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Index Characterization of Differential-Algebraic Equations in Hybrid Analysis for Circuit Simulation

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Abstract

Modern modeling approaches in circuit simulation naturally lead to differential-algebraic equations (DAEs). The index of a DAE is a measure of the degree of numerical difficulty. In general, the higher the index is, the more difficult it is to solve the DAE. The modified nodal analysis (MNA) is known to yield a DAE with index at most two in a wide class of nonlinear time-varying electric circuits.

In this paper, we consider a broader class of analysis method called the hybrid analysis. For linear time-invariant RLC circuits, we prove that the index of the DAE arising from the hybrid analysis is at most one, and give a structural characterization for the index of a DAE in the hybrid analysis. This yields an efficient algorithm for finding an hybrid analysis in which the index of the DAE to be solved attains zero. Finally, for linear time-invariant electric circuits which may contain dependent sources, we prove that the optimal hybrid analysis by no means results in a higher index DAE than MNA.

1 Introduction

The hybrid analysis [15] is a common generalization of the loop analysis and the cutset analysis, which are classical circuit analysis methods. Kron [18] proposed the hybrid analysis in 1939, and Amari [1] and Branin [5] developed it further in 1960s. Among various analysis methods, however, the modified nodal analysis (MNA) has been the most commonly used in recent years. An advantage of MNA is to generate the model equations automatically. In contrast, the hybrid analysis retains flexibility, which can be exploited to find a model description that reduces the numerical difficulties.

Circuit analysis methods lead to differential-algebraic equations (DAEs), which consist of algebraic equations and differential operations. DAEs present numerical and analytical difficulties which do not occur with ordinary differential equations (ODEs).

Several numerical methods have been developed for solving DAEs. For example, Gear [11] proposed the backward difference formulae (BDF), which were implemented in the DASSL code

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by Petzold (cf. [6]). Hairer and Wanner [13] implemented an implicit Runge-Kutta method in their RADAU5 code.

The *index* concept plays an important role in the analysis of DAEs. The index is a measure of the degree of difficulty in the numerical solution. In general, the higher the index is, the more difficult it is to solve the DAE. While many different concepts exist to assign an index to a DAE such as the *differentiation index* [6, 8, 13], the *perturbation index* [7, 13], and the *tractability index* [19, 25], this paper focuses on the *Kronecker index*. In the case of linear DAEs with constant coefficients, all these indices are equal [7, 23].

Consistent initial values of a DAE have to fulfill not only explicit constraints but also *hidden constraints*, which are obtained by differentiation of certain equations and algebraic transformations. Since DAEs with higher index than one have hidden constraints, it is difficult to find consistent initial values of those DAEs.

For nonlinear time-invariant electric circuits containing independent sources, resistors, inductors, and capacitors, Tischendorf [26] showed that the index of a DAE arising from MNA does not exceed two. She also proved that MNA leads to a DAE with index one if and only if a circuit contains neither L-I cutsets nor C-V loops, where an *L-I cutset* means a cutset consisting of inductors and/or current sources only, and a *C-V loop* means a cycle consisting of capacitors and voltage sources only. This means that the index of a DAE arising from MNA is determined uniquely by the network. Furthermore, the results of Tischendorf [26] are generalized for nonlinear time-varying electric circuits that may contain a wide class of dependent sources [25]. Based on this structural characterization for DAEs with index one, some methods for finding consistent initial values and some index reduction methods have been developed [2, 3, 4, 24].

While the procedure of MNA is uniquely determined, that of the hybrid analysis is not. The hybrid analysis starts with selecting a *partition* of elements and a *reference tree* in the network. This selection determines a system of equations, called the *hybrid equations*, to be solved numerically. Thus it is natural to seek for an optimal selection that makes the hybrid equations easy to solve, among the exponential number of possibilities. In fact, the problem of obtaining the minimum size hybrid equations was solved [14, 17, 21] in 1968. Recently, for linear time-invariant electric circuits which may contain dependent sources, an algorithm for finding a partition and a reference tree which minimize the index of the hybrid equations was proposed in [16].

In this paper, for linear time-invariant RLC circuits, we prove that the index of the hybrid equations never exceeds one, which means that it is easy to find consistent initial values of the hybrid equations. Moreover, we give a structural characterization of circuits with index zero. This characterization brings an index minimization algorithm in the hybrid analysis. Focusing on only linear time-invariant RLC circuits, this algorithm is much faster and simpler than the previous one [16], which runs in $O(n^6)$ time, where n is the number of elements in the circuit. The time complexity can be improved to $O(n^3)$ under a genericity assumption that the set of nonzero entries coming from the physical parameters is algebraically independent. In contrast, the new algorithm runs in $O(n)$ time, without relying on the genericity assumption.

In addition, for linear time-invariant electric circuits which may contain dependent sources, we prove that the optimal hybrid analysis never results in a higher index DAE than MNA. This suggests that the hybrid analysis is superior to MNA in numerical accuracy.

The organization of this paper is as follows. In Section 2, we explain matrix pencils and the definition of the index. We describe the procedure of the hybrid analysis in Section 3. Section 4 characterizes the index of the hybrid equations, which leads to an index minimization algorithm in the hybrid analysis. In Section 5, we make comparisons of the optimal hybrid analysis and MNA. Finally, Section 6 concludes this paper.

2 DAEs and Matrix Pencils

For a polynomial $a(s)$, we denote the degree of $a(s)$ by $\deg a$, where $\deg 0 = -\infty$ by convention. A polynomial matrix $A(s) = (a_{pq}(s))$ with $\deg a_{pq} \leq 1$ for all (p, q) is called a *matrix pencil*. Obviously, a matrix pencil $A(s)$ can be represented by $A(s) = A_0 + sA_1$ with a pair of constant matrices A_0 and A_1. A matrix pencil $A(s)$ is said to be *regular* if $A(s)$ is square and $\det A(s)$ is a nonvanishing polynomial.

Consider a linear DAE with constant coefficients

$$A_0 \boldsymbol{x}(t) + A_1 \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{f}(t), \tag{1}$$

where A_0 and A_1 are constant matrices. In the case of $\det A_1 \neq 0$, the DAE (1) reduces to an ODE, and in the case of $A_1 = O$, it reduces to algebraic equations. With the use of the Laplace transformation, the linear DAE with constant coefficients in the form of (1) is expressed by the matrix pencil $A(s) = A_0 + sA_1$ as $A(s)\hat{\boldsymbol{x}}(s) = \hat{\boldsymbol{f}}(s) + A_1\boldsymbol{x}(0)$, where s is the variable for the Laplace transform that corresponds to $\mathrm{d}/\mathrm{d}t$, the differentiation with respect to time.

Theorem 2.1 ([6, Theorem 2.3.1]). *The linear DAE with constant coefficients* (1) *is solvable if and only if $A(s)$ is a regular matrix pencil.*

The reader is referred to [6, Definition 2.2.1] for the precise definition of solvability. By Theorem 2.1, we assume that $A(s)$ is a regular matrix pencil throughout this paper. A regular matrix pencil is known to have the *Kronecker canonical form*, which determines the index. Let N_μ denote a $\mu \times \mu$ matrix pencil defined by

$$N_\mu = \begin{pmatrix} 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & s \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

A matrix pencil $A(s)$ is said to be *strictly equivalent* to $A'(s)$ if $A(s)$ can be brought into $A'(s)$ by an equivalence transformation with nonsingular constant matrices.

Theorem 2.2 ([10, Chapter XII, Theorem 3]). *An $n \times n$ regular matrix pencil $A(s)$ is strictly* *equivalent to its Kronecker canonical form:*

$$\begin{pmatrix} sI_{\mu_0}+J_{\mu_0} & O & O & \cdots & O \\ O & N_{\mu_1} & O & \cdots & O \\ O & O & N_{\mu_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & O \\ O & O & \cdots & O & N_{\mu_b} \end{pmatrix},$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_b$*,* $\mu_0 + \mu_1 + \mu_2 + \cdots + \mu_b = n$*,* I_{μ_0} *is a* $\mu_0 \times \mu_0$ *unit matrix, and* J_{μ_0} *is a* $\mu_0 \times \mu_0$ *constant matrix.*

The matrices N_{μ_i} $(i = 1, \ldots, b)$ are called the *nilpotent blocks*. The maximum size μ_1 of them is the *index*, denoted by $\nu(A)$. Obviously, ODEs have index zero, and algebraic equations have index one.

We denote by $A[K, J]$ the submatrix of $A(s)$ with row set $K \subseteq P$ and column set $J \subseteq Q$, where P and Q are the row set and the column set of $A(s)$, respectively. Let $\delta_r(A)$ denote the highest degree of a minor of order r in $A(s)$:

$$\delta_r(A) = \max_{K,J}\{\deg \det A[K, J] \mid |K| = |J| = r, K \subseteq P, J \subseteq Q\}.$$

The index $\nu(A)$ can be determined from $\delta_r(A)$ as follows.

Theorem 2.3 ([20, Theorem 5.1.8])**.** *Let* $A(s)$ *be an* $n \times n$ *regular matrix pencil. The index* $\nu(A)$ *is given by*

$$\nu(A) = \delta_{n-1}(A) - \delta_n(A) + 1.$$

3 Hybrid Analysis

In this section, we describe the procedure of the hybrid analysis [16]. We focus on linear time-invariant RLC circuits, which are composed of resistors, inductors, capacitors, independent voltage/current sources.

Let $\Gamma = (W, E)$ be the network graph with vertex set W and edge set E. An edge in Γ corresponds to a branch that contains one element in the circuit. We denote the set of edges corresponding to independent voltage sources and independent current sources by E_g and E_h, respectively. We split $E_* := E \setminus (E_g \cup E_h)$ into E_y and E_z, i.e., $E_y \cup E_z = E_*$ and $E_y \cap E_z = \emptyset$. A partition (E_y, E_z) is called an *admissible partition*, if E_y includes all the capacitances, and E_z includes all the inductances.

We now explain the *circuit equations* for a linear time-invariant RLC circuit. Let $\boldsymbol{\xi}$ denote the vector of currents through all branches of the circuit, and $\boldsymbol{\eta}$ the vector of voltages across all branches. We denote the *reduced cutset matrix* by Ψ and the *reduced loop matrix* by Φ. Using *Kirchhoff's current law* (KCL), which states that the sum of currents entering each node is equal to zero, we have $\Psi\boldsymbol{\xi} = \mathbf{0}$. Similarly, using *Kirchhoff's voltage law* (KVL), which states that the sum of voltages in each loop of the network is equal to zero, we have $\Phi\boldsymbol{\eta} = \mathbf{0}$. The physical characteristics of elements determine *constitutive equations*. In this paper, we assume that the constitutive equations of a resistor, a capacitor, and an inductor are described by

$$\xi = \frac{1}{\alpha}\eta, \quad \xi = \beta\frac{\mathrm{d}\eta}{\mathrm{d}t}, \quad \text{and} \quad \eta = \gamma\frac{\mathrm{d}\xi}{\mathrm{d}t},$$

where ξ and η denote a current variable and a voltage variable, and α, β, and γ denote a resistance, a capacitance, and an inductance, which are constant. Given an admissible partition (E_y, E_z), we split $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ into

$$\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\xi}_g \\ \boldsymbol{\xi}_y \\ \boldsymbol{\xi}_z \\ \boldsymbol{\xi}_h \end{pmatrix} \quad \text{and} \quad \boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}_g \\ \boldsymbol{\eta}_y \\ \boldsymbol{\eta}_z \\ \boldsymbol{\eta}_h \end{pmatrix},$$

where the subscripts correspond to the partition of E. After the Laplace transformation, the circuit equations, which consist of KCL, KVL, and constitutive equations, are described by

$$\left(\begin{array}{cccc|cccc} \multicolumn{4}{c|}{\Psi} & \multicolumn{4}{c}{O} \\ \hline \multicolumn{4}{c|}{O} & \multicolumn{4}{c}{\Phi} \\ \hline O & I & O & O & O & Y(s) & O & O \\ O & O & Z(s) & O & O & O & I & O \\ O & O & O & O & I & O & O & O \\ O & O & O & I & O & O & O & O \end{array}\right) \begin{pmatrix} \boldsymbol{\xi}_g \\ \boldsymbol{\xi}_y \\ \boldsymbol{\xi}_z \\ \boldsymbol{\xi}_h \\ \hline \boldsymbol{\eta}_g \\ \boldsymbol{\eta}_y \\ \boldsymbol{\eta}_z \\ \boldsymbol{\eta}_h \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{g}(s) \\ \boldsymbol{h}(s) \end{pmatrix}, \tag{2}$$

where $Y(s)$ and $Z(s)$ are diagonal matrix pencils. The degree of a diagonal entry of $Y(s)$ is equal to zero if its column corresponds to a variable for a resistor, and equal to one if its column corresponds to a capacitor. Similarly, the degree of a diagonal entry of $Z(s)$ is equal to zero if its column corresponds to a resistor, and equal to one if its column corresponds to an inductor. Thus the coefficient matrix $A(s)$ of the circuit equations is a matrix pencil. The row set and the column set of $A(s)$ are denoted by P and Q, respectively.

We call a spanning tree T of Γ a *reference tree* if T contains all edges in E_g, no edges in E_h, and as many edges in E_y as possible. Note that a reference tree T may contain some edges in E_z. The cotree of T is denoted by $\overline{T} = E \setminus T$.

Given an admissible partition (E_y, E_z), we denote the column sets of $A(s)$ corresponding to the current variables and the voltage variables for elements in E_g, E_y, E_z, E_h by I_g, I_y, I_z, I_h, and V_g, V_y, V_z, V_h, respectively. Moreover, given a reference tree T, we denote the column sets of $A(s)$ corresponding to the current variables and the voltage variables for elements in $E_y \cap T$ and $E_y \cap \overline{T}$ by I_y^τ, I_y^λ, and V_y^τ, V_y^λ, respectively. The superscripts τ and λ designate the tree T and the cotree $\overline{T}$. We define I_z^τ, I_z^λ, and V_z^τ, V_z^λ in a similar way. We also use $I^\tau = I_g \cup I_y^\tau \cup I_z^\tau$, $I^\lambda = I_y^\lambda \cup I_z^\lambda \cup I_h$ and $V^\tau = V_g \cup V_y^\tau \cup V_z^\tau$, $V^\lambda = V_y^\lambda \cup V_z^\lambda \cup V_h$ for convenience. The column sets corresponding to current variables and voltage variables are denoted by $Q_I := I^\tau \cup I^\lambda$ and $Q_V := V^\tau \cup V^\lambda$. Let i_e and v_e denote the column corresponding to the current variable and the voltage variable for an element e. For a set of elements $F \subseteq E$, we define $Q_I^F := \{i_e \mid e \in F\}$ and $Q_V^F := \{v_e \mid e \in F\}$.

The row sets of $A(s)$ corresponding to KCL, KVL, and constitutive equations are denoted by P_I, P_V, and S, respectively. Given an admissible partition (E_y, E_z) and a reference tree T, let $A_T(s)$ be the coefficient matrix of the circuit equations, where Ψ is the fundamental cutset matrix and Φ is the fundamental loop matrix with respect to T. This means that we transform $A(s)$ into $A_T(s)$ such that $A_T[P_I, I^\tau] = I$ and $A_T[P_V, V^\lambda] = I$ by row operations in $P_I \cup P_V$. Note that P_I and I^τ as well as P_V and V^λ have one-to-one correspondence. The row sets of $A_T(s)$ corresponding to I_g, I_y^τ, I_z^τ, and V_y^λ, V_z^λ, V_h are denoted by P_g, P_y^τ, P_z^τ, and P_y^λ, P_z^λ, P_h. If $K \subseteq P$ and $J \subseteq Q$ have the same superscript and subscript, $A_T[K, J]$ is the unit matrix. Similarly, the row sets corresponding to I_y, V_z, V_g, and I_h are denoted by S_y, S_z, S_g, and S_h. By the definition of a reference tree, $A_T(s)$ has the following property.

Lemma 3.1 ([16, Lemma 3.1]). *For a reference tree T, we have $A_T[P_z^\tau, I_y^\lambda] = O$ and $A_T[P_y^\lambda, V_z^\tau] = O$.*

Thus $A_T(s)$ is in the form of

$$
A_T(s) = \begin{array}{c} \\ P_g \\ P_y^\tau \\ P_z^\tau \\ P_y^\lambda \\ P_z^\lambda \\ P_h \\ S_y \\ \\ S_z \\ \\ S_g \\ S_h \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} I_g & I_y^\tau & I_y^\lambda & I_z^\tau & I_z^\lambda & I_h & V_g & V_y^\tau & V_y^\lambda & V_z^\tau & V_z^\lambda & V_h \end{array} \\
\begin{pmatrix}
I & O & * & O & * & * & O & O & O & O & O & O \\
O & I & * & O & * & * & O & O & O & O & O & O \\
O & O & O & I & * & * & O & O & O & O & O & O \\
O & O & O & O & O & O & * & * & I & O & O & O \\
O & O & O & O & O & O & * & * & O & * & I & O \\
O & O & O & O & O & O & * & * & O & * & O & I \\
O & I & O & O & O & O & O & Y^\tau(s) & O & O & O & O \\
O & O & I & O & O & O & O & O & Y^\lambda(s) & O & O & O \\
O & O & O & Z^\tau(s) & O & O & O & O & O & I & O & O \\
O & O & O & O & Z^\lambda(s) & O & O & O & O & O & I & O \\
O & O & O & O & O & O & I & O & O & O & O & O \\
O & O & O & O & O & I & O & O & O & O & O & O
\end{pmatrix}
\end{array}, \tag{3}
$$

where $*$ means a constant matrix and $Y^\tau(s)$, $Y^\lambda(s)$, $Z^\tau(s)$ and $Z^\lambda(s)$ are diagonal matrix pencils. We can determine $A_T(s)$ only after being given both an admissible partition (E_y, E_z) and a reference tree T.

Let us denote $P_* = P \setminus (P_y^\tau \cup P_z^\lambda)$ and $Q_* = Q \setminus (I_z^\lambda \cup V_y^\tau)$. We transform A_T into A'_T by row operations:

$$
A_T = \begin{pmatrix} B & G \\ H & M \end{pmatrix} \to A'_T = \begin{pmatrix} I & O \\ -HB^{-1} & I \end{pmatrix} \begin{pmatrix} B & G \\ H & M \end{pmatrix} = \begin{pmatrix} B & G \\ O & M - HB^{-1}G \end{pmatrix}, \tag{4}
$$

where $B = A_T[P_*, Q_*], G = A_T[P_*, Q \setminus Q_*], H = A_T[P \setminus P_*, Q_*]$, and $M = A_T[P \setminus P_*, Q \setminus Q_*]$. We denote $M - HB^{-1}G$ by D.

Let $\check{B}$, $\check{G}$, $\check{H}$, $\check{M}$, and $\check{D}$ denote the matrices obtained by replacing s with $\mathrm{d}/\mathrm{d}t$ in B, G, H, M, and D, respectively. Consider the DAE

$$
\check{B}\boldsymbol{x}_1(t) + \check{G}\boldsymbol{x}_2(t) = \boldsymbol{f}_1(t), \tag{5}
$$
$$
\check{H}\boldsymbol{x}_1(t) + \check{M}\boldsymbol{x}_2(t) = \boldsymbol{f}_2(t). \tag{6}
$$

By applying the transformation shown in (4), we obtain

$$\check{B}\boldsymbol{x}_1(t) = \boldsymbol{f}_1(t) - \check{G}\boldsymbol{x}_2(t), \tag{7}$$

$$\check{D}\boldsymbol{x}_2(t) = \boldsymbol{f}_2(t) - \check{H}\check{B}^{-1}\boldsymbol{f}_1(t). \tag{8}$$

We call the resulting DAE (8) the *hybrid equations*. Let us denote the vectors of currents corresponding to I_g, I_y^τ, I_y^λ, I_z^τ, I_z^λ, I_h by $\boldsymbol{\xi}_g$, $\boldsymbol{\xi}_y^\tau$, $\boldsymbol{\xi}_y^\lambda$, $\boldsymbol{\xi}_z^\tau$, $\boldsymbol{\xi}_z^\lambda$, $\boldsymbol{\xi}_h$, and the vectors of voltages corresponding to V_g, V_y^τ, V_y^λ, V_z^τ, V_z^λ, V_h by $\boldsymbol{\eta}_g$, $\boldsymbol{\eta}_y^\tau$, $\boldsymbol{\eta}_y^\lambda$, $\boldsymbol{\eta}_z^\tau$, $\boldsymbol{\eta}_z^\lambda$, $\boldsymbol{\eta}_h$. The procedure of the hybrid analysis is as follows.

1. The values of $\boldsymbol{\xi}_h$ and $\boldsymbol{\eta}_g$ are obvious from the equations corresponding to S_h and S_g.
2. Compute the values of $\boldsymbol{\xi}_z^\lambda$ and $\boldsymbol{\eta}_y^\tau$ by solving the hybrid equations (8).
3. Compute the values of $\boldsymbol{\xi}_z^\tau$ and $\boldsymbol{\eta}_y^\lambda$ by substituting the values obtained in Steps 1–2 into the equations corresponding to P_z^τ and P_y^λ.
4. Compute the values of $\boldsymbol{\xi}_y^\tau$, $\boldsymbol{\xi}_y^\lambda$, $\boldsymbol{\eta}_z^\tau$, and $\boldsymbol{\eta}_z^\lambda$ by substituting the values obtained in Steps 1–3 into S_y and S_z.
5. Compute the values of $\boldsymbol{\xi}_g$ and $\boldsymbol{\eta}_h$ by substituting the values obtained in Steps 1–4 into P_g and P_h.

In the case of $E_y = \emptyset$, the above procedure is called the *loop analysis* or the *tieset analysis*. In the case of $E_z = \emptyset$, the procedure is called the *cutset analysis*.

All operations in Steps 3–5 are substitutions and differentiations of the obtained solutions. This is because we can transform B into an upper triangular matrix pencil with diagonal ones by permutations [16, Lemma 3.2]. Hence, the numerical difficulty is determined by the index of the hybrid equations (8).

4 Index of Hybrid Equations

In this section, for linear time-invariant RLC circuits, we prove that the index $\nu(D)$ of the hybrid equations is at most one, and we provide a necessary and sufficient condition for $\nu(D) = 0$.

Given an admissible partition (E_y, E_z) and a reference tree T, consider the transformation shown in (4). For each $p \in P$ and $q \in Q$, let d_{pq} denote the degree of $\det A_T[P \setminus \{p\}, Q \setminus \{q\}]$. The index $\nu(D)$ can be rewritten as follows.

Lemma 4.1 ([16, Lemma 4.1])**.** *Given an admissible partition (E_y, E_z) and a reference tree T, the index of D is given by*

$$\nu(D) = \max_{p,q}\{d_{pq} \mid p \in P \setminus P_*, q \in Q \setminus Q_*\} - \deg\det A_T + 1. \tag{9}$$

The index of the hybrid equations has the following property.

Lemma 4.2 ([16, Theorem 4.4])**.** *Given an admissible partition (E_y, E_z), the index $\nu(D)$ is the same for any reference tree.*

The generalized Laplace expansion applied to $A_T(s)$ results in

$$\det A_T = \sum_F \operatorname{sgn} F \det A_T[P_I, Q_I^F] \det A_T[P_V, Q_V^{\overline{F}}] \det A_T[S, Q_I^{\overline{F}} \cup Q_V^F], \tag{10}$$

where the summation is over every spanning tree F of Γ which contains all edges in E_g but no edges in E_h, and $\operatorname{sgn} F$ is equal to $+1$ or -1. This is because $A_T[P_I, Q_I^F]$ and $A_T[P_V, Q_V^{\overline{F}}]$ are nonsingular due to the special structure of $A_T(s)$. It is known that $\det A_T$ has the following property.

Lemma 4.3 ([22, Theorem 2.5.1])**.** *Each expansion term of* $\det A_T$ *in* (10) *has the same sign.*

For RLC circuits, we characterize $\deg \det A_T$ and d_{pq} in terms of the number of inductors and capacitors. Let R, L, and C denote the sets of resistors, inductors, and capacitors, respectively. We introduce a *normal reference tree* as follows.

Definition 4.4. *A reference tree is called normal if it contains as many edges in C and as few edges in L as possible.*

The value of $\deg \det A_T$ for a normal reference tree T is expressed as follows.

Theorem 4.5. *Given an admissible partition (E_y, E_z) and a normal reference tree T, we have*

$$\deg \det A_T = |\overline{T} \cap L| + |T \cap C|. \tag{11}$$

Proof. Each expansion term $a_F(s)$ of $\det A_T$ corresponding to a spanning tree F which contains all edges in E_g but no edges in E_h is given by

$$a_F(s) = \operatorname{sgn} F \det A_T[P_I, Q_I^F] \det A_T[P_V, Q_V^{\overline{F}}] \det A_T[S, Q_I^{\overline{F}} \cup Q_V^F].$$

Since $A_T[S, Q_I]$ and $A_T[S, Q_V]$ are matrix pencils defined by (2), we have

$$\deg a_F(s) = |\overline{F} \cap L| + |F \cap C|.$$

By Definition 4.4, this implies that $\deg a_F(s) \leq \deg a_T(s)$ for any spanning tree F. Then Lemma 4.3 ensures that $\deg \det A_T = \deg a_T(s)$. Thus we obtain (11). □

We now introduce the *Resistor-Acyclic condition* for admissible partition (E_y, E_z), which is proved in Theorem 4.9 to be a necessary and sufficient condition for $\nu(D) = 0$.

[Resistor-Acyclic condition]

- Each $e \in E_y \cap R$ belongs to a cycle consisting of independent voltage sources, capacitors, and e.
- Each $e \in E_z \cap R$ belongs to a cutset consisting of inductors, independent current sources, and e.

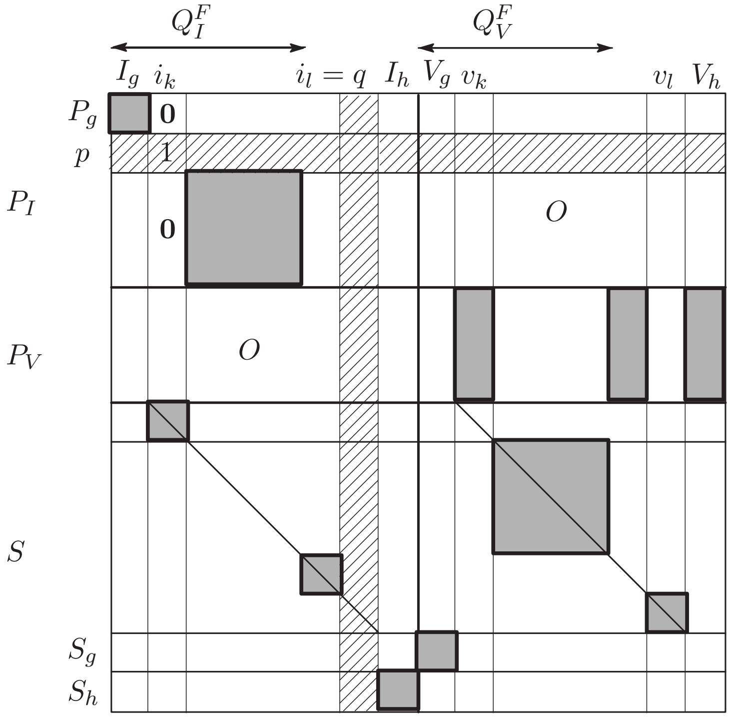

Figure 1: A nonzero expansion term of $\det A_T[P \setminus \{p\}, Q \setminus \{q\}]$ for $p \in P_y^\tau$ and $q \in I_z^\lambda$.

Let $\Gamma = (W, E)$ be a connected network graph. For an edge $e = (u, v) \in E$, *contracting* e means deleting e and identifying u and v. A *coloop* is an edge whose deletion makes the graph disconnected. Let $\Gamma_R = (W_R, R)$ denote the graph obtained from $\Gamma = (W, E)$ by contracting all edges in $E_g \cup C$ and deleting all edges in $L \cup E_h$. The Resistor-Acyclic condition can be expressed as follows by using Γ_R.

Lemma 4.6. *An admissible partition (E_y, E_z) satisfies the Resistor-Acyclic condition if and only if the set of all the selfloops and the set of all the coloops in Γ_R coincide with $E_y \cap R$ and $E_z \cap R$, respectively.* □

In order to characterize $\nu(D)$, we now adopt a normal reference tree T and analyze d_{pq} for $p \in P \setminus P_*$ and $q \in Q \setminus Q_*$ in the following two cases:

(i) $p \in P_y^\tau$, $q \in I_z^\lambda$ or $p \in P_z^\lambda$, $q \in V_y^\tau$,

(ii) $p \in P_y^\tau$, $q \in V_y^\tau$ or $p \in P_z^\lambda$, $q \in I_z^\lambda$.

We first consider the case (i).

Lemma 4.7. *If $p \in P_y^\tau$, $q \in I_z^\lambda$ or $p \in P_z^\lambda$, $q \in V_y^\tau$, we have $d_{pq} \le \deg\det A_T$. If, in addition, the Resister-Acyclic condition is satisfied, then $d_{pq} \le \deg\det A_T - 2$ holds.*

Proof. We prove the claim in the case of $p \in P_y^\tau$ and $q \in I_z^\lambda$. Let k denote the element such that i_k corresponds to p, and l denote the element such that $i_l = q$ as shown in Figure 1. Then it follows that $k \in E_y \cap T$ and $l \in E_z \cap \overline{T}$. Each nonzero expansion term of $\det A_T[P \setminus \{p\}, Q \setminus \{q\}]$

has one-to-one correspondence with spanning tree F of Γ such that F contains all edges in $E_g \cup \{k\}$ but no edges in $E_h \cup \{l\}$, and $F \setminus \{k\} \cup \{l\}$ is a spanning tree. Since $A_T[P_I, \{i_k\}]$ is a unit vector with the (p, i_k) entry being one, $A_T[P_I \setminus \{p\}, Q_I^F \setminus \{i_k\}]$ is nonsingular. Thus, with respect to F, $\det A_T[P \setminus \{p\}, Q \setminus \{q\}]$ has a nonzero expansion term

$$a_F(s) = \det A_T[P_I \setminus \{p\}, Q_I^F \setminus \{i_k\}] \det A_T[P_V, Q_V^{\overline{F}} \cup \{v_k\} \setminus \{v_l\}] \\ \det A_T[S, (Q_I^{\overline{F}} \cup \{i_k\} \setminus \{i_l\}) \cup (Q_V^F \setminus \{v_k\} \cup \{v_l\})],$$

which is depicted in Figure 1. Let F maximize $\deg a_F(s)$. Since $a_F(s)$ possibly disappears in $\det A_T[P \setminus \{p\}, Q \setminus \{q\}]$ due to numerical cancellations, we have

$$d_{pq} \leq \deg a_F(s) = |L \cap (\overline{F} \cup \{k\} \setminus \{l\})| + |C \cap (F \setminus \{k\} \cup \{l\})|.$$

Note that $k \in E_y$ implies that k is not an inductor and $l \in E_z$ implies that l is not a capacitor. Hence we obtain

$$|L \cap (\overline{F} \cup \{k\} \setminus \{l\})| = |L \cap (\overline{F} \setminus \{l\})| \leq |L \cap \overline{F}| \leq |L \cap \overline{T}|,$$

where T is a normal reference tree. Similarly $|C \cap (F \setminus \{k\} \cup \{l\})| \leq |C \cap T|$ holds. Then it follows from Theorem 4.5 that

$$d_{pq} \leq |L \cap \overline{T}| + |C \cap T| = \deg \det A_T.$$

If (E_y, E_z) satisfies the Resistor-Acyclic condition, we have $k \in E_y \cap T \subseteq C$ and $l \in E_z \cap \overline{T} \subseteq L$, which implies that $|L \cap (\overline{F} \setminus \{l\})| = |L \cap \overline{F}| - 1$ and $|C \cap (F \setminus \{k\})| = |C \cap F| - 1$. Therefore, $d_{pq} \leq \deg \det A_T - 2$ holds.

In the case of $p \in P_z^\lambda$, $q \in V_y^\tau$, the claim is proved in a similar way. □

Next, we consider the case (ii).

Lemma 4.8. *If $p \in P_y^\tau$, $q \in V_y^\tau$ or $p \in P_z^\lambda$, $q \in I_z^\lambda$, we have $d_{pq} \leq \deg \det A_T$. If, in addition, the Resister-Acyclic condition is satisfied, then $d_{pq} \leq \deg \det A_T - 1$ holds.*

Proof. We prove the claim in the case of $p \in P_y^\tau$ and $q \in V_y^\tau$. Let k denote the element such that i_k corresponds to p, and l denote the element such that $v_l = q$ as shown in Figure 2. Then it follows that $k \in E_y \cap T$. Each nonzero expansion term of $\det A_T[P \setminus \{p\}, Q \setminus \{q\}]$ has one-to-one correspondence with spanning tree F of Γ such that F contains all edges in $E_g \cup \{k\}$ but no edges in $E_h \cup \{l\}$, and $F \setminus \{k\} \cup \{l\}$ is a spanning tree. With respect to F, $\det A_T[P \setminus \{p\}, Q \setminus \{q\}]$ has a nonzero expansion term

$$a_F(s) = \det A_T[P_I \setminus \{p\}, Q_I^F \setminus \{i_k\}] \det A_T[P_V, Q_V^{\overline{F}} \cup \{v_k\} \setminus \{v_l\}] \\ \det A_T[S, (Q_I^{\overline{F}} \cup \{i_k\}) \cup (Q_V^F \setminus \{v_k\})],$$

which is depicted in Figure 2. Let F maximize $\deg a_F(s)$. Since $a_F(s)$ possibly disappears in $\det A_T[P \setminus \{p\}, Q \setminus \{q\}]$ due to numerical cancellations, we have

$$d_{pq} \leq \deg a_F(s) = |L \cap (\overline{F} \cup \{k\})| + |C \cap (F \setminus \{k\})|.$$

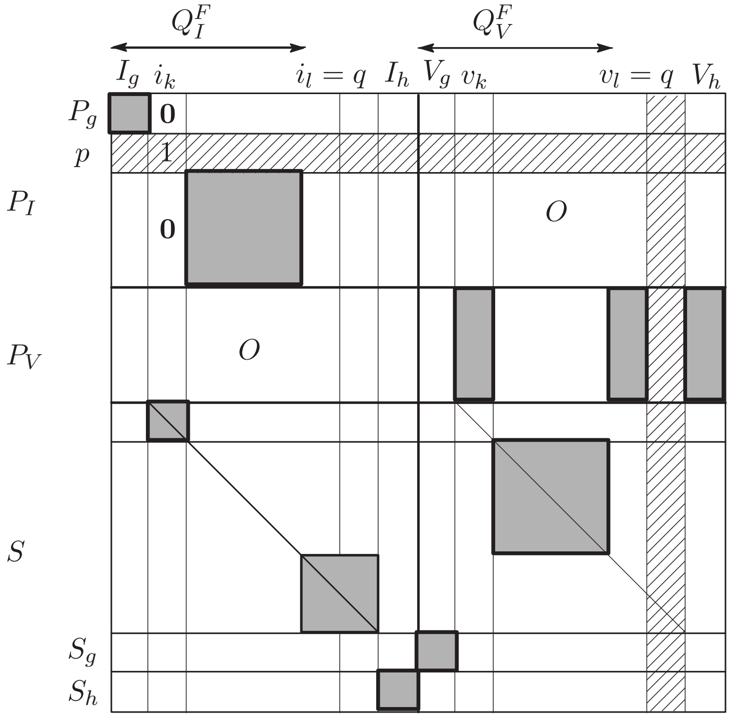

Figure 2: A nonzero expansion term of $\det A_T[P \setminus \{p\}, Q \setminus \{q\}]$ for $p \in P_y^\tau$ and $q \in I_y^\tau$.

Note that $k \in E_y$ implies that k is not an inductor. Hence we have

$$|L \cap (\overline{F} \cup \{k\})| = |L \cap \overline{F}| \leq |L \cap \overline{T}|,$$

where T is a normal reference tree. Similarly $|C \cap (F \setminus \{k\})| \leq |C \cap F| \leq |C \cap T|$ holds. Then it follows from Theorem 4.5 that

$$d_{pq} \leq |L \cap \overline{T}| + |C \cap T| = \deg \det A_T.$$

If (E_y, E_z) satisfies the Resistor-Acyclic condition, we have $k \in E_y \cap T \subseteq C$, which implies that $|C \cap (F \setminus \{k\})| = |C \cap F| - 1$. Therefore, $d_{pq} \leq \deg \det A_T - 1$ holds.

In the case of $p \in P_z^\lambda$, $q \in I_z^\lambda$, the claim is proved in a similar way. □

By Lemmas 4.7 and 4.8, we obtain the following theorem concerning the index $\nu(D)$.

Theorem 4.9. *For any admissible partition (E_y, E_z), the index $\nu(D)$ of the hybrid equations is at most one. Moreover, for a given admissible partition (E_y, E_z), the index $\nu(D)$ is equal to zero if and only if (E_y, E_z) satisfies the Resistor-Acyclic condition.*

Proof. For any admissible partition (E_y, E_z), it follows from Lemmas 4.7 and 4.8 that $d_{pq} \leq \deg \det A_T$ for $p \in P \setminus P_*$ and $q \in Q \setminus Q_*$, which implies that $\nu(D) \leq 1$ by Lemma 4.1.

If (E_y, E_z) satisfies the Resister-Acyclic condition, then it follows from Lemmas 4.7 and 4.8 that $\max\{d_{pq} \mid p \in P \setminus P_*, q \in Q \setminus Q_*\} \leq \deg \det A_T - 1$, which implies $\nu(D) = 0$ by the nonnegativity of the index.

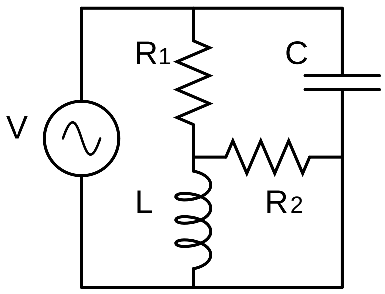

Figure 3: An RLC circuit that has hybrid equations with index one (Example 4.10).

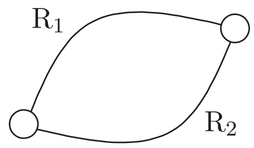

Figure 4: The graph Γ_R of Example 4.10.

Conversely, we show that (E_y, E_z) satisfies the Resistor-Acyclic condition if $\nu(D) = 0$. Let m denote the size of D. Then $\deg\det D = m$ holds. We assume that there exists a resistor $e \in E_y$ which does not belong to any cycle consisting of independent voltage sources, capacitors, and e. Then, there exists a normal reference tree T containing e. The transformation (4) with respect to T makes $D[P \setminus P_*, \{v_e\}]$ to be constant, which implies that $\deg\det D \leq m - 1$. This contradicts $\nu(D) = 0$. Similarly, each resistor $e \in E_z$ belongs to a cutset consisting of inductors, independent current sources, and e. □

For any admissible partition (E_y, E_z), the index $\nu(D)$ of the hybrid equations is at most one by Theorem 4.9. We now describe an algorithm for finding the minimum index $\nu(D)$ and an admissible partition (E_y, E_z) if $\nu(D) = 0$. The correctness of this algorithm, as well as the uniqueness of the admissible partition that attains $\nu(D) = 0$, follows from Lemma 4.6 and Theorem 4.9.

Algorithm for index minimization in RLC circuit

Step 1 Set $E_y \leftarrow \{e \mid e \in C\}$ and $E_z \leftarrow \{e \mid e \in L\}$.

Step 2 Contract all edges in $E_g \cup E_y$ and delete all edges in $E_z \cup E_h$ from $\Gamma = (W, E)$. Then we obtain $\Gamma_R = (W_R, R)$.

Step 3 If Γ_R has a cycle except selfloops, then output $\nu(D) = 1$ and halt.

Step 4 Set $E_y \leftarrow E_y \cup \{e \mid e\text{: selfloop}\}$ and $E_z \leftarrow E_* \setminus E_y$. Output $\nu(D) = 0$ and (E_y, E_z), and halt.

We demonstrate Algorithm for index minimization in RLC circuit in some examples.

Example 4.10 (RLC circuit with index one [9, 20]). Consider a circuit depicted in Figure 3. MNA results in a DAE with index two, because this circuit has a C-V loop. Algorithm for index minimization in RLC circuit finds graph Γ_R shown in Figure 4. Since Γ_R has a cycle which consists of R_1 and R_2, the hybrid analysis results in a DAE with index one for any admissible partition.

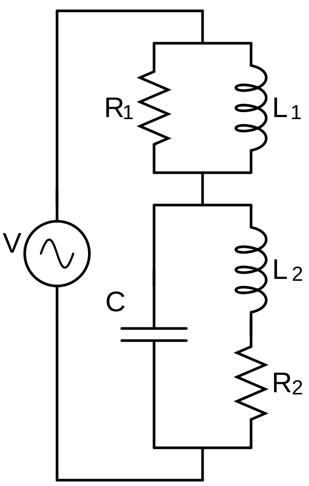

Figure 5: An RLC circuit that has hybrid equations with index zero (Example 4.11).

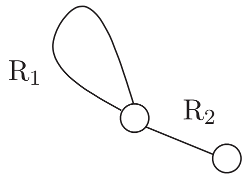

Figure 6: The graph Γ_R of Example 4.11.

Example 4.11 (RLC circuit with index zero)**.** Consider another circuit given in Figure 5. MNA results in a DAE with index one, because this circuit has neither C-V loops nor L-I cutsets. Algorithm for index minimization in RLC circuit finds graph Γ_R shown in Figure 6. Since Γ_R does not have any cycle except selfloops, the hybrid analysis results in a DAE with index zero for admissible partition $(\{C, R_1\}, \{L_1, L_2, R_2\})$.

5 Comparison of Hybrid Analysis with MNA

In this section, for linear time-invariant electric circuits which may contain dependent sources, we prove that the index of a DAE arising from the optimal hybrid analysis does not exceed that from MNA.

We first extend the definition of an admissible partition to electric circuits which contain dependent sources. A partition (E_y, E_z) of $E_* = E \setminus (E_g \cup E_h)$ is called *admissible* if E_y includes all capacitors and dependent current sources, and E_z includes all inductors and dependent voltage sources. Consider the hybrid analysis with admissible partition (E_y, E_z) such that E_y includes all resistors. Let $A_T(s)$ denote the coefficient matrix with respect to this admissible partition (E_y, E_z) and a reference tree T. We prove that this hybrid analysis never results in a higher index DAE than MNA.

We now explain the procedure of MNA. Let us split vector $\boldsymbol{\xi}$ of current variables and vector $\boldsymbol{\eta}$ of voltage variables with respect to independent voltage sources, dependent voltage sources, capacitors, resistors, inductors, dependent current sources, and independent current sources as follows:

$$\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\xi}_{V_\mathrm{i}} \\ \boldsymbol{\xi}_{V_\mathrm{d}} \\ \boldsymbol{\xi}_{C} \\ \boldsymbol{\xi}_{R} \\ \boldsymbol{\xi}_{L} \\ \boldsymbol{\xi}_{I_\mathrm{d}} \\ \boldsymbol{\xi}_{I_\mathrm{i}} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}_{V_\mathrm{i}} \\ \boldsymbol{\eta}_{V_\mathrm{d}} \\ \boldsymbol{\eta}_{C} \\ \boldsymbol{\eta}_{R} \\ \boldsymbol{\eta}_{L} \\ \boldsymbol{\eta}_{I_\mathrm{d}} \\ \boldsymbol{\eta}_{I_\mathrm{i}} \end{pmatrix}.$$

The vector of node potentials except a reference node is denoted by $\boldsymbol{\zeta}$. Instead of the circuit equations, consider the following DAE which consists of KCL, constitutive equations, and relations between voltages and node potentials:

$$\left(\begin{array}{ccccccc|cccccccc|c}
\multicolumn{7}{c|}{\Psi} & \multicolumn{8}{c|}{O} & O \\ \hline
O & I & O & O & \multicolumn{3}{|c|}{} & \multicolumn{5}{c|}{} & O & O & O & \\
O & O & I & O & \multicolumn{3}{|c|}{} & \multicolumn{5}{c|}{} & O & O & O & \\
O & O & O & I & \multicolumn{3}{|c|}{Z(s)} & \multicolumn{5}{c|}{Y(s)} & O & O & O & O \\
O & O & O & O & \multicolumn{3}{|c|}{} & \multicolumn{5}{c|}{} & I & O & O & \\
O & O & O & O & \multicolumn{3}{|c|}{} & \multicolumn{5}{c|}{} & O & I & O & \\ \hline
O & O & O & O & O & O & O & I & O & O & O & O & O & O & O & O \\
O & O & O & O & O & O & I & O & O & O & O & O & O & O & O & O \\ \hline
\multicolumn{7}{c|}{O} & \multicolumn{8}{c|}{I} & *
\end{array}\right)
\left(\begin{array}{c} \boldsymbol{\xi}_{V_\mathrm{i}} \\ \boldsymbol{\xi}_{I_\mathrm{d}} \\ \boldsymbol{\xi}_{C} \\ \boldsymbol{\xi}_{R} \\ \boldsymbol{\xi}_{L} \\ \boldsymbol{\xi}_{V_\mathrm{d}} \\ \boldsymbol{\xi}_{I_\mathrm{i}} \\ \hline \boldsymbol{\eta}_{V_\mathrm{i}} \\ \boldsymbol{\eta}_{I_\mathrm{d}} \\ \boldsymbol{\eta}_{C} \\ \boldsymbol{\eta}_{R} \\ \boldsymbol{\eta}_{L} \\ \boldsymbol{\eta}_{V_\mathrm{d}} \\ \boldsymbol{\eta}_{I_\mathrm{i}} \\ \hline \boldsymbol{\zeta} \end{array}\right)
= \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{g}(s) \\ \boldsymbol{h}(s) \\ \mathbf{0} \end{pmatrix}, \tag{12}$$

where $*$ designates a constant matrix, and $Y(s)$ and $Z(s)$ are matrix pencils. We denote the coefficient matrix of (12) by $\tilde{A}(s)$ with row set $\tilde{P}$ and column set $\tilde{Q}$. The column set of $\tilde{A}(s)$ corresponding to $\boldsymbol{\zeta}$ is denoted by Q_N. We denote the column sets of $\tilde{A}(s)$ corresponding to the current variables for inductors and independent/dependent voltage sources by I_L and I_V, respectively. The row sets corresponding to constitutive equations of inductors and independent/dependent voltage sources are denoted by S_L and S_V. Recall that P_I and S denote the row set corresponding to KCL and constitutive equations. Similarly, P_N denotes the row set corresponding to relations between voltages and node potentials. Let us assume that $\Psi[P_I, Q_I^T] = I$ for the sake of simplicity, where T is a reference tree in the hybrid analysis.

Let us denote $\tilde{P}_* = \tilde{P} \setminus (P_I \cup S_L \cup S_V)$ and $\tilde{Q}_* = \tilde{Q} \setminus (I_L \cup I_V \cup Q_N)$. In MNA, we transform $\tilde{A}(s)$ by (4) with $B = \tilde{A}[\tilde{P}_*, \tilde{Q}_*]$. The rest of the procedure of MNA is similar to that of the hybrid analysis. We call the obtained equations corresponding to (8) the *MNA equations*.

Recall that an L-I cutset is a cutset consisting of inductors and/or current sources only, and a C-V loop is a cycle consisting of capacitors and voltage sources only. For nonlinear time-varying electric circuits, the index of the MNA equations has the following property.

Theorem 5.1 ([25, Theorem 4.1]). *For nonlinear time-varying RLC circuits, MNA leads to* *a DAE with index one if and only if the network contains neither L-I cutsets nor C-V loops. Otherwise, MNA leads to a DAE with index two.*

This theorem is generalized for nonlinear time-varying electric circuits containing dependent sources which satisfy certain conditions [25]. Theorem 5.1 implies that the index of the MNA equations is at least one. On the other hand, for RLC circuits, the index of the hybrid equations is at most one by Theorem 4.9. Hence, the index of the hybrid equations does not exceed that of the MNA equations for linear time-invariant RLC circuits. In the rest of this paper, we generalize this for linear time-invariant electric circuits which may contain dependent sources.

For any square submatrix $A[K, J]$, we write $w(K, J) = \deg\det A[K, J]$, where $w(\emptyset, \emptyset) = 0$ by convention. Then, $w(K, J)$ enjoys the following property.

Lemma 5.2 ([20, pp. 287–289])**.** *Let $A(s)$ be a matrix pencil with row set P and column set Q. For any $(K, J) \in \Lambda$, $(K', J') \in \Lambda$, where $\Lambda = \{(K, J) \mid |K| = |J|, K \subseteq P, J \subseteq Q\}$, both* (VB-1) *and* (VB-2) *below hold:*

(VB-1) *For any $k \in K \setminus K'$, at least one of the following two statements holds:*

- **(1a)** $\exists j \in J \setminus J' : w(K, J) + w(K', J') \leq w(K \setminus \{k\}, J \setminus \{j\}) + w(K' \cup \{k\}, J' \cup \{j\})$,
- **(1b)** $\exists h \in K' \setminus K : w(K, J) + w(K', J') \leq w(K \setminus \{k\} \cup \{h\}, J) + w(K' \setminus \{h\} \cup \{k\}, J')$.

(VB-2) *For any $j \in J \setminus J'$, at least one of the following two statements holds:*

- **(2a)** $\exists k \in K \setminus K' : w(K, J) + w(K', J') \leq w(K \setminus \{k\}, J \setminus \{j\}) + w(K' \cup \{k\}, J' \cup \{j\})$,
- **(2b)** $\exists l \in J' \setminus J : w(K, J) + w(K', J') \leq w(K, J \setminus \{j\} \cup \{l\}) + w(K', J' \setminus \{l\} \cup \{j\})$.

We denote $\tilde{d}_{\tilde{p}\tilde{q}} = \deg\det \tilde{A}[\tilde{P} \setminus \{\tilde{p}\}, \tilde{Q} \setminus \{\tilde{q}\}]$ for $\tilde{p} \in \tilde{P}$ and $\tilde{q} \in \tilde{Q}$. Similarly to Lemma 4.1, the index $\tilde{\nu}$ of the MNA equations is given by

$$\tilde{\nu} = \max_{\tilde{p},\tilde{q}}\{\tilde{d}_{\tilde{p}\tilde{q}} \mid \tilde{p} \in \tilde{P} \setminus \tilde{P}_*, \tilde{q} \in \tilde{Q} \setminus \tilde{Q}_*\} - \deg\det \tilde{A} + 1. \tag{13}$$

In order to compare $\tilde{\nu}$ with $\nu(D)$, we show the relation between $\tilde{A}(s)$ and $A_T(s)$ in the following lemma.

Lemma 5.3. *For the matrix pencils $\tilde{A}(s)$ and $A_T(s)$, it holds that $\det \tilde{A}(s) = \pm \det A_T(s)$. Moreover, we have $d_{pq} = \tilde{d}_{pq}$ for $p \in P \setminus P_*$ and $q \in Q \setminus Q_*$.*

Proof. Let P_V denote the column set such that $P_V \subseteq P_N$ and $\tilde{A}[P_V, Q_V^{\overline{T}}] = I$. We transform $\tilde{A}(s)$ into

$$\tilde{A}'(s) = \begin{array}{c} P_I \\ S \\ P_V \\ P_N \setminus P_V \end{array} \left(\begin{array}{cc|cc|c} \multicolumn{2}{c|}{\Psi} & O & O & O \\ \hline \multicolumn{4}{c|}{\text{constitutive eq.}} & O \\ \hline O & O & * & I & O \\ O & O & * & O & I \end{array} \right)$$

by elementary row operations on P_N without adding multiples of rows in P_V to any rows. Then P_V corresponds to KVL. Since it holds that $\tilde{A}'[\tilde{P} \setminus (P_N \setminus P_V), \tilde{Q} \setminus Q_N] = A_T$, we have $\det \tilde{A}(s) = \pm \det \tilde{A}'(s) = \pm \det A_T(s)$. This transformation does not change the value of $\tilde{d}_{pq}$ for any $p \in P_I \cup P_V$ and $q \in \tilde{Q} \setminus Q_N$. Hence we have $\tilde{d}_{pq} = \deg\det \tilde{A}'[\tilde{P} \setminus \{p\}, \tilde{Q} \setminus \{q\}] = d_{pq}$ for $p \in P \setminus P_*$ and $q \in Q \setminus Q_*$.

□

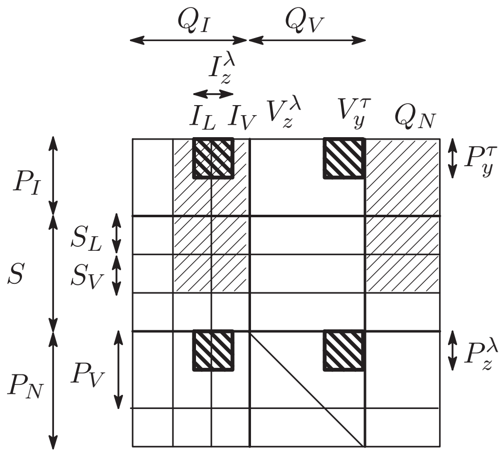

Figure 7: The coefficient matrix $\tilde{A}(s)$ of MNA equations.

In order to prove $\nu(D) \leq \tilde{\nu}$, we make use of the following lemmas.

Lemma 5.4. *For any $p \in P_I \cup S_L \cup S_V$ and $q \in V_y^\tau$, there exists $\tilde{q} \in Q_N$ such that $\tilde{d}_{pq} \leq \tilde{d}_{p\tilde{q}}$.*

Proof. We apply (VB-2) in Lemma 5.2 to (P_N, Q_V) and $(\tilde{P} \setminus \{p\}, \tilde{Q} \setminus \{q\})$ with $j = q$. Since $P_N \subseteq \tilde{P} \setminus \{p\}$, (2a) does not hold. Then there exists $\tilde{q} \in \tilde{Q} \setminus Q_V$ such that

$$\deg\det \tilde{A}[P_N, Q_V] + \tilde{d}_{pq} \leq \deg\det \tilde{A}[P_N, Q_V \setminus \{q\} \cup \{\tilde{q}\}] + \tilde{d}_{p\tilde{q}}.$$

If $\tilde{d}_{pq} > -\infty$, it follows from $\tilde{A}[P_N, Q_I] = O$ that $\tilde{q} \in Q_N$. Since $\tilde{A}[P_N, Q_V \cup Q_N]$ is a constant matrix, we have $\deg\det \tilde{A}[P_N, Q_V] = \deg\det \tilde{A}[P_N, Q_V \setminus \{q\} \cup \{\tilde{q}\}] = 0$, and hence $\tilde{d}_{pq} \leq \tilde{d}_{p\tilde{q}}$. □

Lemma 5.5. *For any $p \in P_z^\lambda$ and $q \in I_z^\lambda \cup V_y^\tau$, there exists $\tilde{p} \in S_L \cup S_V$ such that $\tilde{d}_{pq} \leq \tilde{d}_{\tilde{p}q}$.*

Proof. We apply (VB-1) in Lemma 5.2 to $(P_z^\lambda, V_z^\lambda)$ and $(\tilde{P} \setminus \{p\}, \tilde{Q} \setminus \{q\})$ with $k = p$. Since $V_z^\lambda \subseteq \tilde{Q} \setminus \{q\}$, (1a) does not hold. Then there exists $\tilde{p} \in \tilde{P} \setminus P_z^\lambda$ such that

$$\deg\det \tilde{A}[P_z^\lambda, V_z^\lambda] + \tilde{d}_{pq} \leq \deg\det \tilde{A}[P_z^\lambda \setminus \{p\} \cup \{\tilde{p}\}, V_z^\lambda] + \tilde{d}_{\tilde{p}q}.$$

If $\tilde{d}_{pq} > -\infty$, it follows from $\tilde{A}[\tilde{P} \setminus (S_L \cup S_V \cup P_z^\lambda), V_z^\lambda] = O$ that $\tilde{p} \in S_L \cup S_V$. Since $\deg\det \tilde{A}[P_z^\lambda, V_z^\lambda] = \deg\det \tilde{A}[P_z^\lambda \setminus \{p\} \cup \{\tilde{p}\}, V_z^\lambda] = 0$ holds, we have $\tilde{d}_{pq} \leq \tilde{d}_{\tilde{p}q}$. □

We are now ready to prove the following theorem.

Theorem 5.6. *For linear time-invariant electric circuits which may contain dependent sources, let (E_y, E_z) be an admissible partition such that E_y includes all resistors. Then, the index $\nu(D)$ of the hybrid equation with (E_y, E_z) never exceeds the index $\tilde{\nu}$ of the MNA equations.*

Proof. By (9), (13), and Lemma 5.3, it suffices to show that

$$\max_{p,q}\{\tilde{d}_{pq} \mid p \in P \setminus P_*, q \in Q \setminus Q_*\} \leq \max_{\tilde{p},\tilde{q}}\{\tilde{d}_{\tilde{p}\tilde{q}} \mid \tilde{p} \in \tilde{P} \setminus \tilde{P}_*, \tilde{q} \in \tilde{Q} \setminus \tilde{Q}_*\}. \tag{14}$$

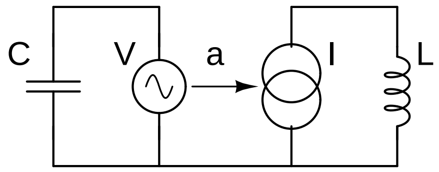

Figure 8: A linear circuit that has hybrid equations with index two.

In Figure 7, the dark shadow region represents $\tilde{A}[P \setminus P_*, Q \setminus Q_*]$, while the light shadow region represents $\tilde{A}[\tilde{P} \setminus \tilde{P}_*, \tilde{Q} \setminus \tilde{Q}_*]$. Recall that $P \setminus P_* = P_y^\tau \cup P_z^\lambda$, $Q \setminus Q_* = I_z^\lambda \cup V_y^\tau$ and $\tilde{P} \setminus \tilde{P}_* = P_I \cup S_L \cup S_V$, $\tilde{Q} \setminus \tilde{Q}_* = I_L \cup I_V \cup Q_N$.

Since $P_y^\tau \subseteq P_I$ and $I_z^\lambda \subseteq I_L \cup I_V$, we have

$$\begin{aligned}\max\{\tilde{d}_{pq} \mid p \in P_y^\tau, q \in I_z^\lambda\} &\leq \max\{\tilde{d}_{\tilde{p}\tilde{q}} \mid \tilde{p} \in P_I, \tilde{q} \in I_L \cup I_V\} \\ &\leq \max\{\tilde{d}_{\tilde{p}\tilde{q}} \mid \tilde{p} \in \tilde{P} \setminus \tilde{P}_*, \tilde{q} \in \tilde{Q} \setminus \tilde{Q}_*\}. \end{aligned} \tag{15}$$

Setting $p \in P_y^\tau$ and $q \in V_y^\tau$ in Lemma 5.4, we have

$$\begin{aligned}\max\{\tilde{d}_{pq} \mid p \in P_y^\tau, q \in V_y^\tau\} &\leq \max\{\tilde{d}_{p\tilde{q}} \mid p \in P_y^\tau, \tilde{q} \in Q_N\} \\ &\leq \max\{\tilde{d}_{\tilde{p}\tilde{q}} \mid \tilde{p} \in \tilde{P} \setminus \tilde{P}_*, \tilde{q} \in \tilde{Q} \setminus \tilde{Q}_*\}. \end{aligned} \tag{16}$$

Setting $p \in P_z^\lambda$ and $q \in I_z^\lambda$ in Lemma 5.5, we have

$$\begin{aligned}\max\{\tilde{d}_{pq} \mid p \in P_z^\lambda, q \in I_z^\lambda\} &\leq \max\{\tilde{d}_{\tilde{p}q} \mid \tilde{p} \in S_L \cup S_V, q \in I_z^\lambda\} \\ &\leq \max\{\tilde{d}_{\tilde{p}\tilde{q}} \mid \tilde{p} \in \tilde{P} \setminus \tilde{P}_*, \tilde{q} \in \tilde{Q} \setminus \tilde{Q}_*\}. \end{aligned} \tag{17}$$

Furthermore, it follows from Lemmas 5.4 and 5.5 that

$$\begin{aligned}\max\{\tilde{d}_{pq} \mid p \in P_z^\lambda, q \in V_y^\tau\} &\leq \max\{\tilde{d}_{\tilde{p}q} \mid \tilde{p} \in S_L \cup S_V, q \in V_y^\tau\} \\ &\leq \max\{\tilde{d}_{\tilde{p}q} \mid \tilde{p} \in \tilde{P} \setminus \tilde{P}_*, q \in V_y^\tau\} \\ &\leq \max\{\tilde{d}_{\tilde{p}\tilde{q}} \mid \tilde{p} \in \tilde{P} \setminus \tilde{P}_*, \tilde{q} \in Q_N\} \\ &\leq \max\{\tilde{d}_{\tilde{p}\tilde{q}} \mid \tilde{p} \in \tilde{P} \setminus \tilde{P}_*, \tilde{q} \in \tilde{Q} \setminus \tilde{Q}_*\}. \end{aligned} \tag{18}$$

By (15)–(18), we obtain (14). □

Theorem 5.6 implies that the hybrid analysis with minimum index by no means results in a higher index DAE than MNA.

Example 5.7 (Electric circuit with index two [12])**.** Consider a circuit depicted in Figure 8, which contains a dependent current source I. While MNA results in a DAE with index three [12], the hybrid analysis with admissible partition

$$E_g = \{V\}, \quad E_h = \emptyset, \quad E_y = \{C, I\}, \quad E_z = \{L\}$$

results in a DAE with index two [16], which is lower than MNA.

6 Conclusion

For linear time-invariant RLC circuits, we have proved that the index of the hybrid equations never exceeds one, and given a structural characterization of circuits with index zero. The proof makes use of the special property of RLC circuits that the coefficient matrix of constitutive equations consists of diagonal matrix pencils. Moreover, the structural characterization has brought an index minimization algorithm, which is much faster and simpler than the previous one developed in [16]. Finally, for linear time-invariant electric circuits which may contain dependent sources, we have shown that the minimum index of hybrid equations does not exceed the index of MNA equations, which suggests that the hybrid analysis is superior to MNA in numerical accuracy. Extending these results to nonlinear/time-varying electric circuits is left for future investigation.

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