boldsymbol{u}^*, \boldsymbol{z}^*)$ of the problem (11) by using Algorithm 2.12.

**Step 3:** If $g(\boldsymbol{x}^*) < 0$, then set $\underline{\lambda}^{k+1} := \lambda$ and $\overline{\lambda}^{k+1} := \overline{\lambda}^k$. Otherwise, set $\overline{\lambda}^{k+1} := \lambda$ and $\underline{\lambda}^{k+1} := \underline{\lambda}^k$.

**Step 4:** Set $k := k + 1$, and go to Step 1.

*Remark 2.16.* If the global optimal solution of (11) is found successfully at Step 2 of each iteration, Algorithm 2.15 converges to the global optimal solution of (10). Before the algorithm terminates, exactly $\lceil \log_2((\overline{\lambda}^0 - \underline{\lambda}^0)/\epsilon) \rceil$ iterations are required, where $\lceil p \rceil$ denotes the minimum integer that is not smaller than $p \in \mathbb{R}$. Note that the optimal solution of (11) obtained in the previous iteration can be used as an initial solution for Algorithm 2.12 at Step 2 of the next iteration. Numerical experiments demonstrate that the usage of the previous solution as the initial solution drastically reduces the number of iterations required by Algorithm 2.12; see section 6. ■

# 3 Structures containing no-compression cables

Consider an elastic finite dimensional structure containing cable members that cannot transmit compressive forces; i.e. the cable member is assumed to consist of *no-compression material*. An example of such structures is illustrated in Figure 1. The structure undergoes large deformation.

Let $n^{\rm m}$ and $n^{\rm d}$, respectively, denote the number of members and the number of degrees of freedom of displacements. The equilibrium point $\boldsymbol{\xi}^0$ is defined by the total displacement vector in this section. The incremental elongation of the $j$th member is denoted by $c_j$. The compatibility condition at $\boldsymbol{\xi}^0$ between $c_j$ $(j = 1, \dots, n^{\rm m})$ and the incremental displacements $\boldsymbol{u}$ is written as

$$\boldsymbol{c}(\boldsymbol{u}) = \boldsymbol{B}(\boldsymbol{\xi}^0)^{\rm T} \boldsymbol{u}, \tag{23}$$

where $\boldsymbol{B}(\boldsymbol{\xi}^0) \in \mathbb{R}^{n^{\rm d} \times n^{\rm m}}$ is a constant matrix.

## 3.1 Stability of structures with cables

We pay particular attention to a cable member $j$, the stiffnesses of which depends on the sign of the incremental elongation $c_j$. Let $\mathcal{J} \subseteq \{1, \dots, n^{\rm m}\}$ denote the set of all indices of cable members with vanishing elongations at $\boldsymbol{\xi}^0$.

For each $j \in \mathcal{J}$, let $k_j^{\rm c}(c_j)$ denote the elongation stiffness at $\boldsymbol{\xi}^0$, which is written as

$$k_j^{\rm c}(c_j) = \begin{cases} d_j, & \text{if } c_j \geq 0, \\ 0, & \text{if } c_j < 0, \end{cases} \tag{24}$$

where $d_j \in \mathbb{R}_{++}$ is constant. The dependence (24) of stiffness of the $j$th cable $(j \in \mathcal{J})$ on the sign of $c_j$ essentially gives the stability determination problem the combinatorial complexity.

Consider the structure obtained by neglecting the cable members belonging to $\mathcal{J}$. The tangential stiffness matrix of the obtained structure is denoted by $\boldsymbol{K}^+(\boldsymbol{\xi}^0) \in \mathcal{S}^{n^{\rm d}}$, which is a constant matrix at the given $\boldsymbol{\xi}^0$. Note that the slack cables at $\boldsymbol{\xi}^0$ do not contribute to $\boldsymbol{K}^+$, while the contributions of tense cables are included in $\boldsymbol{K}^+$. We are interested in a case in which the smallest eigenvalue of $\boldsymbol{K}^+$ is negative due to the geometrical nonlinearity.

From (24) and the definition of $\boldsymbol{K}^{+}$, the definition (1) of $v$ can be reduced to

$$v(\boldsymbol{u}) = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}^{+}(\boldsymbol{\xi}^{0})\boldsymbol{u} + \sum_{j\in\mathcal{J}} k_j^{\mathrm{c}}(c_j)\,[c_j(\boldsymbol{u})]^2 . \tag{25}$$

The stability determination problem (2) is formulated as

$$v^{*} = \min_{\boldsymbol{u}} \left\{ v(\boldsymbol{u}) \mid \|\boldsymbol{u}\|^2 = 1 \right\}, \tag{26}$$

where the subsidiary conditions (23) and (24) should be satisfied. For $c_j \in \mathbb{R}$, consider the optimization problem in the variable $z_j \in \mathbb{R}$

$$w_j^{\mathrm{c}}(c_j) := \min_{z_j} \left\{ d_j z_j^2 \mid z_j \geq c_j \right\}. \tag{27}$$

The following proposition prepares a reformulation of the problem (26) into the form of (5):

**Proposition 3.1.** *Let $\overline{z}_j$ denote the (unique) optimal solution of the problem* (27)*. Then, $c_j$ and $\overline{z}_j$ satisfy*

$$\overline{z}_j = \begin{cases} c_j, & \text{if } c_j \geq 0, \\ 0, & \text{if } c_j < 0. \end{cases} \tag{28}$$

*Furthermore, the optimal value of* (27) *satisfies*

$$w_j^{\mathrm{c}}(c_j) = k_j^{\mathrm{c}}(c_j) c_j^2. \tag{29}$$

*Proof.* See section 8.2.1. □

Define the vector $\boldsymbol{z}_{\mathcal{J}} \in \mathbb{R}^{|\mathcal{J}|}$ by

$$\boldsymbol{z}_{\mathcal{J}} = (z_j \mid j \in \mathcal{J}), \tag{30}$$

which is the sub-vector of $\boldsymbol{z}$ composed of $z_j$ indexed by the set $\mathcal{J}$. Furthermore, let $\boldsymbol{B}_{\mathcal{J}} \in \mathbb{R}^{n^{\mathrm{d}} \times |\mathcal{J}|}$ denote the sub-matrix of $\boldsymbol{B}$ composed of the rows indexed by $\mathcal{J}$. Similarly, we denote by $\boldsymbol{d}_{\mathcal{J}} \in \mathbb{R}^{|\mathcal{J}|}$ the vector composed of $d_j$ indexed by $\mathcal{J}$. Let $\boldsymbol{D}_{\mathcal{J}} = \mathrm{Diag}(\boldsymbol{d}_{\mathcal{J}}) \in \mathcal{S}_{++}^{|\mathcal{J}|}$. Consider the following problem in the variables $(\boldsymbol{u}, \boldsymbol{z}_{\mathcal{J}}) \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^{|\mathcal{J}|}$:

$$\min_{\boldsymbol{u}, \boldsymbol{z}_{\mathcal{J}}} \left\{ \boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}^{+}\boldsymbol{u} + \boldsymbol{z}_{\mathcal{J}}^{\mathrm{T}}\boldsymbol{D}_{\mathcal{J}}\boldsymbol{z}_{\mathcal{J}} \mid \boldsymbol{z}_{\mathcal{J}} \geq \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}}\boldsymbol{u},\ \|\boldsymbol{u}\|^2 = 1 \right\}. \tag{31}$$

**Proposition 3.2.** *The optimal value of* (31) *coincides with $v^{*}$ defined by* (26)*. Furthermore, $\overline{\boldsymbol{u}}$ is an optimal solution of the problem* (26) *if and only if $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}_{\mathcal{J}})$ satisfying* (28) *with*

$$\boldsymbol{c}_{\mathcal{J}} = \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}}\overline{\boldsymbol{u}} \tag{32}$$

*is an optimal solution of the problem* (31)*.*

*Proof.* See section 8.2.2 □

Proposition 3.2 implies that the stability determination problem (26) is reformulated into (31), which can be embedded into the form of (5).

### 3.2 Feasibility of problem (31)

Introducing a parameter $\widetilde{\lambda}$ so that $\boldsymbol{K}^{+} + \widetilde{\lambda}\boldsymbol{I}$ is positive semidefinite in accordance with (7), define $\widetilde{\boldsymbol{K}}$ by

$$\widetilde{\boldsymbol{K}} = \boldsymbol{K}^{+} + \widetilde{\lambda}\boldsymbol{I}. \tag{33}$$

From the problem (31) and (33), the perturbed problem (10) defining $\widetilde{v}$ is explicitly obtained as

$$\widetilde{v} = \min_{\boldsymbol{u},\boldsymbol{z}_{\mathcal{J}}} \left\{ \boldsymbol{u}^{\mathrm{T}}\widetilde{\boldsymbol{K}}\boldsymbol{u} + \boldsymbol{z}_{\mathcal{J}}^{\mathrm{T}}\boldsymbol{D}_{\mathcal{J}}\boldsymbol{z}_{\mathcal{J}} \mid \boldsymbol{z}_{\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}}\boldsymbol{u} \geq \boldsymbol{0},\ \|\boldsymbol{u}\|^2 - 1 \geq 0 \right\}. \tag{34}$$

Proposition 2.6 verifies to solve (34), instead of (31), in order to determine the stability. The explicit form of (11), which defines $g^*$, is given as

$$g^*(\lambda) = \max_{\boldsymbol{u},\boldsymbol{z}_{\mathcal{J}}} \left\{ \|\boldsymbol{u}\|^2 - 1 \mid \boldsymbol{z}_{\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}}\boldsymbol{u} \geq \boldsymbol{0},\ \boldsymbol{u}^{\mathrm{T}}\widetilde{\boldsymbol{K}}\boldsymbol{u} + \boldsymbol{z}_{\mathcal{J}}^{\mathrm{T}}\boldsymbol{D}_{\mathcal{J}}\boldsymbol{z}_{\mathcal{J}} \leq \lambda \right\}. \tag{35}$$

In association with Theorem 2.9, we solve the problem (35) with $\lambda := \widetilde{\lambda}$ by using Algorithm 2.12 in order to determine stability of the given equilibrium point $\boldsymbol{\xi}^0$. The incremental displacement associated with the minimum increment of the total potential energy can be obtained by solving (34) by using Algorithm 2.15. In Algorithm 2.15, $\lambda$ in (35) plays a role of the parameter for a bisection method.

### 3.3 SOCP formulation of subproblem

We conclude this section by investigating an explicit formulation of the subproblem (22) solved at Step 1 of Algorithm 2.12. According to (12) and (13), define $h_1$ and $h_2(\cdot;\lambda)$ by

$$h_1(\boldsymbol{u},\boldsymbol{z}_{\mathcal{J}}) = \frac{\rho}{2}\left(\|\boldsymbol{u}\|^2 + \|\boldsymbol{z}_{\mathcal{J}}\|^2\right) + \|\boldsymbol{u}\|^2 - 1,$$

$$h_2(\boldsymbol{u},\boldsymbol{z}_{\mathcal{J}};\lambda) = \begin{cases} \frac{\rho}{2}\left(\|\boldsymbol{u}\|^2 + \|\boldsymbol{z}_{\mathcal{J}}\|^2\right), & \text{if } \boldsymbol{z}_{\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}}\boldsymbol{u} \geq \boldsymbol{0},\ \boldsymbol{u}^{\mathrm{T}}\widetilde{\boldsymbol{K}}\boldsymbol{u} + \boldsymbol{z}_{\mathcal{J}}^{\mathrm{T}}\boldsymbol{D}_{\mathcal{J}}\boldsymbol{z}_{\mathcal{J}} \leq \lambda, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then the problem (35) is embedded into (14) with $m := |\mathcal{J}|$, on which Algorithm 2.12 works. Consequently, the subproblem (22) of Algorithm 2.12 is explicitly obtained as

$$\max_{\boldsymbol{u},\boldsymbol{z}_{\mathcal{J}}} \left\{ -\left\|\boldsymbol{u} - (1 + (2/\rho))\boldsymbol{u}^k\right\|^2 - \left\|\boldsymbol{z}_{\mathcal{J}} - \boldsymbol{z}_{\mathcal{J}}^k\right\|^2 \,\middle|\, \boldsymbol{z}_{\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}}\boldsymbol{u} \geq \boldsymbol{0},\ \boldsymbol{u}^{\mathrm{T}}\widetilde{\boldsymbol{K}}\boldsymbol{u} + \boldsymbol{z}_{\mathcal{J}}^{\mathrm{T}}\boldsymbol{D}_{\mathcal{J}}\boldsymbol{z}_{\mathcal{J}} \leq \lambda \right\}. \tag{36}$$

The following proposition prepares the reformulation of the problem (36) into the standard form of SOCP problem.

**Proposition 3.3.** *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a constant matrix. Then, $\boldsymbol{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$ satisfy $y \geq \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})\boldsymbol{x}$ if and only if*

$$y + (1/4) \geq \left\| \begin{pmatrix} y - (1/4) \\ \boldsymbol{A}\boldsymbol{x} \end{pmatrix} \right\|$$

*is satisfied.*

Since $\widetilde{\boldsymbol{K}}$ is not singular from its definition (33), there exists a matrix $\boldsymbol{R}_u \in \mathbb{R}^{n^{\mathrm{d}} \times n^{\mathrm{d}}}$ satisfying

$$\boldsymbol{R}_u^{\mathrm{T}} \boldsymbol{R}_u = \widetilde{\boldsymbol{K}}. \tag{37}$$

For instance, we may choose the Cholesky factor of $\widetilde{\boldsymbol{K}}$ as $\boldsymbol{R}_u$. Similarly, we denote by $\boldsymbol{R}_z \in \mathbb{R}^{|\mathcal{J}| \times |\mathcal{J}|}$ a constant matrix satisfying

$$\boldsymbol{R}_z^{\mathrm{T}} \boldsymbol{R}_z = \boldsymbol{D}_{\mathcal{J}}.$$

By introducing an auxiliary variable $t \in \mathbb{R}$ and utilizing Proposition 3.3, the problem (36) is embedded into the dual standard form of SOCP as

$$\left.\begin{array}{ll}
\displaystyle\min_{\boldsymbol{u}, \boldsymbol{z}_{\mathcal{J}}, t} & t \\
\mathrm{s.t.} & \boldsymbol{z}_{\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}} \boldsymbol{u} \geq \boldsymbol{0}, \\
& \lambda + \dfrac{1}{4} \geq \left\| \begin{pmatrix} \lambda - (1/4) \\ \boldsymbol{R}_u \boldsymbol{u} \\ \boldsymbol{R}_z \boldsymbol{z}_{\mathcal{J}} \end{pmatrix} \right\|, \\
& t + \dfrac{1}{4} \geq \left\| \begin{pmatrix} t - (1/4) \\ \boldsymbol{u} - (1 + (2/\rho)) \boldsymbol{u}^k \\ \boldsymbol{z}_{\mathcal{J}} - \boldsymbol{z}_{\mathcal{J}}^k \end{pmatrix} \right\|.
\end{array}\right\} \tag{38}$$

We solve (38) by using the primal-dual interior-point method at Step 1 of Algorithm 2.12.

# 4 Frictionless unilateral contact

Letting $\dim \in \{2, 3\}$, consider a finite-element discretization of an elastic structure in the $\mathbb{R}^{\dim}$ space, which possibly contact with fixed rigid obstacles without friction. The configuration of the structure is described by $\boldsymbol{\xi} \in \mathbb{R}^{n^{\mathrm{d}}}$, which is the position vector of the nodes with respect to the global coordinate system. Some nodes are supposed to be subjected to the unilateral contact constraints, where we denote by $\mathcal{P}_{\mathrm{C}}$ the set of indices of contact candidate nodes. In this section we follow the standard notation and assumptions of contact mechanics [19, 26].

## 4.1 Stability of frictionless contact

Let $\boldsymbol{x}^p \in \mathbb{R}^{\dim}$ denote the position vector of the $p$th node with respect to an appropriately defined reference frame. For each $p \in \mathcal{P}_{\mathrm{C}}$, the surface of the corresponding obstacle is identified as

$$\left\{ \boldsymbol{x} \in \mathbb{R}^{\dim} \mid \phi^p(\boldsymbol{x}) = 0 \right\},$$

where $\phi^p : \mathbb{R}^{\dim} \to \mathbb{R}$ is assumed to be twice continuously differentiable. Suppose that $\phi$ satisfies $\nabla \phi^p(\boldsymbol{x}) \neq \boldsymbol{0}$ at the points on or sufficiently close to the surface of the obstacles. The admissible region of the position vector is written as

$$\left\{ \boldsymbol{\xi} \in \mathbb{R}^{n^{\mathrm{d}}} \mid \phi^p(\boldsymbol{x}^p(\boldsymbol{\xi})) \leq 0 \ (p \in \mathcal{P}_{\mathrm{C}}) \right\}. \tag{39}$$

On each point of the surface, define the vector $\boldsymbol{n}^p(\boldsymbol{x}) \in \mathbb{R}^{\dim}$ by

$$\boldsymbol{n}^p(\boldsymbol{x}) = \frac{\nabla \phi^p(\boldsymbol{x})}{\|\nabla \phi^p(\boldsymbol{x})\|},$$

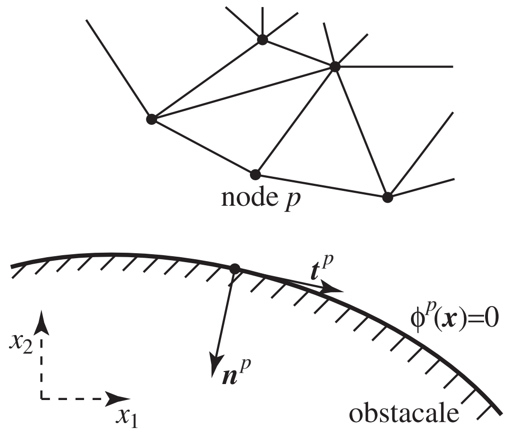



Figure 2: Rigid obstacle and definition of the normal vector $\boldsymbol{n}^p$ in a two-dimensional case.

which is the unit inner normal vector of the surface; see Figure 2. The reaction at the $p$th node, $r_{\rm n}^p \in \mathbb{R}$, is restricted to be in the direction opposite to $\boldsymbol{n}^p$. For each $p \in \mathcal{P}_{\rm C}$, the unilateral contact condition is written as

$$\phi^p(\boldsymbol{x}^p(\boldsymbol{\xi})) \leq 0, \quad r_{\rm n}^p \leq 0, \quad \phi^p(\boldsymbol{x}^p(\boldsymbol{\xi}))r_{\rm n}^p = 0. \tag{40}$$

Define a partition $\mathcal{P}_{\rm f}$, $\mathcal{P}_0$, and $\mathcal{P}_{\rm r}$ of the set $\mathcal{P}_{\rm C}$ as

$$\begin{aligned} \mathcal{P}_{\rm f}(\boldsymbol{\xi}) &= \{p \in \mathcal{P}_{\rm C} \mid \Phi^p(\boldsymbol{\xi}) < 0\}, && \text{[currently not in contact (free)]}, \\ \mathcal{P}_0(\boldsymbol{\xi}) &= \{p \in \mathcal{P}_{\rm C} \mid \Phi^p(\boldsymbol{\xi}) = 0,\ r_{\rm n}^p = 0\}, && \text{[currently in contact without reaction]}, \\ \mathcal{P}_{\rm r}(\boldsymbol{\xi}) &= \{p \in \mathcal{P}_{\rm C} \mid \Phi^p(\boldsymbol{\xi}) = 0,\ r_{\rm n}^p < 0\}, && \text{[currently in contact with reaction]}. \end{aligned}$$

We next investigate the kinematic conditions on the incremental displacements at the given $\boldsymbol{\xi}$. Let $\boldsymbol{u} \in \mathbb{R}^{n^{\rm d}}$ denote the infinitesimal incremental displacement vector defined with respect to the global coordinate system. For each $p \in \mathcal{P}_{\rm C}$ we denote by $u_{\rm n}^p$ the projection of the incremental nodal displacement of the $p$th node onto the direction of $\boldsymbol{n}^p$. Define the vector-valued function $\boldsymbol{g}_{\rm n}^p : \mathbb{R}^{n^{\rm d}} \to \mathbb{R}^{n^{\rm d}}$ by

$$\boldsymbol{g}_{\rm n}^p(\boldsymbol{\xi}) = \left[\frac{\partial \boldsymbol{x}^p}{\partial \boldsymbol{\xi}}(\boldsymbol{\xi})\right]^{\rm T} \boldsymbol{n}^p(\boldsymbol{x}^p(\boldsymbol{\xi})),$$

where

$$\frac{\partial \boldsymbol{x}^p}{\partial \boldsymbol{\xi}}(\boldsymbol{\xi}) = \left(\frac{\partial \boldsymbol{x}^p}{\partial \xi_j}(\boldsymbol{\xi}) \mid j = 1, \dots, n^{\rm d}\right).$$

Then the relation between $u_{\rm n}^p$ and $\boldsymbol{u}$ is written as

$$u_{\rm n}^p = \boldsymbol{g}_{\rm n}^p(\boldsymbol{\xi})^{\rm T}\boldsymbol{u}. \tag{41}$$

See, e.g. [19], for more details.

Suppose that the equilibrium point is given as $\boldsymbol{\xi} = \boldsymbol{\xi}^0$. Define the matrices $\boldsymbol{T}_0$ and $\boldsymbol{T}_{\rm r}$ by

$$\begin{aligned} \boldsymbol{T}_0^{\rm T} &= \left(\boldsymbol{g}_{\rm n}^p(\boldsymbol{\xi}^0) \mid p \in \mathcal{P}_0(\boldsymbol{\xi}^0)\right), \\ \boldsymbol{T}_{\rm r}^{\rm T} &= \left(\boldsymbol{g}_{\rm n}^p(\boldsymbol{\xi}^0) \mid p \in \mathcal{P}_{\rm r}(\boldsymbol{\xi}^0)\right). \end{aligned}$$

From (39), (40), and (41) it follows that the admissible set of the infinitesimal incremental displacements vector $\boldsymbol{u}$ is written as

$$\mathcal{A}(\boldsymbol{\xi}^0) = \left\{ \boldsymbol{u} \in \mathbb{R}^{n^{\mathrm{d}}} \mid \boldsymbol{T}_0 \boldsymbol{u} \leq \boldsymbol{0}, \ \boldsymbol{T}_{\mathrm{r}} \boldsymbol{u} = \boldsymbol{0} \right\}. \tag{42}$$

At the given equilibrium configuration $\boldsymbol{\xi}^0$, let $\boldsymbol{K} \in \mathcal{S}^{n^{\mathrm{d}}}$ denote the tangential stiffness matrix obtained by neglecting the contact conditions. Note that $\boldsymbol{K}$ depends on the curvature of the obstacle surface [16]. Because of the geometrical nonlinearity, $\boldsymbol{K}$ is indefinite in general. It follows from (42) that the stability determination problem (2) for frictionless contact problems is formulated as

$$v^* = \min_{\boldsymbol{u}} \left\{ \boldsymbol{u}^{\mathrm{T}} \boldsymbol{K} \boldsymbol{u} \mid \boldsymbol{u} \in \mathcal{A}(\boldsymbol{\xi}^0), \ \|\boldsymbol{u}\|^2 = 1 \right\}. \tag{43}$$

Explicit SOCP formulations that are to be solved in our algorithm can be obtained in a manner similar to section 3.3. Details of the reduction of those formulations appear in Appendix A.

## 5 Elastic-plastic trusses

Consider an elastic-plastic truss in the two- or three-dimensional space. Let $n^{\mathrm{m}}$ and $n^{\mathrm{d}}$, respectively, denote the number of members and the number of degrees of freedom of displacements. We denote by $q_j$ the axial force of the $j$th member. At the given equilibrium point $\boldsymbol{\xi}^0$, the yield function of the $j$th member is dented by $\phi_j(\cdot; \boldsymbol{\xi}^0) : \mathbb{R} \to \mathbb{R}$, where we assume an associated yielding law for simplicity. Note that $\boldsymbol{\xi}^0$ consist of the current nodal coordinates and yield stresses that are path-dependent. At $\boldsymbol{\xi}^0$, each member satisfies

$$\phi_j(q_j; \boldsymbol{\xi}^0) \leq 0.$$

The compatibility relation between the incremental elongation $c_j$ ($j = 1, \dots, n^{\mathrm{m}}$) and the incremental displacements $\boldsymbol{u}$ is written by (23). Define $\nu_j$ as

$$\nu_j = \frac{\mathrm{d}\phi_j(q_j; \boldsymbol{\xi}^0)}{\mathrm{d}q_j}.$$

At the yielding state $\phi_j(q_j; \boldsymbol{\xi}^0) = 0$ loading and unloading are characterized by $\nu_j c_j > 0$ and $\nu_j c_j < 0$, respectively, for the given $c_j$. Note that $\phi_j(\cdot; \boldsymbol{\xi}^0)$ is not necessarily continuously differentiable function. We assume only that $\phi_j(\cdot; \boldsymbol{\xi}^0)$ is continuously differentiable at $q_j$ satisfying $\phi_j(q_j; \boldsymbol{\xi}^0) = 0$ (as is usual with an yielding law).

The large deformation is considered in general. The tangential stiffness is defined as the sum of the linear and the geometrical stiffness matrices.

### 5.1 Directional stability of elastic-plastic trusses

Let $q_j^0$ denote the axial force at the given equilibrium point. Define the partition $\mathcal{J}$ and $\overline{\mathcal{J}}$ of the set of member indices, $\{1, \dots, n^{\mathrm{m}}\}$, by

$$\mathcal{J} = \left\{ j \in \{1, \dots, n^{\mathrm{m}}\} \mid \phi_j(q_j^0; \boldsymbol{\xi}^0) = 0 \right\},$$
$$\overline{\mathcal{J}} = \left\{ j \in \{1, \dots, n^{\mathrm{m}}\} \mid \phi_j(q_j^0; \boldsymbol{\xi}^0) < 0 \right\},$$

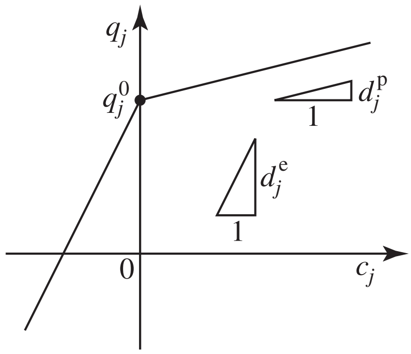



Figure 3: Constitutive relation between the axial force $q_j$ and the incremental elongation $c_j$ in the case of $\nu_j > 0$.

i.e. $j \in \mathcal{J}$ implies that the $j$-th member is at yielding. Let $k_j(c_j)$ denote the tangential elongation stiffness at $\boldsymbol{\xi}^0$, which is written in the form of

$$k_j(c_j; q_j^0) = \begin{cases} d_j^{\mathrm{p}}, & \text{if } \nu_j c_j \geq 0, \\ d_j^{\mathrm{e}}, & \text{if } \nu_j c_j < 0, \end{cases} \tag{44}$$

where $d_j^{\mathrm{e}} \in \mathbb{R}_{++}$ and $d_j^{\mathrm{p}} \in \mathbb{R}_{++}$ are the constants. Figure 3 depicts the relation between $c_j$ and $q_j$ for tensile state (44). The case of elastic-perfectly plastic behavior, $d_j^{\mathrm{p}} = 0$, is discussed in Remark B.1. Thus, the elongation stiffness of the $j$th member ($j \in \mathcal{J}$) depends on the sign of the incremental elongation, which essentially gives a stability determination problem combinatorial complexity.

Consider the truss obtained by neglecting the linear stiffnesses of members belonging to $\mathcal{J}$. The tangential stiffness matrix of the obtained structure is denoted by $\boldsymbol{K}^{\mathrm{e}}(\boldsymbol{\xi}^0) \in \mathcal{S}^{n^{\mathrm{d}}}$, which is a constant matrix at the given $\boldsymbol{\xi}^0$. Note that $\boldsymbol{K}^{\mathrm{e}}$ incorporates the contributions from the linear stiffnesses of members $\overline{\mathcal{J}}$ as well as the geometrical stiffnesses of members $\mathcal{J} \cup \overline{\mathcal{J}}$. Certainly, we are interested in the case in which the smallest eigenvalue of $\boldsymbol{K}^{\mathrm{e}}$ is negative due to the geometrical nonlinearity.

From (44) and the definition of $\boldsymbol{K}^{\mathrm{e}}$, we see that $v$ defined by (1) can be written as

$$v(\boldsymbol{u}) = \boldsymbol{u}^{\mathrm{T}} \boldsymbol{K}^{\mathrm{e}}(\boldsymbol{\xi}^0) \boldsymbol{u} + \sum_{j \in \mathcal{J}} k_j(c_j; q_j^0) c_j(\boldsymbol{u})^2. \tag{45}$$

Then the stability determination problem, (2), is formulated as

$$v^* = \min_{\boldsymbol{u}} \left\{ v(\boldsymbol{u}) \mid \|\boldsymbol{u}\|^2 = 1 \right\}, \tag{46}$$

where the subsidiary conditions (23) and (44) should be satisfied.

For $c_j \in \mathbb{R}$, consider the optimization problem in the variables $z_{\mathrm{e}j} \in \mathbb{R}$ and $z_{\mathrm{p}j} \in \mathbb{R}$

$$w_j(c_j; q_j^0) := \min_{z_{\mathrm{e}j}, z_{\mathrm{p}j}} \left\{ d_j^{\mathrm{e}} z_{\mathrm{e}j}^2 + d_j^{\mathrm{p}} z_{\mathrm{p}j}^2 \mid \nu_j z_{\mathrm{e}j} \leq \nu_j c_j, \ \nu_j z_{\mathrm{p}j} \geq \nu_j c_j \right\}. \tag{47}$$

The following proposition prepares a reformulation of the stability determination problem (46) into the form of (5).

**Proposition 5.1.** *Let* $(\overline{z}_{\mathrm{e}j}, \overline{z}_{\mathrm{p}j})$ *denote the optimal solution of* (47)*. Then,* $(\overline{z}_{\mathrm{e}j}, \overline{z}_{\mathrm{p}j})$ *is given by*

$$(\overline{z}_{\mathrm{e}j}, \overline{z}_{\mathrm{p}j}) = \begin{cases} (0, c_j), & \text{if } \nu_j c_j \geq 0, \\ (c_j, 0), & \text{if } \nu_j c_j < 0. \end{cases} \tag{48}$$

*Furthermore, the optimal value of the problem* (47) *satisfies*

$$w_j(c_j; q_j^0) = k_j(c_j; q_j^0) c_j^2 \tag{49}$$

*for any* $q_j^0 \neq 0$*.*

The proof appears in section 8.3.

In a manner similar to (30) in section 3.1, define the vectors $\boldsymbol{c}_{\mathcal{J}} \in \mathbb{R}^{|\mathcal{J}|}$, $\boldsymbol{z}_{\mathrm{e}\mathcal{J}} \in \mathbb{R}^{|\mathcal{J}|}$, and $\boldsymbol{z}_{\mathrm{p}\mathcal{J}} \in \mathbb{R}^{|\mathcal{J}|}$. Let

$$\boldsymbol{D}_{\mathcal{J}}^{\mathrm{e}} = \mathrm{Diag}(\boldsymbol{d}_{\mathcal{J}}^{\mathrm{e}}), \quad \boldsymbol{D}_{\mathcal{J}}^{\mathrm{p}} = \mathrm{Diag}(\boldsymbol{d}_{\mathcal{J}}^{\mathrm{p}}), \quad \boldsymbol{N}_{\mathcal{J}} = \mathrm{Diag}(\boldsymbol{\nu}_{\mathcal{J}})$$

for $\boldsymbol{d}_{\mathcal{J}}^{\mathrm{e}} \in \mathbb{R}^{|\mathcal{J}|}$, $\boldsymbol{d}_{\mathcal{J}}^{\mathrm{p}} \in \mathbb{R}^{|\mathcal{J}|}$, and $\boldsymbol{\nu}_{\mathcal{J}} \in \mathbb{R}^{|\mathcal{J}|}$, respectively. Consider the following problem in the variables $(\boldsymbol{u}, \boldsymbol{z}_{\mathrm{e}\mathcal{J}}, \boldsymbol{z}_{\mathrm{p}\mathcal{J}}) \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^{|\mathcal{J}|} \times \mathbb{R}^{|\mathcal{J}|}$:

$$\left. \begin{array}{ll} \displaystyle\min_{\boldsymbol{u}, \boldsymbol{z}_{\mathrm{e}\mathcal{J}}, \boldsymbol{z}_{\mathrm{p}\mathcal{J}}} & \boldsymbol{u}^{\mathrm{T}} \boldsymbol{K}^{\mathrm{e}} \boldsymbol{u} + \boldsymbol{z}_{\mathrm{e}\mathcal{J}}^{\mathrm{T}} \boldsymbol{D}_{\mathcal{J}}^{\mathrm{e}} \boldsymbol{z}_{\mathrm{e}\mathcal{J}} + \boldsymbol{z}_{\mathrm{p}\mathcal{J}}^{\mathrm{T}} \boldsymbol{D}_{\mathcal{J}}^{\mathrm{p}} \boldsymbol{z}_{\mathrm{p}\mathcal{J}} \\ \mathrm{s.t.} & \boldsymbol{N}_{\mathcal{J}} \boldsymbol{z}_{\mathrm{e}\mathcal{J}} \leq \boldsymbol{N}_{\mathcal{J}} \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}} \boldsymbol{u}, \\ & \boldsymbol{N}_{\mathcal{J}} \boldsymbol{z}_{\mathrm{p}\mathcal{J}} \geq \boldsymbol{N}_{\mathcal{J}} \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}} \boldsymbol{u}, \\ & \|\boldsymbol{u}\|^2 = 1. \end{array} \right\} \tag{50}$$

The following relation between the problems (46) and (50) can be shown in a manner similar to Proposition 3.2, and hence the proof is omitted:

**Proposition 5.2.** *The optimal value of* (50) *coincides with* $v^*$ *defined by* (46)*. Furthermore,* $\overline{\boldsymbol{u}}$ *is an optimal solution of* (46) *if and only if* $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}_{\mathrm{e}\mathcal{J}}, \overline{\boldsymbol{z}}_{\mathrm{p}\mathcal{J}})$ *satisfying* (48) *with* (32) *is an optimal solution of* (50)*.*

Proposition 3.2 implies that the stability determination problem (46) is reformulated into (50), which can be embedded into the form of (5).

A feasibility problem for the problem (50) as well as SOCP formulations that are to be solved can be obtained in a manner analogous to sections 3.2 and 3.3. All details appear in appendix B.

# 6 Numerical experiments

The stability determination problems (10) and (11) are solved for various structures by using Algorithms 2.15 and 2.12, respectively. We reformulate the subproblem (22) as an SOCP problem, and solve it by using SeDuMi Ver. 1.1 [27, 30], which implements the primal-dual interior-point method for the linear programming problem over symmetric cones. Computation has been carried out on Pentium M (1.2 GHz with 1.0 GB memory) with MATLAB Ver. 7.0.1 [38].

In sections 6.1, 6.2, and 6.3, respectively, we solve numerical examples of the problems investigated in sections 3, 4, and 5. In each example, the elastic modulus of structures is 200 GPa;

an initial solution $(\boldsymbol{u}^0, \boldsymbol{z}^0_{\mathcal{J}})$ for Algorithm 2.12 is generated randomly by using MATLAB built-in-function '*rand*' so that $\|(\boldsymbol{u}^0, \boldsymbol{z}^0_{\mathcal{J}})\|_\infty \le 0.5$ is satisfied. At Step 2 of Algorithm 2.15, the optimal solution obtained in the previous iteration is used as an initial solution for Algorithm 2.12 as discussed in Remark 2.16. The termination tolerance is chosen as $\epsilon = 10^{-3}$ at Step 0 of Algorithm 2.12. At Step 0 of Algorithm 2.15, we choose $\epsilon = 10^{-4}\widetilde{\lambda}$, $\overline{\lambda}^0 = 2\widetilde{\lambda}$, and $\underline{\lambda}^0 = 0$. The parameter $\rho$ introduced in (12) and (13) is chosen as $\rho = 0.1$ for all examples.

### 6.1 Cable-strut system

Consider a plane cable-strut system illustrated in Figure 4, where $W = 1.0$ m, $H = 1.0$ m, $n^{\rm d} = 70$, and $n^{\rm m} = 82$. The nodes (a1)–(a5), (c1)–(c7), and (d1)–(d7) are pin-supported. The displacements of the nodes (b1)–(b5) are constrained in the $y$-direction. The stability determination problems for the cable-strut structures have been investigated in section 3.

The members in the $x$-direction are struts modeled as truss members, while the members in the $y$-direction are cables that do not transmit compressive forces. The cross-sectional areas of struts and cables are $5 \times 10^{-3}$ m$^2$ and $0.32 \times 10^{-3}$ m$^2$, respectively. As for the external force, 1.5 MN is applied in the negative direction of the $x$-axis at nodes (b1)–(b5). Note that Figure 4 illustrates the deformed equilibrium configuration corresponding to the applied load, i.e. the initial unstressed length of each cable member is equal to $H$. Therefore, the elongation of each cable member vanishes at the given equilibrium point, and hence $|\mathcal{J}| = 42$. Accordingly, the number of possible combinations in the formulation of the tangential stiffness matrix is $2^{42}$.

At the equilibrium point, the smallest eigenvalue of the tangential stiffness matrix $\boldsymbol{K}^+$ obtained by neglecting all the cable members is $\lambda_1 = -5.772$. Hence, we choose $\widetilde{\lambda} = 6.060$ (i.e. $\widetilde{\lambda} = 1.05|\lambda_1|$) in (33). For the sake of the stability determination, the problem (35) is solved with $\lambda := \widetilde{\lambda}$ by using Algorithm 2.12 to obtain $g^*(\widetilde{\lambda}) = 1.831 \times 10^{-3}$. Hence, from Corollary 2.14 we can conclude that the equilibrium point is unstable. The CPU time required by Algorithm 2.12 is 8.61 seconds, and 48 SOCP problems are solved.

We next solve (34) in order to obtain the incremental displacements corresponding to the minimal incremental total potential energy. By using Algorithm 2.15, we obtain $\widetilde{v} = 6.049$ and $v^* = -1.147 \times 10^{-2}$ defined in (26). This result verifies that the equilibrium point is unstable in association with Definition 2.3. Figure 5 illustrates the optimal solution obtained, where the slackening cable members have been removed. Algorithm 2.15 requires 15 iterations and 12.8 seconds of the CPU time. In total, 71 SOCP problems are solved, and the average CPU time for solving one SOCP problem is 0.18 seconds. Note that 48 SOCP problems are solved in the first iteration of Algorithm 2.15, while 1.64 SOCP problems in average are required for the remaining iterative steps. This is because the optimal solution of (11) obtained in the previous iteration is used as an initial solution of the next iteration; see Remark 2.16. Notice again that it is sufficient to solve (35) with $\lambda := \widetilde{\lambda}$ in order to determine the stability of the given equilibrium point.

As an alternative case, we choose a slightly larger cross-sectional area $0.33 \times 10^{-3}$ m$^2$ for each cable member. The optimal value of (35) with $\lambda := \widetilde{\lambda}$ is computed by using Algorithm 2.12 as $g^*(\widetilde{\lambda}) = -2.459 \times 10^{-2}$. Provided that the obtained solution is globally optimal, Theorem 2.9 implies that the equilibrium point of this case is stable. The problem (34) is solved by using Algorithm 2.15 to obtain $\widetilde{v} = 6.213$ and $v^* = 1.528 \times 10^{-3}$. Thus, the cable-strut system is stabilized by slightly

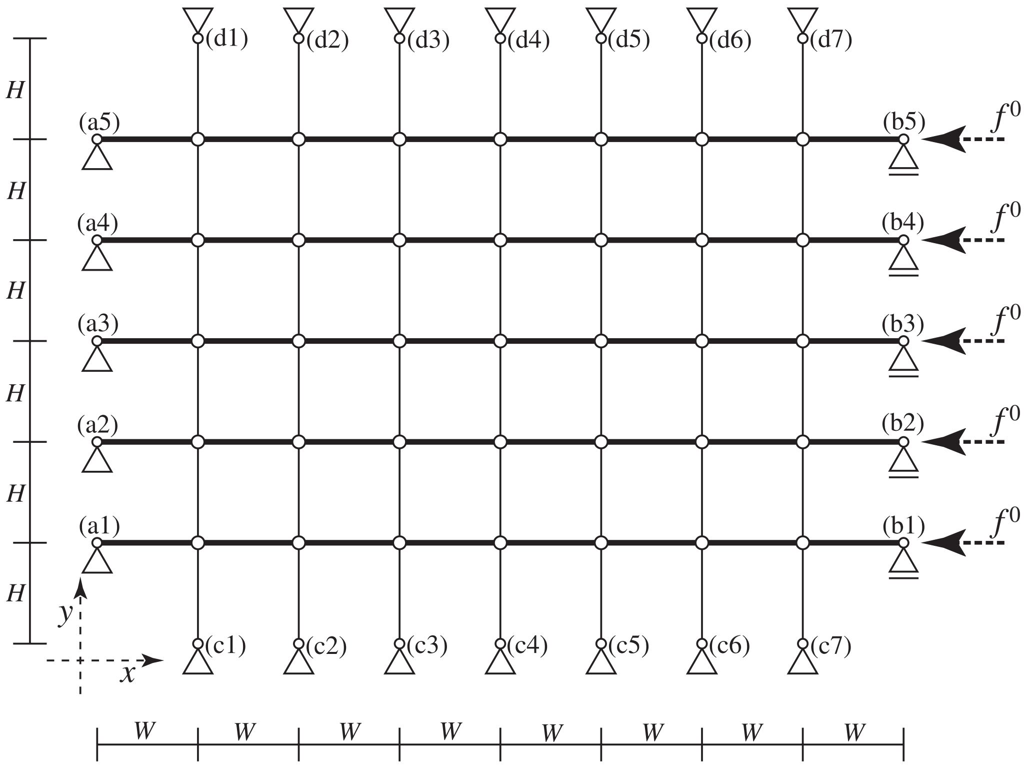



Figure 4: A cable-strut system.

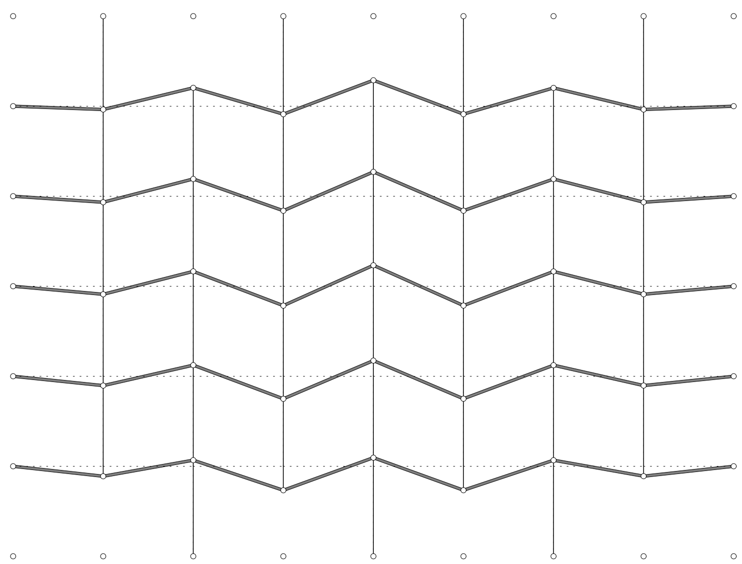


Figure 5: Optimum solution of (34) for the cable-strut system.

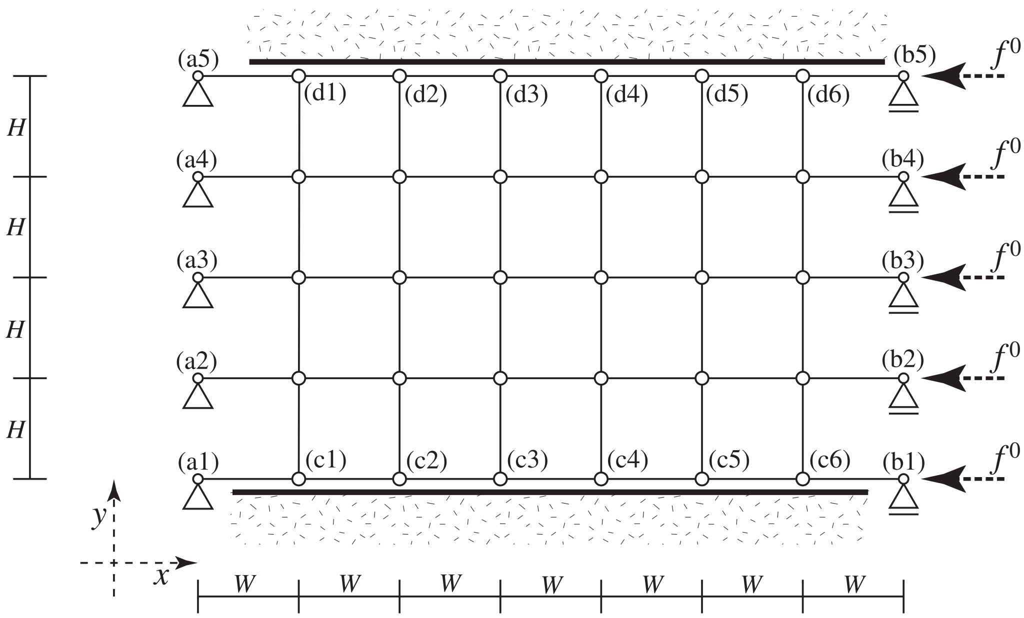



Figure 6: An elastic truss with the two rigid obstacles.

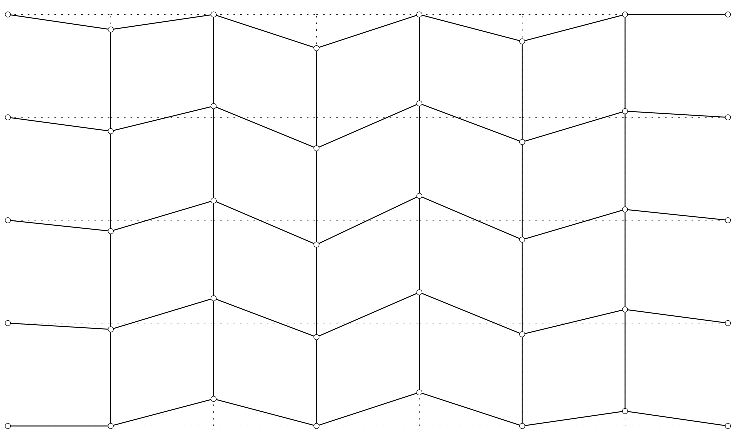


Figure 7: Optimum solution of (72) for the truss with the obstacles.

increasing the cross-sectional areas of cables. Moreover, from these two results, we may conjecture that the global optimal solutions of (34) are successfully found.

### 6.2 Frictionless contact with rigid obstacles

Consider a plane truss with the rigid obstacles illustrated in Figure 6, where $W = 1.0$ m, $H = 1.0$ m, $n^{\mathrm{d}} = 60$, and $n^{\mathrm{m}} = 59$. The nodes (a1)–(a5) are pin-supported. The displacements of the nodes (b1)–(b5) are constrained in the $y$-direction. The stability determination problems for the frictionless contact have been investigated in section 4.

The cross-sectional areas of members in the $x$- and $y$-directions, respectively, are $8.0 \times 10^{-3}$ m$^2$ and $1.0 \times 10^{-3}$ m$^2$. The external force 8.0 MN is applied in the negative direction of the $x$-axis at nodes (b1)–(b5). Note that Figure 6 illustrates the deformed equilibrium configuration corresponding to the applied forces. The members in the $y$-direction have vanishing axial forces at the given equilibrium point.

The nodes (c1)–(c6) and (d1)–(d6) are supposed to be contact candidates, which contact with the obstacles without reactions at the given equilibrium point, i.e. $\mathcal{P}_0$ consists of nodes (c1)–(c6) and (d1)–(d6), $|\mathcal{P}_0| = 12$, and $\mathcal{P}_\mathrm{r} = \emptyset$. Thus, there exist $2^{12}$ possible combinations of contact conditions.

At the given equilibrium point, the smallest eigenvalue of $\boldsymbol{K}$ in (43) is $\lambda_1 = -30.42$. Hence, we choose $\widetilde{\lambda} = 31.94$ (i.e. $\widetilde{\lambda} = 1.05|\lambda_1|$) in (71). For the stability determination, the problem (73) with $\lambda := \widetilde{\lambda}$ is solved by using Algorithm 2.12 to obtain $g^*(\widetilde{\lambda}) = -3.844 \times 10^{-2}$. Accordingly, from Theorem 2.9 we see that the given equilibrium point is stable, provided that the solution obtained is globally optimal. The CPU time required by Algorithm 2.12 is 6.25 seconds, and 65 SOCP problems are solved.

We next solve (72) in order to obtain the incremental displacements with the minimal incremental total potential energy. By using Algorithm 2.15, we obtain $\widetilde{v} = 33.21$ and $v^* = 1.277$ defined in (43). This result and Definition 2.3 verify that the given equilibrium point is stable. The optimal solution is shown in Figure 7. Algorithm 2.15 requires 15 iterations and 9.59 seconds of the CPU time.

In order to verify the convergence to the global optimal solution, we solve a slightly modified case, in which the cross-sectional areas of members in the $y$-direction are $0.93 \times 10^{-3}$ m$^2$. In this case, we obtain $g^*(\widetilde{\lambda}) = 1.402 \times 10^{-2}$, $\widetilde{v} = 31.49$, and $v^* = -4.425 \times 10^{-1}$. From Corollary 2.14 we can conclude that this equilibrium point is unstable. The obtained values of $v^*$ (and also the values of $g^*(\widetilde{\lambda})$) of these two cases are very close, and hence it seems that the global optimal solutions are successfully found.

### 6.3 Elastic-plastic truss

Consider an elastic-plastic plane truss illustrated in Figure 8, where $W = 1.0$ m, $H = 1.5$ m, $n^\mathrm{d} = 38$, and $n^\mathrm{m} = 56$. The nodes (a1) and (a2) are pin-supported. At the equilibrium point some members are supposed to be in plastic range. The stability determination problems for the elastic-plastic trusses have been investigated in section 5. The ratio between the elastic and the plastic moduli is given as $d_j^\mathrm{p}/d_j^\mathrm{e} = 0.2$ in (44).

The cross-sectional areas of members in the $y$-direction are $2.25 \times 10^{-3}$ m$^2$. The remaining members have $0.15 \times 10^{-3}$ m$^2$. Letting $p \in \mathbb{R}_+$ be a load parameter, the external force $4p$ MN is applied in the negative direction of the $y$-axis at the nodes (b1) and (b3), while $5p$ MN is applied at the node (b2). In the following examples we assume that the deformation is infinitesimal and that the load parameter $p$ is supposed to be monotonically increased to the specified value. In this case, the state variables such as stresses and straits are not path-dependent, and the equilibrium point is found by using the linear stiffness matrix, because it is known that the effect of pre-buckling deformation is negligibly small for this kind of slender structures. The tangential stiffness at the equilibrium point is defined as the sum of the linear stiffness matrix and the geometrical stiffness matrix.

As for the distribution of the members in plastic range, we consider the two cases: the members in the plastic range are depicted by thick lines in Figure 9 (a) and Figure 10 (a), respectively, for the cases (A) and (B). Note that $|\mathcal{J}| = 16$ for both cases.

We first consider the symmetric case (A). The external load is given by putting $p = 0.560$. At

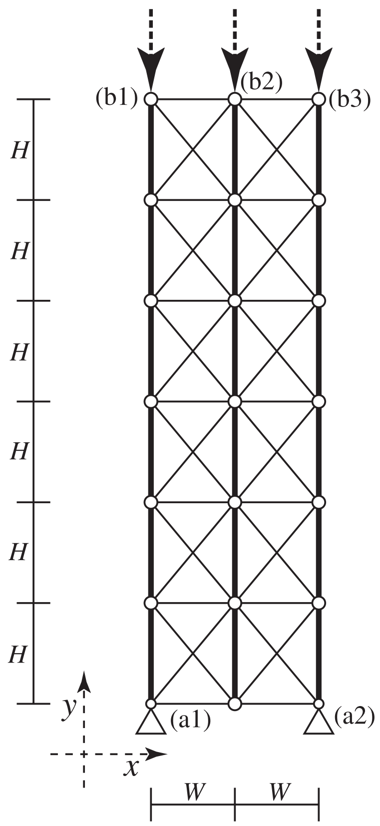
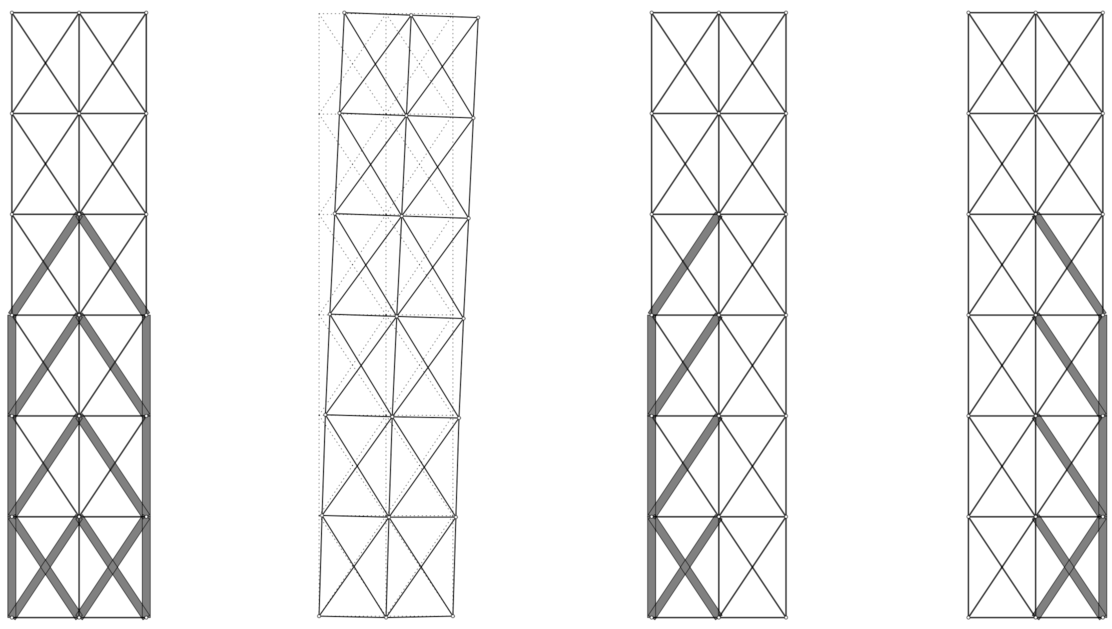
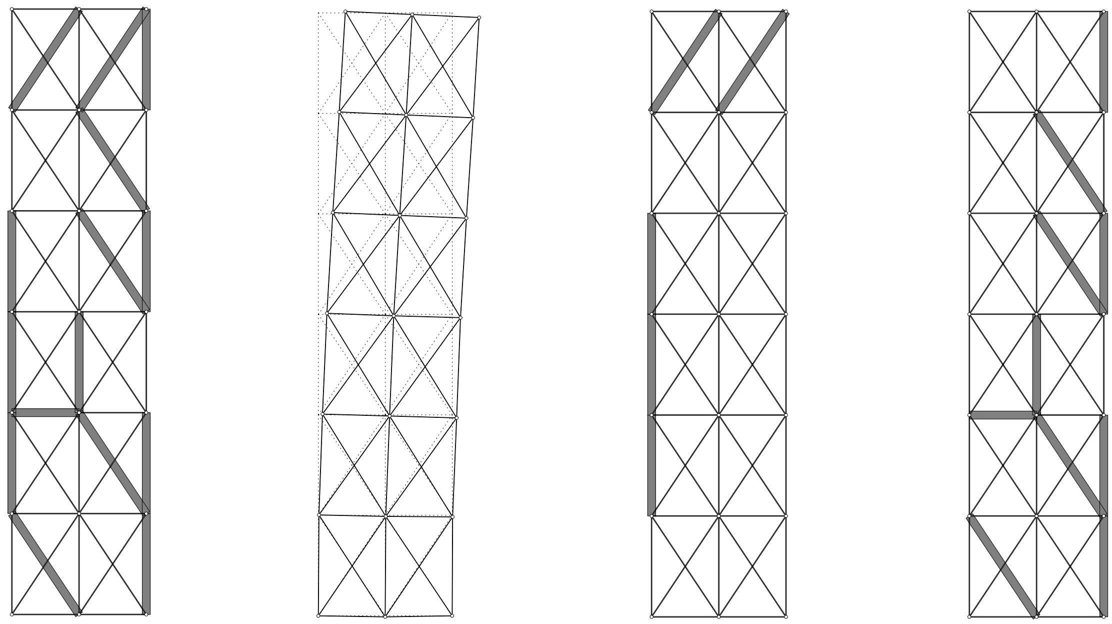



Figure 8: An elastic/plastic tower-type truss.



(a) yielding members (b) optimal solution (c) unloading members (d) loading members

Figure 9: Optimal solution of the tower-type truss: case (A).



(a) yielding members (b) optimal solution (c) unloading members (d) loading members

Figure 10: Optimal solution of the tower-type truss: case (B).

the given equilibrium point, the smallest eigenvalue of $\boldsymbol{K}^{\rm e}$ in (45) is $\lambda_1 = -2.901$. Hence, we choose $\widetilde{\lambda} = 3.046$ (i.e. $\widetilde{\lambda} = 1.05|\lambda_1|$) in (77). For the stability determination, Problem (79) with $\lambda := \widetilde{\lambda}$ is solved by using Algorithm 2.12 to obtain $g^*(\widetilde{\lambda}) = -1.202 \times 10^{-4}$. Hence, from Theorem 2.9 we see that the given equilibrium point is stable, provided that the solution obtained is globally optimal. The CPU time required by Algorithm 2.12 is 8.11 seconds, and 40 SOCP problems are solved. If we assume that all members at yielding are unloading, then the minimum eigenvalue of the tangential stiffness matrix is $1.379 \times 10^{-1}$. Conversely, if we assume that all members at yielding are loading, then the minimum eigenvalue of the tangential stiffness matrix is $-2.081 \times 10^{-1}$. From Hill's sufficient condition for the uniqueness of the equilibrium state, it follows that there exists a bifurcation point before reaching this equilibrium point.

We examine a slightly larger load $p = 0.565$. At the corresponding equilibrium point, the smallest eigenvalue of $\boldsymbol{K}^{\rm e}$ is $\lambda_1 = -2.930$, and we choose $\widetilde{\lambda} = 3.076$. The optimal value of Problem (79) with $\lambda := \widetilde{\lambda}$ is found as $g^*(\widetilde{\lambda}) = 2.251 \times 10^{-4}$. Hence, Corollary 2.14 implies that this equilibrium point is unstable. Next Problem (78) is solved in order to obtain the incremental displacements corresponding to the minimal incremental total potential energy. By using Algorithm 2.15, we obtain $\widetilde{v} = 3.076$ and $v^* = -5.633 \times 10^{-4}$, where $v^*$ has been defined in (46). This result and Definition 2.3 verify that the given equilibrium point is unstable. The optimal incremental displacements obtained is shown in Figure 9 (b). Algorithm 2.15 requires 15 iterations and 8.39 seconds of the CPU time. In total, 43 SOCP problems are solved. At the optimal solution, unloading and loading members are depicted by thick lines in Figure 9 (c) and (d), respectively.

For the case (B) defined by Figure 10 (a), it is difficult to estimate loading or unloading member intuitively. First, we choose $p = 0.620$. At the corresponding equilibrium point, the smallest eigenvalue of $\boldsymbol{K}^{\rm e}$ is $\lambda_1 = -9.511 \times 10^{-1}$. Hence, we choose $\widetilde{\lambda} = 9.987 \times 10^{-1}$ (i.e. $\widetilde{\lambda} = 1.05|\lambda_1|$) in (77). For the stability determination, (79) with $\lambda := \widetilde{\lambda}$ is solved by using Algorithm 2.12 to obtain $g^*(\widetilde{\lambda}) = -8.601 \times 10^{-4}$, from which and Theorem 2.9 we see that the given equilibrium point is stable, provided that the solution obtained is globally optimal. If we assume that all members in plastic range are unloading, then the minimum eigenvalue of the tangential stiffness matrix is $1.255 \times 10^{-1}$. Conversely, assuming that all members in the plastic range are loading, the minimum eigenvalue of the tangential stiffness matrix is $-6.976 \times 10^{-2}$. Hence, from Hill's sufficient condition for uniqueness, the truss undergoes bifurcation before reaching this equilibrium point.

We next consider a slightly larger load $p = 0.625$. At the corresponding equilibrium point, the smallest eigenvalue of $\boldsymbol{K}^{\mathrm{e}}$ is $\lambda_1 = -9.627 \times 10^{-1}$, and we choose $\widetilde{\lambda} = 1.011$ $(= 1.05|\lambda_1|)$. The optimal value of (79) with $\lambda := \widetilde{\lambda}$ is found as $g^*(\widetilde{\lambda}) = 5.453 \times 10^{-4}$, which implies that implies that this equilibrium point is unstable from Corollary 2.14 . The CPU time required by Algorithm 2.12 is 2.094 seconds, and 11 SOCP problems are solved. The problem (78) is solved in order to obtain the incremental displacements corresponding to the minimal incremental total potential energy. By using Algorithm 2.15, we obtain $\widetilde{v} = 1.010$ and $v^* = -5.553 \times 10^{-4}$, where $v^*$ has been defined in (46). This result and Definition 2.3 verify that the given equilibrium point is unstable. The optimal solution obtained is shown in Figure 10 (b).
refalgalg.bisection requires 15 iterations and 6.19 seconds of the CPU time. In total, 33 SOCP problems are solved. The unloading and loading members in the optimal incremental displacement are depicted by thick lines in Figure 10 (c) and (d), respectively. If we assume that all members at yielding are unloading, then the tangent stiffness matrix is positive definite, and the minimum eigenvalue is $1.245 \times 10^{-1}$.

## 7 Conclusions

In this paper, we have proposed a numerical technique for determining the stability of the given equilibrium point of structures subjected to the unilateral constraints. As for situations of unilateral constraints, we have investigated three typical problems in nonsmooth mechanics: cable structures, frictionless contact, and plasticity. For these problems which possess combinatorial complexity essentially, we have provided the unified formulations and methodology for stability analysis which is applicable to large-scale problems.

We have introduced a nonconvex quadratic programming problem whose optimal value determines the stability of the given equilibrium point. In association with the feasibility of this problem, we have proposed another formulation for the stability determination, which is a maximization problem of a convex quadratic function over a convex homogeneous quadratic inequality and some linear inequalities. It is shown that the stability is determined by the sign of the optimal value of this problem. Thus, we have shown that the stability analysis is one of important applications of the convex maximization over the convex set, which has been studied extensively in computational optimization.

In order to solve the proposed nonconvex optimization problem, we embed it to a *DC* (difference of convex functions) *programming* problem, and the obtained problem is solved by using the so-called DC algorithm. The DC algorithm is one of a few algorithms based on a local approach which has been successfully applied to large-scale DC programming problems, and quite often converges to the global optimal solution. The explicit formulations of the subproblem to be solved are presented for cable systems, contact problem, and elastic-plastic trusses. In each case, it is shown that the subproblem is reduced to the second-order cone programming problem, which can be solved efficiently by using the primal-dual interior-point method.

It has been shown in the numerical examples of various structures subjected to the unilateral constraints that the algorithm presented can find if the given equilibrium point is directionally stable or not. For nonconvex programming problems, there is no practicable global optimal conditions in general, which makes it difficult to check the global optimality of solutions obtained by the proposed algorithm. However, throughout parametric studies it has been shown that the solutions obtained seem to be globally optimal for our numerical examples.

Regarding the elastic-plastic problem, we have investigated only truss structures in this paper. It is of interest to note that yielding laws for various structures can be written as convex inequalities typically. This observation implies that the stability determination problem proposed in this paper may inherit some important properties if it is applied to structures other than trusses, i.e. the stability determination problem may be written in the form of the maximization problem of a convex function over a convex set for various classes of structures. This intuitive observation suggests that a sequential convex programming approach developed in this paper can be extended to various structures, which remains as our promising future work.

# 8 Proofs of rechnical results

## 8.1 For section 2

### 8.1.1 Proof of Proposition 2.6

We start with the claim that the problem (10) can be rewritten equivalently as

$$\min_{\boldsymbol{u},\boldsymbol{z}} \left\{ \boldsymbol{u}^{\mathrm{T}} \widetilde{\boldsymbol{Q}}_0 \boldsymbol{u} + \boldsymbol{z}^{\mathrm{T}} \boldsymbol{Q}_1 \boldsymbol{z} \mid (\boldsymbol{u},\boldsymbol{z}) \in \mathcal{F},\ \|\boldsymbol{u}\|^2 = 1 \right\}, \tag{51}$$

i.e. the inequality constraint $\|\boldsymbol{u}\|^2 \geq 1$ can be replaced with the equality constraint $\|\boldsymbol{u}\|^2 = 1$, without changing both the optimal value and the optimal solution (more precisely, the set of optimal solutions does not change, if the optimal solution of (10) is not unique). In order to see the contradiction, assume that $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}})$ satisfying

$$\|\overline{\boldsymbol{u}}\|^2 > 1$$

is an optimal solution of (10). It follows from the definition (6) of $\mathcal{F}$ that

$$(\overline{\boldsymbol{u}}/\|\overline{\boldsymbol{u}}\|^2, \overline{\boldsymbol{z}}/\|\overline{\boldsymbol{u}}\|^2) \in \mathcal{F}$$

is satisfied. Hence, we see that $(\overline{\boldsymbol{u}}/\|\overline{\boldsymbol{u}}\|^2, \overline{\boldsymbol{z}}/\|\overline{\boldsymbol{u}}\|^2)$ is a feasible solution of (10), at which the objective function of (10) satisfies

$$\widetilde{f}(\overline{\boldsymbol{u}}/\|\overline{\boldsymbol{u}}\|^2, \overline{\boldsymbol{z}}/\|\overline{\boldsymbol{u}}\|^2) = \frac{1}{\|\overline{\boldsymbol{u}}\|^2} \widetilde{f}(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}) < \widetilde{f}(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}). \tag{52}$$

Note that $\widetilde{f}(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}) > 0$ because $\widetilde{\boldsymbol{Q}}_0$ and $\boldsymbol{Q}_1$ in (9) are positive definite. The inequality of (52) contradicts with the assumption that $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}})$ is optimum, which implies that the problems (10) and (51) share the same optimal value and the same set of optimal solutions. It follows from the definition (8) of $\widetilde{\boldsymbol{Q}}_0$ that the condition

$$\boldsymbol{u}^{\mathrm{T}}\widetilde{\boldsymbol{Q}}_0\boldsymbol{u} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{Q}_0\boldsymbol{u} + \widetilde{\lambda}$$

holds for any $\boldsymbol{u}$ satisfying

$$\|\boldsymbol{u}\|^2 = 1.$$

Finally, observe that the problems (5) and (10) share the same feasible set and have the objective functions with the difference of constant $\widetilde{\lambda}$, which concludes the proof. □

#### 8.1.2 Proof of Proposition 2.8

In order to show Proposition 2.8, we first prepare the following lemma:

**Lemma 8.1.** *For any* $\lambda \in \mathbb{R}_{++}$,

$$\lim_{\epsilon \to +0} \inf_{\boldsymbol{x}} \left\{ \widetilde{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{F},\ g(\boldsymbol{x}) \geq \epsilon \right\} = \widetilde{v} < +\infty, \tag{53}$$

$$\lim_{\lambda' \to \lambda - 0} g^*(\lambda') = g^*(\lambda) > -\infty. \tag{54}$$

*Proof.* We show this lemma by using the result of Lemma 4.1 in Tuy [34]. In order to prove (53), it suffices to show (i) $\widetilde{v} < +\infty$ and (ii) 0 is not a local maximum of $g$ over $\mathcal{F}$. Recall that we have assumed that the problem (5) has a feasible solution, which guarantees the condition (i) is satisfied. Provided that $\boldsymbol{x}_1$ denotes a feasible solution of (10), we easily see that $\mu\boldsymbol{x}_1$ is also feasible for any $\mu \in \mathbb{R}_{++}$. Hence, $\max\{g(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{F}\}$ is unbounded above and has no local maximum, which implies that the condition (ii) is satisfied. Similarly, the assertion (54) can be proved by showing (iii) $g^*(\lambda) > -\infty$ and (iv) $\lambda$ is not a local minimum of $\widetilde{f}$ over $\mathcal{F}$. To see (iii), observe that $\widetilde{f}(\boldsymbol{0}) = 0$ holds from the definition (9) of $\widetilde{f}$. Furthermore, the definition (6) of $\mathcal{F}$ implies $\boldsymbol{0} \in \mathcal{F}$. Hence, $\boldsymbol{x} = \boldsymbol{0}$ is feasible for the problem (11), and thus the condition (iii) is verified. Recall that $\widetilde{\boldsymbol{Q}}_0$ and $\boldsymbol{Q}_1$ in (9) are positive definite. Hence, the convex quadratic minimization $\min\{\widetilde{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{F}\} = 0$ has no local minimum, which guarantees that the condition (iv) is satisfied. □

**Proof of Proposition 2.8** If $\widetilde{v}$ defined by (10) satisfies $\widetilde{v} \geq \lambda$, then the set

$$\left\{ \boldsymbol{x} \in \mathcal{F} \mid g(\boldsymbol{x}) \geq 0,\ \widetilde{f}(\boldsymbol{x}) \leq \lambda' \right\}$$

is empty for any $\lambda'$ satisfying $\lambda' < \lambda$. Hence, for any $\lambda' < \lambda$, we see that the inequality

$$g^*(\lambda') = \max_{\boldsymbol{x}} \left\{ g(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{F},\ \widetilde{f}(\boldsymbol{x}) \leq \lambda' \right\} \leq 0$$

holds, from which and Lemma 8.1 we obtain $g^*(\lambda) \leq 0$. Similarly, if $g^*$ defined by (11) satisfies $g^*(\lambda) \leq 0$, then the set

$$\left\{ \boldsymbol{x} \in \mathcal{F} \mid \widetilde{f}(\boldsymbol{x}) \leq \lambda,\ g(\boldsymbol{x}) > \epsilon \right\}$$

is empty for any $\epsilon > 0$. Hence, for any $\epsilon > 0$, the inequality

$$\min_{\boldsymbol{x}} \left\{ \widetilde{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{F},\ g(\boldsymbol{x}) \geq \epsilon \right\} \geq \lambda$$

holds, from which and Lemma 8.1 we obtain $\widetilde{v} \geq \lambda$. □

#### 8.1.3 Proof of Theorem 2.9

From Corollary 2.7 we obtain the assertions (i) and (ii), respectively, by showing that (a) $g^*(\lambda) < 0$ implies $\widetilde{v} > \lambda$; (b) $g^*(\lambda) > 0$ implies $\widetilde{v} < \lambda$. Furthermore, the assertion (a) follows (iii) and Proposition 2.8 (ii); the assertion (b) follows (iii) and the fact that (c) $g^*(\lambda) \geq 0$ implies $\widetilde{v} \leq \lambda$; Thus, it suffices to show the assertions (iii) and (c). We start with showing (c). Observe that $g^*(\lambda) \geq 0$ implies that there exists a feasible solution $\boldsymbol{x}'$ of (11) satisfying $g(\boldsymbol{x}') \geq 0$. Then, we easily see that $\boldsymbol{x}'$ is also feasible for the problem (10) satisfying $\widetilde{f}(\boldsymbol{x}') \leq \lambda$, which yields the assertion (c). We next show the assertion (iii). Note that 'if' part of (iii) follows (c) and Proposition 2.8 (ii). Hence, suppose $\widetilde{v} = \lambda$. Then there exists a feasible solution $\boldsymbol{x}''$ of (10) satisfying $\widetilde{f}(\boldsymbol{x}'') = \lambda$. Since $\boldsymbol{x}''$ is also feasible for the problem (11), $g(\boldsymbol{x}'') \geq 0$ implies $g^*(\lambda) \geq 0$. On the other hand, Proposition 2.8 (i) and the condition $\widetilde{v} = \lambda$ imply $g^*(\lambda) \leq 0$. Thus, we obtain $g^*(\lambda) = 0$. □

#### 8.1.4 Proof of Proposition 2.13

Observe that $\widetilde{f}$ defined by (9) is strongly convex, because $\widetilde{\boldsymbol{Q}}_0$ and $\boldsymbol{Q}_1$ are positive semidefinite. Hence, the set

$$\left\{(\boldsymbol{u}, \boldsymbol{z}) \in \mathbb{R}^{n^{\mathrm{d}}} \times \mathbb{R}^{n^{\mathrm{m}}} \mid \widetilde{f}(\boldsymbol{u}, \boldsymbol{z}) \leq \lambda\right\}$$

is nonempty, bounded, and convex. Accordingly, (22) is the minimization of the strongly convex function over the nonempty bounded convex set, which implies that the optimal solution of (22) exists uniquely. □

### 8.2 For section 3

#### 8.2.1 Proof of Proposition 3.1

Since (29) immediately follows (27) and (28), it suffices to show that (28) holds. Observing that Problem (27) is a convex quadratic optimization problem, we see that $\overline{z}_j$ is an optimal solution if and only if it satisfies the KKT conditions

$$2d_j\overline{z}_j - \sigma_j = 0, \tag{55}$$

$$\overline{z}_j \geq c_j, \tag{56}$$

$$\sigma_j \geq 0, \tag{57}$$

$$\sigma_j(\overline{z}_j - c_j) = 0, \tag{58}$$

where $\sigma_j$ is the Lagrange multiplier. By substituting $\sigma_j$ in (55) into (58) and using $d_j > 0$, we obtain $\overline{z}_j(\overline{z}_j - c_j) = 0$, i.e.

$$\overline{z}_j \in \{0, c_j\}. \tag{59}$$

Firstly, suppose $c_j \geq 0$. Then (56) and (59) imply

$$\overline{z}_j = c_j.$$

Alternatively, suppose $c_j < 0$. By using (55) and (57), we see

$$\overline{z}_j \geq 0,$$

from which and (59) we conclude $\overline{z}_j = 0$. □

### 8.2.2 Proof of Proposition 3.2

Clearly, $\overline{\boldsymbol{u}}$ is an optimal solution of (26) if and only if $(\overline{\boldsymbol{u}}, \boldsymbol{c}_{\mathcal{J}})$ satisfying (32) is an optimal solution of the problem

$$\min_{\boldsymbol{u},\boldsymbol{c}_{\mathcal{J}}} \left\{ \boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}^{+}\boldsymbol{u} + \sum_{j\in\mathcal{J}} k_j^{\mathrm{c}}(c_j)c_j^2 \;\middle|\; \boldsymbol{c}_{\mathcal{J}} = \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}}\boldsymbol{u},\ \|\boldsymbol{u}\|^2 = 1 \right\}. \tag{60}$$

From Proposition 3.1 it follows that (60) is equivalently rewritten as

$$\min_{\boldsymbol{u},\boldsymbol{c}_{\mathcal{J}}} \left\{ \boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}^{+}\boldsymbol{u} + \sum_{j\in\mathcal{J}} \min_{z_j} \{ d_j z_j^2 \mid z_j \geq c_j \} \;\middle|\; \boldsymbol{c}_{\mathcal{J}} = \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}}\boldsymbol{u},\ \|\boldsymbol{u}\|^2 = 1 \right\}, \tag{61}$$

without changing the optimal value and the set of optimal solutions, where (28) holds at an optimal solution of (61). Observe that, in (61), only the objective function includes $\boldsymbol{z}_{\mathcal{J}}$, which justifies that (61) is equivalently rewritten as

$$\left.\begin{array}{ll} \displaystyle\min_{\boldsymbol{u},\boldsymbol{c}_{\mathcal{J}},\boldsymbol{z}_{\mathcal{J}}} & \displaystyle\boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}^{+}\boldsymbol{u} + \sum_{j\in\mathcal{J}} d_j z_j^2 \\ \mathrm{s.t.} & z_j \geq c_j, \quad j \in \mathcal{J}, \\ & \boldsymbol{c}_{\mathcal{J}} = \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}}\boldsymbol{u}, \\ & \|\boldsymbol{u}\|^2 = 1, \end{array}\right\} \tag{62}$$

without changing the optimal value and the set of optimal solutions. Elimination of $\boldsymbol{c}_{\mathcal{J}}$ from (62) results in (31), which concludes the proof. □

## 8.3 For section 5

### 8.3.1 Proof of Proposition 5.1

Since (49) immediately follows (47) and (48), it suffices to show that (48) holds. Observing that Problem (47) is a convex quadratic optimization problem, we see that $(\overline{z}_{\mathrm{e}j}, \overline{z}_{\mathrm{p}j})$ is an optimal solution if and only if it satisfies the KKT condition

$$2d_j^{\mathrm{e}}\overline{z}_{\mathrm{e}j} + \nu_j\sigma_{1j} = 0, \tag{63}$$
$$2d_j^{\mathrm{p}}\overline{z}_{\mathrm{p}j} - \nu_j\sigma_{2j} = 0, \tag{64}$$
$$\nu_j\overline{z}_{\mathrm{e}j} \leq \nu_j c_j, \quad \sigma_{1j} \geq 0, \tag{65}$$
$$\nu_j\overline{z}_{\mathrm{p}j} \geq \nu_j c_j, \quad \sigma_{2j} \geq 0, \tag{66}$$
$$\sigma_{1j}\nu_j(\overline{z}_{\mathrm{e}j} - c_j) = 0, \tag{67}$$
$$\sigma_{2j}\nu_j(\overline{z}_{\mathrm{p}j} - c_j) = 0, \tag{68}$$

where $\sigma_{1j}$ and $\sigma_{2j}$ are the Lagrange multipliers. By substituting (63) and (64) into (67) and (68), respectively, we obtain

$$\overline{z}_{\mathrm{e}j}(\overline{z}_{\mathrm{e}j} - c_j) = 0,$$
$$\overline{z}_{\mathrm{p}j}(\overline{z}_{\mathrm{p}j} - c_j) = 0,$$

i.e.

$$\overline{z}_{\mathrm{e}j} \in \{0, c_j\}, \tag{69}$$

$$\overline{z}_{\mathrm{p}j} \in \{0, c_j\}. \tag{70}$$

In the remainder of the proof, we assume $\nu_j > 0$ for simplicity. The case of $\nu_j < 0$ can be shown similarly. First, suppose $c_j < 0$. Then, the first inequality of (65) and $\nu_j c_j < 0$ imply

$$\nu_j \overline{z}_{\mathrm{e}j} < 0,$$

from which and (69) we obtain $\overline{z}_{\mathrm{e}j} = c_j$. Alternatively, suppose $c_j \geq 0$. The second inequality of (65) and $\nu_j > 0$ imply

$$\nu_j \sigma_{1j} \geq 0,$$

from which and (63) it follows that $\overline{z}_{\mathrm{e}j} \leq 0$ should be satisfied. Accordingly, from (69) and $c_j \geq 0$ we obtain $\overline{z}_{\mathrm{e}j} = 0$. Similarly, we can show that $\overline{z}_{\mathrm{p}j}$ satisfies

$$\overline{z}_{\mathrm{p}j} = \begin{cases} c_j, & \text{if } c_j \geq 0, \\ 0, & \text{if } c_j < 0, \end{cases}$$

for $\nu_j > 0$, which concludes the proof. □

## Acknowledgment

The work of the first author is partially supported by a grant from the Inamori Foundation.

## References

[1] Alizadeh, F. and Goldfarb, D., Second-order cone programming, *Mathematical Programming*, **B95** (2003) 3–51.

[2] An, L.T.H., An efficient algorithm for globally minimizing a quadratic function under convex quadratic constraints, *Mathematical Programming*, **87** (2000) 401–426.

[3] An, L.T.H. and Tao, P.D., Solving a class of linearly constrained indefinite quadratic problems by D.C. Algorithms, *Journal of Global Optimization*, **11** (1997) 253–285.

[4] An, L.T.H. and Tao, P.D., The DC (difference of convex functions) programming and DCA revisited with DC models of real world nonconvex optimization problems, *Annals of Operations Research*, **133** (2005) 23–46.

[5] Atai, A.A. and Steigmann, D.J., 1997. On the nonlinear mechanics of discrete networks, *Archive of Applied Mechanics*, **67** (1987) 303–319.

[6] Bignoni, D., Bifurcation and Instability of Non-Associative Elastoplastic Solids. In: *Material Instabilities in Elastic and Plastic Solids*, H. Petryk (ed), Springer–Verlag, Wien, pp. 1–52.

[7] Casciaro, R. and Mancusi, G., Imperfection sensitivity due to coupled local instability: a non-convex QP solution algorithm, *International Journal for Numerical Methods in Engineering*, **67** (2006) 815–840.

[8] Do, C., Bifurcation theory for elastic plates subjected to unilateral conditions, *Journal of Mathematical Analysis and Applications*, **60** (1977) 435–448.

[9] Duvaut, G. and Lions, J.L., *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[10] Eisner, J., Kučera, M. and Recke, L., Direction and stability of bifurcation branches for variational inequalities, *Journal of Mathematical Analysis and Applications*, **301** (2005) 276–294.

[11] Elishakoff, I., Essay on the contributors to the elastic stability theory, *Meccanica*, **40** (2005) 75–110.

[12] Gander, W., Golub, G.H. and von Mat, U., A constrained eigenvalue problem, *Linear Algebra and its Applications*, **114** (1989) 815–839.

[13] Hill, R., A general theory of uniqueness and stability in elastic-plastic solids, *Journal of the Mechanics and Physics of Solids*, **6** (1958) 236–249.

[14] Holmes, P., Domokos, G., Schmitt, J. and Szeberényi, I., Constrained Euler buckling: an interplay of computation and analysis, *Computer Methods in Applied Mechanics and Engineering*, **170** (1999) 175–207.

[15] Kanno, Y. and Ohsaki, M., Minimum principle of complementary energy of cable networks by using second-order cone programming, *International Journal of Solids and Structures*, **40** (2003) 4437–4460.

[16] Klarbring, A., On discrete and discretized non-linear elastic structures in unilateral contact (stability, uniqueness and variational principles), *International Journal of Solids and Structures*, **24** (1988) 459–479.

[17] Klarbring, A., Stability and critical points in large displacement frictionless contact problems. In: *Friction and Instabilities*, J.A.C. Martins and M. Raous (eds), Springer–Verlag, Wien, pp. 39–64, 2002.

[18] LaSalle, J.P. and Lefschetz, S., *Stability by Liapunov's Direct Method*, Academic Press, New York, 1961.

[19] Martins, J.A.C, Pinto da Costa, A. and Simões, F.M.F., Some notes on friction and instabilities. In: *Friction and Instabilities*, J.A.C. Martins and M. Raous (eds), Springer–Verlag, Wien, pp. 65–136, 2002.

[20] Miersemann, E. and Mittelmann, H.D., Stability and continuation of solutions to obstacle problems, *Journal of Computational and Applied Mathematics*, **35** (1991) 5–31.

[21] Nesterov, Yu., Semidefinite relaxation and nonconvex quadratic optimization, *Optimization Methods and Software*, **9** (1998) 141–160.

[22] Nocedal, J. and Wright, S.J., *Numerical Optimization (2nd ed.)*, Springer, New York, 2006.

[23] Ohsaki, M. and Zhang, J.Y., Stability conditions of prestressed pin-jointed structures, *International Journal of Non-Linear Mechanics*, to appear.

[24] Panagiotopoulos, P.D., A variational inequality approach to the inelastic stress-unilateral analysis of cable-structures, *Computers & Structures*, **6** (1976) 133–139.

[25] Petryk, H. and Thermann, K., On discretized plasticity problems with bifurcations, *International Journal of Solids and Structures*, **29** (1992) 745–765.

[26] Pinto da Costa, A., Martins, J.A.C., Figueiredo, I.N. and Júdice, J.J., The directional instability problem in systems with frictional contacts, *Computer Methods in Applied Mechanics and Engineering*, **193** (2004) 357–384.

[27] Pólik, I., Addendum to the SeDuMi user guide: version 1.1, Advanced Optimization Laboratory, McMaster University, Ontario, Canada. `http://sedumi.mcmaster.ca`.

[28] Salle, J.L. and Lefscetz, S., *Stability by Liapunov's Direct Method*, Academic Press, City??, 1961.

[29] Simo, J.C. and Hughes, T.J.R., *Computational Inelasticity*, Springer-Verlag, New York, 1998.

[30] Sturm, J.F., Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, *Optimization Methods and Software*, **11/12** (1999) 625–653.

[31] Tao, P.D. and An, L.T.H., Difference of convex functions optimization algorithms (DCA) for globally minimizing nonconvex quadratic forms on Euclidean balls and spheres, *Optimization Research Letters*, **19** (1996) 207–216.

[32] Thompson, J.M.T. and Hunt, G.W., *A General Theory of Elastic Stability*, John Wiley, New York, 1973.

[33] Tschöpe, H., Oñate, E. and Wriggers, P., Direct computation of instability points for contact problems, *Computational Mechanics*, **31** (2003) 173–178.

[34] Tuy, H., Convex programs with an additional reverse convex constraint, *Journal of Optimization Theory and Applications*, **52** (1987) 463–486.

[35] Villaggio, P., Buckling under unilateral constraints, *International Journal of Solids and Structures*, **15** (1979) 193–201.

[36] Volokh, K.Yu. and Vilnay, O., Why pre-tensioning stiffens cable systems, *International Journal of Solids and Structures*, **37** (2000) 1809–1816.

[37] Xiao, S., Chen, B. and Yang, M., Bifurcation and buckling analysis of a unilaterally confined self-rotating cantilever beam, *Acta Mechanica Sinica*, **22** (2006) 177–184.

[38] —, *Using MATLAB*, The MathWorks, Inc., Natick, 2002.

# A Explicit formulations for contact problems

Explicit formulations of problems relevant to (43) in section 4 are investigated.

Introducing $\widetilde{\lambda}$ so that $\boldsymbol{K} + \widetilde{\lambda}\boldsymbol{I}$ is positive semidefinite in accordance with (8), define $\widetilde{\boldsymbol{K}}$ by

$$\widetilde{\boldsymbol{K}} = \boldsymbol{K} + \widetilde{\lambda}\boldsymbol{I}. \tag{71}$$

The perturbed problem (10) defining $\widetilde{v}$ is explicitly formulated as

$$\widetilde{v} = \min_{\boldsymbol{u}} \left\{ \boldsymbol{u}^{\mathrm{T}}\widetilde{\boldsymbol{K}}\boldsymbol{u} \mid \boldsymbol{T}_0\boldsymbol{u} \leq \boldsymbol{0},\ \boldsymbol{T}_{\mathrm{r}}\boldsymbol{u} = \boldsymbol{0},\ \|\boldsymbol{u}\|^2 - 1 \geq 0 \right\}. \tag{72}$$

Accordingly, the problem (11) defining $g^*$ is obtained as

$$g^*(\lambda) = \max_{\boldsymbol{u}} \left\{ \|\boldsymbol{u}\|^2 - 1 \mid \boldsymbol{T}_0\boldsymbol{u} \leq \boldsymbol{0},\ \boldsymbol{T}_{\mathrm{r}}\boldsymbol{u} = \boldsymbol{0},\ \boldsymbol{u}^{\mathrm{T}}\widetilde{\boldsymbol{K}}\boldsymbol{u} \leq \lambda \right\}, \tag{73}$$

which is to be solved by using Algorithm 2.12. The subproblem (22) in Step 1 of Algorithm 2.12 is formulated as

$$\max_{\boldsymbol{u}} \left\{ -\left\| \boldsymbol{u} - (1 + \frac{2}{\rho})\boldsymbol{u}^k \right\|^2 \mid \boldsymbol{T}_0\boldsymbol{u} \leq \boldsymbol{0},\ \boldsymbol{T}_{\mathrm{r}}\boldsymbol{u} = \boldsymbol{0},\ \boldsymbol{u}^{\mathrm{T}}\widetilde{\boldsymbol{K}}\boldsymbol{u} \leq \lambda \right\}. \tag{74}$$

In a manner similar to section 3.3, we solve (74) by embedding it into the dual standard form of the SOCP problem. We start with observing the fact that the problem (74) is equivalently rewritten as

$$\left.\begin{array}{ll} \displaystyle\max_{\boldsymbol{u},t} & -t \\ \text{s.t.} & \boldsymbol{T}_0\boldsymbol{u} \leq \boldsymbol{0}, \\ & \boldsymbol{T}_{\mathrm{r}}\boldsymbol{u} = \boldsymbol{0}, \\ & \boldsymbol{u}^{\mathrm{T}}\widetilde{\boldsymbol{K}}\boldsymbol{u} \leq \lambda, \\ & -t \leq -\left\| \boldsymbol{u} - (1 + \dfrac{2}{\rho})\boldsymbol{u}^k \right\|^2, \end{array}\right\} \tag{75}$$

where $t \in \mathbb{R}$ is an auxiliary variable. Let $\boldsymbol{R}_u \in \mathbb{R}^{n^{\mathrm{d}} \times n^{\mathrm{d}}}$ denote a constant matrix satisfying (37). It follows from Proposition 3.3 that (75) is equivalently rewritten as

$$\left.\begin{array}{ll} \displaystyle\max_{\boldsymbol{u},t} & -t \\ \text{s.t.} & \boldsymbol{T}\boldsymbol{u} \leq \boldsymbol{0}, \\ & \lambda + \dfrac{1}{4} \geq \left\| \begin{pmatrix} \lambda - (1/4) \\ \boldsymbol{R}_u\boldsymbol{u} \end{pmatrix} \right\|, \\ & t + \dfrac{1}{4} \geq \left\| \begin{pmatrix} t - (1/4) \\ \boldsymbol{u} - (1 + (2/\rho))\boldsymbol{u}^k \end{pmatrix} \right\|, \end{array}\right\} \tag{76}$$

which is a dual-standard form of SOCP.

# B Explicit formulations for elastic-plastic problems

## B.1 Feasibility problem

The feasibility problem for the problem (50) associated with the elastic-plastic trusses in section 5 is investigated. By introducing a parameter $\widetilde{\lambda}$ satisfying

$$\widetilde{\boldsymbol{K}} := \boldsymbol{K}^{\rm e} + \widetilde{\lambda}\boldsymbol{I} \in \mathcal{S}^{n^{\rm d}}_{++}, \tag{77}$$

the perturbed problem defining $\widetilde{v}$ is explicitly obtained as

$$\widetilde{v} = \min_{\boldsymbol{u},\boldsymbol{z}_{{\rm e}\mathcal{J}},\boldsymbol{z}_{{\rm p}\mathcal{J}}} \Big\{ \boldsymbol{u}^{\rm T}\widetilde{\boldsymbol{K}}\boldsymbol{u} + \boldsymbol{z}_{{\rm e}\mathcal{J}}^{\rm T}\boldsymbol{D}^{\rm e}_{\mathcal{J}}\boldsymbol{z}_{{\rm e}\mathcal{J}} + \boldsymbol{z}_{{\rm p}\mathcal{J}}^{\rm T}\boldsymbol{D}^{\rm p}_{\mathcal{J}}\boldsymbol{z}_{{\rm p}\mathcal{J}} \mid \boldsymbol{N}_{\mathcal{J}}(\boldsymbol{B}_{\mathcal{J}}^{\rm T}\boldsymbol{u} - \boldsymbol{z}_{{\rm e}\mathcal{J}}) \ge \boldsymbol{0},\ \boldsymbol{N}_{\mathcal{J}}(\boldsymbol{z}_{{\rm p}\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}}^{\rm T}\boldsymbol{u}) \ge \boldsymbol{0},\ \|\boldsymbol{u}\|^2 - 1 \ge 0 \Big\}. \tag{78}$$

Observe that (78) is embedded into the form of the problem (10). It follows from Proposition 2.6 that we solve (78) instead of (50) for stability determination. The explicit form of (11), which defines $g^*$, is obtained as

$$g^*(\lambda) = \max_{\boldsymbol{u},\boldsymbol{z}_{{\rm e}\mathcal{J}},\boldsymbol{z}_{{\rm p}\mathcal{J}}} \Big\{ \|\boldsymbol{u}\|^2 - 1 \mid \boldsymbol{N}_{\mathcal{J}}(\boldsymbol{B}_{\mathcal{J}}^{\rm T}\boldsymbol{u} - \boldsymbol{z}_{{\rm e}\mathcal{J}}) \ge \boldsymbol{0},\ \boldsymbol{N}_{\mathcal{J}}(\boldsymbol{z}_{{\rm p}\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}}^{\rm T}\boldsymbol{u}) \ge \boldsymbol{0},\ \boldsymbol{u}^{\rm T}\boldsymbol{K}^{\rm e}\boldsymbol{u} + \boldsymbol{z}_{{\rm e}\mathcal{J}}^{\rm T}\boldsymbol{D}^{\rm e}_{\mathcal{J}}\boldsymbol{z}_{{\rm e}\mathcal{J}} + \boldsymbol{z}_{{\rm p}\mathcal{J}}^{\rm T}\boldsymbol{D}^{\rm p}_{\mathcal{J}}\boldsymbol{z}_{{\rm p}\mathcal{J}} \le \lambda \Big\}. \tag{79}$$

*Remark B.1.* As a particular case, suppose that $d^{\rm p}_j = 0$ $(j \in \mathcal{J})$ which implies elastic-perfectly plastic truss. Then the stability determination problem, (50), can be simplified as the following problem in the variables $(\boldsymbol{u}, \boldsymbol{z}_{{\rm e}\mathcal{J}}) \in \mathbb{R}^{n^{\rm d}} \times \mathbb{R}^{|\mathcal{J}|}$:

$$\min_{\boldsymbol{u},\boldsymbol{z}_{{\rm e}\mathcal{J}}} \left\{ \boldsymbol{u}^{\rm T}\boldsymbol{K}^{\rm e}\boldsymbol{u} + \boldsymbol{z}_{{\rm e}\mathcal{J}}^{\rm T}\boldsymbol{D}^{\rm e}_{\mathcal{J}}\boldsymbol{z}_{{\rm e}\mathcal{J}} \mid \boldsymbol{N}_{\mathcal{J}}\boldsymbol{z}_{{\rm e}\mathcal{J}} \le \boldsymbol{N}_{\mathcal{J}}\boldsymbol{B}_{\mathcal{J}}^{\rm T}\boldsymbol{u},\ \|\boldsymbol{u}\|^2 = 1 \right\}. \tag{80}$$

The optimal value of the problem (80) coincides with $v^*$ defined by (46) with $\boldsymbol{d}^{\rm p} := \boldsymbol{0}$. Furthermore, $\overline{\boldsymbol{u}}$ is an optimal solution of (46) with $\boldsymbol{d}^{\rm p} := \boldsymbol{0}$ if and only if $(\overline{\boldsymbol{u}}, \overline{\boldsymbol{z}}_{{\rm e}\mathcal{J}})$ satisfying (32) and (48) is an optimal solution of (50). The proof is analogous to Proposition 3.1 and Proposition 3.2, and thus is omitted. As a consequence, we see that the problem (78) is simplified as

$$\widetilde{v} = \min_{\boldsymbol{u},\boldsymbol{z}_{{\rm e}\mathcal{J}}} \Big\{ \boldsymbol{u}^{\rm T}\widetilde{\boldsymbol{K}}\boldsymbol{u} + \boldsymbol{z}_{{\rm e}\mathcal{J}}^{\rm T}\boldsymbol{D}^{\rm e}_{\mathcal{J}}\boldsymbol{z}_{{\rm e}\mathcal{J}} \mid \boldsymbol{N}_{\mathcal{J}}(\boldsymbol{B}_{\mathcal{J}}^{\rm T}\boldsymbol{u} - \boldsymbol{z}_{{\rm e}\mathcal{J}}) \ge \boldsymbol{0},\ \|\boldsymbol{u}\|^2 - 1 \ge 0 \Big\}; \tag{81}$$

and, accordingly, the problem (79) is obtained as

$$g^*(\lambda) = \max_{\boldsymbol{u},\boldsymbol{z}_{{\rm e}\mathcal{J}}} \left\{ \|\boldsymbol{u}\|^2 - 1 \mid \boldsymbol{N}_{\mathcal{J}}(\boldsymbol{B}_{\mathcal{J}}^{\rm T}\boldsymbol{u} - \boldsymbol{z}_{{\rm e}\mathcal{J}}) \ge \boldsymbol{0},\ \boldsymbol{u}^{\rm T}\boldsymbol{K}^{\rm e}\boldsymbol{u} + \boldsymbol{z}_{{\rm e}\mathcal{J}}^{\rm T}\boldsymbol{D}^{\rm e}_{\mathcal{J}}\boldsymbol{z}_{{\rm e}\mathcal{J}} \le \lambda \right\}. \tag{82}$$

■

## B.2 SOCP formulation of subproblem

In a manner similar to section 3.3, we shall see that the subproblem (22) of Algorithm 2.12 solving (79) can be formulated as an SOCP problem.

Analogously to (36), the subproblem (22) of Algorithm 2.12 solving (79) is obtained as

$$\left.\begin{array}{ll} \displaystyle\max_{\boldsymbol{u},\boldsymbol{z}_{{\rm e}\mathcal{J}},\boldsymbol{z}_{{\rm p}\mathcal{J}}} & -\left\|\boldsymbol{u} - (1 + (2/\rho))\boldsymbol{u}^k\right\|^2 - \left\|\boldsymbol{z}_{\mathcal{J}} - \boldsymbol{z}^k_{{\rm e}\mathcal{J}}\right\|^2 - \left\|\boldsymbol{z}_{\mathcal{J}} - \boldsymbol{z}^k_{{\rm p}\mathcal{J}}\right\|^2 \\ \text{s.t.} & \boldsymbol{N}_{\mathcal{J}}(\boldsymbol{B}_{\mathcal{J}}^{\rm T}\boldsymbol{u} - \boldsymbol{z}_{{\rm e}\mathcal{J}}) \ge \boldsymbol{0}, \\ & \boldsymbol{N}_{\mathcal{J}}(\boldsymbol{z}_{{\rm p}\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}}^{\rm T}\boldsymbol{u}) \ge \boldsymbol{0}, \\ & \boldsymbol{u}^{\rm T}\widetilde{\boldsymbol{K}}\boldsymbol{u} + \boldsymbol{z}_{{\rm e}\mathcal{J}}^{\rm T}\boldsymbol{D}^{\rm e}_{\mathcal{J}}\boldsymbol{z}_{{\rm e}\mathcal{J}} + \boldsymbol{z}_{{\rm p}\mathcal{J}}^{\rm T}\boldsymbol{D}^{\rm p}_{\mathcal{J}}\boldsymbol{z}_{{\rm p}\mathcal{J}} \le \lambda, \end{array}\right\} \tag{83}$$

because the function $h_1$ in (12) is defined as

$$h_1(\boldsymbol{u}, \boldsymbol{z}_{\mathrm{e}\mathcal{J}}, \boldsymbol{z}_{\mathrm{p}\mathcal{J}}) = \frac{\rho}{2}\left(\|\boldsymbol{u}\|^2 + \|\boldsymbol{z}_{\mathrm{e}\mathcal{J}}\|^2 + \|\boldsymbol{z}_{\mathrm{p}\mathcal{J}}\|^2\right) + \|\boldsymbol{u}\|^2 - 1.$$

Let $\boldsymbol{R}_u \in \mathbb{R}^{n^{\mathrm{d}} \times n^{\mathrm{d}}}$, $\boldsymbol{R}_z^{\mathrm{e}} \in \mathbb{R}^{|\mathcal{J}| \times |\mathcal{J}|}$, and $\boldsymbol{R}_z^{\mathrm{p}} \in \mathbb{R}^{|\mathcal{J}| \times |\mathcal{J}|}$, respectively, denote constant matrices satisfying (37) and

$$\begin{aligned}(\boldsymbol{R}_z^{\mathrm{e}})^{\mathrm{T}} \boldsymbol{R}_z^{\mathrm{e}} &= \boldsymbol{D}_{\mathcal{J}}^{\mathrm{e}}, \\ (\boldsymbol{R}_z^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{R}_z^{\mathrm{p}} &= \boldsymbol{D}_{\mathcal{J}}^{\mathrm{p}}.\end{aligned}$$

In a manner similar to (38), by introducing an auxiliary variable $t \in \mathbb{R}$ and utilizing Proposition 3.3, the problem (83) can be embedded into the dual-standard form of SOCP as

$$\left.\begin{array}{ll} \displaystyle\min_{\boldsymbol{u}, \boldsymbol{z}_{\mathrm{e}\mathcal{J}}, \boldsymbol{z}_{\mathrm{p}\mathcal{J}}, t} & t \\ \mathrm{s.t.} & \boldsymbol{N}_{\mathcal{J}}(\boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}} \boldsymbol{u} - \boldsymbol{z}_{\mathrm{e}\mathcal{J}}) \geq \boldsymbol{0}, \\ & \boldsymbol{N}_{\mathcal{J}}(\boldsymbol{z}_{\mathrm{p}\mathcal{J}} - \boldsymbol{B}_{\mathcal{J}}^{\mathrm{T}} \boldsymbol{u}) \geq \boldsymbol{0}, \\ & \lambda + \dfrac{1}{4} \geq \left\| \begin{pmatrix} \lambda - (1/4) \\ \boldsymbol{R}_u \boldsymbol{u} \\ \boldsymbol{R}_z^{\mathrm{e}} \boldsymbol{z}_{\mathrm{e}\mathcal{J}} \\ \boldsymbol{R}_z^{\mathrm{p}} \boldsymbol{z}_{\mathrm{p}\mathcal{J}} \end{pmatrix} \right\|, \\ & t + \dfrac{1}{4} \geq \left\| \begin{pmatrix} t - (1/4) \\ \boldsymbol{u} - (1 + (2/\rho))\boldsymbol{u}^k \\ \boldsymbol{z}_{\mathrm{e}\mathcal{J}} - \boldsymbol{z}_{\mathrm{e}\mathcal{J}}^k \\ \boldsymbol{z}_{\mathrm{p}\mathcal{J}} - \boldsymbol{z}_{\mathrm{p}\mathcal{J}}^k \end{pmatrix} \right\|, \end{array}\right\} \qquad (84)$$

which can be solved efficiently by using the primal-dual interior-point method.