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# A Numerical Algorithm for Block-Diagonal Decomposition of Matrix $*$-Algebras with General Irreducible Components

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**Abstract**

An algorithm is proposed for finding the finest simultaneous block-diagonalization of a finite number of square matrices, or equivalently the irreducible decomposition of a matrix $*$-algebra given in terms of its generators. This is an extension of the algorithm of Murota–Kanno–Kojima–Kojima, which is targeted to a special case of the problem. The algorithm, composed of numerical-linear algebraic computations, does not require any algebraic structure to be known in advance. The main ingredient of the algorithm is the Schur decomposition and its skew-Hamiltonian variant for eigenvalue computation.

**Keywords**: matrix $*$-algebra, block-diagonalization, group symmetry, Schur decomposition, skew-Hamiltonian Schur decomposition,

**AMS Classifications**: 15A21, 65F15, 68W20, 90C22

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## 1 Introduction

In this paper we consider the following problem: Given a finite set of $n \times n$ real matrices $A_1, \ldots, A_N$, find an $n \times n$ orthogonal matrix $P$ that provides them with a simultaneous block-diagonal decomposition, i.e., such that $P^\top A_1 P, \ldots, P^\top A_N P$ become block-diagonal matrices with a common block-diagonal structure. For this problem two different but closely related theoretical frameworks are available. One is group representation theory [11, 14] and the other matrix $*$-algebra [15]. They are not only necessary to answer the fundamental theoretical question about such block-diagonal decomposition but also useful in its actual computation.

In the literature of semidefinite programming, the above problem has recently been studied quite intensively with its application to efficient solution of semidefinite programs (SDPs) with group symmetry, where $A_1, \ldots, A_N$ are symmetric matrices representing the objective function and the constraints of an SDP. If $A_1, \ldots, A_N$ are block-diagonalized, the associated SDP is decomposed accordingly into smaller SDPs, and therefore can be solved efficiently. Kanno, Ohsaki, Murota and Katoh [9] introduced a class of group symmetric SDPs, which arise from topology optimization problems of trusses. Gatermann and Parrilo [8] investigated the problem of minimizing a group symmetric polynomial. They proposed to reduce the size of SOS and SDP relaxations for the problem by exploiting the group symmetry and decomposing the SDP. On the other hand, de Klerk, Pasechnik and Schrijver [5] applied the theory of matrix $*$-algebra to reduce the size of a class of group symmetric SDPs. Very recently, de Klerk and Sotirov [6] dealt with quadratic assignment problems, and showed how to exploit the group symmetry to reduce the size of the SDP relaxations.

As for the block-diagonal decomposition itself, Murota, Kanno, Kojima and Kojima [12] have proposed an algorithm that is composed solely of numerical linear-algebraic computations such as eigenvalue computations. Their idea is to consider the matrix $*$-algebra $\mathcal{T}$ generated by the given symmetric matrices $A_1, \ldots, A_N$ and to make use of the standard structure theorem (see Theorem 2.1). Then the finest block-diagonal decomposition corresponds to the decomposition of $\mathcal{T}$ into irreducible components. Though under a restrictive assumption that each irreducible component of $\mathcal{T}$ is isomorphic to a full matrix algebra of some order, the algorithm of [12] successfully constructs an eligible orthogonal matrix $P$ for a given family of symmetric matrices $A_1, \ldots, A_N$. The key algorithmic observation is that the decomposition into simple components can be computed from the eigenvalue decomposition of a single generic element of $\mathcal{T}$.

This fact was also observed earlier by Eberly and Giesbrecht [7] in designing an algorithm for the simple-component decomposition of a separable matrix algebra (not a $*$-algebra) over an arbitrary infinite field. To be specific, the "self-centralizing element" in [7] corresponds to the "generic element" in [12]. Treating a general matrix algebra, however, they considered a transformation of the form $S^{-1}AS$ with a nonsingular matrix $S$ instead of an orthogonal transformation of the form $P^\top AP$, and they used companion forms and factorization of minimum polynomials instead of eigenvalue decomposition.

The objective of this paper is to extend the algorithm of [12] to cope with all possible types of irreducible components. According to the structure theorem, an irreducible component of a matrix $*$-algebra $\mathcal{T}$ is isomorphic to an algebra of the following three types: a full matrix algebra over the field of real numbers (Case $\mathbb{R}$), a faithful real $*$-representation of a full matrix algebra over the field of complex numbers (Case $\mathbb{C}$) and a faithful real $*$-representation of a full matrix algebra over the (noncommutative) field of quaternion numbers (Case $\mathbb{H}$); see Section 2 for details. The algorithm of [12] is targeted exclusively to Case $\mathbb{R}$, while the other two cases, Case $\mathbb{C}$ and Case $\mathbb{H}$, do occur even when $\mathcal{T}$ is generated by symmetric matrices. In engineering applications, Case $\mathbb{C}$ occurs, for example, in the stiffness matrix of a cyclically-symmetric truss dome such as the Schwedler dome; see [13] for various domes.

The proposed algorithm consists of two stages. One is for the simple-component decomposition and the other for the irreducible-component decomposition. The notion of genericity in [12] is refined here to S-genericity for the simple-component decomposition and I-genericity for the irreducible-component decomposition. Our simple-component decomposition algorithm is essentially the same as that of [12]. As for the irreducible-component decomposition, the algorithm of [12] based on the diagonalization of a generic element of $\mathcal{T}$ works also in our Case $\mathbb{R}$, although some minor modifications are needed. New algorithms are devised for Case $\mathbb{C}$ and Case $\mathbb{H}$. We resort to the Schur decomposition for Case $\mathbb{C}$ and the Schur decomposition and its skew-Hamiltonian variant for Case $\mathbb{H}$.

Our algorithm, similarly to [12], finds the finest block-diagonalization without knowing any algebraic structure, such as group symmetry, in advance and is based on purely linear algebraic computations such as eigenvalue computation. In other words, our algorithm will automatically exploit the underlying algebraic structure, which is often an outcome of physical or geometrical symmetry, sparsity, and structural or numerical degeneracy in the given matrices.

This paper is organized as follows. Section 2 describes the theoretical background of our algorithm based on matrix $*$-algebra. The proposed algorithm for the simple-component decomposition is given in Section 3, and that for the irreducible-component decomposition in Section 4. The algorithm is demonstrated in Section 5. Section 6 concludes the paper.

## 2 Matrix $*$-Algebras

Let $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ be the real number field, the complex field, and the quaternion field, respectively. The quaternion field $\mathbb{H}$ is a vector space $\{a + \imath b + \jmath c + kd : a, b, c, d \in \mathbb{R}\}$ over $\mathbb{R}$ with basis $1, \imath, \jmath$ and $k$, equipped with the multiplication defined as follows:

$$\imath = \jmath k = -k\jmath, \quad \jmath = k\imath = -\imath k, \quad k = \imath\jmath = -\jmath\imath, \quad \imath^2 = \jmath^2 = k^2 = -1$$

and for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $x, y, u, v \in \mathbb{H}$,

$$(\alpha x + \beta y)(\gamma u + \delta v) = \alpha\gamma xu + \alpha\delta xv + \beta\gamma yu + \beta\delta yv.$$

For a quaternion $h = a+\imath b+\jmath c+kd$, its conjugate is defined as $\bar{h} = a-\imath b-\jmath c-kd$, and the norm of $h$ is defined as $|h| = \sqrt{h\bar{h}} = \sqrt{\bar{h}h} = \sqrt{a^2 + b^2 + c^2 + d^2}$. We can consider $\mathbb{C}$ as a subset of $\mathbb{H}$ by identifying the generator $\imath$ of the quaternion field $\mathbb{H}$ with the imaginary unit of the complex field $\mathbb{C}$.

Let $\mathcal{M}_n$ denote the set of $n \times n$ matrices over $\mathbb{R}$. A subset $\mathcal{T}$ of $\mathcal{M}_n$ is said to be a $*$-subalgebra (or a matrix $*$-algebra) over $\mathbb{R}$ if $I_n \in \mathcal{T}$ and

$$A, B \in \mathcal{T}; \alpha, \beta \in \mathbb{R} \implies \alpha A + \beta B, AB, A^\top \in \mathcal{T}.$$

Obviously, $\mathcal{M}_n$ itself is a matrix $*$-algebra. There are two other basic matrix $*$-algebras: the real representation of complex matrices $\mathcal{C}_n \subset \mathcal{M}_{2n}$ defined by

$$\mathcal{C}_n = \left\{ \begin{bmatrix} C(z_{11}) & \cdots & C(z_{1n}) \\ \vdots & \ddots & \vdots \\ C(z_{n1}) & \cdots & C(z_{nn}) \end{bmatrix} : z_{11}, z_{12}, \ldots, z_{nn} \in \mathbb{C} \right\}$$

with

$$C(a + \imath b) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

and the real representation of quaternion matrices $\mathcal{H}_n \subset \mathcal{M}_{4n}$ defined by

$$\mathcal{H}_n = \left\{ \begin{bmatrix} H(h_{11}) & \cdots & H(h_{1n}) \\ \vdots & \ddots & \vdots \\ H(h_{n1}) & \cdots & H(h_{nn}) \end{bmatrix} : h_{11}, h_{12}, \ldots, h_{nn} \in \mathbb{H} \right\}$$

with

$$H(a + \imath b + \jmath c + kd) = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

For two matrices $A$ and $B$, their direct sum, denoted as $A \oplus B$, is defined as

$$A \oplus B = \begin{bmatrix} A & O \\ O & B \end{bmatrix},$$

and their tensor product, denoted as $A \otimes B$, is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix},$$

where $A$ is assumed to be $n \times n$. Note that $A \otimes B = \Pi^\top (B \otimes A)\Pi$ for some permutation matrix $\Pi$.

We say that a matrix $*$-algebra $\mathcal{T}$ is simple if $\mathcal{T}$ has no ideal other than $\{O\}$ and $\mathcal{T}$ itself, where an ideal of $\mathcal{T}$ means a subalgebra $\mathcal{I}$ of $\mathcal{T}$ such that

$$A \in \mathcal{T}, B \in \mathcal{I} \implies AB \in \mathcal{I}.$$

A linear subspace $W$ of $\mathbb{R}^n$ is said to be invariant with respect to $\mathcal{T}$, or $\mathcal{T}$-invariant, if $AW \subseteq W$ for every $A \in \mathcal{T}$. We say that $\mathcal{T}$ is irreducible if no $\mathcal{T}$-invariant subspace other than $\{\mathbf{0}\}$ and $\mathbb{R}^n$ exists. It is mentioned that $\mathcal{M}_n$, $\mathcal{C}_n$ and $\mathcal{H}_n$ are typical examples of irreducible matrix $*$-algebras.

We say that matrix $*$-algebras $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic if there exists a bijection $\phi$ from $\mathcal{T}_1$ to $\mathcal{T}_2$ with the following properties:

$$\phi(\alpha A + \beta B) = \alpha\phi(A) + \beta\phi(B), \quad \phi(AB) = \phi(A)\phi(B), \quad \phi(A^\top) = \phi(A)^\top.$$

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic, we write $\mathcal{T}_1 \simeq \mathcal{T}_2$. For a matrix $*$-algebra $\mathcal{T}$ and an orthogonal matrix $P$, the set

$$P^\top \mathcal{T} P = \{P^\top AP : A \in \mathcal{T}\}$$

forms another matrix $*$-algebra isomorphic to $\mathcal{T}$. For a matrix $*$-algebra $\mathcal{T}'$, the set

$$\mathcal{T} = \{\mathrm{diag}\,(B, B, \dots, B) : B \in \mathcal{T}'\}$$

forms another matrix $*$-algebra isomorphic to $\mathcal{T}'$.

From a standard result of the theory of matrix $*$-algebra (e.g., [15, Chapter X], [10, Theorem 5.4]) we can see the following structure theorem.

**Theorem 2.1.** Let $\mathcal{T}$ be a $*$-subalgebra of $\mathcal{M}_n$.

(A) There exist an orthogonal matrix $\hat{Q} \in \mathcal{M}_n$ and simple $*$-subalgebras $\mathcal{T}_j$ of $\mathcal{M}_{\hat{n}_j}$ for some $\hat{n}_j$ $(j = 1, 2, \dots, \ell)$ such that

$$\hat{Q}^\top \mathcal{T} \hat{Q} = \{\mathrm{diag}\,(S_1, S_2, \dots, S_\ell) : S_j \in \mathcal{T}_j \ (j = 1, 2, \dots, \ell)\}.$$

(B) If $\mathcal{T}$ is simple, there exist an orthogonal matrix $P \in \mathcal{M}_n$ and an irreducible $*$-subalgebra $\mathcal{T}'$ of $\mathcal{M}_{\bar{n}}$ for some $\bar{n}$ such that

$$P^\top \mathcal{T} P = \{\mathrm{diag}\,(B, B, \dots, B) : B \in \mathcal{T}'\}.$$

(C) If $\mathcal{T}$ is irreducible, there exists an orthogonal matrix $P \in \mathcal{M}_n$ such that $P^\top \mathcal{T} P = \mathcal{M}_n, \mathcal{C}_{n/2}$ or $\mathcal{H}_{n/4}$. ■

It follows from the above theorem that, with a single orthogonal matrix $P$, all the matrices in $\mathcal{T}$ can be transformed simultaneously to a block-diagonal form as

$$P^\top AP = \bigoplus_{j=1}^{\ell} \bigoplus_{i=1}^{\bar{m}_j} B_j = \bigoplus_{j=1}^{\ell} (I_{\bar{m}_j} \otimes B_j) \tag{2.1}$$

with $B_j \in \mathcal{T}'_j$, where $\mathcal{T}'_j$ denotes the irreducible $*$-subalgebra corresponding to the simple subalgebra $\mathcal{T}_j$; we have $\mathcal{T}'_j = \mathcal{M}_{\bar{n}_j}, \mathcal{C}_{\bar{n}_j/2}$ or $\mathcal{H}_{\bar{n}_j/4}$ for some $\bar{n}_j$, where the structural indices $\ell$, $\bar{n}_j$, $\bar{m}_j$ and the algebraic structure of $\mathcal{T}'_j$ for $j = 1, \ldots, \ell$ are uniquely determined by $\mathcal{T}$. It may be noted that $\hat{n}_j$ in Theorem 2.1 (A) is equal to $\bar{m}_j \bar{n}_j$ in the present notation. Conversely, for any choice of $B_j \in \mathcal{T}'_j$ for $j = 1, \ldots, \ell$, the matrix of (2.1) belongs to $P^\top \mathcal{T} P$.

We denote by

$$\mathbb{R}^n = \bigoplus_{j=1}^{\ell} U_j \tag{2.2}$$

the decomposition of $\mathbb{R}^n$ that corresponds to the simple components. In other words, $U_j = \mathrm{Im}(\hat{Q}_j)$ for the $n \times \hat{n}_j$ submatrix $\hat{Q}_j$ of $\hat{Q}$ that corresponds to $\mathcal{T}_j$ in Theorem 2.1 (A). Although the matrix $\hat{Q}$ is not unique, the subspace $U_j$ is determined uniquely and $\dim U_j = \hat{n}_j = \bar{m}_j \bar{n}_j$ for $j = 1, \ldots, \ell$.

## 3 Decomposition into Simple Components

An algorithm for the decomposition into simple components has been proposed by Murota et al. [12] for the special case where $\mathcal{T}$ is generated by symmetric matrices. It turns out that this algorithm also works in our general case.

The idea of [12] is that the decomposition into simple components can be computed from the eigenvalue decomposition of a single matrix $A$ if it is free from degeneracy in eigenvalues. To extend this idea to our general case, it is convenient to make a refinement of the notion of nondegeneracy. Let us say that $A \in \mathcal{T}$ is S-generic (generic in eigenvalue structure with respect to simple components) if all the matrices $B_1, \ldots, B_\ell$ appearing in the decomposition (2.1) of $A$ does not share a common eigenvalue. It is emphasized that each $B_j$ is allowed to have multiple eigenvalues. Note that the S-genericity of $A$ does not depend on the choice of $P$ in (2.1), although the matrices $B_1, \ldots, B_\ell$ themselves vary with $P$. By the structure theorem, there exists a symmetric S-generic matrix $A$ in $\mathcal{T}$. To be specific, choose distinct $\alpha_j$ and set $B_j = \alpha_j I_{\bar{n}_j}$ for $j = 1, \ldots, \ell$ in (2.1).

Let $A$ be a symmetric S-generic matrix in $\mathcal{T}$. Let $\alpha_1, \ldots, \alpha_k$ be the distinct (real) eigenvalues of $A$ with multiplicities denoted as $m_1, \ldots, m_k$, and $Q = [Q_1, \ldots, Q_k]$ be an orthogonal matrix consisting of the eigenvectors, where $Q_i$ is an $n \times m_i$ matrix for $i = 1, \ldots, k$. Then we have

$$Q^\top A Q = \mathrm{diag}\,(\alpha_1 I_{m_1}, \ldots, \alpha_k I_{m_k}) = \begin{array}{|c|c|c|c|}\hline \alpha_1 I_{m_1} & O & O & O \\ \hline O & \alpha_2 I_{m_2} & O & O \\ \hline O & O & \ddots & O \\ \hline O & O & O & \alpha_k I_{m_k} \\ \hline \end{array}\,. \tag{3.3}$$

Let $K = \{1, \ldots, k\}$ and $V_i = \mathrm{Im}(Q_i)$, which is the eigenspace for the eigenvalue $\alpha_i$, where $i = 1, \ldots, k$. Each eigenvalue $\alpha_i$ of $A$ is an eigenvalue of $B_j$ for some (uniquely determined) $j$ and the multiplicity of $\alpha_i$ in $A$ is equal to $\bar{m}_j$ times the multiplicity of $\alpha_i$ in $B_j$.

The eigenvalue decomposition of a symmetric S-generic matrix $A$ is consistent with the decomposition into simple components of $\mathcal{T}$ as follows.

**Proposition 3.1.** Let $A \in \mathcal{T}$ be symmetric and S-generic. For each $i \in \{1, \ldots, k\}$, there exists $j \in \{1, \ldots, \ell\}$ such that $V_i \subseteq U_j$. Hence there exists a partition of $K = \{1, \ldots, k\}$ into $\ell$ disjoint subsets:

$$K = K_1 \cup \cdots \cup K_\ell \tag{3.4}$$

such that

$$U_j = \bigoplus_{i \in K_j} V_i, \qquad j = 1, \ldots, \ell. \tag{3.5}$$

■

The partition (3.4) of $K$ can be determined as follows. Let $\sim$ be the equivalence relation on $K$ defined as the symmetric and transitive closure of the binary relation:

$$i \sim i' \quad \Longleftrightarrow \quad \exists p\ (1 \leq p \leq m) : Q_i^\top A_p Q_{i'} \neq O, \tag{3.6}$$

where $i \sim i$ for all $i \in K$ by convention.

**Proposition 3.2.** The partition (3.4) coincides with the partition of $K$ into equivalence classes induced by $\sim$.

*Proof.* This is not difficult to see from the general theory of matrix $*$-algebra, but a proof is given here for completeness. It is also mentioned that the proof below is almost identical with the proof of Proposition 3.3 in [12]. Denote by $\{L_1, \ldots, L_{\ell'}\}$ the equivalence classes induced from $\sim$.

If $i \sim i'$, then $Q_i^\top A_p Q_{i'} \neq O$ for some $p$. This means that for any $I \subseteq K$ with $i \in I$ and $i' \in K \setminus I$, the subspace $\bigoplus_{i'' \in I} V_{i''}$ is not invariant under $A_p$ or $A_p^\top$. Hence $V_{i'}$ must be contained in the same simple component as $V_i$. Therefore each $L_j$ must be contained in some $K_{j'}$.

To show the converse, define a matrix $\tilde{Q}_j = (Q_i \mid i \in L_j)$, which is of size $n \times \sum_{i \in L_j} m_i$, and an $n \times n$ matrix $E_j = \tilde{Q}_j \tilde{Q}_j^\top$ for $j = 1, \ldots, \ell'$.

Each matrix $E_j$ belongs to $\mathcal{T}$, as shown below, and it is idempotent (i.e., ${E_j}^2 = E_j$) and $E_1 + \cdots + E_{\ell'} = I_n$. On the other hand, for distinct $j$ and $j'$ we have $\tilde{Q}_j^\top A_p \tilde{Q}_{j'} = O$ for all $p$, and hence $\tilde{Q}_j^\top M \tilde{Q}_{j'} = O$ for all $M \in \mathcal{T}$. This implies that $E_j M = M E_j$ for all $M \in \mathcal{T}$. Therefore $\mathrm{Im}(E_j)$ is a union of simple components, and hence $L_j$ is a union of some $K_{j'}$'s.

It remains to show that $E_j \in \mathcal{T}$. Since $\alpha_i$'s are distinct, for any real numbers $u_1, \ldots, u_k$ there exists a polynomial $f$ such that $f(\alpha_i) = u_i$ for $i = 1, \ldots, k$. Let $f_j$ be such $f$ for $(u_1, \ldots, u_k)$ defined as $u_i = 1$ for $i \in L_j$ and $u_i = 0$ for $i \in K \setminus L_j$. Then $E_j = \tilde{Q}_j \tilde{Q}_j^\top = Q \cdot f_j(\mathrm{diag}\,(\alpha_1 I_{m_1}, \ldots, \alpha_k I_{m_k})) \cdot Q^\top = Q \cdot f_j(Q^\top A Q) \cdot Q^\top = f_j(A)$. This shows $E_j \in \mathcal{T}$. □

A symmetric S-generic matrix $A$ can be obtained from a random linear combination of generators, as follows. For a real vector $r = (r_1, \ldots, r_N)$ put

$$A(r) = r_1 A_1 + \cdots + r_N A_N.$$

We denote by $\mathrm{span}\{\cdots\}$ the set of linear combinations of the matrices in the braces.

**Proposition 3.3.** If $\mathrm{span}\{I_n, A_1, \ldots, A_N\} = \mathcal{T}$, there exists an open dense subset $R$ of $\mathbb{R}^N$ such that $A(r)^\top + A(r)$ is S-generic for every $r \in R$.

*Proof.* Let $B_{pj}$ denote the matrix $B_j$ in the decomposition (2.1) of $A = A_p^\top + A_p$ for $p = 1, \ldots, N$. For $j = 1, \ldots, \ell$ define $f_j(\lambda) = f_j(\lambda; r) = \det(\lambda I - (r_1 B_{1j} + \cdots + r_N B_{Nj}))$, which is a polynomial in $\lambda, r_1, \ldots, r_N$.

The matrix $A(r)^\top + A(r)$ is S-generic if and only if $f_j(\lambda)$ and $f_{j'}(\lambda)$ with $j \neq j'$ have no common root, and the latter condition is equivalent to the resultant of $f_j(\lambda)$ and $f_{j'}(\lambda)$ being nonzero. Each resultant is a nonzero polynomial in $r_1, \ldots, r_N$, since $\mathcal{T}$ has at least one symmetric S-generic matrix $A$, which can be represented as $A = r_0 I_n + A(r)^\top + A(r)$ for some $r_0 \in \mathbb{R}$ and $r \in \mathbb{R}^N$ by the assumption on the linear span of the generators. Letting $\Sigma_{jj'}$ be the zero set of the resultant of $f_j(\lambda)$ and $f_{j'}(\lambda)$, we can take $R = \mathbb{R}^N \setminus \left(\cup_{j \neq j'} \Sigma_{jj'}\right)$. □

# 4 Decomposition into Irreducible Components

Each simple component $\mathcal{T}_j$ is to be decomposed further into irreducible components. In this section, we focus on a single $\mathcal{T}_j$ and omit the subscript $j$ for notational simplicity. In other words, we assume that $\mathcal{T}$ is a simple $*$-algebra of $n \times n$ matrices.

By the structure theorem, we have three cases with some $\bar{m}$ and $\check{n}$:

$$\text{Case } \mathbb{R}: \quad P^\top \mathcal{T} P = \{B \otimes I_{\bar{m}} : B \in \mathcal{M}_{\check{n}}\} \quad (n = \check{n}\bar{m}), \tag{4.7}$$

$$\text{Case } \mathbb{C}: \quad P^\top \mathcal{T} P = \{B \otimes I_{\bar{m}} : B \in \mathcal{C}_{\check{n}}\} \quad (n = 2\check{n}\bar{m}), \tag{4.8}$$

$$\text{Case } \mathbb{H}: \quad P^\top \mathcal{T} P = \{B \otimes I_{\bar{m}} : B \in \mathcal{H}_{\check{n}}\} \quad (n = 4\check{n}\bar{m}). \tag{4.9}$$

In other words, for each $A \in \mathcal{T}$, there exists $B$ such that

$$P^\top A P = B \otimes I_{\bar{m}}, \tag{4.10}$$

where $B \in \mathcal{M}_{\check{n}}$ in Case $\mathbb{R}$, $B \in \mathcal{C}_{\check{n}}$ in Case $\mathbb{C}$ and $B \in \mathcal{H}_{\check{n}}$ in Case $\mathbb{H}$. Conversely, in Case $\mathbb{R}$, for every $B \in \mathcal{M}_{\check{n}}$, there exists $A \in \mathcal{T}$ such that (4.10) is true, and similarly in Case $\mathbb{C}$ and Case $\mathbb{H}$. Note that in the structure theorem we have $I_{\bar{m}} \otimes B$ in contrast to $B \otimes I_{\bar{m}}$ here, which is more convenient for subsequent arguments. Note that $I_{\bar{m}} \otimes B$ and $B \otimes I_{\bar{m}}$ are connected by a permutation, as explained in Section 2.

### 4.1 I-generic matrix

Our algorithm for the decomposition into irreducible components makes full use of the eigenvalue structures of $\mathcal{M}_{\check{n}}$, $\mathcal{C}_{\check{n}}$ and $\mathcal{H}_{\check{n}}$, which have the following characteristics. The eigenvalues of a matrix in $\mathcal{M}_{\check{n}}$ consist of a number of reals and pairs of complex conjugates, both possibly with multiplicities. The eigenvalues of a matrix in $\mathcal{C}_{\check{n}}$ consist of pairs of complex conjugates, which implies in particular that the multiplicity of a real eigenvalue is even. The eigenvalues of a matrix in $\mathcal{H}_{\check{n}}$ consist of pairs of complex conjugates appearing twice, which implies in particular that the multiplicity of a real eigenvalue is even and at least four.

We introduce another kind of genericity notion as follows, where $\mathcal{T}$ is assumed to be a simple $*$-algebra. Let us say that $A$ is I-generic (generic in eigenvalue structure with respect to irreducible components) if the following is true for the matrix $B$ such that $P^\top A P = B \otimes I_{\bar{m}}$ in (4.10): in Case $\mathbb{R}$ or in Case $\mathbb{C}$, all the eigenvalues of $B$ are simple, and in Case $\mathbb{H}$, all the eigenvalues of $B$ have multiplicity two.

In Case $\mathbb{R}$, an I-generic matrix has distinct $\check{n}$ real or complex eigenvalues, each with multiplicity $\bar{m}$. In this case, there exists a symmetric I-generic matrix. In Case $\mathbb{C}$, an I-generic matrix has distinct $2\check{n}$ complex (nonreal) eigenvalues. Half of them are conjugate to the remaining half and the multiplicities of the eigenvalues are all $\bar{m}$. In Case $\mathbb{H}$, an I-generic matrix has distinct $2\check{n}$ complex (nonreal) eigenvalues. Half of them are conjugate to the remaining half and the multiplicities of the eigenvalues are all $2\bar{m}$.

The following proposition can be proven by the same argument as the proof of Proposition 3.3.

**Proposition 4.1.** If $\mathrm{span}\{I_n, A_1, \ldots, A_N\} = \mathcal{T}$, there exists an open dense subset $R$ of $\mathbb{R}^N$ such that $A(r)$ is I-generic for every $r \in R$. In Case $\mathbb{R}$, this statement remains true when $A(r)$ above is replaced by $A(r)^\top + A(r)$.

*Proof.* Let $B_p$ denote the matrix $B$ in the decomposition (4.10) of $A = A_p$, i.e., $P^\top A_p P = B_p \otimes I_{\bar{m}}$. Define $f(\lambda) = f(\lambda; r) = \det(\lambda I - (r_1 B_1 + \cdots + r_N B_N))$, which is a polynomial in $\lambda, r_1, \ldots, r_N$.

In Case $\mathbb{R}$ or in Case $\mathbb{C}$, the matrix $A(r)$ is I-generic if and only if $f(\lambda)$ does not have multiple root, and the latter condition is equivalent to the resultant of $f(\lambda)$ and $f'(\lambda)$ being nonzero. The resultant is a nonzero polynomial in $r_1, \ldots, r_N$, since $\mathcal{T}$ has at least one I-generic matrix $A$, which can be represented as $A = r_0 I_n + A(r)$ for some $r_0 \in \mathbb{R}$ and $r \in \mathbb{R}^N$ by the assumption on the linear span of the generators. Letting $\Sigma$ be the zero set of the resultant of $f(\lambda)$ and $f'(\lambda)$, we can take $R = \mathbb{R}^N \setminus \Sigma$.

In Case $\mathbb{H}$, by the eigenvalue structure of $\mathcal{H}_{\check{n}}$, there exists a polynomial $g$ in $\lambda$ such that $f(\lambda) = g(\lambda)^2$. The matrix $A(r)$ is I-generic if and only if $g(\lambda)$ has no multiple root, and the latter condition is equivalent to the resultant of $g(\lambda)$ and $g'(\lambda)$ being nonzero. This is further equivalent to the resultant of $f(\lambda)$ and $f''(\lambda)$ being nonzero. The resultant of $f(\lambda)$ and $f''(\lambda)$ is a nonzero polynomial in $r_1, \ldots, r_N$, since $\mathcal{T}$ has at least one I-generic matrix $A$, which can be represented as $A = r_0 I_n + A(r)$ for some $r_0 \in \mathbb{R}$ and $r \in \mathbb{R}^N$ by the assumption on the linear span of the generators. Letting $\Sigma$ be the zero set of the resultant of $f(\lambda)$ and $f''(\lambda)$, we can take $R = \mathbb{R}^N \setminus \Sigma$.

In Case $\mathbb{R}$, the above argument remains true when we replace $A(r)$ by $A(r)^\top + A(r)$, since there exists a symmetric I-generic matrix. $\square$

### 4.2 Identifying the case

In this section, we propose an algorithm that identifies the type of a simple algebra $\mathcal{T}$ as Case $\mathbb{R}$, Case $\mathbb{C}$ or Case $\mathbb{H}$. By adding transposes or products of some of the given generators, if necessary, we can assume that $\mathrm{span}\{I_n, A_1, \ldots, A_N\} = \mathcal{T}$.

Our algorithm consists of two stages. At the first stage, we decide whether $\mathcal{T}$ is in Case $\mathbb{R}$ or not. We conclude that $\mathcal{T}$ is in Case $\mathbb{R}$, if the multiplicity of the eigenvalues of a random linear combination, say, $A$ of the generators is equal to the multiplicity of the eigenvalues of $A^\top + A$. At the second stage, we identify $\mathcal{T}$ as Case $\mathbb{C}$ or Case $\mathbb{H}$. Let $d$ be the dimension of $\mathcal{T}$ as a linear space, and $\mu$ be the multiplicity the eigenvalues of some I-generic matrix. In Case $\mathbb{C}$, we have $n^2 = 2d\mu^2$ because $n = 2\check{n}\bar{m}$, $d = 2\check{n}^2$ and $\mu = \bar{m}$, whereas in Case $\mathbb{H}$, we have $n^2 = d\mu^2$ because $n = 4\check{n}\bar{m}$, $d = 4\check{n}^2$ and $\mu = 2\bar{m}$. We can thus distinguish between Case $\mathbb{C}$ and Case $\mathbb{H}$.

### 4.3 Case $\mathbb{R}$: $\mathcal{T} \simeq \mathcal{M}_{\check{n}}$

In this section, we consider Case $\mathbb{R}$, where we have $n = \check{n}\bar{m}$. In this case, there exists a symmetric I-generic matrix and such a matrix can be obtained through random linear combination of symmetric matrices. By virtue of this fact the algorithm of [12], with a slight modification, can be used in our general case. We describe the algorithm in our present notation for completeness and also for readers' convenience. It also serves as a prototype for the more complicate procedures for Case $\mathbb{C}$ and Case $\mathbb{H}$ to be treated in

Section 4.4 and Section 4.5.

For a matrix $A$ with rows and columns partitioned into $\check{n}$ blocks of size $\bar{m}$, we denote by $A_{[i,j]}$ the $\bar{m}\times\bar{m}$ submatrix in $(i,j)$ block of $A$ $(1 \le i, j \le \check{n})$. It follows from (4.7) that for any $A \in \mathcal{T}$, and for all $(i,j)$, there exists a real number $b_{ij}$ such that

$$(P^\top AP)_{[i,j]} = (B \otimes I_{\bar{m}})_{[i,j]} = b_{ij} I_{\bar{m}}.$$

We construct an orthogonal matrix $P$ that satisfies (4.7). In view of the nonuniqueness of such $P$ we impose two additional conditions. The first is that $P$ diagonalizes a particular $A$. The second is that $P$ normalizes some blocks of $P^\top A_1 P, \ldots, P^\top A_N P$ to scalar matrices. For the normalization of blocks, we consider a tree $T$ with vertex $V = \{1, \ldots, \check{n}\}$. Each edge of $T$ is directed and has a label from the set $\{1, \ldots, N\}$; accordingly $(i, j; p)$ denotes an edge from $i$ to $j$ with label $p \in \{1, \ldots, N\}$. The condition we impose is that $(P^\top A_p P)_{[i,j]}$ should be a scalar matrix for every edge $(i, j; p)$ of $T$.

Proposition 4.2 below states that there exists such an orthogonal matrix $P$. We write $\mathbb{R}_{\ge 0}$ for the set of nonnegative real numbers.

**Proposition 4.2.** Let $\mathcal{T}$ be a simple matrix $*$-algebra isomorphic to $\mathcal{M}_{\check{n}}$ generated by $n \times n$ matrices $A_1, \ldots, A_N$, where $n = \check{n}\bar{m}$ for some $\bar{m}$, and let $A \in \mathcal{T}$ be a symmetric I-generic matrix.

(1) Let $(\alpha_1, \ldots, \alpha_{\check{n}})$ be an ordering of distinct eigenvalues of $A$. There exists an orthogonal matrix $P$ that satisfies (4.7) and the condition

$$P^\top AP = \mathrm{diag}\,(\alpha_1 I_{\bar{m}}, \ldots, \alpha_{\check{n}} I_{\bar{m}}). \tag{4.11}$$

(2) Furthermore, let $T$ be a tree with the vertex-set $V = \{1, \ldots, \check{n}\}$, each edge of which is directed and labelled from $\{1, \ldots, N\}$. There exists an orthogonal matrix $P$ that satisfies (4.7), (4.11) and the condition

$$\forall (i, j; p) \in T, \exists c_{pij} \in \mathbb{R}_{\ge 0} : \ (P^\top A_p P)_{[i,j]} = c_{pij} I_{\bar{m}}. \tag{4.12}$$

*Proof.* (1) Let $R$ be any orthogonal matrix $P$ in (4.7). Then $R^\top AR = B \otimes I_{\bar{m}}$ for some $B \in \mathcal{M}_{\check{n}}$, which is symmetric because $A$ is symmetric. Let $S^\top BS = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_{\check{n}})$ be a diagonalization of $B$, where $S$ is the orthogonal matrix consisting of the eigenvectors of $B$. The matrix $Q = R(S \otimes I_{\bar{m}})$ satisfies (4.7) and also

$$Q^\top AQ = \mathrm{diag}\,(\alpha_1 I_{\bar{m}}, \ldots, \alpha_{\check{n}} I_{\bar{m}}).$$

Hence this matrix $Q$ serves as $P$ in the statement (1).

(2) Since $Q$ satisfies (4.7), there exists, for each $(i, j; p) \in T$, a real number $b_{pij}$ such that $(Q^\top A_p Q)_{[i,j]} = b_{pij} I_{\bar{m}}$ . For $c_{pij} = |b_{pij}| \in \mathbb{R}_{\ge 0}$, we

have $(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top = c_{pij}^2 I_{\bar{m}}$. With reference to the tree $T$, we can choose $\bar{m} \times \bar{m}$ matrices $P_1, \ldots, P_{\check{n}}$ such that

$$
\begin{aligned}
&P_1 = I_{\bar{m}}, \\
&P_j = ((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij} \quad ((i,j;p) \in T),
\end{aligned}
$$

where we define $P_i = P_j$ if $c_{pij} = 0$. We then have

$$
P_j^\top P_j = P_i^\top (Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij}^2 = P_i^\top P_i.
$$

From $P_1 = I_{\bar{m}}$ and the induction with respect to the distance from the vertex 1 on the tree $T$, we see that $P_1, \ldots, P_{\check{n}}$ are orthogonal matrices. Hence $P = Q \cdot \mathrm{diag}\,(P_1, \ldots, P_{\check{n}})$ is an orthogonal matrix satisfying (4.7), (4.11) and

$$
\begin{aligned}
(P^\top A_p P)_{[i,j]} &= P_i^\top (Q^\top A_p Q)_{[i,j]} P_j \\
&= P_i^\top (Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij} \\
&= c_{pij} I_{\bar{m}}.
\end{aligned}
$$

This is (4.12). □

Next, we describe the algorithm for constructing the matrix $P$ in Proposition 4.2 above. The idea is that we first diagonalize a particular $A \in \mathcal{T}$ to get an orthogonal matrix that satisfies (4.11), and normalize it as in the proof of Proposition 4.2 to make it satisfy (4.12). The correctness of our algorithm is guaranteed by Proposition 4.4. It is a key to the algorithm that if two orthogonal diagonalizations $Q^\top AQ$ and $P^\top AP$ of a symmetric I-generic matrix $A$ have the diagonal elements (eigenvalues) in the same order, there exist some $\bar{m} \times \bar{m}$ orthogonal matrices $P_1, \ldots, P_{\check{n}}$ such that $P = Q \cdot \mathrm{diag}\,(P_1, \ldots, P_{\check{n}})$.

**Algorithm 4.3.**

**Step 1:** Let $A \in \mathcal{T}$ be a symmetric I-generic matrix.

**Step 2:** Compute an orthogonal matrix $Q$ such that $Q^\top AQ = \mathrm{diag}\,(\alpha_1 I_{\bar{m}}, \ldots, \alpha_{\check{n}} I_{\bar{m}})$ for some $\alpha_1, \ldots, \alpha_{\check{n}} \in \mathbb{R}$.

**Step 3:** Let $G = (V, E)$ be a directed graph with vertex-set $V = \{1, \ldots, \check{n}\}$ and edge-set $E = \{(i,j;p) : (Q^\top A_p Q)_{[i,j]} \neq O\}$. Fix a spanning tree $T$ of $G$.

**Step 4:** For the tree $T$, let $P_1, \ldots, P_{\check{n}}$ be the $\bar{m} \times \bar{m}$ matrices that satisfy

$$
\begin{aligned}
&P_1 = I_{\bar{m}}, \\
&P_j = ((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij} \quad ((i,j;p) \in T),
\end{aligned}
$$

where $c_{pij}$ is the positive number such that $(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top = c_{pij}^2 I_{\bar{m}}$.

**Step 5:** Output $P = Q \cdot \mathrm{diag}\,(P_1, \ldots, P_{\check{n}})$.

**Proposition 4.4.** The following are true for Algorithm 4.3.

(1) In Step 3, there exists a spanning tree of $G$.

(2) In Step 4, for each edge $(i, j; p)$ of the tree $T$, there exists a positive number $c_{pij}$ such that $(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top = c_{pij}^2 I_{\bar{m}}$.

(3) In Step 4, $P_1, \ldots, P_{\check{n}}$ are orthogonal matrices.

(4) In Step 5, $P$ is an orthogonal matrix.

(5) In Step 5, $P$ satisfies (4.11) and (4.12) with respect to the ordering of eigenvalues in Step 2 and the tree $T$ in Step 3.

(6) In Step 5, $P$ satisfies (4.7).

*Proof.* (1) Let $R$ denote the orthogonal matrix $P$ in Proposition 4.2 (1) with respect to the ordering of the eigenvalues $(\alpha_1, \ldots, \alpha_{\check{n}})$ in Step 2. We first claim the following.
Claim: $(Q^\top A_p Q)_{[i,j]} \neq O$ if and only if $(R^\top A_p R)_{[i,j]} \neq O$.

Proof of Claim: Since $Q^\top A Q = R^\top A R$, there exist $\bar{m} \times \bar{m}$ orthogonal matrices $R_1, \ldots, R_{\check{n}}$ such that $R = Q \cdot \mathrm{diag}\,(R_1, \ldots, R_{\check{n}})$. Therefore we have $R_i^\top (Q^\top A_p Q)_{[i,j]} R_j = (R^\top A_p R)_{[i,j]}$, which implies the claim.

Suppose that $G$ does not have a spanning tree, i.e., that $G$ is disconnected. Let $W$ be one of the connected components of $G$. Then, for all $i \in W$ and $j \in V \setminus W$, we have $(R^\top A_p R)_{[i,j]} = O$ for all $p$. Since $\mathcal{T}$ is generated by $A_1, \ldots, A_N$ we have $(R^\top A' R)_{[i,j]} = O$ for all $A' \in \mathcal{T}$. However, there exists a $A' \in \mathcal{T}$ that satisfies $(R^\top A' R)_{[i,j]} \neq O$ because $R^\top \mathcal{T} R = \mathcal{M}_{\check{n}} \otimes I_{\bar{m}}$. This is a contradiction.

(2) Let $R$ denote the orthogonal matrix $P$ in Proposition 4.2 (2) with respect to the ordering of the eigenvalues $(\alpha_1, \ldots, \alpha_{\check{n}})$ in Step 2 and the tree $T$ in Step 3. By the same argument for Claim in (1), there exist $\bar{m} \times \bar{m}$ orthogonal matrices $R_1, \ldots, R_{\check{n}}$ such that $R = Q \cdot \mathrm{diag}\,(R_1, \ldots, R_{\check{n}})$. For each edge $(i, j; p)$ of $T$, we have from (4.12) that

$$\begin{aligned}(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top &= R_i (R^\top A_p R)_{[i,j]}((R^\top A_p R)_{[i,j]})^\top R_i^\top \\ &= R_i (c_{pij}^2 I_{\bar{m}}) R_i^\top = c_{pij}^2 I_{\bar{m}},\end{aligned}$$

where $c_{pij} \neq 0$ by the definition of the edges of $G$.

(3) From $P_1 = I_{\bar{m}}$ and

$$P_j^\top P_j = P_i^\top ((Q^\top A_p Q)_{[i,j]})((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij}^2 = P_i^\top P_i,$$

we see that $P_1, \ldots, P_{\check{n}}$ are orthogonal matrices by the induction with respect to the distance from the vertex 1 on the tree $T$.

(4) Since $Q$ and $P_i$ $(i = 1, \ldots, \check{n})$ are orthogonal, $P = Q \cdot \mathrm{diag}\,(P_1, \ldots, P_{\check{n}})$ is an orthogonal matrix.

(5) $P$ satisfies (4.11) by the definition of $Q$, and (4.12) by the definition of $P_1, \ldots, P_{\check{n}}$.

(6) For $A$ in Step 1, we have $P^\top AP = R^\top AR$. Hence there exist $\bar{m} \times \bar{m}$ orthogonal matrices $\hat{R}_1, \ldots, \hat{R}_{\check{n}}$ such that $R = P \cdot \mathrm{diag}\,(\hat{R}_1, \ldots, \hat{R}_{\check{n}})$. For each $(i, j; p)$ in $T$, evaluating $(i, j)$ block of $R^\top A_p R$, we have

$$c_{pij} I_{\bar{m}} = (R^\top A_p R)_{[i,j]} = \hat{R}_i^\top (P^\top A_p P)_{[i,j]} \hat{R}_j = c_{pij} \hat{R}_i^\top \hat{R}_j.$$

Hence $\hat{R}_1 = \cdots = \hat{R}_{\check{n}} =: \hat{R}$ by the induction with respect to the distance from the vertex 1 on the tree $T$. Then for each $i, j$ and $p$, there exists a real number $b_{pij}$ such that

$$(P^\top A_p P)_{[i,j]} = \hat{R}(R^\top A_p R)_{[i,j]} \hat{R}^\top = \hat{R}(b_{pij} I_{\bar{m}}) \hat{R}^\top = b_{pij} I_{\bar{m}}.$$

This states that $P^\top A_p P \in \mathcal{M}_{\check{n}} \otimes I_{\bar{m}}$ holds for all generators $A_p$. This is (4.7). □

Two facts are noteworthy in the above arguments: (i) when we normalize the blocks of the generators with respect to the tree, all other blocks of the generators automatically become scalar matrices, and (ii) the orthogonal matrix $P$ in Proposition 4.2 (2) has the degree of freedom represented by the $\bar{m} \times \bar{m}$ orthogonal matrix $\hat{R}$. These two properties do not carry over to Case $\mathbb{C}$ or Case $\mathbb{H}$, i.e., (i) fails in Case $\mathbb{H}$, and (ii) fails both in Case $\mathbb{C}$ and in Case $\mathbb{H}$. Then we have to design more complicated algorithms.

### 4.4 Case $\mathbb{C}$: $\mathcal{T} \simeq \mathcal{C}_{\check{n}}$

In this section, we consider Case $\mathbb{C}$, where we have $n = 2\check{n}\bar{m}$. For a matrix $A$ with rows and columns partitioned into $\check{n}$ blocks of size $2\bar{m}$, we denote by $A_{[i,j]}$ the $2\bar{m} \times 2\bar{m}$ submatrix in $(i, j)$ block of $A$ $(1 \leq i, j \leq \check{n})$. It follows from (4.8) that for all $A \in \mathcal{T}$, and for all $(i, j)$, there exists a complex number $b'_{ij}$ such that

$$(P^\top AP)_{[i,j]} = (B \otimes I_{\bar{m}})_{[i,j]} = C(b'_{ij}) \otimes I_{\bar{m}}.$$

In this case, there does not exist a symmetric I-generic matrix. Accordingly we use the Schur decomposition instead of the diagonalization.

The Schur decomposition of an $\check{n} \times \check{n}$ complex matrix $Z$ is defined as

$$U^* Z U = \begin{bmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1\check{n}} \\ 0 & \zeta_{22} & \cdots & \zeta_{2\check{n}} \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \zeta_{\check{n}\check{n}} \end{bmatrix},$$

where $U$ is a unitary matrix and $U^*$ is its conjugate transpose. The diagonal elements of $U^*ZU$ are the eigenvalues of $Z$. The Schur decomposition gives a nested sequence of $Z$-invariant subspaces $\{\mathbf{0}\} = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{\check{n}} = \mathbb{C}^{\check{n}}$ where $V_i = W_i/W_{i-1}$ is the eigenspace corresponding to the eigenvalue $\zeta_{ii}$ for $i = 1, \dots, \check{n}$. Any square matrix can be transformed into a Schur form, but the decomposition is not unique in two ways. It depends on the ordering of eigenspaces and also on the choice of orthogonal bases within eigenspaces. The Schur decomposition can also be defined for a real matrix. It is a matrix of quasi-upper triangular form, called the real Schur form, of which the diagonal elements are $1 \times 1$ blocks corresponding to real eigenvalues and $2 \times 2$ blocks expressed as $C(z)$ for complex eigenvalues $z$.

We can construct an orthogonal matrix $P$ that satisfies (4.8) as follows. The idea is that we replace the diagonalization in Case $\mathbb{R}$ to the Schur decomposition.

Proposition 4.5 below, to be compared with Proposition 4.2 in Case $\mathbb{R}$, states that there exists an orthogonal matrix $P$ in (4.8) that meets two additional conditions: (i) it brings a particular I-generic $A$ to a Schur form and (ii) it is normalized with respect to a tree. We write $\operatorname{eig}(\cdot)$ for the eigenvalues of a matrix.

**Proposition 4.5.** Let $\mathcal{T}$ be a simple matrix $*$-algebra isomorphic to $\mathcal{C}_{\check{n}}$ generated by $n \times n$ matrices $A_1, \dots, A_N$, where $n = 2\check{n}\bar{m}$ for some $\bar{m}$, and let $A \in \mathcal{T}$ be an I-generic matrix.

(1) Let $(\lambda_1, \bar{\lambda}_1, \dots, \lambda_{\check{n}}, \bar{\lambda}_{\check{n}})$ be an ordering of distinct eigenvalues of $A$. There exists an orthogonal matrix $P$ that satisfies (4.8) and the conditions

$$(P^\top AP)_{[i,j]} = O \quad (i > j), \tag{4.13}$$

$$(P^\top AP)_{[1,1]} = C(\lambda_1) \otimes I_{\bar{m}}, \tag{4.14}$$

$$\operatorname{eig}((P^\top AP)_{[i,i]}) = \{\lambda_i, \bar{\lambda}_i\} \quad (i = 2, \dots, \check{n}). \tag{4.15}$$

(2) Furthermore, let $T$ be a tree with vertex-set $V = \{1, \dots, \check{n}\}$, each edge of which is directed and labelled from $\{1, \dots, N\}$. There exists an orthogonal matrix $P$ that satisfies (4.8), (4.13), (4.14), (4.15) and the condition

$$\forall (i, j; p) \in T, \exists c_{pij} \in \mathbb{R}_{\geq 0} : \ (P^\top A_p P)_{[i,j]} = C(c_{pij}) \otimes I_{\bar{m}} = c_{pij} I_{2\bar{m}}. \tag{4.16}$$

*Proof.* (1) Let $R$ be any orthogonal matrix $P$ in (4.8). Then $R^\top AR = B \otimes I_{\bar{m}}$ for some $B \in \mathcal{C}_{\check{n}}$. We have $B = C(B')$ for some $\check{n} \times \check{n}$ complex matrix $B'$, where $C(B')$ denotes the $2\check{n} \times 2\check{n}$ real matrix that is obtained from $B'$ by replacing each entry of $B' = (b'_{ij})$ by $C(b'_{ij})$. Let $U^*B'U$ be the Schur decomposition of $B'$, where the diagonal elements are ordered as $(\lambda_1, \dots, \lambda_{\check{n}})$. Since $C(U)^\top C(U) = C(U^* U) = C(I_{\check{n}}) = I_{2\check{n}}$, $C(U)$ is an orthogonal matrix. Then the matrix $Q = R(C(U) \otimes I_{\bar{m}})$ satisfies (4.8) and also

$$(Q^\top AQ)_{[i,j]} = O \quad (i > j),$$
$$(Q^\top AQ)_{[i,i]} = C(\lambda_i) \otimes I_{\bar{m}} \quad (i = 1, \dots, \check{n}).$$

Hence this matrix $Q$ serves as $P$ in the statement (1).

(2) Since $Q$ satisfies (4.8), there exists, for each $(i, j; p) \in T$, a complex number $b'_{pij}$ such that $(Q^\top A_p Q)_{[i,j]} = C(b'_{pij}) \otimes I_{\bar{m}}$ . For $c_{pij} = |b'_{pij}| \in \mathbb{R}_{\geq 0}$, we have $(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top = c_{pij}^2 I_{2\bar{m}}$. With reference to the tree $T$, we can choose $2\bar{m} \times 2\bar{m}$ matrices $P_1, \dots, P_{\check{n}}$ such that

$$P_1 = I_{2\bar{m}},$$
$$P_j = ((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij} \quad ((i, j; p) \in T),$$

where we define $P_i = P_j$ if $c_{pij} = 0$. We then have

$$P_j^\top P_j = P_i^\top (Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij}^2 = P_i^\top P_i.$$

From $P_1 = I_{2\bar{m}}$ and the induction with respect to the distance from the vertex 1 on the tree $T$, we see that $P_1, \dots, P_{\check{n}}$ are orthogonal matrices. Hence $P = Q \cdot \mathrm{diag}\,(P_1, \dots, P_{\check{n}})$ is an orthogonal matrix satisfying (4.8), (4.13), (4.14), (4.15) and

$$\begin{aligned}(P^\top A_p P)_{[i,j]} &= P_i^\top (Q^\top A_p Q)_{[i,j]} P_j \\ &= P_i^\top (Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij} \\ &= c_{pij} I_{2\bar{m}}.\end{aligned}$$

This is (4.16). □

Next, we describe the algorithm for constructing the matrix $P$ in Proposition 4.5 above. The idea is that we first decompose a particular $A \in \mathcal{T}$ into the Schur form to get an orthogonal matrix that satisfies (4.13), (4.14), (4.15) and normalize it as in the proof of Proposition 4.5 to make it satisfy (4.16). The correctness of our algorithm is guaranteed by Proposition 4.7. It is a key to the algorithm that if two Schur decompositions $Q^\top AQ$ and $P^\top AP$ of an I-generic matrix $A$ have the diagonal elements (eigenvalues) in the same order, there exist some $\bar{m} \times \bar{m}$ orthogonal matrices $P_1, \dots, P_{\check{n}}$ such that $P = Q \cdot \mathrm{diag}\,(P_1, \dots, P_{\check{n}})$.

**Algorithm 4.6.**

**Step 1:** Let $A \in \mathcal{T}$ be an I-generic matrix.

**Step 2:** Compute an orthogonal matrix $Q$ such that

$$(Q^\top AQ)_{[i,j]} = O \quad (i > j),$$
$$(Q^\top AQ)_{[i,i]} = C(\lambda_i) \otimes I_{\bar{m}} \quad (i = 1, \ldots, \check{n})$$

by decomposing $A$ into the real Schur form and using the permutation. (It should be noted here that the diagonal blocks of the real Schur form are $I_{\bar{m}} \otimes C(\lambda_i)$.)

**Step 3:** Let $G = (V, E)$ be a directed graph with vertex-set $V = \{1, \ldots, \check{n}\}$ and edge-set $E = \{(i, j; p) : (Q^\top A_p Q)_{[i,j]} \neq O\}$. Fix a spanning tree $T$ of $G$.

**Step 4:** For the tree $T$, let $P_1, \ldots, P_{\check{n}}$ be the $2\bar{m} \times 2\bar{m}$ matrices that satisfy

$$P_1 = I_{2\bar{m}},$$
$$P_j = ((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij} \quad ((i, j; p) \in T),$$

where $c_{pij}$ is the positive number such that $(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top = c_{pij}^2 I_{2\bar{m}}$.

**Step 5:** Output $P = Q \cdot \mathrm{diag}\,(P_1, \ldots, P_{\check{n}})$.

**Proposition 4.7.** The following are true for Algorithm 4.6.

(1) In Step 3, there exists a spanning tree of $G$.

(2) In Step 4, for each edge $(i, j; p)$ of the tree $T$, there exists a positive number $c_{pij}$ such that $(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top = c_{pij}^2 I_{2\bar{m}}$.

(3) In Step 4, $P_1, \ldots, P_{\check{n}}$ are orthogonal matrices.

(4) In Step 5, $P$ is an orthogonal matrix.

(5) In Step 5, $P$ satisfies (4.13), (4.14), (4.15) and (4.16) with respect to the ordering of eigenvalues in Step 2 and the tree $T$ in Step 3.

(6) In Step 5, $P$ satisfies (4.8).

*Proof.* (1) Let $R$ denote the orthogonal matrix $P$ in Proposition 4.5 (1) with respect to the ordering of the eigenvalues $(\lambda_1, \bar{\lambda}_1, \ldots, \lambda_{\check{n}}, \bar{\lambda}_{\check{n}})$ in Step 2. We first claim the following.
Claim: $(Q^\top A_p Q)_{[i,j]} \neq O$ if and only if $(R^\top A_p R)_{[i,j]} \neq O$.

Proof of Claim: Since $Q^\top AQ = R^\top AR$, there exist $2\bar{m} \times 2\bar{m}$ orthogonal matrices $R_1, \ldots, R_{\check{n}}$ such that $R = Q \cdot \mathrm{diag}\,(R_1, \ldots, R_{\check{n}})$. Therefore we have $R_i^\top (Q^\top A_p Q)_{[i,j]} R_j = (R^\top A_p R)_{[i,j]}$, which implies the claim.

Suppose that $G$ does not have a spanning tree, i.e., that $G$ is disconnected. Let $W$ be one of the connected components of $G$. Then, for all

$i \in W$ and $j \in V \setminus W$, we have $(R^\top A_p R)_{[i,j]} = O$ for all $p$. Since $\mathcal{T}$ is generated by $A_1, \dots, A_N$ we have $(R^\top A' R)_{[i,j]} = O$ for all $A' \in \mathcal{T}$. However, there exists a $A' \in \mathcal{T}$ that satisfies $(R^\top A' R)_{[i,j]} \neq O$ because $R^\top \mathcal{T} R = \mathcal{C}_{\check{n}} \otimes I_{\bar{m}}$. This is a contradiction.

(2) Let $R$ denote the orthogonal matrix $P$ in Proposition 4.5 (2) with respect to the ordering of the eigenvalues $(\lambda_1, \bar{\lambda}_1, \dots, \lambda_{\check{n}}, \bar{\lambda}_{\check{n}})$ in Step 2 and the tree $T$ in Step 3. By the same argument for Claim in (1), there exist $2\bar{m} \times 2\bar{m}$ orthogonal matrices $R_1, \dots, R_{\check{n}}$ such that $R = Q \cdot \mathrm{diag}\,(R_1, \dots, R_{\check{n}})$. For each edge $(i, j; p)$ of $T$, we have from (4.16) that

$$\begin{aligned}(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top &= R_i (R^\top A_p R)_{[i,j]}((R^\top A_p R)_{[i,j]})^\top R_i^\top \\ &= R_i (c_{pij}^2 I_{2\bar{m}}) R_i^\top = c_{pij}^2 I_{2\bar{m}},\end{aligned}$$

where $c_{pij} \neq 0$ by the definition of the edges of $G$.

(3) From $P_1 = I_{2\bar{m}}$ and

$$P_j^\top P_j = P_i^\top (Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij}^2 = P_i^\top P_i$$

we see that $P_1, \dots, P_{\check{n}}$ are orthogonal matrices by the induction with respect to the distance from the vertex 1 on the tree $T$.

(4) Since $Q$ and $P_i$ $(i = 1, \dots, \check{n})$ are orthogonal, $P = Q \cdot \mathrm{diag}\,(P_1, \dots, P_{\check{n}})$ is an orthogonal matrix.

(5) $P$ satisfies (4.13), (4.14) and (4.15) by the definition of $Q$, and (4.16) by the definition of $P_1, \dots, P_{\check{n}}$.

(6) For $A$ in Step 1, we see that $P^\top A P$ and $R^\top A R$ have the diagonal elements in the same order. Hence there exist $2\bar{m} \times 2\bar{m}$ orthogonal matrices $\hat{R}_1, \dots, \hat{R}_{\check{n}}$ such that $R = P \cdot \mathrm{diag}\,(\hat{R}_1, \dots, \hat{R}_{\check{n}})$. For each $(i, j; p)$ in $T$, evaluating $(i, j)$ block of $R^\top A_p R$, we have

$$c_{pij} I_{2\bar{m}} = (R^\top A_p R)_{[i,j]} = \hat{R}_i^\top (P^\top A_p P)_{[i,j]} \hat{R}_j = c_{pij} \hat{R}_i^\top \hat{R}_j.$$

Hence $\hat{R}_1 = \cdots = \hat{R}_{\check{n}} =: \hat{R}$ by the induction with respect to the distance from the vertex 1 on the tree $T$. By (4.14), the definition of $P$ and $P_1 = I_{2\bar{m}}$, we have

$$\begin{aligned}C(\lambda_1) \otimes I_{\bar{m}} &= (R^\top A R)_{[1,1]} = \hat{R}^\top (P^\top A P)_{[1,1]} \hat{R} \\ &= \hat{R}^\top (Q^\top A Q)_{[1,1]} \hat{R} = \hat{R}^\top (C(\lambda_1) \otimes I_{\bar{m}}) \hat{R}.\end{aligned} \tag{4.17}$$

Since $\lambda_1$ is an eigenvalue of I-generic matrix $A$, we have $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$. Therefore every complex number $z$ can be written as $z = \alpha + \beta \lambda_1$ for some $\alpha, \beta \in \mathbb{R}$, and hence we have

$$\begin{aligned}\hat{R}^\top (C(z) \otimes I_{\bar{m}}) \hat{R} &= \hat{R}^\top (\alpha I_{2\bar{m}} + \beta C(\lambda_1) \otimes I_{\bar{m}}) \hat{R} \\ &= \alpha I_{2\bar{m}} + \beta C(\lambda_1) \otimes I_{\bar{m}} = C(z) \otimes I_{\bar{m}}\end{aligned}$$

by (4.17). Then for each $i, j$ and $p$, there exists a complex number $b'_{pij}$ such that

$$(P^\top A_p P)_{[i,j]} = \hat{R}(R^\top A_p R)_{[i,j]}\hat{R}^\top = \hat{R}(C(b'_{pij}) \otimes I_{\bar{m}})_{[i,j]}\hat{R}^\top = C(b'_{pij}) \otimes I_{\bar{m}}.$$

This states that $P^\top A_p P \in \mathcal{C}_{\check{n}} \otimes I_{\bar{m}}$ holds for all generators $A_p$. This is (4.8). □

According to this proof, the orthogonal matrix $P$ in Proposition 4.5 (2) has the degree of freedom represented by a $2\bar{m} \times 2\bar{m}$ orthogonal matrix $\hat{R}$ such that $C(\lambda_1) \otimes I_{\bar{m}}$ is invariant under the transformation by $\hat{R}$. Recall that, in Case $\mathbb{R}$, the degree of freedom of $P$ in Proposition 4.2 (2) is also described by an orthogonal matrix $\hat{R}$, on which no additional restrictions are imposed. This difference may be ascribed, as we see, to the fact that the complex field $\mathbb{C}$ is generated by two elements 1 and $\imath$, whereas $\mathbb{R}$ is generated by a single element. In Case $\mathbb{H}$, to be considered in the next section, we have even smaller degree of freedom as a consequence of the fact that the quaternion field $\mathbb{H}$ is generated by three elements $1, \imath$ and $\jmath$.

### 4.5 Case $\mathbb{H}$: $\mathcal{T} \simeq \mathcal{H}_{\check{n}}$

In this section, we consider Case $\mathbb{H}$, where we have $n = 4\check{n}\bar{m}$. For a matrix $A$ with rows and columns partitioned into $\check{n}$ blocks of size $4\bar{m}$, we denote by $A_{[i,j]}$ the $4\bar{m} \times 4\bar{m}$ submatrix in $(i, j)$ block of $A$ $(1 \leq i, j \leq \check{n})$. It follows from (4.9) that for any $A \in \mathcal{T}$, and for each $(i, j)$, there exists a quaternion number $b'_{ij}$ such that

$$(P^\top AP)_{[i,j]} = (B \otimes I_{\bar{m}})_{[i,j]} = H(b'_{ij}) \otimes I_{\bar{m}}.$$

The argument of this case is basically parallel to Case $\mathbb{C}$. Let us say that $B'$ is a quaternion matrix if each entry of $B'$ is a quaternion number. An $\check{n} \times \check{n}$ quaternion matrix $U$ is called unitary if $U^*U = UU^* = I_{\check{n}}$, where $U^*$ is the quaternion conjugate transpose of $U$. A quaternion variant of the Schur decomposition is known [4] to exist. That is, for any $\check{n} \times \check{n}$ quaternion matrix $B'$, there exists a quaternion unitary matrix $U$ such that $U^*B'U$ is an upper triangular form with quaternion entries. We can choose diagonal elements of $U^*B'U$ to be quaternions that are free from $\jmath$ and $k$ components.

Proposition 4.8 below, to be compared with Proposition 4.2 in Case $\mathbb{R}$ and Proposition 4.5 in Case $\mathbb{C}$, states that there exists an orthogonal matrix $P$ in (4.9) that meets two additional conditions: (i) it brings a particular I-generic $A$ to a Schur form and (ii) it is normalized with respect to a tree.

**Proposition 4.8.** Let $\mathcal{T}$ be a simple matrix $*$-algebra isomorphic to $\mathcal{H}_{\check{n}}$ generated by $n \times n$ matrices $A_1, \ldots, A_N$, where $n = 4\check{n}\bar{m}$ for some $\bar{m}$, and let $A \in \mathcal{T}$ be an I-generic matrix.

(1) Let $(\lambda_1, \bar{\lambda}_1, \ldots, \lambda_{\check{n}}, \bar{\lambda}_{\check{n}})$ be an ordering of distinct eigenvalues of $A$. There exists an orthogonal matrix $P$ that satisfies (4.9) and the conditions

$$(P^\top AP)_{[i,j]} = O \quad (i > j), \tag{4.18}$$

$$(P^\top AP)_{[1,1]} = H(\lambda_1) \otimes I_{\bar{m}}, \tag{4.19}$$

$$\mathrm{eig}((P^\top AP)_{[i,i]}) = \{\lambda_i, \bar{\lambda}_i\} \quad (i = 2, \ldots, \check{n}). \tag{4.20}$$

(2) Furthermore, let $T$ be a tree with vertex-set $V = \{1, \ldots, \check{n}\}$, each edge of which is directed and labelled from $\{1, \ldots, N\}$. There exists an orthogonal matrix $P$ that satisfies (4.9), (4.18), (4.19), (4.20) and the condition

$$\forall (i, j; p) \in T, \exists c_{pij} \in \mathbb{R}_{\geq 0} : \ (P^\top A_p P)_{[i,j]} = H(c_{pij}) \otimes I_{\bar{m}} = c_{pij} I_{4\bar{m}}. \tag{4.21}$$

*Proof.* (1) Let $R$ be any orthogonal matrix $P$ in (4.9). Then $R^\top AR = B \otimes I_{\bar{m}}$ for some $B \in \mathcal{H}_{\check{n}}$. We have $B = H(B')$ for some $\check{n} \times \check{n}$ quaternion matrix $B'$, where $H(B')$ denotes the $4\check{n} \times 4\check{n}$ real matrix that is obtained from $B'$ by replacing each entry of $B' = (b'_{ij})$ by $H(b'_{ij})$. Let $U^* B' U$ be the quaternion variant of the Schur decomposition of $B'$, where the diagonal elements are free from $\jmath$ and $k$ components and ordered as $(\lambda_1, \ldots, \lambda_{\check{n}})$. Since $H(U)^\top H(U) = H(U^* U) = H(I_{\check{n}}) = I_{4\check{n}}$, $H(U)$ is an orthogonal matrix. Then the matrix $Q = R(H(U) \otimes I_{\bar{m}})$ satisfies (4.9) and also

$$(Q^\top AQ)_{[i,j]} = O \quad (i > j),$$
$$(Q^\top AQ)_{[i,i]} = H(\lambda_i) \otimes I_{\bar{m}} \quad (i = 1, \ldots, \check{n}).$$

Hence this matrix $Q$ serves as $P$ in the statement (1).

(2) Since $Q$ satisfies (4.9), there exists, for each $(i, j; p) \in T$, a quaternion number $b'_{pij}$ such that $(Q^\top A_p Q)_{[i,j]} = H(b'_{pij}) \otimes I_{\bar{m}}$ . For $c_{pij} = |b'_{pij}| \in \mathbb{R}_{\geq 0}$, we have $(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top = c_{pij}^2 I_{4\bar{m}}$. With reference to the tree $T$, we can choose $4\bar{m} \times 4\bar{m}$ matrices $P_1, \ldots, P_{\check{n}}$ such that

$$P_1 = I_{4\bar{m}},$$
$$P_j = ((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij} \quad ((i, j; p) \in T),$$

where we define $P_i = P_j$ if $c_{pij} = 0$. We then have

$$P_j^\top P_j = P_i^\top (Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij}^2 = P_i^\top P_i.$$

From $P_1 = I_{4\bar{m}}$ and the induction with respect to the distance from the vertex 1 on the tree $T$, we see that $P_1, \ldots, P_{\check{n}}$ are orthogonal matrices that can be written as $P_j = H(U_j)$ for some quaternion unitary matrix $U_j$. Hence

$P = Q \cdot \mathrm{diag}\,(P_1, \dots, P_{\check{n}})$ is an orthogonal matrix satisfying (4.9), (4.18), (4.19), (4.20) and

$$\begin{aligned}(P^\top A_p P)_{[i,j]} &= P_i^\top (Q^\top A_p Q)_{[i,j]} P_j \\ &= P_i^\top (Q^\top A_p Q)_{[i,j]} ((Q^\top A_p Q)_{[i,j]})^\top P_i / c_{pij} \\ &= c_{pij} I_{4\bar{m}}.\end{aligned}$$

This is (4.21). □

We now propose an algorithm which constructs the matrix $P$ in Proposition 4.8 above. In Case $\mathbb{C}$, we have seen that an orthogonal matrix satisfying the conditions (4.13), (4.14), (4.15) and (4.16) in Proposition 4.5 is good for (4.8). In Case $\mathbb{H}$, however, the conditions (4.18), (4.19), (4.20) and (4.21) specified in Proposition 4.8 are not sufficient for (4.9), and we use the skew-Hamiltonian Schur decomposition [3] to make the matrix satisfy (4.9).

Let $J$ be a $4\bar{m} \times 4\bar{m}$ matrix defined as

$$J = \begin{bmatrix} O & -I_{2\bar{m}} \\ I_{2\bar{m}} & O \end{bmatrix}.$$

We call a matrix $S$ symplectic if $S^\top J S = S J S^\top = J$, and a matrix $W$ skew-Hamiltonian if $WJ = -(WJ)^\top$. Note that $S^{-1}WS$ is skew-Hamiltonian for a symplectic matrix $S$ and a skew-Hamiltonian matrix $W$.

The next proposition states that every skew-Hamiltonian matrix can be transformed to the so-called skew-Hamiltonian Schur form (see (4.22) below) by an orthogonal symplectic matrix. See [2, 3] for an algorithm for computing the skew-Hamiltonian Schur decomposition.

**Proposition 4.9** ([3])**.** For every skew-Hamiltonian matrix $W$, there exists an orthogonal symplectic matrix $S$ such that

$$S^\top W S = \begin{bmatrix} W_{11} & W_{12} \\ O & W_{11}^\top \end{bmatrix}, \tag{4.22}$$

where $W_{11}$ is a quasi-upper triangular matrix (the real Schur form) and $W_{12}$ is a skew-symmetric matrix ($W_{12} = -W_{12}^\top$). ■

Let $\Pi$ denote the $4\bar{m} \times 4\bar{m}$ permutation matrix representing the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & 2\bar{m} & 2\bar{m}+1 & 2\bar{m}+2 & 2\bar{m}+3 & \cdots & 4\bar{m} \\ 1 & 3 & 5 & \cdots & 4\bar{m}-1 & 2 & 4 & 6 & \cdots & 4\bar{m} \end{pmatrix}.$$

We have

$$J' := \Pi J \Pi^\top = H(\imath) \otimes I_{\bar{m}} = \begin{bmatrix} O & -I_{\bar{m}} & O & O \\ I_{\bar{m}} & O & O & O \\ O & O & O & -I_{\bar{m}} \\ O & O & I_{\bar{m}} & O \end{bmatrix},$$

and for a matrix $W$ in the skew-Hamiltonian Schur form (4.22) with $W_{11} = I_{\bar{m}} \otimes C(\imath)$ and $W_{12} = O$, we have

$$\Pi \begin{bmatrix} I_{\bar{m}} \otimes C(\imath) & O \\ O & -I_{\bar{m}} \otimes C(\imath) \end{bmatrix} \Pi^\top = H(\jmath) \otimes I_{\bar{m}}.$$

By using $J'$ instead of $J$ we say that $S$ is a permuted symplectic matrix if $S^\top J' S = SJ'S^\top = J'$, and $W$ is a permuted skew-Hamiltonian matrix if $WJ' = -(WJ')^\top$. Note that $\Pi^\top S \Pi$ is symplectic if $S$ is permuted symplectic and $\Pi^\top W \Pi$ is skew-Hamiltonian if $W$ is permuted skew-Hamiltonian.

**Proposition 4.10.** For a matrix $X = \hat{R}(H(h) \otimes I_{\bar{m}})\hat{R}^\top$ with an orthogonal permuted symplectic $\hat{R}$ and a quaternion $h = a + \imath b + \jmath c + kd$, there exists an orthogonal permuted symplectic matrix $\hat{S}$ such that

$$\hat{S}^\top X \hat{S} = H(a + \imath b + \jmath\sqrt{c^2 + d^2}) \otimes I_{\bar{m}}.$$

*Proof.* Put $X = X^{(1)} + X^{(2)}$, where $X^{(1)} = \hat{R}(H(a + \imath b) \otimes I_{\bar{m}})\hat{R}^\top$ and $X^{(2)} = \hat{R}(H(\jmath c + kd) \otimes I_{\bar{m}})\hat{R}^\top$. Since $J' = H(\imath) \otimes I_{\bar{m}}$, we have $X^{(2)}J' = \hat{R}(H(-kc + \jmath d) \otimes I_{\bar{m}})\hat{R}^\top = -(X^{(2)}J')^\top$, and therefore $X^{(2)}$ is a permuted skew-Hamiltonian matrix. Since $X^{(2)}$ is normal (i.e., $X^{(2)\top} X^{(2)} = X^{(2)} X^{(2)\top}$) and the eigenvalues of $X^{(2)}$ are $\pm\imath\sqrt{c^2 + d^2}$, the skew-Hamiltonian Schur decomposition of $\Pi^\top X^{(2)} \Pi$ is given by

$$S^\top \Pi^\top X^{(2)} \Pi S = \sqrt{c^2 + d^2} \begin{bmatrix} I_{\bar{m}} \otimes C(\imath) & O \\ O & -I_{\bar{m}} \otimes C(\imath) \end{bmatrix}$$

with some orthogonal symplectic matrix $S$. Then we have $\hat{S}^\top X^{(2)} \hat{S} = \sqrt{c^2 + d^2} H(\jmath) \otimes I_{\bar{m}}$, where $\hat{S} = \Pi S \Pi^\top$. Since $\hat{R}$ and $\hat{S}$ are orthogonal permuted symplectic, we have $\hat{S}^\top X^{(1)} \hat{S} = H(a + \imath b) \otimes I_{\bar{m}}$. Therefore we have $\hat{S}^\top X \hat{S} = H(a + \imath b + \jmath\sqrt{c^2 + d^2}) \otimes I_{\bar{m}}$ . ☐

The above proof shows how to compute the orthogonal permuted symplectic matrix $\hat{S}$ when we know that $X = \hat{R}(H(h) \otimes I_{\bar{m}})\hat{R}^\top$ with some (unknown) permuted symplectic matrix $\hat{R}$ and some (unknown) $h = a + \imath b + \jmath c + kd$. The value of $a$ is obtained as the $(1, 1)$ component of $X$, since $H(\imath), H(\jmath)$ and $H(k)$ are skew-symmetric matrices. The value of $b$ is obtained as the negative of the $(1, 1)$ component of $XJ'$, since $J' = H(\imath) \otimes I_{\bar{m}} = \hat{R}(H(\imath) \otimes I_{\bar{m}})\hat{R}^\top$ and $XJ' = \hat{R}(H(\jmath a - b - kc + \imath d) \otimes I_{\bar{m}})\hat{R}^\top$. Then we let

$$\begin{aligned} X^{(1)} &= aI_{4\bar{m}} + bJ' = \hat{R}(H(a + \imath b) \otimes I_{\bar{m}})\hat{R}^\top, \\ X^{(2)} &= X - X^{(1)}, \end{aligned}$$

where $X^{(2)}$ is a permuted skew-Hamiltonian. Let $S^\top \Pi^\top X^{(2)} \Pi S$ be a skew-Hamiltonian Schur decomposition of $\Pi^\top X^{(2)} \Pi$. Then we obtain $\hat{S}$ in Proposition 4.10 by $\hat{S} = \Pi S \Pi^\top$.

If $\hat{S}$ is an orthogonal permuted symplectic matrix in Proposition 4.10 for $X = \hat{R}(H(h) \otimes I_{\bar{m}})\hat{R}^\top$ with a particular $h = a + \imath b + \jmath c + kd$ with $c \neq 0$ or $d \neq 0$, then, for every matrix $Y = \hat{R}(H(h') \otimes I_{\bar{m}})\hat{R}^\top$ with $h' \in \mathbb{H}$, we have $\hat{S}^\top Y \hat{S} = H(h'') \otimes I_{\bar{m}}$ for some quaternion $h''$. This follows from the fact that the quaternion field is generated by three elements $1, \imath$ and $h$ as algebra, and that

$$\begin{aligned}
\hat{S}^\top \hat{R}(H(1) \otimes I_{\bar{m}})\hat{R}^\top \hat{S} &= H(1) \otimes I_{\bar{m}}, \\
\hat{S}^\top \hat{R}(H(\imath) \otimes I_{\bar{m}})\hat{R}^\top \hat{S} &= H(\imath) \otimes I_{\bar{m}}, \\
\hat{S}^\top \hat{R}(H(h) \otimes I_{\bar{m}})\hat{R}^\top \hat{S} &= H(a + \imath b + \jmath\sqrt{c^2 + d^2}) \otimes I_{\bar{m}}.
\end{aligned}$$

Next, we propose an algorithm for constructing the matrix $P$ in (4.9). The algorithm can be divided into two major stages. The first stage, consisting of Steps 1–5 below, is the same as Algorithm 4.6 for Case $\mathbb{C}$, and constructs an orthogonal matrix $\tilde{P} = Q \cdot \mathrm{diag}\,(\tilde{P}_1, \ldots, \tilde{P}_{\check{n}})$. This matrix $\tilde{P}$ is then modified to $P = \tilde{P} \cdot \mathrm{diag}\,(\hat{S}, \ldots, \hat{S})$ at the second stage consisting of Steps 6–7, where we make use of the skew-Hamiltonian Schur decomposition [3].

**Algorithm 4.11.**

**Step 1:** Let $A \in \mathcal{T}$ be an I-generic matrix.

**Step 2:** Compute an orthogonal matrix $Q$ such that

$$\begin{aligned}
(Q^\top A Q)_{[i,j]} &= O \quad (i > j), \\
(Q^\top A Q)_{[i,i]} &= H(\lambda_i) \otimes I_{\bar{m}} \quad (i = 1, \ldots, \check{n})
\end{aligned}$$

by decomposing $A$ into the real Schur form and using the permutation. (It should be noted here that the diagonal blocks of the real Schur form are $I_{\bar{m}} \otimes H(\lambda_i)$.)

**Step 3:** Let $G = (V, E)$ be a directed graph with vertex-set $V = \{1, \ldots, \check{n}\}$ and edge-set $E = \{(i, j; p) : (Q^\top A_p Q)_{[i,j]} \neq O\}$. Fix a spanning tree $T$ of $G$.

**Step 4:** For the tree $T$, let $\tilde{P}_1, \ldots, \tilde{P}_{\check{n}}$ be the $4\bar{m} \times 4\bar{m}$ matrices that satisfy

$$\begin{aligned}
\tilde{P}_1 &= I_{4\bar{m}}, \\
\tilde{P}_j &= ((Q^\top A_p Q)_{[i,j]})^\top \tilde{P}_i / c_{pij} \quad ((i, j; p) \in T),
\end{aligned}$$

where $c_{pij}$ is the positive number such that $(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top = c_{pij}^2 I_{4\bar{m}}$.

**Step 5:** Let $\tilde{P} = Q \cdot \mathrm{diag}\,(\tilde{P}_1, \ldots, \tilde{P}_{\check{n}})$.

**Step 6:** Take $i, j, p$ (where $1 \leq i, j \leq \check{n}$, $1 \leq p \leq N$) such that $X = (\tilde{P}^\top A_p \tilde{P})_{[i,j]}$ cannot be written as $H(z) \otimes I_{\bar{m}}$ for any $z \in \mathbb{C}$ and let $X^{(2)} = X - aI_{4\bar{m}} - bJ'$, where $a$ is the $(1,1)$ entry of $X$ and $-b$ is the $(1,1)$ entry of $XJ'$. Let $S$ be an orthogonal symplectic matrix which decomposes $\Pi^\top X^{(2)} \Pi$ into the skew-Hamiltonian Schur form.

**Step 7:** Output $P = \tilde{P} \cdot \mathrm{diag}\,(\hat{S}, \ldots, \hat{S})$, where $\hat{S} = \Pi^\top S \Pi$.

**Proposition 4.12.** The following are true for Algorithm 4.11.

(1) In Step 3, there exists a spanning tree of $G$.

(2) In Step 4, for each edge $(i, j; p)$ of the tree $T$, there exists a positive number $c_{pij}$ such that $(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top = c_{pij}^2 I_{4\bar{m}}$.

(3) In Step 4, $\tilde{P}_1, \ldots, \tilde{P}_{\check{n}}$ are orthogonal matrices.

(4) In Step 5, $\tilde{P}$ is an orthogonal matrix.

(5) In Step 5, $\tilde{P}$ satisfies (4.18), (4.19), (4.20) and (4.21) with $P$ replaced by $\tilde{P}$ with respect to the ordering of eigenvalues in Step 2 and the tree $T$ in Step 3.

(6) In Step 5, there exists an orthogonal permuted symplectic matrix $\hat{R}$ such that, for each $i, j$ and $p$,

$$(\tilde{P}^\top A_p \tilde{P})_{[i,j]} = \hat{R}^\top (H(b'_{pij}) \otimes I_{\bar{m}}) \hat{R}$$

for some quaternion $b'_{pij}$, where $1 \leq i, j \leq \check{n}$, $1 \leq p \leq N$.

(7) In Step 6, there exists $X = (\tilde{P}^\top A_p \tilde{P})_{[i,j]}$ that cannot be written as $H(z) \otimes I_{\bar{m}}$ for any $z \in \mathbb{C}$.

(8) In Step 7, $P$ is an orthogonal matrix.

(9) In Step 7, $P$ satisfies (4.18), (4.19), (4.20) and (4.21) with respect to the ordering of eigenvalues in Step 2 and the tree $T$ in Step 3.

(10) In Step 7, $P$ satisfies (4.9).

*Proof.* (1)–(5) can be proven in the similar way as Proposition 4.7.

(1) Let $R$ denote the orthogonal matrix $P$ in Proposition 4.8 (1) with respect to the ordering of the eigenvalues $(\lambda_1, \bar{\lambda}_1, \ldots, \lambda_{\check{n}}, \bar{\lambda}_{\check{n}})$ in Step 2. We first claim the following.
Claim: $(Q^\top A_p Q)_{[i,j]} \neq O$ if and only if $(R^\top A_p R)_{[i,j]} \neq O$.

Proof of Claim: Since $Q^\top A Q = R^\top A R$, there exist $4\bar{m} \times 4\bar{m}$ orthogonal matrices $R_1, \ldots, R_{\check{n}}$ such that $R = Q \cdot \mathrm{diag}\,(R_1, \ldots, R_{\check{n}})$. Therefore we have $R_i^\top (Q^\top A_p Q)_{[i,j]} R_j = (R^\top A_p R)_{[i,j]}$, which implies the claim.

Suppose that $G$ does not have a spanning tree, i.e., that $G$ is disconnected. Let $W$ be one of the connected components of $G$. Then, for all $i \in W$ and $j \in V \setminus W$, we have $(R^\top A_p R)_{[i,j]} = O$ for all $p$. Since $\mathcal{T}$ is generated by $A_1, \ldots, A_N$ we have $(R^\top A' R)_{[i,j]} = O$ for all $A' \in \mathcal{T}$. However, there exists a $A' \in \mathcal{T}$ that satisfies $(R^\top A' R)_{[i,j]} \neq O$ because $R^\top \mathcal{T} R = \mathcal{H}_{\check{n}} \otimes I_{\bar{m}}$. This is a contradiction.

(2) Let $R$ denote the orthogonal matrix $P$ in Proposition 4.8 (2) with respect to the ordering of the eigenvalues $(\lambda_1, \bar{\lambda}_1, \ldots, \lambda_{\check{n}}, \bar{\lambda}_{\check{n}})$ in Step 2 and the tree $T$ in Step 3. By the same argument for Claim in (1), there exist $4\bar{m} \times 4\bar{m}$ orthogonal matrices $R_1, \ldots, R_{\check{n}}$ such that $R = Q \cdot \mathrm{diag}\,(R_1, \ldots, R_{\check{n}})$. For each edge $(i, j; p)$ of $T$, we have from (4.21) that

$$\begin{aligned}(Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top &= R_i (R^\top A_p R)_{[i,j]}((R^\top A_p R)_{[i,j]})^\top R_i^\top \\ &= R_i (c_{pij}^2 I_{4\bar{m}}) R_i^\top = c_{pij}^2 I_{4\bar{m}},\end{aligned}$$

where $c_{pij} \neq 0$ by the definition of the edges of $G$.

(3) From $\tilde{P}_1 = I_{4\bar{m}}$ and

$$\tilde{P}_j^\top \tilde{P}_j = \tilde{P}_i^\top (Q^\top A_p Q)_{[i,j]}((Q^\top A_p Q)_{[i,j]})^\top \tilde{P}_i / c_{pij}^2 = \tilde{P}_i^\top \tilde{P}_i$$

we see that $\tilde{P}_1, \ldots, \tilde{P}_{\check{n}}$ are orthogonal matrices by the induction with respect to the distance from the vertex 1 on the tree $T$.

(4) Since $Q$ and $\tilde{P}_i$ $(i = 1, \ldots, \check{n})$ are orthogonal, $\tilde{P} = Q \cdot \mathrm{diag}\,(\tilde{P}_1, \ldots, \tilde{P}_{\check{n}})$ is an orthogonal matrix.

(5) $\tilde{P}$ satisfies (4.18), (4.19) and (4.20) by the definition of $Q$, and (4.21) by the definition of $\tilde{P}_1, \ldots, \tilde{P}_{\check{n}}$.

For $A$ in Step 1, we see that $\tilde{P}^\top A \tilde{P}$ and $R^\top A R$ have the diagonal elements in the same order. Hence there exist $4\bar{m} \times 4\bar{m}$ orthogonal matrices $\hat{R}_1, \ldots, \hat{R}_{\check{n}}$ such that $R = \tilde{P} \cdot \mathrm{diag}\,(\hat{R}_1, \ldots, \hat{R}_{\check{n}})$. For each $(i, j; p)$ in $T$, evaluating $(i, j)$ block of $R^\top A_p R$, we have

$$c_{pij} I_{4\bar{m}} = (R^\top A_p R)_{[i,j]} = \hat{R}_i^\top (\tilde{P}^\top A_p \tilde{P})_{[i,j]} \hat{R}_j = c_{pij} \hat{R}_i^\top \hat{R}_j.$$

Hence $\hat{R}_1 = \cdots = \hat{R}_{\check{n}} =: \hat{R}$ by the induction with respect to the distance from the vertex 1 on the tree $T$. Therefore, for all $i, j, p$ with $1 \leq i, j \leq \check{n}$, $1 \leq p \leq N$, there exists a quaternion number $b'_{pij}$ such that

$$(\tilde{P}^\top A_p \tilde{P})_{[i,j]} = \hat{R} (R^\top A_p R)_{[i,j]} \hat{R}^\top = \hat{R} (H(b'_{pij}) \otimes I_{\bar{m}}) \hat{R}^\top.$$

The $(1, 1)$ block of $R^\top A R$ is evaluated as

$$\begin{aligned}H(\lambda_1) \otimes I_{\bar{m}} &= (R^\top A R)_{[1,1]} = \hat{R}^\top (\tilde{P}^\top A \tilde{P})_{[1,1]} \hat{R} \\ &= \hat{R}^\top (Q^\top A Q)_{[1,1]} \hat{R} = \hat{R}^\top (H(\lambda_1) \otimes I_{\bar{m}}) \hat{R} \qquad (4.23)\end{aligned}$$

by (4.19), the definition of $R$ and $\tilde{P}_1 = I_{4\bar{m}}$. Since $\lambda_1$ is an eigenvalue of I-generic matrix $A$, we have $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$. Therefore the imaginary unit $\imath$ can be written as $\imath = \alpha + \beta\lambda_1$ for some $\alpha, \beta \in \mathbb{R}$, and hence we have

$$\begin{aligned}\hat{R}^\top J' \hat{R} &= \hat{R}^\top (H(\imath) \otimes I_{\bar{m}})\hat{R} = \hat{R}^\top (\alpha I_{4\bar{m}} + \beta H(\lambda_1) \otimes I_{\bar{m}})\hat{R} \\ &= \alpha I_{4\bar{m}} + \beta H(\lambda_1) \otimes I_{\bar{m}} = J'\end{aligned}$$

by (4.23). This says that $\hat{R}$ is an orthogonal permuted symplectic matrix.

(7) This follows from (6) and $\mathcal{T} \simeq \mathcal{H}_{\check{n}}$.

(8) Since $\tilde{P}$ and $\hat{S}$ are orthogonal, $P = \tilde{P} \cdot \mathrm{diag}\,(\hat{S}, \ldots, \hat{S})$ is an orthogonal matrix.

(9) This follows from (5) and that $\hat{S}$ is orthogonal permuted symplectic.

(10) The construction of $\hat{S}$ is consistent with the description given after Proposition 4.10. By Proposition 4.10, which is applicable by (6), we see that for all $i, j, p$ with $1 \le i, j \le \check{n}$, $1, \le p \le N$, there exists a quaternion number $b''_{pij}$ such that

$$(P^\top A_p P)_{[i,j]} = (\hat{S}^\top \tilde{P}^\top A_p \tilde{P} \hat{S})_{[i,j]} = H(b''_{pij}) \otimes I_{\bar{m}}.$$

This states that $P^\top A_p P \in \mathcal{H}_{\check{n}} \otimes I_{\bar{m}}$ holds for all generators $A_p$. This is (4.9). □

# 5 Numerical Examples

In this section we demonstrate the proposed algorithm for the irreducible decomposition in Case $\mathbb{C}$ and Case $\mathbb{H}$. Case $\mathbb{R}$ is not included here as it is almost the same as the algorithm of [12]. Recall also that our algorithm for the simple decomposition is essentially the same as that of [12].

## 5.1 Example for Case $\mathbb{C}$

We consider the $*$-algebra $\mathcal{T}$ generated by $A_1$ and $A_2$ below:

$$A_1 = \begin{bmatrix} 3.98 & 1.55 & 2.14 & 0.69 & 0.94 & 0.43 & 2.96 & -1.49 & -0.60 & 1.22 & 3.91 & 1.28 \\ -3.38 & 3.40 & 0.18 & -3.09 & 0.93 & -0.16 & 2.23 & -1.52 & -0.81 & -0.21 & -2.23 & -4.25 \\ 2.54 & -0.32 & 4.36 & -4.10 & 2.94 & 0.91 & 0.50 & 0.35 & 0.51 & 1.60 & 0.52 & 0.09 \\ -0.29 & 0.20 & 1.15 & 4.78 & -4.44 & -3.16 & 2.49 & 1.00 & 0.86 & 3.90 & -0.55 & 0.44 \\ -2.64 & -0.17 & -0.05 & -0.60 & 3.02 & 1.53 & -1.71 & -0.82 & -1.49 & -1.96 & 2.13 & 2.54 \\ 1.48 & -1.29 & -2.72 & 0.71 & -1.06 & 1.48 & -1.64 & -2.42 & 1.97 & -1.55 & 3.27 & 0.53 \\ 1.81 & 2.91 & -0.47 & -2.43 & 1.39 & -1.56 & 2.03 & -2.21 & -1.60 & 1.24 & 0.55 & -0.28 \\ 3.10 & -3.32 & -0.04 & -1.10 & 2.10 & 0.46 & 0.99 & 4.27 & 1.57 & 0.14 & -1.19 & 1.86 \\ 1.67 & 1.91 & -0.51 & -0.96 & -1.38 & -0.79 & 0.37 & 2.22 & 0.40 & 1.76 & -0.84 & 1.22 \\ 0.23 & 5.02 & 1.61 & 0.43 & 0.71 & 0.56 & -0.19 & -0.30 & 2.16 & 2.47 & -1.06 & -0.84 \\ -0.70 & 2.35 & -3.61 & 3.48 & 1.71 & 1.68 & 1.14 & -1.29 & -1.57 & 0.61 & 4.06 & -1.23 \\ 2.66 & 1.48 & -0.64 & -0.10 & -0.17 & 0.85 & 0.63 & -1.51 & -1.64 & 1.87 & 1.38 & 3.98 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 7.17 & 3.14 & 1.39 & -0.50 & 0.78 & 1.50 & 3.84 & -1.62 & -0.91 & 1.73 & 3.22 & 2.93 \\ -4.26 & 5.89 & -0.33 & -3.45 & 1.67 & -0.13 & 0.70 & -2.83 & -0.31 & 0.44 & -1.87 & -1.38 \\ 3.33 & 1.32 & 4.58 & -3.93 & 4.43 & 0.62 & 0.85 & 0.25 & -0.83 & 0.62 & 0.06 & 1.36 \\ 1.47 & 2.34 & 0.61 & 5.75 & -3.78 & -1.36 & 0.76 & 0.15 & 2.09 & 5.40 & -0.95 & 0.18 \\ 0.52 & -0.13 & -1.93 & -3.95 & 4.10 & 1.97 & -1.14 & -0.09 & 0.03 & -3.21 & 2.45 & 0.16 \\ -0.59 & 0.30 & -3.22 & 2.70 & 0.20 & 2.15 & 1.28 & -0.50 & -1.11 & 0.30 & 3.59 & 0.03 \\ -1.40 & 4.06 & 1.62 & -0.43 & 1.80 & -0.84 & 3.23 & -3.08 & -1.53 & 0.53 & 1.11 & -1.30 \\ 3.31 & -4.18 & 2.04 & 0.20 & -0.12 & -0.83 & -0.97 & 3.98 & -0.14 & 0.03 & 0.11 & 4.46 \\ 0.65 & -0.90 & 2.17 & -1.02 & -0.93 & -0.70 & 0.32 & 0.84 & 1.58 & 0.68 & -0.98 & -1.71 \\ 1.42 & 2.58 & 1.39 & -0.89 & -1.49 & -3.30 & 3.43 & -0.55 & -0.21 & 3.80 & -3.45 & -0.77 \\ -0.46 & 1.24 & -4.02 & 3.07 & -2.20 & 1.98 & -0.46 & -4.10 & -1.76 & -0.33 & 5.30 & 1.21 \\ 3.05 & -0.39 & 0.63 & 0.38 & 0.01 & 1.58 & 0.74 & -2.78 & 0.10 & 0.02 & 2.84 & 2.39 \end{bmatrix}.$$

According to the procedure of Section 4.2 we can recognize this case as Case $\mathbb{C}$ with $\check{n} = 3$ and $\bar{m} = 2$. This means that we have $P^\top \mathcal{T} P = \mathcal{C}_3 \otimes I_2$ with a suitable choice of an orthogonal matrix $P$.

**Step 1:** As an I-generic matrix $A$, we take the following matrix:

$$A = \begin{bmatrix} 3.61 & 1.48 & 1.42 & 0.26 & 0.66 & 0.54 & 2.37 & -1.13 & -0.51 & 1.01 & 2.75 & 1.29 \\ -2.68 & 3.04 & 0.03 & -2.36 & 0.84 & -0.11 & 1.32 & -1.40 & -0.49 & -0.02 & -1.57 & -2.54 \\ 2.04 & 0.11 & 3.27 & -3.00 & 2.49 & 0.61 & 0.44 & 0.24 & 0.10 & 0.98 & 0.29 & 0.33 \\ 0.15 & 0.59 & 0.74 & 3.73 & -3.14 & -1.96 & 1.48 & 0.56 & 0.89 & 3.20 & -0.49 & 0.27 \\ -1.29 & -0.11 & -0.43 & -1.14 & 2.46 & 1.22 & -1.15 & -0.45 & -0.78 & -1.71 & 1.64 & 1.38 \\ 0.66 & -0.62 & -2.12 & 0.94 & -0.52 & 1.24 & -0.60 & -1.39 & 0.81 & -0.76 & 2.49 & 0.29 \\ 0.67 & 2.39 & 0.09 & -1.38 & 1.11 & -1.00 & 1.75 & -1.81 & -1.17 & 0.77 & 0.52 & -0.42 \\ 2.34 & -2.63 & 0.41 & -0.54 & 1.09 & 0.07 & 0.32 & 3.09 & 0.80 & 0.08 & -0.61 & 1.92 \\ 1.02 & 0.82 & 0.19 & -0.72 & -0.92 & -0.56 & 0.26 & 1.35 & 0.55 & 1.07 & -0.65 & 0.29 \\ 0.42 & 3.20 & 1.14 & 0.04 & 0.06 & -0.39 & 0.62 & -0.27 & 1.10 & 2.10 & -1.29 & -0.61 \\ -0.47 & 1.50 & -2.75 & 2.48 & 0.44 & 1.30 & 0.50 & -1.54 & -1.20 & 0.25 & 3.26 & -0.40 \\ 2.05 & 0.70 & -0.21 & 0.03 & -0.09 & 0.78 & 0.49 & -1.38 & -0.85 & 1.00 & 1.32 & 2.61 \end{bmatrix}.$$

**Step 2:** By decomposing $A$ into the real Schur form and using the permutation, we obtain a quasi-upper triangular matrix

$$Q^\top A Q = \left[\begin{array}{rrrr|rrrr|rrrr}
6.53 & 0 & -4.92 & 0 & 0.37 & 0.79 & -0.48 & -0.44 & 0.25 & -0.27 & 0.32 & 1.28 \\
0 & 6.53 & 0 & 4.92 & -0.39 & 0.58 & 0.81 & -0.18 & 0.57 & 0.17 & 1.18 & -0.37 \\
4.92 & 0 & 6.53 & 0 & 0.48 & -0.44 & 0.37 & -0.79 & -0.32 & -1.28 & 0.25 & -0.27 \\
0 & -4.92 & 0 & 6.53 & 0.81 & 0.18 & 0.39 & 0.58 & 1.18 & -0.37 & -0.57 & -0.17 \\
\hline
0 & 0 & 0 & 0 & 1.85 & 0 & 1.25 & 0 & -0.46 & 0.29 & -0.36 & 0.63 \\
0 & 0 & 0 & 0 & 0 & 1.85 & 0 & -1.25 & 0.38 & 0.51 & -0.58 & -0.28 \\
0 & 0 & 0 & 0 & -1.25 & 0 & 1.85 & 0 & 0.36 & -0.63 & -0.46 & 0.29 \\
0 & 0 & 0 & 0 & 0 & 1.25 & 0 & 1.85 & -0.58 & -0.28 & -0.38 & -0.51 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.70 & 0 & -1.01 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.70 & 0 & -1.01 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.01 & 0 & -0.70 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.01 & 0 & -0.70
\end{array}\right],$$

where $Q$ is an orthogonal matrix given as

$$Q = \left[\begin{array}{rrrr|rrrr|rrrr}
0.32 & -0.20 & -0.59 & -0.14 & -0.31 & -0.13 & 0.14 & -0.15 & 0.34 & 0.37 & -0.21 & 0.23 \\
-0.42 & 0.52 & -0.20 & -0.06 & 0.05 & 0.08 & -0.02 & -0.28 & -0.34 & 0.46 & -0.31 & -0.08 \\
0.31 & 0.23 & -0.40 & 0.32 & -0.28 & 0.27 & 0.03 & 0.48 & -0.33 & -0.16 & -0.06 & -0.27 \\
0.31 & 0.01 & 0.21 & -0.67 & -0.00 & -0.09 & -0.29 & 0.26 & -0.07 & 0.24 & -0.12 & -0.42 \\
-0.25 & -0.14 & -0.17 & 0.36 & 0.28 & 0.04 & -0.53 & 0.27 & 0.46 & 0.23 & -0.10 & -0.24 \\
-0.16 & -0.44 & -0.06 & -0.04 & -0.02 & 0.46 & 0.35 & -0.16 & -0.05 & 0.28 & 0.35 & -0.47 \\
-0.10 & 0.19 & -0.44 & -0.13 & -0.00 & -0.44 & -0.16 & -0.26 & 0.04 & -0.30 & 0.49 & -0.36 \\
0.48 & -0.11 & 0.11 & 0.32 & 0.02 & 0.06 & -0.48 & -0.47 & -0.32 & 0.22 & 0.18 & 0.05 \\
0.19 & 0.13 & 0.00 & -0.07 & 0.09 & 0.39 & -0.01 & -0.45 & 0.33 & -0.44 & -0.44 & -0.30 \\
0.09 & 0.33 & -0.16 & -0.31 & 0.20 & 0.54 & -0.17 & 0.10 & 0.20 & 0.03 & 0.44 & 0.39 \\
-0.37 & -0.43 & -0.21 & -0.26 & -0.29 & 0.19 & -0.44 & -0.02 & -0.30 & -0.31 & -0.15 & 0.20 \\
0.15 & -0.25 & -0.34 & -0.09 & 0.78 & -0.08 & 0.15 & 0.06 & -0.32 & -0.10 & -0.16 & 0.07
\end{array}\right].$$

**Step 3:** Since

$$Q^\top A_1 Q = \left[\begin{array}{rrrr|rrrr|rrrr}
8.19 & 0 & -6.29 & 0 & 0.81 & 1.15 & -0.67 & -1.04 & 0.29 & -0.63 & 0.50 & 1.78 \\
0 & 8.19 & 0 & 6.29 & -0.98 & 0.91 & 1.21 & -0.53 & 0.59 & 0.30 & 1.79 & -0.49 \\
6.29 & 0 & 8.19 & 0 & 0.67 & -1.04 & 0.81 & -1.15 & -0.50 & -1.78 & 0.29 & -0.63 \\
0 & -6.29 & 0 & 8.19 & 1.21 & 0.53 & 0.98 & 0.91 & 1.79 & -0.49 & -0.59 & -0.30 \\
\hline
-0.14 & 0.14 & -0.22 & -0.36 & 3.43 & 0 & 1.99 & 0 & -1.31 & 1.12 & -0.49 & 1.23 \\
-0.35 & -0.25 & 0.16 & -0.06 & 0 & 3.43 & 0 & -1.99 & 0.45 & 0.96 & -1.60 & -1.02 \\
0.22 & -0.36 & -0.14 & -0.14 & -1.99 & 0 & 3.43 & 0 & 0.49 & -1.23 & -1.31 & 1.12 \\
0.16 & 0.06 & 0.35 & -0.25 & 0 & 1.99 & 0 & 3.43 & -1.60 & -1.02 & -0.45 & -0.96 \\
\hline
-0.13 & -0.28 & 0.19 & -0.71 & 0.69 & 0.29 & 0.19 & 0.80 & -2.06 & 0 & -2.06 & 0 \\
0.21 & -0.11 & 0.73 & 0.21 & -0.84 & -0.09 & 0.18 & 0.71 & 0 & -2.06 & 0 & -2.06 \\
-0.19 & -0.71 & -0.13 & 0.28 & -0.19 & 0.80 & 0.69 & -0.29 & 2.06 & 0 & -2.06 & 0 \\
-0.73 & 0.21 & 0.21 & 0.11 & -0.18 & 0.71 & -0.84 & 0.09 & 0 & 2.06 & 0 & -2.06
\end{array}\right],$$

$$Q^\top A_2 Q = \left[\begin{array}{rrrr|rrrr|rrrr}
10.50 & 0 & -7.58 & 0 & -0.28 & 0.85 & -0.63 & 0.55 & 0.44 & 0.28 & 0.26 & 1.62 \\
0 & 10.50 & 0 & 7.58 & 0.59 & 0.49 & 0.82 & 0.48 & 1.21 & 0.02 & 1.11 & -0.51 \\
7.58 & 0 & 10.50 & 0 & 0.63 & 0.55 & -0.28 & -0.85 & -0.26 & -1.62 & 0.44 & 0.28 \\
0 & -7.58 & 0 & 10.50 & 0.82 & -0.48 & -0.59 & 0.49 & 1.11 & -0.51 & -1.21 & -0.02 \\
\hline
0.36 & -0.36 & 0.55 & 0.90 & 0.16 & 0 & 0.92 & 0 & 1.12 & -1.45 & -0.48 & -0.10 \\
0.88 & 0.64 & -0.41 & 0.15 & 0 & 0.16 & 0 & -0.92 & 0.68 & 0.01 & 1.29 & 1.22 \\
-0.55 & 0.90 & 0.36 & 0.36 & -0.92 & 0 & 0.16 & 0 & 0.48 & 0.10 & 1.12 & -1.45 \\
-0.41 & -0.15 & -0.88 & 0.64 & 0 & 0.92 & 0 & 0.16 & 1.29 & 1.22 & -0.68 & -0.01 \\
\hline
0.33 & 0.70 & -0.49 & 1.78 & -1.75 & -0.74 & -0.48 & -2.03 & 1.86 & 0 & 0.37 & 0 \\
-0.53 & 0.28 & -1.84 & -0.52 & 2.11 & 0.23 & -0.45 & -1.80 & 0 & 1.86 & 0 & 0.37 \\
0.49 & 1.78 & 0.33 & -0.70 & 0.48 & -2.03 & -1.75 & 0.74 & -0.37 & 0 & 1.86 & 0 \\
1.84 & -0.52 & -0.53 & -0.28 & 0.45 & -1.80 & 2.11 & -0.23 & 0 & -0.37 & 0 & 1.86
\end{array}\right],$$

we can take $T = \{(1,2;1),(1,3;1)\}$ as a spanning tree of $G = (V, E)$.

**Step 4:** With reference to the tree $T$, we choose $P_1 = I_4$,

$$P_2 = \begin{bmatrix} 0.43 & 0.61 & -0.36 & -0.55 \\ 0.36 & -0.55 & 0.43 & -0.61 \\ -0.52 & 0.48 & 0.64 & -0.28 \\ 0.64 & 0.28 & 0.52 & 0.48 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0.15 & -0.32 & 0.25 & 0.90 \\ -0.25 & -0.90 & 0.15 & -0.32 \\ 0.30 & 0.15 & 0.91 & -0.25 \\ 0.91 & -0.25 & -0.30 & -0.15 \end{bmatrix}.$$

**Step 5:** The orthogonal matrix $P = Q \cdot \operatorname{diag}(P_1, P_2, P_3)$ gives the decomposition $P^\top \mathcal{T} P = \mathcal{C}_3 \otimes I_2$ as follows:

$$P^\top A_1 P = \left[\begin{array}{cccc|cccc|cccc}
8.19 & 0 & -6.29 & 0 & 1.88 & 0 & 0 & 0 & 1.97 & 0 & 0 & 0 \\
0 & 8.19 & 0 & 6.29 & 0 & 1.88 & 0 & 0 & 0 & 1.97 & 0 & 0 \\
6.29 & 0 & 8.19 & 0 & 0 & 0 & 1.88 & 0 & 0 & 0 & 1.97 & 0 \\
0 & -6.29 & 0 & 8.19 & 0 & 0 & 0 & 1.88 & 0 & 0 & 0 & 1.97 \\
\hline
-0.44 & 0 & -0.14 & 0 & 3.43 & 0 & 1.99 & 0 & -0.68 & 0 & -2.06 & 0 \\
0 & -0.44 & 0 & 0.14 & 0 & 3.43 & 0 & -1.99 & 0 & -0.68 & 0 & 2.06 \\
0.14 & 0 & -0.44 & 0 & -1.99 & 0 & 3.43 & 0 & 2.06 & 0 & -0.68 & 0 \\
0 & -0.14 & 0 & -0.44 & 0 & 1.99 & 0 & 3.43 & 0 & -2.06 & 0 & -0.68 \\
\hline
-0.79 & 0 & -0.05 & 0 & 0.91 & 0 & -0.66 & 0 & -2.06 & 0 & -2.06 & 0 \\
0 & -0.79 & 0 & 0.05 & 0 & 0.91 & 0 & 0.66 & 0 & -2.06 & 0 & 2.06 \\
0.05 & 0 & -0.79 & 0 & 0.66 & 0 & 0.91 & 0 & 2.06 & 0 & -2.06 & 0 \\
0 & -0.05 & 0 & -0.79 & 0 & -0.66 & 0 & 0.91 & 0 & -2.06 & 0 & -2.06
\end{array}\right],$$

$$P^\top A_2 P = \left[\begin{array}{cccc|cccc|cccc}
10.50 & 0 & -7.58 & 0 & 0.32 & 0 & -1.18 & 0 & 1.50 & 0 & -0.83 & 0 \\
0 & 10.50 & 0 & 7.58 & 0 & 0.32 & 0 & 1.18 & 0 & 1.50 & 0 & 0.83 \\
7.58 & 0 & 10.50 & 0 & 1.18 & 0 & 0.32 & 0 & 0.83 & 0 & 1.50 & 0 \\
0 & -7.58 & 0 & 10.50 & 0 & -1.18 & 0 & 0.32 & 0 & -0.83 & 0 & 1.50 \\
\hline
1.12 & 0 & 0.35 & 0 & 0.16 & 0 & 0.92 & 0 & 1.68 & 0 & 0.89 & 0 \\
0 & 1.12 & 0 & -0.35 & 0 & 0.16 & 0 & -0.92 & 0 & 1.68 & 0 & -0.89 \\
-0.35 & 0 & 1.12 & 0 & -0.92 & 0 & 0.16 & 0 & -0.89 & 0 & 1.68 & 0 \\
0 & 0.35 & 0 & 1.12 & 0 & 0.92 & 0 & 0.16 & 0 & 0.89 & 0 & 1.68 \\
\hline
2.00 & 0 & 0.11 & 0 & -2.29 & 0 & 1.66 & 0 & 1.86 & 0 & 0.37 & 0 \\
0 & 2.00 & 0 & -0.11 & 0 & -2.29 & 0 & -1.66 & 0 & 1.86 & 0 & -0.37 \\
-0.11 & 0 & 2.00 & 0 & -1.66 & 0 & -2.29 & 0 & -0.37 & 0 & 1.86 & 0 \\
0 & 0.11 & 0 & 2.00 & 0 & 1.66 & 0 & -2.29 & 0 & 0.37 & 0 & 1.86
\end{array}\right].$$

With a permutation matrix $\Pi$, explicit block diagonal forms can be obtained:

$$\Pi^\top P^\top A_1 P \Pi = \left[\begin{array}{cccccc|cccccc}
8.19 & -6.29 & 1.88 & 0 & 1.97 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6.29 & 8.19 & 0 & 1.88 & 0 & 1.97 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.44 & -0.14 & 3.43 & 1.99 & -0.68 & -2.06 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.14 & -0.44 & -1.99 & 3.43 & 2.06 & -0.68 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.79 & -0.05 & 0.91 & -0.66 & -2.06 & -2.06 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.05 & -0.79 & 0.66 & 0.91 & 2.06 & -2.06 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 8.19 & 6.29 & 1.88 & 0 & 1.97 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -6.29 & 8.19 & 0 & 1.88 & 0 & 1.97 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.44 & 0.14 & 3.43 & -1.99 & -0.68 & 2.06 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.14 & -0.44 & 1.99 & 3.43 & -2.06 & -0.68 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.79 & 0.05 & 0.91 & 0.66 & -2.06 & 2.06 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.05 & -0.79 & -0.66 & 0.91 & -2.06 & -2.06
\end{array}\right],$$

$$\Pi^\top P^\top A_2 P \Pi = \left[\begin{array}{cccccc|cccccc}
10.50 & -7.58 & 0.32 & -1.18 & 1.50 & -0.83 & 0 & 0 & 0 & 0 & 0 & 0 \\
7.58 & 10.50 & 1.18 & 0.32 & 0.83 & 1.50 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.12 & 0.35 & 0.16 & 0.92 & 1.68 & 0.89 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.35 & 1.12 & -0.92 & 0.16 & -0.89 & 1.68 & 0 & 0 & 0 & 0 & 0 & 0 \\
2.00 & 0.11 & -2.29 & 1.66 & 1.86 & 0.37 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.11 & 2.00 & -1.66 & -2.29 & -0.37 & 1.86 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 10.50 & 7.58 & 0.32 & 1.18 & 1.50 & 0.83 \\
0 & 0 & 0 & 0 & 0 & 0 & -7.58 & 10.50 & -1.18 & 0.32 & -0.83 & 1.50 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.12 & -0.35 & 0.16 & -0.92 & 1.68 & -0.89 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.35 & 1.12 & 0.92 & 0.16 & 0.89 & 1.68 \\
0 & 0 & 0 & 0 & 0 & 0 & 2.00 & -0.11 & -2.29 & -1.66 & 1.86 & -0.37 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.11 & 2.00 & 1.66 & -2.29 & 0.37 & 1.86
\end{array}\right].$$

## 5.2 Example for Case $\mathbb{H}$

We consider the $*$-algebra $\mathcal{T}$ generated by $A_1$ and $A_2$ below:

$A_1 =$

$$\begin{bmatrix}
5.87 & 0.11 & 6.42 & -2.93 & -7.48 & 3.57 & 3.12 & 3.22 & -1.61 & -4.63 & 0.33 & 4.24 & 1.89 & 3.25 & -1.29 & -1.86 \\
6.84 & 0.08 & 0.53 & 1.77 & 1.43 & -1.41 & 4.25 & 4.75 & -2.88 & 1.13 & 3.76 & -3.95 & -1.75 & 3.71 & 0.92 & 1.10 \\
-2.54 & -0.21 & 0.96 & -2.57 & -2.95 & 4.04 & -1.29 & 0.90 & -5.99 & -4.18 & 1.99 & -1.85 & 1.36 & 1.80 & -0.07 & -1.49 \\
5.95 & 0.84 & 3.45 & 0.77 & 2.57 & 5.65 & -3.46 & -0.94 & -2.17 & 0.38 & -5.52 & -2.05 & -3.66 & 0.47 & -4.40 & 1.66 \\
4.93 & -1.00 & -1.37 & 3.31 & 4.90 & 3.58 & -3.90 & -4.18 & 0.46 & -2.67 & 4.13 & 3.89 & -2.36 & -0.29 & -0.11 & 7.93 \\
-2.09 & -1.38 & 1.38 & -3.54 & 2.62 & 3.81 & -0.10 & -2.74 & -5.57 & -2.97 & 0.51 & 6.05 & -1.15 & -2.59 & 0.62 & 0.75 \\
-5.91 & -1.56 & 2.83 & 0.17 & 3.85 & 5.07 & 3.68 & 2.87 & -2.64 & 1.71 & 9.31 & -1.63 & -0.32 & -0.01 & -0.48 & 0.53 \\
-1.36 & -5.80 & -1.84 & -0.84 & 5.47 & 4.51 & -2.57 & 6.33 & 0.67 & 4.13 & -4.26 & 5.31 & 4.24 & 5.23 & 3.19 & -1.78 \\
3.87 & 5.63 & -1.32 & 4.03 & 1.98 & 3.86 & 5.22 & -4.42 & 3.42 & 2.15 & 2.86 & 2.34 & 0.70 & 3.26 & 0.40 & -3.77 \\
2.39 & -1.45 & 2.39 & 3.49 & 1.62 & 0.50 & -4.02 & 1.62 & -0.12 & 2.40 & -1.78 & -1.77 & -3.51 & 2.78 & -1.40 & -7.7 \\
-2.04 & 0.92 & -0.36 & 5.09 & -4.26 & -1.99 & -2.58 & 2.59 & -4.38 & 5.38 & 4.31 & 1.19 & -5.47 & 4.02 & 1.49 & 2.88 \\
-3.07 & -1.05 & 1.43 & -0.91 & -4.74 & -1.66 & -1.84 & -8.85 & -0.62 & 2.13 & 0.19 & 4.96 & -2.20 & 6.71 & -0.89 & -0.85 \\
0.53 & 1.92 & -5.09 & -2.22 & 1.01 & 1.45 & 5.12 & -0.06 & 4.41 & -0.11 & 2.83 & 3.76 & 1.13 & 1.77 & -3.44 & -3.23 \\
-1.58 & -3.94 & -1.20 & 1.26 & -1.92 & 4.39 & 0.27 & -3.75 & -0.34 & 4.18 & -0.22 & -1.06 & -2.75 & -0.08 & 0.10 & -4.96 \\
1.58 & -1.59 & -1.63 & 1.64 & 0.78 & -1.64 & -0.65 & 0.24 & -1.61 & 1.39 & 0.46 & 4.57 & 5.46 & 0.22 & -1.99 & 2.33 \\
4.03 & 1.31 & -0.60 & -3.39 & -2.79 & -4.01 & -3.67 & 2.57 & 0.08 & 4.16 & 2.57 & 3.55 & 3.31 & 2.06 & 2.52 & 2.79
\end{bmatrix},$$

$A_2 =$

$$\begin{bmatrix}
9.03 & 0.06 & 2.91 & -2.26 & -5.08 & 0.96 & 4.80 & 2.89 & -3.07 & 0.66 & 1.22 & 7.12 & 2.49 & 3.64 & -3.75 & -0.31 \\
3.94 & 4.39 & 1.41 & 3.17 & 1.38 & 4.35 & 5.16 & 3.03 & -4.63 & -0.36 & 3.30 & -1.37 & -5.11 & -1.08 & 0.47 & -3.01 \\
-4.84 & -2.67 & 3.01 & -3.33 & 0.11 & -0.75 & -0.39 & -1.99 & -4.30 & -2.19 & 0.44 & 1.56 & -0.55 & 0.62 & -3.10 & -2.16 \\
5.73 & -0.02 & 2.24 & 4.33 & -3.03 & 1.33 & 0.67 & -4.01 & -3.95 & -0.64 & -5.92 & -3.23 & -2.11 & 3.41 & -1.05 & -2.70 \\
4.07 & -3.79 & -1.95 & 2.14 & 8.16 & 3.62 & -6.00 & -5.51 & -3.72 & 0.23 & 1.28 & 3.49 & -1.25 & -0.70 & 2.95 & 3.92 \\
0.44 & -1.99 & 3.31 & -2.00 & 3.73 & 5.62 & 0.72 & -4.17 & -2.78 & -1.93 & 2.04 & 2.57 & 0.31 & 0.36 & -1.11 & 1.89 \\
-3.08 & -0.92 & 0.18 & -4.91 & 5.84 & 3.57 & 5.36 & 0.58 & -0.70 & -1.73 & 6.55 & 2.80 & 0.05 & 0.74 & 0.29 & -2.54 \\
-0.46 & -5.48 & 1.18 & 0.96 & 6.34 & 1.14 & 1.47 & 9.45 & 3.91 & 1.77 & -5.88 & 3.01 & 3.42 & 2.85 & 3.78 & -4.72 \\
0.97 & 6.99 & -1.52 & 5.21 & 4.37 & 0.49 & 3.15 & -5.95 & 7.17 & 4.41 & 1.43 & -0.07 & 0.75 & -0.79 & -0.60 & -3.90 \\
-1.89 & 0.90 & 0.50 & 3.72 & 1.49 & 1.35 & 1.65 & 0.34 & -0.58 & 4.02 & -1.20 & 1.19 & -3.60 & 4.34 & 0.09 & -4.53 \\
-3.12 & -0.59 & -1.28 & 5.66 & -1.77 & -2.95 & -4.27 & 5.31 & -5.11 & 4.96 & 9.80 & -0.34 & -3.62 & 2.74 & 1.34 & -0.84 \\
-5.23 & -2.58 & 0.58 & 2.01 & -4.69 & -1.77 & -1.96 & -5.35 & -0.11 & 2.01 & 0.31 & 7.43 & 3.19 & 4.74 & -1.87 & -4.23 \\
-1.76 & 3.70 & 0.07 & -2.39 & -0.75 & -0.90 & 4.25 & -4.04 & 1.88 & 3.00 & 3.36 & 2.89 & 3.79 & -1.12 & -0.32 & -0.52 \\
-2.40 & 0.61 & 1.91 & 0.36 & 0.19 & 0.91 & -3.38 & -3.10 & -1.84 & -1.33 & -3.20 & -0.73 & -0.90 & 3.17 & 0.41 & -3.6 \\
5.16 & -2.74 & 0.62 & 3.01 & 0.65 & -0.55 & -2.96 & -1.92 & 2.52 & 3.90 & 0.22 & 4.84 & 0.17 & 1.12 & 2.92 & 0.05 \\
4.19 & 0.25 & -1.07 & 0.89 & -1.35 & -4.58 & -0.32 & 1.51 & 2.59 & 0.62 & 3.15 & 3.76 & 2.37 & -0.84 & 1.88 & 6.43
\end{bmatrix}.$$

According to the procedure of Section 4.2 we can recognize this case as Case $\mathbb{H}$ with $\check{n} = 2$ and $\bar{m} = 2$. This means that we have $P^\top \mathcal{T} P = \mathcal{H}_2 \otimes I_2$ with a suitable choice of an orthogonal matrix $P$.

**Steps 1–5:** We obtain the following matrices by the same procedure as in Case $\mathbb{C}$.

$\tilde{P}^\top A_1 \tilde{P} =$

$$\left[\begin{array}{cccccccc|cccccccc}
8.16 & -0.34 & 13.40 & 0.65 & 1.61 & 0.50 & -0.42 & 0.01 & 5.68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.34 & 8.16 & -0.65 & 13.40 & 0.12 & -0.84 & -0.49 & 1.44 & 0 & 5.68 & 0 & 0 & 0 & 0 & 0 & 0 \\
-13.40 & 0.65 & 8.16 & 0.34 & -0.42 & 0.01 & -1.61 & -0.50 & 0 & 0 & 5.68 & 0 & 0 & 0 & 0 & 0 \\
-0.65 & -13.40 & -0.34 & 8.16 & -0.49 & 1.44 & -0.12 & 0.84 & 0 & 0 & 0 & 5.68 & 0 & 0 & 0 & 0 \\
-1.61 & -0.12 & 0.42 & 0.49 & 8.16 & -0.72 & 13.40 & 0.15 & 0 & 0 & 0 & 0 & 5.68 & 0 & 0 & 0 \\
-0.50 & 0.84 & -0.01 & -1.44 & 0.72 & 8.16 & -0.15 & 13.40 & 0 & 0 & 0 & 0 & 0 & 5.68 & 0 & 0 \\
0.42 & 0.49 & 1.61 & 0.12 & -13.40 & 0.15 & 8.16 & 0.72 & 0 & 0 & 0 & 0 & 0 & 0 & 5.68 & 0 \\
-0.01 & -1.44 & 0.50 & -0.84 & -0.15 & -13.40 & -0.72 & 8.16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.68 \\
\hline
0.29 & -0.26 & 1.25 & -0.76 & -0.90 & -0.40 & 1.60 & 0.39 & -2.74 & -0.29 & 2.87 & -2.07 & -3.06 & -1.20 & 3.64 & 0.80 \\
0.26 & 0.29 & 0.76 & 1.25 & 0.29 & -0.50 & 0.47 & -1.76 & 0.29 & -2.74 & 2.07 & 2.87 & 0.52 & -0.43 & 1.34 & -4.73 \\
-1.25 & -0.76 & 0.29 & 0.26 & 1.60 & 0.39 & 0.90 & 0.40 & -2.87 & -2.07 & -2.74 & 0.29 & 3.64 & 0.80 & 3.06 & 1.20 \\
0.76 & -1.25 & -0.26 & 0.29 & 0.47 & -1.76 & -0.29 & 0.50 & 2.07 & -2.87 & -0.29 & -2.74 & 1.34 & -4.73 & -0.52 & 0.43 \\
0.90 & -0.29 & -1.60 & -0.47 & 0.29 & 0.43 & 1.25 & -0.68 & 3.06 & -0.52 & -3.64 & -1.34 & -2.74 & 1.43 & 2.87 & -1.53 \\
0.40 & 0.50 & -0.39 & 1.76 & -0.43 & 0.29 & 0.68 & 1.25 & 1.20 & 0.43 & -0.80 & 4.73 & -1.43 & -2.74 & 1.53 & 2.87 \\
-1.60 & -0.47 & -0.90 & 0.29 & -1.25 & -0.68 & 0.29 & -0.43 & -3.64 & -1.34 & -3.06 & 0.52 & -2.87 & -1.53 & -2.74 & -1.43 \\
-0.39 & 1.76 & -0.40 & -0.50 & 0.68 & -1.25 & 0.43 & 0.29 & -0.80 & 4.73 & -1.20 & -0.43 & 1.53 & -2.87 & 1.43 & -2.74
\end{array}\right],$$

$\tilde{P}^\top A_2 \tilde{P} =$

$$
\left[\begin{array}{rrrrrrrr|rrrrrrrr}
10.30 & 0.53 & 13.20 & -1.02 & -2.54 & -0.79 & 0.65 & -0.02 & 0.91 & -1.07 & -3.95 & -0.44 & 0.88 & 0.05 & 2.48 & 0.80 \\
-0.53 & 10.30 & 1.02 & 13.20 & -0.19 & 1.32 & 0.77 & -2.26 & 1.07 & 0.91 & 0.44 & -3.95 & 0.79 & -2.38 & 0.13 & -1.12 \\
-13.20 & -1.02 & 10.30 & -0.53 & 0.65 & -0.02 & 2.54 & 0.79 & 3.95 & -0.44 & 0.91 & 1.07 & 2.48 & 0.80 & -0.88 & -0.05 \\
1.02 & -13.20 & 0.53 & 10.30 & 0.77 & -2.26 & 0.19 & -1.32 & 0.44 & 3.95 & -1.07 & 0.91 & 0.13 & -1.12 & -0.79 & 2.38 \\
2.54 & 0.19 & -0.65 & -0.77 & 10.30 & 1.13 & 13.20 & -0.23 & -0.88 & -0.79 & -2.48 & -0.13 & 0.91 & -0.33 & -3.95 & -1.11 \\
0.79 & -1.32 & 0.02 & 2.26 & -1.13 & 10.30 & 0.23 & 13.20 & -0.05 & 2.38 & -0.80 & 1.12 & 0.33 & 0.91 & 1.11 & -3.95 \\
-0.65 & -0.77 & -2.54 & -0.19 & -13.20 & -0.23 & 10.30 & -1.13 & -2.48 & -0.13 & 0.88 & 0.79 & 3.95 & -1.11 & 0.91 & 0.33 \\
0.02 & 2.26 & -0.79 & 1.32 & 0.23 & -13.20 & 1.13 & 10.30 & -0.80 & 1.12 & 0.05 & -2.38 & 1.11 & 3.95 & -0.33 & 0.91 \\
\hline
-0.45 & 0.41 & -1.96 & 1.20 & 1.41 & 0.62 & -2.52 & -0.61 & 1.44 & -0.14 & 1.75 & 0.89 & 1.73 & 0.60 & -1.11 & -0.18 \\
-0.41 & -0.45 & -1.20 & -1.96 & -0.46 & 0.79 & -0.74 & 2.77 & 0.14 & 1.44 & -0.89 & 1.75 & -0.05 & -0.43 & -0.62 & 2.01 \\
1.96 & 1.20 & -0.45 & -0.41 & -2.52 & -0.61 & -1.41 & -0.62 & -1.75 & 0.89 & 1.44 & 0.14 & -1.11 & -0.18 & -1.73 & -0.60 \\
-1.20 & 1.96 & 0.41 & -0.45 & -0.74 & 2.77 & 0.46 & -0.79 & -0.89 & -1.75 & -0.14 & 1.44 & -0.62 & 2.01 & 0.05 & 0.43 \\
-1.41 & 0.46 & 2.52 & 0.74 & -0.45 & -0.68 & -1.96 & 1.07 & -1.73 & 0.05 & 1.11 & 0.62 & 1.44 & -0.78 & 1.75 & 0.45 \\
-0.62 & -0.79 & 0.61 & -2.77 & 0.68 & -0.45 & -1.07 & -1.96 & -0.60 & 0.43 & 0.18 & -2.01 & 0.78 & 1.44 & -0.45 & 1.75 \\
2.52 & 0.74 & 1.41 & -0.46 & 1.96 & 1.07 & -0.45 & 0.68 & 1.11 & 0.62 & 1.73 & -0.05 & -1.75 & 0.45 & 1.44 & 0.78 \\
0.61 & -2.77 & 0.62 & 0.79 & -1.07 & 1.96 & -0.68 & -0.45 & 0.18 & -2.01 & 0.60 & -0.43 & -0.45 & -1.75 & -0.78 & 1.44
\end{array}\right],
$$

where $\tilde{P}$ is an orthogonal matrix given as
$\tilde{P} =$

$$
\left[\begin{array}{rrrrrrrr|rrrrrrrr}
0.06 & -0.41 & -0.03 & -0.41 & -0.11 & -0.33 & -0.57 & -0.02 & -0.34 & -0.15 & -0.02 & -0.12 & -0.12 & 0.06 & 0.13 & 0.16 \\
-0.08 & -0.24 & -0.28 & -0.02 & 0.10 & -0.03 & -0.17 & 0.49 & 0.28 & -0.08 & 0.29 & -0.06 & -0.16 & -0.11 & -0.27 & -0.54 \\
0.02 & 0.29 & -0.09 & -0.07 & 0.19 & -0.24 & -0.22 & -0.07 & -0.06 & -0.21 & -0.05 & 0.49 & 0.31 & -0.59 & -0.12 & -0.02 \\
-0.31 & 0.04 & 0.01 & 0.01 & -0.34 & -0.39 & -0.04 & 0.29 & 0.38 & 0.04 & -0.38 & 0.25 & 0.24 & 0.35 & -0.06 & 0.11 \\
0.46 & 0.16 & -0.26 & 0.15 & -0.56 & -0.04 & 0.20 & 0.22 & -0.14 & -0.44 & 0.01 & 0.04 & -0.12 & -0.04 & 0.20 & -0.04 \\
0.24 & 0.32 & -0.35 & -0.19 & -0.17 & -0.18 & -0.11 & -0.06 & 0.18 & 0.60 & -0.12 & -0.37 & -0.04 & -0.24 & 0.01 & 0.01 \\
0.17 & 0.22 & -0.48 & -0.32 & 0.33 & 0.24 & -0.05 & 0.12 & 0.02 & -0.01 & 0.16 & 0.29 & 0.04 & 0.48 & 0.03 & 0.25 \\
0.14 & -0.29 & -0.05 & -0.50 & 0.13 & -0.27 & 0.65 & -0.13 & 0.04 & -0.08 & -0.21 & 0.05 & -0.09 & -0.07 & -0.18 & -0.07 \\
-0.25 & -0.08 & -0.00 & -0.30 & -0.38 & 0.56 & 0.02 & 0.25 & -0.24 & 0.17 & -0.14 & 0.15 & -0.05 & -0.26 & -0.29 & 0.19 \\
-0.18 & 0.07 & 0.24 & -0.26 & 0.03 & -0.11 & 0.20 & 0.37 & 0.02 & 0.19 & 0.34 & 0.12 & -0.03 & -0.19 & 0.67 & 0.03 \\
0.39 & 0.05 & 0.30 & 0.06 & 0.38 & 0.04 & -0.04 & 0.58 & -0.14 & 0.01 & -0.42 & -0.22 & 0.13 & -0.02 & -0.05 & 0.08 \\
0.23 & 0.37 & 0.52 & -0.34 & -0.14 & 0.06 & -0.20 & -0.12 & 0.12 & 0.00 & 0.01 & 0.20 & -0.30 & 0.21 & -0.14 & -0.37 \\
-0.01 & -0.09 & -0.05 & -0.27 & -0.02 & 0.41 & -0.12 & -0.16 & 0.41 & -0.31 & -0.29 & -0.22 & 0.34 & -0.09 & 0.38 & -0.20 \\
-0.13 & 0.27 & 0.17 & -0.19 & -0.03 & -0.09 & 0.05 & 0.07 & 0.26 & -0.39 & 0.34 & -0.44 & 0.02 & -0.09 & -0.30 & 0.46 \\
0.27 & -0.16 & 0.13 & -0.10 & -0.22 & -0.02 & 0.06 & -0.00 & -0.15 & 0.19 & 0.40 & -0.03 & 0.73 & 0.15 & -0.17 & -0.15 \\
0.44 & -0.41 & 0.15 & 0.14 & -0.05 & 0.10 & -0.11 & -0.03 & 0.51 & 0.14 & 0.12 & 0.30 & -0.13 & -0.17 & -0.01 & 0.38
\end{array}\right].
$$

**Step 6:** Put $X = (\tilde{P}^\top A_1 \tilde{P})_{[1,1]}$, which contains $\jmath$ and $k$ components:

$$
X = \begin{bmatrix}
8.16 & -0.34 & 13.40 & 0.65 & 1.61 & 0.50 & -0.42 & 0.01 \\
0.34 & 8.16 & -0.65 & 13.40 & 0.12 & -0.84 & -0.49 & 1.44 \\
-13.40 & 0.65 & 8.16 & 0.34 & -0.42 & 0.01 & -1.61 & -0.50 \\
-0.65 & -13.40 & -0.34 & 8.16 & -0.49 & 1.44 & -0.12 & 0.84 \\
-1.61 & -0.12 & 0.42 & 0.49 & 8.16 & -0.72 & 13.40 & 0.15 \\
-0.50 & 0.84 & -0.01 & -1.44 & 0.72 & 8.16 & -0.15 & 13.40 \\
0.42 & 0.49 & 1.61 & 0.12 & -13.40 & 0.15 & 8.16 & 0.72 \\
-0.01 & -1.44 & 0.50 & -0.84 & -0.15 & -13.40 & -0.72 & 8.16
\end{bmatrix}.
$$

We have $a = 8.16$, $b = 13.40$, and hence

$$
X^{(2)} = X - aI_8 - bJ' = \begin{bmatrix}
0 & -0.34 & 0 & 0.65 & 1.61 & 0.50 & -0.42 & 0.01 \\
0.34 & 0 & -0.65 & 0 & 0.12 & -0.84 & -0.49 & 1.44 \\
0 & 0.65 & 0 & 0.34 & -0.42 & 0.01 & -1.61 & -0.50 \\
-0.65 & 0 & -0.34 & 0 & -0.49 & 1.44 & -0.12 & 0.84 \\
-1.61 & -0.12 & 0.42 & 0.49 & 0 & -0.72 & 0 & 0.15 \\
-0.50 & 0.84 & -0.01 & -1.44 & 0.72 & 0 & -0.15 & 0 \\
0.42 & 0.49 & 1.61 & 0.12 & 0 & 0.15 & 0 & 0.72 \\
-0.01 & -1.44 & 0.50 & -0.84 & -0.15 & 0 & -0.72 & 0
\end{bmatrix}.
$$

By using the permutation and the skew-Hamiltonian Schur decomposition, we obtain

$$\hat{S}^\top X^{(2)} \hat{S} = \begin{bmatrix} 0 & -1.89 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.89 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.89 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.89 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.89 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.89 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.89 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.89 & 0 \end{bmatrix},$$

where $\hat{S}$ is a orthogonal permuted symplectic matrix given by

$$\hat{S} = \begin{bmatrix} 0.81 & 0.41 & 0.00 & 0.00 & -0.08 & 0.41 & 0.00 & 0.00 \\ -0.47 & 0.27 & 0.15 & -0.03 & -0.34 & 0.61 & -0.29 & -0.33 \\ 0.00 & 0.00 & 0.81 & 0.41 & 0.00 & 0.00 & -0.08 & 0.41 \\ -0.15 & 0.03 & -0.47 & 0.27 & 0.29 & 0.33 & -0.34 & 0.61 \\ 0.02 & -0.34 & 0.26 & -0.42 & 0.66 & 0.43 & 0.08 & -0.09 \\ -0.09 & 0.07 & -0.11 & 0.68 & 0.28 & 0.17 & 0.52 & -0.35 \\ -0.26 & 0.42 & 0.02 & -0.34 & -0.08 & 0.09 & 0.66 & 0.43 \\ 0.11 & -0.68 & -0.09 & 0.07 & -0.52 & 0.35 & 0.28 & 0.17 \end{bmatrix}.$$

**Step 7:** The orthogonal matrix $P = \tilde{P} \cdot \operatorname{diag}(\hat{S}, \hat{S})$ gives the decomposition $P^\top \mathcal{T} P = \mathcal{H}_2 \otimes I_2$ as follows:

$P^\top A_1 P =$

$$\left[\begin{array}{rrrrrrrr|rrrrrrrr} 8.16 & 0 & 13.40 & 0 & 1.89 & 0 & 0 & 0 & 5.68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8.16 & 0 & 13.40 & 0 & 1.89 & 0 & 0 & 0 & 5.68 & 0 & 0 & 0 & 0 & 0 & 0 \\ -13.40 & 0 & 8.16 & 0 & 0 & 0 & -1.89 & 0 & 0 & 0 & 5.68 & 0 & 0 & 0 & 0 & 0 \\ 0 & -13.40 & 0 & 8.16 & 0 & 0 & 0 & -1.89 & 0 & 0 & 0 & 5.68 & 0 & 0 & 0 & 0 \\ -1.89 & 0 & 0 & 0 & 8.16 & 0 & 13.40 & 0 & 0 & 0 & 0 & 0 & 5.68 & 0 & 0 & 0 \\ 0 & -1.89 & 0 & 0 & 0 & 8.16 & 0 & 13.40 & 0 & 0 & 0 & 0 & 0 & 5.68 & 0 & 0 \\ 0 & 0 & 1.89 & 0 & -13.40 & 0 & 8.16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.68 & 0 \\ 0 & 0 & 0 & 1.89 & 0 & -13.40 & 0 & 8.16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5.68 \\ \hline 0.29 & 0 & 1.25 & 0 & -1.44 & 0 & 1.50 & 0 & -2.74 & 0 & 2.87 & 0 & -4.39 & 0 & 3.12 & 0 \\ 0 & 0.29 & 0 & 1.25 & 0 & -1.44 & 0 & 1.50 & 0 & -2.74 & 0 & 2.87 & 0 & -4.39 & 0 & 3.12 \\ -1.25 & 0 & 0.29 & 0 & 1.50 & 0 & 1.44 & 0 & -2.87 & 0 & -2.74 & 0 & 3.12 & 0 & 4.39 & 0 \\ 0 & -1.25 & 0 & 0.29 & 0 & 1.50 & 0 & 1.44 & 0 & -2.87 & 0 & -2.74 & 0 & 3.12 & 0 & 4.39 \\ 1.44 & 0 & -1.50 & 0 & 0.29 & 0 & 1.25 & 0 & 4.39 & 0 & -3.12 & 0 & -2.74 & 0 & 2.87 & 0 \\ 0 & 1.44 & 0 & -1.50 & 0 & 0.29 & 0 & 1.25 & 0 & 4.39 & 0 & -3.12 & 0 & -2.74 & 0 & 2.87 \\ -1.50 & 0 & -1.44 & 0 & -1.25 & 0 & 0.29 & 0 & -3.12 & 0 & -4.39 & 0 & -2.87 & 0 & -2.74 & 0 \\ 0 & -1.50 & 0 & -1.44 & 0 & -1.25 & 0 & 0.29 & 0 & -3.12 & 0 & -4.39 & 0 & -2.87 & 0 & -2.74 \end{array}\right],$$

$P^\top A_2 P =$

$$\left[\begin{array}{rrrrrrrr|rrrrrrrr} 10.30 & 0 & 13.20 & 0 & -2.97 & 0 & 0 & 0 & 0.91 & 0 & -3.95 & 0 & 0.26 & 0 & 2.97 & 0 \\ 0 & 10.30 & 0 & 13.20 & 0 & -2.97 & 0 & 0 & 0 & 0.91 & 0 & -3.95 & 0 & 0.26 & 0 & 2.97 \\ -13.20 & 0 & 10.30 & 0 & 0 & 0 & 2.97 & 0 & 3.95 & 0 & 0.91 & 0 & 2.97 & 0 & -0.26 & 0 \\ 0 & -13.20 & 0 & 10.30 & 0 & 0 & 0 & 2.97 & 0 & 3.95 & 0 & 0.91 & 0 & 2.97 & 0 & -0.26 \\ 2.97 & 0 & 0 & 0 & 10.30 & 0 & 13.20 & 0 & -0.26 & 0 & -2.97 & 0 & 0.91 & 0 & -3.95 & 0 \\ 0 & 2.97 & 0 & 0 & 0 & 10.30 & 0 & 13.20 & 0 & -0.26 & 0 & -2.97 & 0 & 0.91 & 0 & -3.95 \\ 0 & 0 & -2.97 & 0 & -13.20 & 0 & 10.30 & 0 & -2.97 & 0 & 0.26 & 0 & 3.95 & 0 & 0.91 & 0 \\ 0 & 0 & 0 & -2.97 & 0 & -13.20 & 0 & 10.30 & 0 & -2.97 & 0 & 0.26 & 0 & 3.95 & 0 & 0.91 \\ \hline -0.45 & 0 & -1.96 & 0 & 2.26 & 0 & -2.36 & 0 & 1.44 & 0 & 1.75 & 0 & 2.21 & 0 & -0.73 & 0 \\ 0 & -0.45 & 0 & -1.96 & 0 & 2.26 & 0 & -2.36 & 0 & 1.44 & 0 & 1.75 & 0 & 2.21 & 0 & -0.73 \\ 1.96 & 0 & -0.45 & 0 & -2.36 & 0 & -2.26 & 0 & -1.75 & 0 & 1.44 & 0 & -0.73 & 0 & -2.21 & 0 \\ 0 & 1.96 & 0 & -0.45 & 0 & -2.36 & 0 & -2.26 & 0 & -1.75 & 0 & 1.44 & 0 & -0.73 & 0 & -2.21 \\ -2.27 & 0 & 2.36 & 0 & -0.45 & 0 & -1.96 & 0 & -2.21 & 0 & 0.73 & 0 & 1.44 & 0 & 1.75 & 0 \\ 0 & -2.26 & 0 & 2.36 & 0 & -0.45 & 0 & -1.96 & 0 & -2.21 & 0 & 0.73 & 0 & 1.44 & 0 & 1.75 \\ 2.36 & 0 & 2.26 & 0 & 1.96 & 0 & -0.45 & 0 & 0.73 & 0 & 2.21 & 0 & -1.75 & 0 & 1.44 & 0 \\ 0 & 2.36 & 0 & 2.26 & 0 & 1.96 & 0 & -0.45 & 0 & 0.73 & 0 & 2.21 & 0 & -1.75 & 0 & 1.44 \end{array}\right].$$

With a permutation matrix $\Pi$, explicit block diagonal forms can be obtained:

$\Pi^\top P^\top A_1 P \Pi =$

$$
\left[\begin{array}{rrrrrrrr|rrrrrrrr}
8.16 & -1.89 & 13.40 & 0 & 5.68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.89 & 8.16 & 0 & 13.40 & 0 & 5.68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-13.40 & 0 & 8.16 & 1.89 & 0 & 0 & 5.68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -13.40 & -1.89 & 8.16 & 0 & 0 & 0 & 5.68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.29 & 1.44 & 1.25 & -1.50 & -2.74 & 4.39 & 2.87 & -3.12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.44 & 0.29 & 1.50 & 1.25 & -4.39 & -2.74 & 3.12 & 2.87 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.25 & -1.50 & 0.29 & -1.44 & -2.87 & -3.12 & -2.74 & -4.39 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.50 & -1.25 & 1.44 & 0.29 & 3.12 & -2.87 & 4.39 & -2.74 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8.16 & -1.89 & 13.40 & 0 & 5.68 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.89 & 8.16 & 0 & 13.40 & 0 & 5.68 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -13.40 & 0 & 8.16 & 1.89 & 0 & 0 & 5.68 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -13.40 & -1.89 & 8.16 & 0 & 0 & 0 & 5.68 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.29 & 1.44 & 1.25 & -1.50 & -2.74 & 4.39 & 2.87 & -3.12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.44 & 0.29 & 1.50 & 1.25 & -4.39 & -2.74 & 3.12 & 2.87 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.25 & -1.50 & 0.29 & -1.44 & -2.87 & -3.12 & -2.74 & -4.39 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.50 & -1.25 & 1.44 & 0.29 & 3.12 & -2.87 & 4.39 & -2.74
\end{array}\right],
$$

$\Pi^\top P^\top A_2 P \Pi =$

$$
\left[\begin{array}{rrrrrrrr|rrrrrrrr}
10.30 & 13.20 & -2.97 & 0 & 0.91 & -3.95 & 0.26 & 2.97 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-13.20 & 10.30 & 0 & 2.97 & 3.95 & 0.91 & 2.97 & -0.26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2.97 & 0 & 10.30 & 13.20 & -0.26 & -2.97 & 0.91 & -3.95 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2.97 & -13.20 & 10.30 & -2.97 & 0.26 & 3.95 & 0.91 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.45 & -1.96 & 2.26 & -2.36 & 1.44 & 1.75 & 2.21 & -0.73 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.96 & -0.45 & -2.36 & -2.26 & -1.75 & 1.44 & -0.73 & -2.21 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2.27 & 2.36 & -0.45 & -1.96 & -2.21 & 0.73 & 1.44 & 1.75 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2.36 & 2.26 & 1.96 & -0.45 & 0.73 & 2.21 & -1.75 & 1.44 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10.30 & 13.20 & -2.97 & 0 & 0.91 & -3.95 & 0.26 & 2.97 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -13.20 & 10.30 & 0 & 2.97 & 3.95 & 0.91 & 2.97 & -0.26 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.97 & 0 & 10.30 & 13.20 & -0.26 & -2.97 & 0.91 & -3.95 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.97 & -13.20 & 10.30 & -2.97 & 0.26 & 3.95 & 0.91 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.45 & -1.96 & 2.26 & -2.36 & 1.44 & 1.75 & 2.21 & -0.73 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.96 & -0.45 & -2.36 & -2.26 & -1.75 & 1.44 & -0.73 & -2.21 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.26 & 2.36 & -0.45 & -1.96 & -2.21 & 0.73 & 1.44 & 1.75 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.36 & 2.26 & 1.96 & -0.45 & 0.73 & 2.21 & -1.75 & 1.44
\end{array}\right].
$$

## 6 Conclusion

We have considered the problem of simultaneous block-diagonal decomposition, which is to find an orthogonal matrix $P$ in the structure theorem for the matrix $*$-algebra generated by a given set of real square matrices $A_1, \ldots, A_N$. For this problem, we have proposed an algorithm, which is an extension of the algorithm given by [12]. While the algorithm of [12] is targeted to a special case (Case $\mathbb{R}$), our algorithm can cope with all possible cases, Case $\mathbb{R}$, Case $\mathbb{C}$ and Case $\mathbb{H}$.

In [12] a variant of the algorithm is suggested for practical efficiency in relation to the following two technical conditions:

1. $\mathcal{T} = \mathrm{span}\{I_n, A_1, \ldots, A_N\}$,
2. $r \in R$, where $R$ is an open dense set,

which also appear in Proposition 3.3 and Proposition 4.1 of the present paper to ensure genericity of $A(r) = r_1 A_1 + \cdots + r_N A_N$ and $A(r)^\top + A(r)$. The variant suggested in [12] executes the original algorithm without regard to the first condition, and in case of any inconsistency during the execution, restarts by adding transposes or products of some of the generators to the current set $\{A_1, \ldots, A_N\}$. A similar variant is conceivable for our algorithm, which we report elsewhere along with applications to practical problems.

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