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# An Implicit Formulation of Robust Structural Optimization under Load Uncertainties

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**Abstract**

This paper discusses an implicit reformulation of the MPEC (mathematical program with complementarity constraints) problem in order to solve a robust structural optimization with a non-probabilistic uncertainty model of the static load. We first show the relation among the robust constraint satisfaction, worst scenario detection, and robust structural optimization, and derive the MPEC formulation of the robust structural optimization. Since MPEC does not satisfy a standard constraint qualification, we propose a reformulation based on the smoothed Fischer–Burmeister function, in which the smoothing parameter is treated as an independent variable. Numerical examples of robust truss design are presented in order to demonstrate that the presented formulation can be solved by using a standard nonlinear programming approach without any difficulty.

**Keywords**

Robust optimization; Structural optimization; Mathematical program with equilibrium constraints; Complementarity condition; Fischer–Burmeister function.

## 1 Introduction

Recently, methodologies as well as numerical techniques for robust structural design have received increasing attention in structural and mechanical design. Since structures built in the real-world always have various uncertainties caused by manufacture errors, limitation of knowledge of input disturbance, observation errors, etc., the notion of robust structural design is required naturally in which structures should always satisfy the given constraints on mechanical performance under various uncertainties [2, 5, 8, 11, 21, 26, 29, 33, 34, 39].

There exist two different frameworks with which we consider the uncertain property of a structural system. The one is probabilistic uncertainty modeling and the other non-probabilistic uncertainty modeling. Based on the probabilistic uncertainty model, various methods have been well-developed for reliability-based optimization (see, e.g., [8, 33, 39], and the references therein). Probabilistic robust design approaches require information on stochastic variation of the uncertain

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parameters, e.g. parameters of the probability density function as well as an appropriate model of the probability density function itself. However, it is often difficult to estimate those parameters accurately, especially when the number of samples of the uncertain parameters is limited.

In contrast to probabilistic approaches, the non-probabilistic uncertainty framework treats the uncertain parameters as the so-called *unknown-but-bounded* parameters, and hence it is not necessary to estimate the probability distribution of the uncertain parameters. One of well-known approaches with a non-probabilistic uncertainty model is the so-called *convex model* method [4]; based on this method, numerical algorithms were suggested for robust structural optimization [2, 29]. Note that these convex model approaches are valid only if the magnitude of uncertainty is small enough, because the convex model method is essentially based on the first-order approximation of the response of a structure with respect to the uncertain parameters. Lee and Park [26] also presented a robust structural optimization based on the first-order approximation.

There exist two closely related methodologies with which we can address arbitrary large magnitude of uncertainty in a structural system. The one is the notion of *robust counterpart* of optimization problem proposed by Ben-Tal and Nemirovski [6, 7], and another the *info-gap decision theory* proposed by Ben-Haim [3]. A unified methodology of robust counterpart was presented for a broader class of convex optimization problems [7], in which the given data of an optimization problem possess non-probabilistic uncertainty. This methodology was applied to the compliance minimization problem of a truss considering load uncertainty [5]. A min-max formulation of a robust compliance design was presented for continua [11]. Kočvara *et al.* [24] performed a free-material design under multiple loadings by using a cascading technique.

In the info-gap decision theory, the *robustness function* plays a key role which represents the greatest level of uncertainty at which any failure cannot occur [3]. By using the robustness function, Kanno and Takewaki [21] proposed a robustness maximization problem of a truss subjected to the stress constraints considering the load uncertainty.

In this paper, we consider a non-probabilistic uncertainty of static external load in the context of structural optimization. In accordance with the notion of robust constraint satisfaction which are shared by the robust optimization methodology [7] and the info-gap theory [3], we present a robust structural optimization, in which any constraint of mechanical performance cannot be violated at the given magnitude of uncertainty. As the first contribution of the paper, we clarify the relation among the robust constraint satisfaction, worst scenario detection problem, and robust structural optimization problem (see section 2).

It is shown that the robust structural optimization problem can be reformulated as an MPEC (*mathematical program with complementarity constraints*) problem. Since the MPEC problem does not satisfy any standard constraint qualification [25], standard nonlinear programming approaches are likely to fail for this problem. This motivates us to propose an implicit reformulation scheme of MPEC by using the smoothed Fischer–Burmeister function [16, 23], which is the second contribution of this paper (see section 4).

It is known that various problems in structural engineering can be formulated as MPECs [1, 14, 20, 36, 37]. Numerous smoothing methods, as well as regularization schemes, were also proposed for MPEC [10, 12, 13, 16, 18–20, 32, 36, 38]. An MPEC problem includes the complementarity conditions

$$g_i(\boldsymbol{x}) \geq 0, \quad h_i(\boldsymbol{x}) \geq 0, \quad g_i(\boldsymbol{x})h_i(\boldsymbol{x}) = 0, \qquad i = 1, \dots, M \tag{1}$$

as its constraint conditions, where $g_i$, $h_i : \mathbb{R}^N \to \mathbb{R}$ $(i = 1, \dots, M)$. In a typical regularization scheme for MPEC, the complementarity constraints, (1), are relaxed as [32]

$$g_i(\boldsymbol{x}) \geq 0, \quad h_i(\boldsymbol{x}) \geq 0, \quad g_i(\boldsymbol{x})h_i(\boldsymbol{x}) \leq \varepsilon, \qquad i = 1, \dots, M, \tag{2}$$

where $\varepsilon > 0$ is a constant. Then a sequence of the relaxed optimization problem with (2) is solved by using a standard nonlinear programming approach, e.g. the SQP method, by gradually decreasing $\epsilon \searrow 0$. In contrast, in a smoothing method for MPEC, the complementarity constraints (1) are replaced with [16, 36]

$$\psi(g_i(\boldsymbol{x}), h_i(\boldsymbol{x}); \varepsilon) = 0, \qquad i = 1, \dots, M, \tag{3}$$

where $\varepsilon > 0$ is a constant, and $\psi(\,\cdot\,; 0) : \mathbb{R}^2 \to \mathbb{R}$ is a complementarity function, i.e. $\psi(a, b; 0) = 0$ holds if and only if $a \geq 0$, $b \geq 0$, and $ab = 0$. In a manner similar to the relaxation method, the smooth optimization problem with (3) is solved sequentially by decreasing $\varepsilon$.

It seems that there exists neither guideline for the choice of an initial value for $\varepsilon$ nor decreasing strategy of $\varepsilon$. Note that the convergence of algorithms can be proved theoretically irrelevant to the initial value of $\varepsilon$ and the decreasing strategy. However, the computational efficiency certainly depends on those choices; too rapid reduction of $\varepsilon$ is not adequate in order to avoid the nonsmooth property of the complementarity function, but unnecessary iterations might be spent if we decrease $\varepsilon$ too slowly; see Remark 4.4 for details.

This observation motivates us to propose a new reformulation of MPEC in which the smoothing parameter $\varepsilon$ is automatically adjusted. Then the presented formulation is solved by using a standard nonlinear optimization method. In our formulation, we treat $\varepsilon$ in (3) as an independent variable which approximately represents the residual of complementarity conditions. As the residual of the complementarity constraints becomes smaller, the smoothing parameter $\varepsilon$ becomes smaller. At the convergent solution, it is guaranteed that $\varepsilon$ vanishes automatically, and hence the complementarity constraints are satisfied exactly; see section 4 for details.

This paper is organized as follows. Section 2 describes the notions of the robust structural optimization and the worst-case detection as well as their relations. In section 3, the robust structural optimization problem is reformulated as an MPEC. We present an implicit reformulation of the MPEC in section 4. Numerical results are shown in section 5; truss optimization problems with stress constraints are in sections 5.1 and 5.2 and moderately large problem with displacement constraints in section 5.3. Finally, conclusions are drawn in section 6.

A few words regarding our notation: all vectors are assumed to be column vectors. The $(m+n)$-dimensional column vector $(\boldsymbol{u}^{\mathrm{T}}, \boldsymbol{v}^{\mathrm{T}})^{\mathrm{T}}$ consisting of $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{v} \in \mathbb{R}^n$ is often written simply as $(\boldsymbol{u}, \boldsymbol{v})$. We denote by $\mathcal{S}^n \subset \mathbb{R}^{n \times n}$ the set of all $n \times n$ real symmetric matrices. For two sets $\mathcal{A} \subseteq \mathbb{R}^m$ and $\mathcal{B} \subseteq \mathbb{R}^n$, their Cartesian product is defined by $\mathcal{A} \times \mathcal{B} = \{(\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{m+n} \mid \boldsymbol{a} \in \mathcal{A},\ \boldsymbol{b} \in \mathcal{B}\}$. Particularly, we write $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. For a vector $\boldsymbol{p} = (p_i) \in \mathbb{R}^n$, we denote by $\|\boldsymbol{p}\|_\infty$ the $\ell_\infty$-norm of $\boldsymbol{p}$ defined by $\|\boldsymbol{p}\|_\infty = \max_{i \in \{1,\dots,n\}} |p_i|$. We write $\boldsymbol{p} \geq \boldsymbol{0}$ if $p_i \geq 0$ $(i = 1, \dots, n)$. We denote by $I$ and $\boldsymbol{1}$ the $n \times n$ identity matrix and the vector $(1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^n$, respectively, without specifying $n$, unless it is not clear from the context.

## 2 Robust structural optimization

Consider a finite-dimensional linear elastic structure subjected to the nodal loads $\boldsymbol{f} \in \mathbb{R}^d$, where $d$ is the number of degrees of freedom of displacements. Small displacements and small strains are assumed. Let $\boldsymbol{x} \in \mathbb{R}^m$ denote the vector of design variables, where $m$ denotes the number of design variables. For example, we may choose $\boldsymbol{x}$ as the cross-sectional areas of a truss, the thickness of a plate discretized into finite elements, etc. We denote by $K(\boldsymbol{x}) \in \mathcal{S}^d$ the stiffness matrix, which is a (matrix-valued) function of $\boldsymbol{x}$. The displacements vector $\boldsymbol{u} \in \mathbb{R}^d$ is found from the system of equilibrium equations

$$K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}, \tag{4}$$

where $\boldsymbol{f}$ is assumed to be independent of $\boldsymbol{x}$.

### 2.1 Structural optimization without uncertainty

We recall the conventional structural optimization problem before discussing the robust optimization.

Consider the mechanical performance of structures that can be expressed by the constraints in terms of the displacements. In this paper, we restrict ourselves to the linear constraints written as

$$\boldsymbol{a}_j^{\mathrm{T}}\boldsymbol{u} \le b_j, \qquad j = 1, \dots, n^{\mathrm{c}}, \tag{5}$$

where $\boldsymbol{a}_j \in \mathbb{R}^d$ and $b_j \in \mathbb{R}$ $(j = 1, \dots, n^{\mathrm{c}})$ are constant, $n^{\mathrm{c}}$ is the number of constraints, and $\boldsymbol{u}$ satisfies (4).

Let $v : \boldsymbol{x} \mapsto v(\boldsymbol{x})$ denote the function which gives the structural volume. We denote by $\mathcal{X} \subseteq \mathbb{R}^m$ the set of the admissible design variables, e.g. the lower and upper bound constraints for $\boldsymbol{x}$ is represented as $\boldsymbol{x} \in \mathcal{X}$. Consider the conventional structural optimization problem which attempts to minimize the structural volume $v(\boldsymbol{x})$ over the constraints (5). This problem is formulated as

$$\left.\begin{array}{ll} \displaystyle\min_{\boldsymbol{x} \in \mathcal{X}, \boldsymbol{u}} & v(\boldsymbol{x}) \\ \mathrm{s.t.} & K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}, \\ & \boldsymbol{a}_j^{\mathrm{T}}\boldsymbol{u} \le b_j, \qquad j = 1, \dots, n^{\mathrm{c}}, \end{array}\right\} \tag{6}$$

where $\boldsymbol{x}$ and $\boldsymbol{u}$ are the variables.

### 2.2 Non-stochastic model of uncertainty

We define a non-probabilistic uncertainty model of the external load.

Suppose that the external load $\boldsymbol{f}$ in (4) is not known precisely, or is uncertain. Throughout the paper we assume that the uncertainty exists only in $\boldsymbol{f}$, and that the other parameters in a structural system are known precisely. Let $\tilde{\boldsymbol{f}} \in \mathbb{R}^d$ denote the nominal value, or the best estimate, of the external load $\boldsymbol{f}$. We describe the uncertainty of $\boldsymbol{f} = (f_j) \in \mathbb{R}^d$ by using unknown parameters $\boldsymbol{\zeta} = (\zeta_p) \in \mathbb{R}^k$, where $k \le d$. Assume that $\boldsymbol{f}$ depend on $\boldsymbol{\zeta}$ affinely as

$$\boldsymbol{f} \in \mathcal{F}(\alpha) := \left\{ \boldsymbol{f} \mid \boldsymbol{f} = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta},\ \alpha \ge \|\boldsymbol{\zeta}\|_\infty \right\}, \tag{7}$$

where $F_0 \in \mathbb{R}^{d \times k}$ is a constant matrix.

Note that the matrix $F_0$ represents the magnitude of the uncertainty of $f_j$ and the relationship of the uncertainties among $f_1, \dots, f_d$. Throughout the paper we assume that $\mathrm{rank}(F_0) = k \leq d$ for simplicity.

### 2.3 Robust structural optimization

When the external load $\boldsymbol{f}$ takes any value in the uncertainty set $\mathcal{F}(\alpha)$ defined in (7), the displacements vector $\boldsymbol{u}$ is running through the set $\{\boldsymbol{u} \mid K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f},\ \boldsymbol{f} \in \mathcal{F}(\alpha)\}$. Now, it is natural to require that the constraints on the mechanical performance, (5), should be satisfied by all realizations of $\boldsymbol{u}$. Thus the robust counterparts to the constraints (5) are introduced as

$$\forall \boldsymbol{u} \in \{\boldsymbol{u} : K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f},\ \boldsymbol{f} \in \mathcal{F}(\alpha)\} : \quad \boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{u} \leq b_j, \qquad j = 1, \dots, n^{\mathrm{c}}. \tag{8}$$

By eliminating $\boldsymbol{u}$, the robust constraints, (8), are reduced to

$$\forall \boldsymbol{f} \in \mathcal{F}(\alpha) : \quad \boldsymbol{a}_j^{\mathrm{T}} K(\boldsymbol{x})^{-1} \boldsymbol{f} \leq b_j, \qquad j = 1, \dots, n^{\mathrm{c}}. \tag{9}$$

Recall the optimization problem (6), and a robust structural optimization problem is defined by replacing the constraint condition (5) with its robust counterpart (9) as

$$[\textsc{SemiInf}] : \quad \left. \begin{array}{ll} \displaystyle\min_{\boldsymbol{x} \in \mathcal{X}} & v(\boldsymbol{x}) \\ \mathrm{s.t.} & \boldsymbol{a}_j^{\mathrm{T}} K(\boldsymbol{x})^{-1} \boldsymbol{f} \leq b_j \ (\forall \boldsymbol{f} \in \mathcal{F}(\alpha)), \qquad j = 1, \dots, n^{\mathrm{c}}. \end{array} \right\} \tag{10}$$

The essential difficulty of the robust structural optimization is captured clearly in (10): the problem (10) has a finite number of variables and an infinite number of inequality constraints, and hence it is called a *semi-infinite programming* problem.

For the fixed $\boldsymbol{x} \in \mathcal{X}$, consider the optimization problem

$$[\textsc{WorstCase}] : \quad \left. \begin{array}{ll} \displaystyle\max_{\boldsymbol{u}, \boldsymbol{f}} & \boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{u} \\ \mathrm{s.t.} & K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}, \\ & \boldsymbol{f} \in \mathcal{F}(\alpha), \end{array} \right\} \tag{11}$$

where $\boldsymbol{u}$ and $\boldsymbol{f}$ are the variables. We denote by $(\boldsymbol{u}^{\mathrm{wc}}, \boldsymbol{f}^{\mathrm{wc}})$ an optimal solution of the problem (11). From the definition, we see that the robust satisfaction of the $j$th constraint of (8) becomes most critical at $\boldsymbol{u}^{\mathrm{wc}}$. Hence, we call $\boldsymbol{f}^{\mathrm{wc}}$ and $\boldsymbol{u}^{\mathrm{wc}}$ the *worst case* (or the *worst scenario*) load and displacement, respectively, for the $j$th constraint condition. The problem (11) is called the worst-case detection problem [17, 22], or the anti-optimization problem [27, 30]. It is obvious that the worst scenario depends on $\boldsymbol{x}$.

In other words, the robust constraint (8) requires that the performance constraint (5) should be satisfied even in the worst case. Namely, (8) is equivalent to

$$\boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{u}^{\mathrm{wc}} \leq b_j, \qquad j = 1, \dots, n^{\mathrm{c}}.$$

Recalling the problem defining $\boldsymbol{u}^{\mathrm{wc}}$, (11), we see that the problem (10) is reduced to

$$\left.\begin{array}{ll}\min\limits_{\boldsymbol{x}\in\mathcal{X}} & v(\boldsymbol{x}) \\ \text{s.t.} & \max\limits_{\boldsymbol{u},\boldsymbol{\zeta}} \left\{\boldsymbol{a}_j^{\mathrm{T}}\boldsymbol{u} : K(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f},\ \boldsymbol{f} \in \mathcal{F}(\alpha)\right\} \le b_j, \qquad j = 1,\dots,n^{\mathrm{c}}.\end{array}\right\} \tag{12}$$

By substituting (7), the problem (12) is equivalently rewritten as

[BiLevel] :

$$\left.\begin{array}{ll}\min\limits_{\boldsymbol{x}\in\mathcal{X}} & v(\boldsymbol{x}) \\ \text{s.t.} & \max\limits_{\boldsymbol{u},\boldsymbol{\zeta}} \left\{\boldsymbol{a}_j^{\mathrm{T}}\boldsymbol{u} : K(\boldsymbol{x})\boldsymbol{u} = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta},\ \boldsymbol{\zeta} \in \mathcal{Z}(\alpha)\right\} \le b_j, \qquad j = 1,\dots,n^{\mathrm{c}}.\end{array}\right\} \tag{13}$$

The problem (13) includes the optimizations in its constraints, and hence it is called the *bi-level optimization* problem.

# 3 MPEC formulation

We investigate the optimality condition of the lower-level problem in the constraints of the bi-level problem (13):

$$\max_{\boldsymbol{u},\boldsymbol{\zeta}} \left\{\boldsymbol{a}_j^{\mathrm{T}}\boldsymbol{u} : K(\boldsymbol{x})\boldsymbol{u} = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta},\ \alpha \ge \|\boldsymbol{\zeta}\|_\infty\right\}, \tag{14}$$

where $\boldsymbol{u}$ and $\boldsymbol{\zeta}$ are the variables.

**Proposition 3.1.** *Assume that there exists a vector $\boldsymbol{u}$ satisfies $K(\boldsymbol{x})\boldsymbol{u} = \tilde{\boldsymbol{f}}$. Then $(\boldsymbol{u}^*, \boldsymbol{\zeta}^*)$ is an optimal solution of the problem* (14) *if and only if there exists a Lagrange multipliers vector $(\boldsymbol{\mu}^*, \boldsymbol{\lambda}^*, \boldsymbol{\tau}^*)$ satisfying*

$$\begin{aligned}
&K(\boldsymbol{x})\boldsymbol{u}^* = \tilde{\boldsymbol{f}} + F_0\boldsymbol{\zeta}^*, \\
&\boldsymbol{a}_j + K(\boldsymbol{x})\boldsymbol{\mu}^* = \boldsymbol{0}, \\
&F_0^{\mathrm{T}}\boldsymbol{\mu}^* + \boldsymbol{\lambda}^* - \boldsymbol{\tau}^* = \boldsymbol{0}, \\
&\alpha - \zeta_p^* \ge 0, \quad \lambda_p^* \ge 0, \quad \lambda_p^*(\alpha - \zeta_p^*) = 0, \qquad p = 1,\dots,k, \\
&\alpha + \zeta_p^* \ge 0, \quad \tau_p^* \ge 0, \quad \tau_p^*(\alpha + \zeta_p^*) = 0, \qquad p = 1,\dots,k.
\end{aligned}$$

*Proof.* Observe that, for the fixed $\boldsymbol{x}$, the problem (14) is a linear programming (LP) problem in the variables $\boldsymbol{u}$ and $\boldsymbol{\zeta}$. The assumption of the proposition implies that the problem (14) has a feasible solution. Then the assertion of the proposition follows from the standard result of the Karush–Kuhn–Tucker conditions of LP (see, e.g., [9]). □

It follows from Proposition 3.1 that the problem (13) is equivalently rewritten as

$$
\begin{aligned}
&[\mathrm{MPEC}]: \\
&\left.
\begin{array}{ll}
\min & v(\boldsymbol{x}) \\
\mathrm{s.t.} & \boldsymbol{x} \in \mathcal{X}, \\
& \forall j = 1, \dots, n^{\mathrm{c}} : \left\{
\begin{array}{l}
K(\boldsymbol{x})\boldsymbol{u}_j - \tilde{\boldsymbol{f}} - F_0 \boldsymbol{\zeta}_j = \boldsymbol{0}, \\
\boldsymbol{a}_j + K(\boldsymbol{x})\boldsymbol{\mu}_j = \boldsymbol{0}, \\
\boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{u}_j \le b_j, \\
F_0^{\mathrm{T}} \boldsymbol{\mu}_j + \boldsymbol{\lambda}_j - \boldsymbol{\tau}_j = \boldsymbol{0}, \\
\alpha - \zeta_{pj} \ge 0, \ \lambda_{pj} \ge 0, \ \lambda_{pj}(\alpha - \zeta_{pj}) = 0, \quad p = 1, \dots, k, \\
\alpha + \zeta_{p} \ge 0, \ \tau_{pj} \ge 0, \ \tau_{pj}(\alpha + \zeta_{pj}) = 0, \quad p = 1, \dots, k,
\end{array}
\right.
\end{array}
\right\} \qquad (15)
\end{aligned}
$$

where $\boldsymbol{x} \in \mathbb{R}^m$, $\boldsymbol{u}_j \in \mathbb{R}^d$, $\boldsymbol{\zeta}_j = (\zeta_{pj}) \in \mathbb{R}^k$, $\boldsymbol{\mu}_j \in \mathbb{R}^d$, $\boldsymbol{\lambda}_j = (\lambda_{pj}) \in \mathbb{R}^k$, and $\boldsymbol{\tau}_j = (\tau_{pj}) \in \mathbb{R}^k$ $(j = 1, \dots, n^{\mathrm{c}})$ are the variables.

The problem (15) has complementarity conditions in its constraints, and hence it is called the *mathematical program with complementarity constraints*, or *mathematical program with equilibrium constraints* (MPEC) [25]. Note that (15) is a single-level optimization problem with a finite number of constraint conditions, while the problem (10) has infinitely many constraints, and the problem (13) is a bi-level optimization problem.

*Remark* 3.2. The problem (15) has auxiliary variables $\boldsymbol{u}_j^*$ and $\boldsymbol{\zeta}_j^*$ $(j = 1, \dots, n^{\mathrm{c}})$. It is of interest to note that these variables have physical interpretations. At the optimal solution, $\boldsymbol{u}_j^*$ and $\boldsymbol{\zeta}_j^*$ correspond to the worst scenario for the $j$th constraint condition defined in (11): the worst load is obtained as $\boldsymbol{f}^{\mathrm{wc}} = \tilde{\boldsymbol{f}} + F_0 \boldsymbol{\zeta}_j^*$, and $\boldsymbol{u}_j^*$ is the displacements vector corresponding to $\boldsymbol{f}^{\mathrm{wc}}$. Note that the worst load depends on the constraint condition, i.e. $\boldsymbol{\zeta}_{j_1}^* \neq \boldsymbol{\zeta}_{j_2}^*$ and $\boldsymbol{u}_{j_1}^* \neq \boldsymbol{u}_{j_2}^*$ $(j_1 \neq j_2)$ in general. We say that the $j$th constraint condition $\boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{u} \le b_j$ is active if $\boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{u}_j^* = b_j$ holds. For robust structural optimization, optimal solutions often have some worst cases in which their corresponding constraints become active. Section 5 illustrates some examples of robust optimal designs with multiple active worst cases. ■

*Remark* 3.3. It is known that there exists the worst scenario load $\boldsymbol{f}^{\mathrm{wc}}$, defined in (11), at an extreme point of $\mathcal{F}(\alpha)$ [17, 28]. Hence, for the problem (15) there exists an optimal solution satisfying

$$
\zeta_{pj}^* \in \{-1, 1\}, \qquad p = 1, \dots, k; \ j = 1, \dots, n^{\mathrm{c}},
$$

This explains the combinatorial property of the robust structural optimization, i.e. the problem (10) is solved if we consider the constraints at all extreme points of $\mathcal{F}(\alpha)$. Obviously, it is not acceptable to enumerate the constraints at extreme points from the practical point of view, because there exist $2^{n^{\mathrm{c}} k}$ extreme points. This is the reason why we attempt to solve the MPEC formulation (15). ■

Unfortunately, it is still difficult to solve the problem (15), because any feasible solution of (15) does not satisfy a standard constraint qualification, e.g. *linear independence constraint qualification* (LICQ), *Mangasarian–Fromovitz constraint qualification*, etc, and hence standard nonlinear programming approaches are likely to fail for this problem (see, e.g., [25]). Hence, in section 4, we propose a reformulation which satisfies the LICQ at almost all feasible solutions.

## 4 Implicit reformulation of MPEC

In order to reformulate the MPEC problem (15) into a tractable form, we introduce a function $\phi : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\phi(y, z, \rho) = y + z - \sqrt{y^2 + z^2 + 2\rho^2}, \tag{16}$$

which is the smoothed Fischer–Burmeister function proposed by [23] for solving linear complementarity problems. Subsequently, smooth nonlinear programming approaches for MPEC were proposed by using the function (16) [16, 18, 19, 36].

Let $e$ denote Euler's constant. The following proposition plays a crucial role in our reformulation.

**Proposition 4.1.** *$\boldsymbol{y} = (y_i) \in \mathbb{R}^n$, $\boldsymbol{z} = (z_i) \in \mathbb{R}^n$, and $\rho$ satisfy*

$$\boldsymbol{y} \geq \boldsymbol{0}, \quad \boldsymbol{z} \geq \boldsymbol{0}, \quad \boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = 0, \quad \rho = 0 \tag{17}$$

*if and only if the conditions*

$$\phi(y_i, z_i, \rho) = 0, \qquad i = 1, \dots, n, \tag{18}$$

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = n(e^{\rho} - 1) \tag{19}$$

*are satisfied.*

*Proof.* Observe that, for each $i = 1, \dots, n$, the equation

$$\phi(y_i, z_i, \rho) = 0$$

holds if and only if $y_i$, $z_i$, and $\rho$ satisfy

$$y_i \geq 0, \quad z_i \geq 0, \quad \rho^2 = y_i z_i.$$

Hence, the 'only if' part can be shown easily.

If $\rho = 0$ in (19), then $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = 0$. Hence, it remains to show that (18) and (19) imply $\rho = 0$. Since $\rho^2 = y_i z_i$ $(i = 1, \dots, n)$ hold if (18) is satisfied, we see that (18) implies

$$n\rho^2 = \sum_{i=1}^{n} y_i z_i = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{z}. \tag{20}$$

Substitution of (20) into (19) yields

$$\rho^2 = e^{\rho} - 1,$$

which holds if and only if $\rho = 0$. □

It follows from Proposition 4.1 that the MPEC problem (15) is equivalently rewritten as the following implicit formulation:

$$
\begin{aligned}
&[\mathrm{NLP}]: \\
&\left.\begin{array}{ll}
\min & v(\boldsymbol{x}) \\
\text{s.t.} & \boldsymbol{x} \in \mathcal{X}, \\
& \forall j = 1, \dots, n^{\mathrm{c}} : \begin{cases}
K(\boldsymbol{x})\boldsymbol{u}_j - \tilde{\boldsymbol{f}} - F_0 \boldsymbol{\zeta}_j = \boldsymbol{0}, \\
\boldsymbol{a}_j + K(\boldsymbol{x})\boldsymbol{\mu}_j = \boldsymbol{0}, \\
\boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{u}_j \leq b_j, \\
F_0^{\mathrm{T}} \boldsymbol{\mu}_j + \boldsymbol{\lambda}_j - \boldsymbol{\tau}_j = \boldsymbol{0}, \\
\phi(\alpha - \zeta_{pj}, \lambda_{pj}, \rho) = 0, \qquad p = 1, \dots, k, \\
\phi(\alpha + \zeta_p, \tau_{pj}, \rho) = 0, \qquad p = 1, \dots, k,
\end{cases} \\
& \displaystyle\sum_{j=1}^{n^{\mathrm{c}}} \sum_{p=1}^{k} [(\alpha - \zeta_{pj})\lambda_{pj} + (\alpha + \zeta_p)\tau_{pj}] + (2n^{\mathrm{c}}k)(1 - e^{\rho}) = 0,
\end{array}\right\}
\end{aligned}
\tag{21}
$$

where $\boldsymbol{x}$, $\rho$, $\boldsymbol{u}_j$, $\boldsymbol{\zeta}_j$, $\boldsymbol{\mu}_j$, $\boldsymbol{\lambda}_j$, and $\boldsymbol{\tau}_j$ $(j = 1, \dots, n^{\mathrm{c}})$ are the variables. The problem (21) is regarded as a conventional nonlinear programming (NLP) problem in the sense that it has no complementarity conditions in its constraint conditions. We call (21) an *implicit reformulation* of the MPEC problem (15), because the smoothing parameter $\rho$ is included as one of the variables in (21) and is updated simultaneously with the other variables at each iteration of the optimization algorithm, e.g. the SQP method.

*Remark* 4.2. In the problem (21), the complementarity conditions in the problem (15) are rewritten by using the smoothed Fischer–Burmeister functions and the additional equation in the form of (19). We may understand more clearly the role of the subsidiary variable $\rho$ by rewriting (19) as

$$
\log\left(\frac{1}{n}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} + 1\right) = \rho. \tag{22}
$$

A key observation in (22) is that $\rho$ can be regarded as a measure of residual of the complementarity conditions, $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = 0$. An alternative candidate for a measure of the residual may be

$$
\frac{1}{n}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = \rho, \tag{23}
$$

which seems to be simpler than (22), but the function (23) does not give an implicit reformulation. Indeed, the system of equations (18) and (23) is not equivalent to (17). More precisely, (17) is not a necessary condition for (18) and (23). Thus, in order to obtain the result of Proposition 4.1, we choose the logarithm function (22) as a measure of the residual. ■

Remark 4.2 also suggests an appropriate choice of the initial value $\rho_{(0)}$ for the variable $\rho$: let $\boldsymbol{y}_{(0)}$ and $\boldsymbol{z}_{(0)}$ be the initial values of $\boldsymbol{y}$ and $\boldsymbol{z}$ satisfying $0 \neq \boldsymbol{y}_{(0)}^{\mathrm{T}}\boldsymbol{z}_{(0)} > -1/n$, then it is certainly reasonable to put $\rho_{(0)} := \log[(\boldsymbol{y}_{(0)}^{\mathrm{T}}\boldsymbol{z}_{(0)}/n) + 1]$, where $\boldsymbol{y}$ and $\boldsymbol{z}$ correspond to $\boldsymbol{y} = (\alpha\boldsymbol{1} - \boldsymbol{\zeta}_1, \dots, \alpha\boldsymbol{1} - \boldsymbol{\zeta}_k, \alpha\boldsymbol{1} + \boldsymbol{\zeta}_1, \dots, \alpha\boldsymbol{1} + \boldsymbol{\zeta}_k)$ and $\boldsymbol{z} = (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, \boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_k)$, respectively, in the problem (21).

*Remark* 4.3. For understanding the role which the smoothing parameter $\rho$ plays, it is interesting to compare the proposed formulation, (21), with the following problem:

$$\left.\begin{array}{ll} \min & c(\boldsymbol{x}) \\ \text{s.t.} & \boldsymbol{x} \in \mathcal{X}, \\ & \forall j = 1,\dots,n^{\mathrm{c}} : \left\{ \begin{array}{ll} K(\boldsymbol{x})\boldsymbol{u}_j - \tilde{\boldsymbol{f}} - F_0 \boldsymbol{\zeta}_j = \mathbf{0}, & \\ \boldsymbol{a}_j + K(\boldsymbol{x})\boldsymbol{\mu}_j = \mathbf{0}, & \\ \boldsymbol{a}_j^{\mathrm{T}} \boldsymbol{u}_j \le b_j, & \\ F_0^{\mathrm{T}} \boldsymbol{\mu}_j + \boldsymbol{\lambda}_j - \boldsymbol{\tau}_j = \mathbf{0}, & \\ \phi(\alpha - \zeta_{pj}, \lambda_{pj}, 0) = 0, & p = 1,\dots,k, \\ \phi(\alpha + \zeta_{p}, \tau_{pj}, 0) = 0, & p = 1,\dots,k, \end{array}\right. \end{array}\right\} \quad (24)$$

which is also equivalent to the MPEC problem (15). Our numerical experiments demonstrates that the SQP method cannot converge to a solution of (24); see section 5.3. This is due to the nonsmooth property of $\phi(\alpha - \zeta_{pj}, \lambda_{pj}, 0)$ and $\phi(\alpha + \zeta_{p}, \tau_{pj}, 0)$ in the constraint conditions of (24). In contrast, in (21), we overcome the nonsmooth property of constraint conditions by introducing the smoothing variable, $\rho$. As discussed in Remark 4.2, $\rho$ in (21) corresponds to the residual of the complementarity conditions, and it is usual that the complementarity constraints are not satisfied exactly during the optimization procedure before it converges. Hence, it is reasonable to expect that $\rho \neq 0$ holds at intermediate solutions of the optimization procedure. If this assumption is satisfied, then the constraint conditions of (21) are differentiable at intermediate solutions, which explains the advantage of the proposed formulation (21) over (24). ■

*Remark* 4.4. The idea of regarding a smoothing parameter $\rho$ as an independent variable can also be found in Jian [18], Jiang and Ralph [19]. Their formulations are different from ours from in the following point. The methods of [18, 19] utilizes the fact that the complementarity conditions (17) in Proposition 4.1 are equivalent to

$$\phi(y_i, z_i, \rho) = 0, \qquad i = 1,\dots,n, \quad (25)$$

$$e^{\rho} = 1. \quad (26)$$

Then the smooth SQP method was applied to the implicit formulation including (25) and (26) in its constraints, instead of the complementarity conditions. Thus, the amount of the smoothing parameter $\rho$ is irrelevant to the variables $\boldsymbol{y}$ and $\boldsymbol{z}$ in the methods of [18, 19], whereas in ours we attempt to adjust $\rho$ to the residual of the complementarity constraints, $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{z} = 0$, as discussed in Remark 4.2. In other words, we expect that $\rho$ becomes larger in the earlier stage of the optimization procedure, and $\rho$ can be smaller as the residual of the other constraints is reduced considerably. In contrast, in the method using (25) and (26), there may possibly exist two disadvantageous situations: (i) $\rho$ is almost equal to 0 even if the residual of complementarity conditions is still relatively large, then the smoothing effect of the Fischer–Burmeister function might not work properly, which may cause the divergence of the optimization algorithm (see Remark 4.3); (ii) $\rho$ is still far from 0 when the residual of complementarity conditions is sufficiently small, then unnecessary iterations might be spent in order to reduce the residual of (26). ■

Recall that the difficulty of the MPEC formulation (15) arises from the fact that MPEC fails to satisfy any standard constraint qualification such as LICQ. We next investigate the constraint qualification of the implicit reformulation (21). For simplicity, we denote by $\boldsymbol{\xi}$ the variables vector of the problem (21), i.e.

$$\boldsymbol{\xi} = \left(\boldsymbol{x}, \rho, \left((\boldsymbol{u}_j, \boldsymbol{\zeta}_j, \boldsymbol{\mu}_j, \boldsymbol{\lambda}_j, \boldsymbol{\tau}_j) \mid j = 1, \dots, n^{\mathrm{c}}\right)\right).$$

Under the assumption that the MPEC problem (15) satisfies MPEC-LICQ [25], the following proposition investigates LICQ of the problem (21).

**Proposition 4.5.** *Let $\bar{\boldsymbol{\xi}}$ denote a feasible solution of the problem* (15) *satisfying the strict complementarity, i.e.*

$$(\alpha - \bar{\zeta}_{pj}) + \bar{\lambda}_{pj} > 0, \quad (\alpha + \bar{\zeta}_p) + \bar{\tau}_{pj} > 0, \qquad p = 1, \dots, k; \; j = 1, \dots, n^{\mathrm{c}}.$$

*If the problem* (15) *satisfies MPEC-LICQ at $\bar{\boldsymbol{\xi}}$, then the problem* (21) *satisfies LICQ at $\bar{\boldsymbol{\xi}}$.*

*Proof.* For simplicity, we consider the complementarity conditions

$$\mathbb{R}^2 \ni \boldsymbol{y} \geq \boldsymbol{0}, \quad \mathbb{R}^2 \ni \boldsymbol{z} \geq \boldsymbol{0}, \quad \boldsymbol{y}^{\mathrm{T}} \boldsymbol{z} = 0, \tag{27}$$

i.e. we put $n = 2$ in Proposition 4.1. Define $\hat{\phi}_1$, $\hat{\phi}_2$, $\delta : \mathbb{R}^5 \to \mathbb{R}$ by

$$\begin{aligned} \hat{\phi}_i(\boldsymbol{y}, \boldsymbol{z}, \rho) &= \phi(y_i, z_i, \rho), \qquad i = 1, 2, \\ \delta(\boldsymbol{y}, \boldsymbol{z}, \rho) &= \boldsymbol{y}^{\mathrm{T}} \boldsymbol{z} - 2(e^{\rho} + 1). \end{aligned}$$

Then Proposition 4.1 implies that $(\boldsymbol{y}, \boldsymbol{z}, \rho)$ satisfies (27) and $\rho = 0$ if and only if

$$\hat{\phi}_1(\boldsymbol{y}, \boldsymbol{z}, \rho) = \hat{\phi}_2(\boldsymbol{y}, \boldsymbol{z}, \rho) = \delta(\boldsymbol{y}, \boldsymbol{z}, \rho) = 0. \tag{28}$$

Suppose that $(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho})$ satisfies (28) and the strict complementarity. Particularly, we assume

$$\bar{y}_1 > 0, \quad \bar{z}_2 > 0, \quad \bar{y}_2 = \bar{z}_1 = \bar{\rho} = 0.$$

Without loss of generality, it suffices to show that $\nabla\hat{\phi}_1(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho})$, $\nabla\hat{\phi}_2(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho})$, and $\nabla\delta(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho})$ are linearly independent. Simple calculation yields

$$\nabla\hat{\phi}_1(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho}) = \begin{bmatrix} 1 - \bar{y}_1/|\bar{y}_1| \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \nabla\hat{\phi}_2(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho}) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 - \bar{z}_2/|\bar{z}_2| \\ 0 \end{bmatrix}, \quad \nabla\delta(\bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}, \bar{\rho}) = \begin{bmatrix} 0 \\ \bar{z}_2 \\ \bar{y}_1 \\ 0 \\ -2 \end{bmatrix},$$

which are linearly independent. The proof in more general case described in the assertion of this proposition can be obtained similarly, but is omitted for simple presentation. □

Proposition 4.5 guarantees that the implicit problem (21) satisfies LICQ, which allows to solve (21) by using a standard nonlinear optimization approach. In contrast, the constraint conditions are not continuously differentiable at a feasible solution which does not satisfy the strict complementarity. However, as discussed in Remark 4.3, we may expect that $\rho \neq 0$ at any intermediate solution of the optimization procedure. If this assumption is satisfied, then the constraint conditions are differentiable at any intermediate solution, and are not continuously differentiable only at the convergent solution. Throughout numerical experiments, it is confirmed that a standard smooth nonlinear optimization approach can solve the problem (21) without any difficulty arising from the nonsmooth property of the constraint conditions of (21); see section 5 for details.

## 5 Numerical experiments

The robust optimal designs under the load uncertainties are found for various structures by solving the problem (21). Computation has been carried out on Core 2 Duo (1.2 GHz with 2.0 GB memory) with MATLAB Ver. 7.5.0 [35]. We solve the problem (21) by using the MATLAB built-in function `fmincon`, which implements the sequential quadratic programming (SQP) method for nonlinearly constrained optimization. The gradients of the objective and constraint functions are provided as the user-defined functions.

It is known that `fmincon` is not superior to other available nonlinear programming solvers. In fact, some nonlinear programming solvers can solve a large class of MPECs [15], while `fmincon` usually fails in our preliminary numerical experiments. However, this drawback is adequate for our purpose, because we attempt to confirm that our implicit reformulation enhances the robustness of a standard nonlinear programming solver when it is applied to MPECs [32].

In the following examples, the elastic modulus is 200 GPa. The design variables vector $\boldsymbol{x} \in \mathbb{R}^m$ in the problem (21) is chosen as the vector of member cross-sectional areas of a truss, where $m$ is the number of members. The set $\mathcal{X}$ in (21) represents the nonnegative constraints of $\boldsymbol{x}$, i.e. $\mathcal{X} = \{\boldsymbol{x} \in \mathbb{R}^m \mid x_i \geq 0 \ (i = 1, \dots, m)\}$.

### 5.1 2-bar truss

Consider a two-bar truss illustrated in Figure 1. The nodes (b) and (c) are pin-supported at $(x, y) = (0, 1000)$ and $(0, 0)$ in mm, respectively, while the node (a) is free, i.e. $d = 2$. The initial lengths of members (1) and (2) are 1000 mm and $1000\sqrt{2}$ mm, respectively.

As the nominal load $\tilde{\boldsymbol{f}}$, the external force $\tilde{\boldsymbol{f}} = (10.0, 0)$ kN is applied at the node (a). In accordance with (7), the uncertainty model of the load is defined as

$$\boldsymbol{f} = \tilde{\boldsymbol{f}} + \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}, \quad \alpha \geq |\zeta_j|, \qquad j = 1, 2, \tag{29}$$

where we put $F^0 = 1.0$ (kN) $\times I$ and $k = 2$. Consequently, the uncertain load $\boldsymbol{f}$ is running through the square depicted with the dashed lines in Figure 1.

As the constraints on the mechanical performance, (5), consider the stress constraint of each member which is written as

$$|\sigma_i| \leq \sigma^{\mathrm{c}}, \qquad i = 1, \dots, m, \tag{30}$$

where $m = 2$. The upper bound of the stress is given as $\sigma^{\mathrm{c}} = 10.0$ MPa.

The robust optimization problem (21) is solved for $\alpha = 1.0$. The obtained optimal cross-sectional areas are $\boldsymbol{x}^* = (1200.0, 200.0)$ mm$^2$. If we do not consider any uncertainty, i.e. $\alpha = 0$, then the robust structural optimization (21) coincides with the conventional optimization over stress constraints at the nominal load. The optimal solution without uncertainty is $\boldsymbol{x}' = (1000.0, 0.0)$ mm$^2$, which corresponds to the so-called fully-stressed design. Note that the robust optimal design is kinematically (and statically) determinate, while the fully-stressed design without uncertainty is kinematically indeterminate.

We next randomly generate a number of loads $\boldsymbol{f}$ satisfying (29) with $\alpha = 1.0$, and compute the corresponding stresses of the robust optimal design. Figure 2 depicts the obtained member

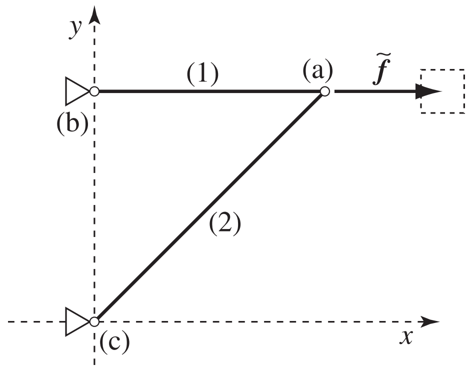



Figure 1: A 2-bar truss.

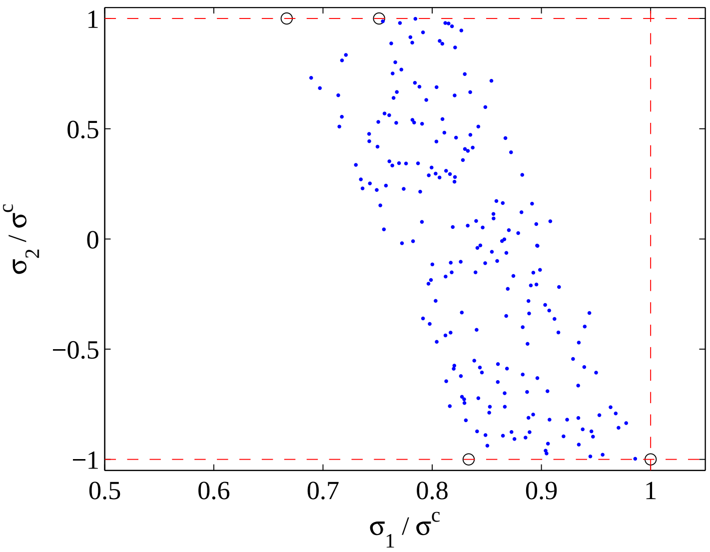



Figure 2: Stress states of the robust optimal design of the 2-bar truss. '∘': obtained worst cases; '·': computed from randomly generated loads.

stresses $(\sigma_1/\sigma^{\rm c}, \sigma_2/\sigma^{\rm c})$. It is observed from Figure 2 that the stress constraints, (30), are satisfied for all randomly generated loads. By solving the problem (21), we also obtain the worst scenarios $\boldsymbol{\zeta}_j$ as the optimal solution; see Remark 3.2. The stress states corresponding to the obtained worst scenarios are shown in Figure 2 by the open circles. Note that the stress constraints (30) for the two members are rewritten as the four linear inequalities, and hence there exist four extremal cases regarding stresses, i.e. the maximum and minimum stresses of members (1) and (2). It is observed from Figure 2 that the obtained worst scenarios correspond to the extremal cases of the randomly generated loads, and the stress constraints are satisfied in those worst scenarios.

### 5.2 10-bar truss

Next we consider a $2 \times 1$ plane truss shown in Figure 3. The nodes (a) and (b) are pin-supported, and the truss has 10 members, i.e. $d = 8$ and $m = 10$. The lengths of members in the directions of the $x$- and $y$-axes are 1000 mm and 600 mm, respectively.

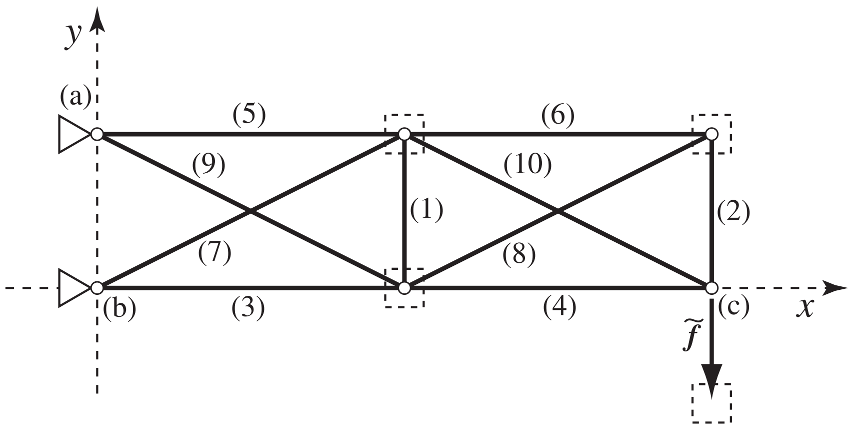



Figure 3: A 10-bar truss.

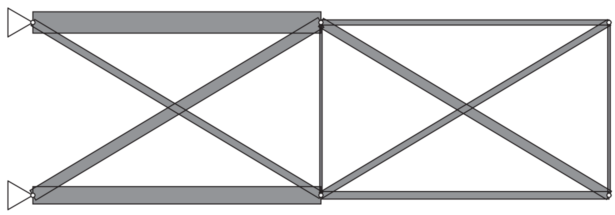


Figure 4: Robust optimal solution of the 10-bar truss.

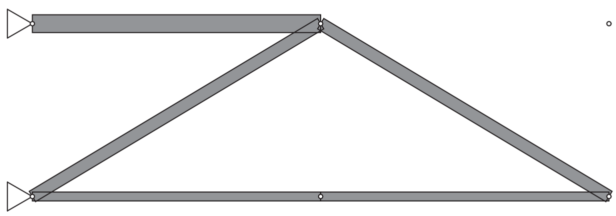


Figure 5: Optimal solution of the 10-bar truss without uncertain loads.

As the nominal load $\tilde{\boldsymbol{f}}$, the external force 2.5 kN is applied at the node (c) in the negative direction of the $y$-axis. The uncertainty of the load is modeled in (7), where we choose $F_0 = 1.0\ (\mathrm{kN}) \times I$ and $k = d = 8$. Consequently, the external forces are running through the squares depicted with the dotted lines in Figure 3. Note that uncertain forces may possible exist at all nodes, and any two components of the uncertain load have no relation. For each member, we consider the stress constraint in the form of (30), where $\sigma^{\mathrm{c}} = 10.0$ MPa and $m = 10$.

The robust optimization problem (21) is solved for $\alpha = 0.3$. The robust optimal solution obtained is shown in Figure 4, where the width of each member is proportional to its cross-sectional area. In contrast, Figure 5 depicts the optimal solution without considering uncertainty, which is obtained by solving the conventional optimization problem (6). Note that the optimal solution without uncertainty in Figure 5 is the so-called fully-stressed design corresponding to the nominal load $\tilde{\boldsymbol{f}}$. The structural volume of the robust optimal solution (Figure 4) is $4.433 \times 10^6$ mm$^3$, while that of the optimal solution without uncertainty (Figure 5) is $2.988 \times 10^6$ mm$^3$; the optimal volume increases by

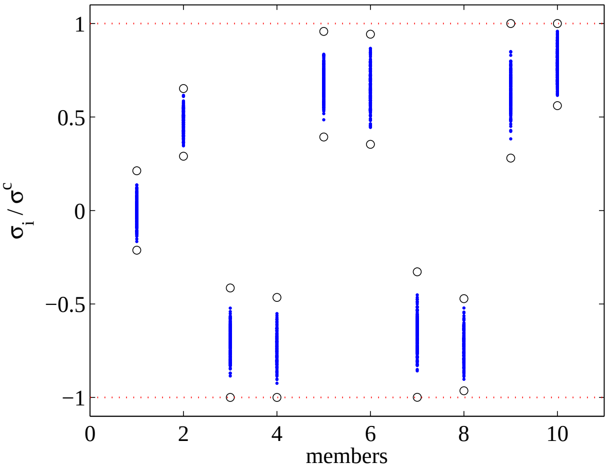



Figure 6: Member stresses of the robust optimal design of the 10-bar truss. '∘': worst cases obtained by solving (21); '·': computed from randomly generated loads.

48.3% by considering uncertainty. Note that the robust optimal design is kinematically determinate, whereas the fully-stressed design without uncertainty possesses an infinitesimal mechanism.

We next randomly generate a number of loads $\boldsymbol{f}$ satisfying (7) with $\alpha = 0.3$, and compute the corresponding stresses of the robust optimal design. Figure 6 depicts the obtained member stresses divided by the specified upper bound, i.e. $\sigma_i/\sigma^{\mathrm{c}}$. The stress states corresponding to the worst loading scenarios, which are obtained by solving (21), are also shown in Figure 6 by the open circles. It is observed from Figure 6 that the obtained worst cases correspond to the extremal cases of the randomly generated loads, and the stress constraints, (30), are satisfied even for those worst cases. This supports that a robust design is obtained successfully. Moreover, the stress constraints of members (3)–(10) become active at their worst cases. Note that the optimal cross-sectional areas of members (1) and (2) are very small; at the exact optimal solution it is expected that there exist worst cases in which the stress constraints of members (1) and (2) also become active, and hence the exact optimal solution may have smaller cross-sectional areas for those members. It is very difficult for the SQP method to find such a solution because of the numerical instability. Note again that members (1) and (2) have already very small cross-sectional areas at the current solution shown in Figure 4, and hence we regard that an approximate optimal solution has been obtained successfully. It is also observed in Figure 6 that it is not easy to detect the worst loading scenario by generating a large number of samples randomly. For example, for members (7) and (9) it is seen that estimates of the worst cases obtained from random samples are too optimistic.

For the fully-stressed design in Figure 5, the stress constraints of all members become active at the nominal load $\tilde{\boldsymbol{f}}$. In contrast, for the robust optimal design in Figure 4, there exists the worst case in which the stress constraint of each member becomes active, but the worst cases for the different members are not common in general. Indeed, the worst scenario loadings for members (3)–(10) are collected in Figure 7 and Figure 8. Thus, in the robust structural optimization, it is necessary to consider the worst scenarios for all constraint conditions, and the worst scenario itself is the function of the design variables. This is the essential difficulty in the robust structural optimization compared

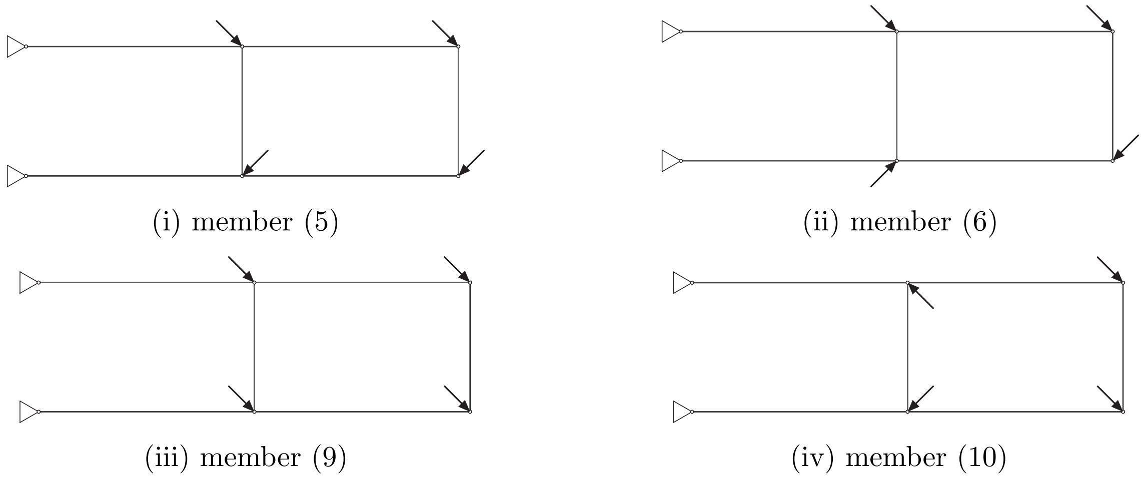



Figure 7: Uncertain external loads in the worst scenarios for the robust optimal solution of the 10-bar truss: the cases in which stress constraints of members (5), (6), (9), and (10) become active in tensile states.

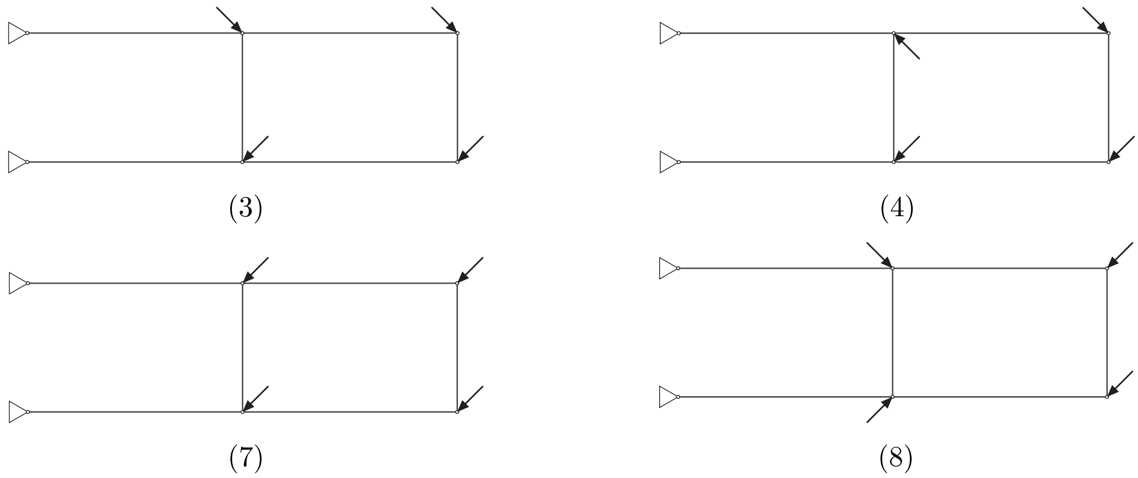



Figure 8: Uncertain external loads in the worst scenarios for the robust optimal solution of the 10-bar truss: the cases in which stress constraints of members (3), (4), (7), and (8) become active in compressive states.

with the conventional structural optimization.

### 5.3 Grid truss under displacement constraints

As a moderately large example, consider a $4 \times 3$ plane truss shown in Figure 9. The nodes (a)–(d) are pin-supported, and the ground structure has 70 members, i.e. $d = 32$ and $m = 70$. The lengths of members both in the directions of the $x$- and $y$-axes are 1000 mm.

As the nominal load $\tilde{\boldsymbol{f}}$, the vertical force of 150.0 kN, 300.0 kN, 450.0 kN, and 600.0 kN are applied at the nodes (e), (f), (g), and (h), respectively, in the negative direction of the $y$-axis. The uncertainty model of the load is defined in (7), where we choose $k = 24$ and

$$F_0 = 1.0 \text{ (kN)} \times \begin{bmatrix} I \\ O \end{bmatrix} \in \mathbb{R}^{32 \times 24}.$$

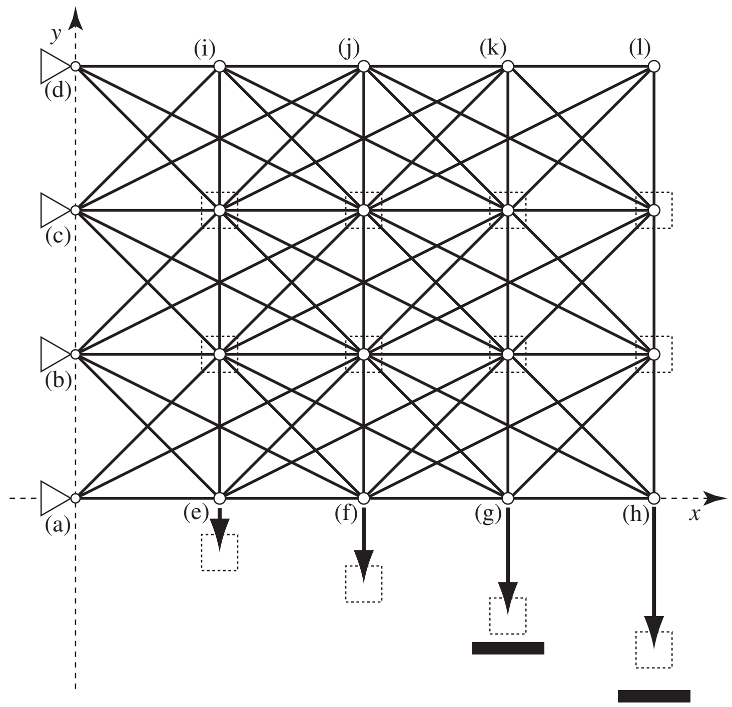



Figure 9: A $4 \times 3$ grid truss.

Consequently, the external forces are running through the squares depicted with the dotted lines in Figure 9. Note that no uncertain loads are applied at the nodes (i)–(l).

As the constraints on the mechanical performance, (5), we consider the displacement constraints written as

$$u_j \geq u^{\mathrm{c}}, \qquad j = 1, \dots, n^{\mathrm{c}} \tag{31}$$

for the vertical displacements of the nodes (g) and (h), i.e. $n^{\mathrm{c}} = 2$. The lower bound $u^{\mathrm{c}}$ of the displacement is $-30$ mm for the node (g) and $-40$ mm for the node (h).

The robust optimization problem (21) is solved for $\alpha = 0.25$, and the obtained robust optimal solution is shown in Figure 10. At this robust optimal solution there exists the worst case corresponding to each of (31), i.e. the constraints on both nodes can happen to be active as a result of robust optimization. The uncertainty loads $F_0 \boldsymbol{\zeta}_j^*$ in the two worst scenarios are illustrated in Figure 11.

For comparison, the conventional optimal solution without considering uncertainty is illustrated in Figure 12, which is obtained by solving the problem (6). The structural volume of the robust optimal solution (Figure 10) is $3.0857 \times 10^7$ mm$^3$, while that of the optimal solution without uncertainty (Figure 12) is $2.2666 \times 10^7$ mm$^3$; the optimal volume increases by 36.1% by considering uncertainty.

It is of interest to note that we also apply the SQP method to the formulation (24) in Remark 4.3 with various values of $\alpha$, then the algorithm diverges and cannot find solutions. This supports that the smoothed scheme proposed in (21) is necessary and effective to avoid the numerical instability arising from the nonsmooth property of Fischer–Burmeister function.

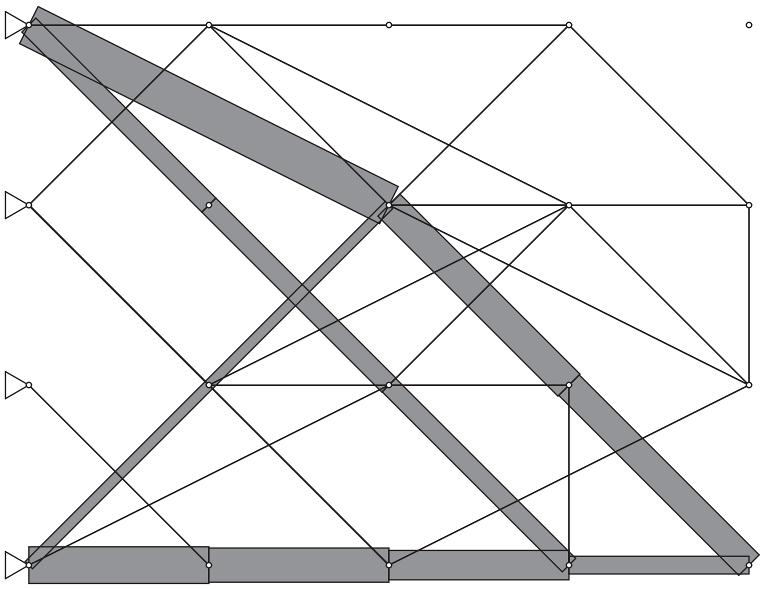
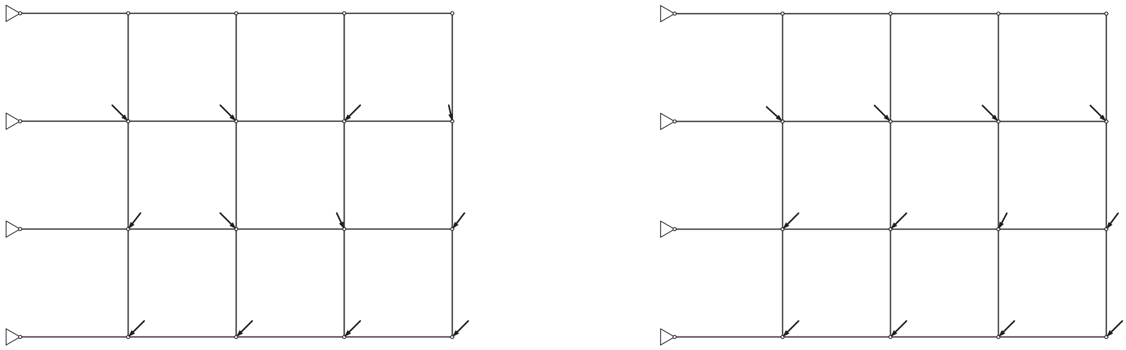


Figure 10: Robust optimal solution of the grid truss.



(i) (ii)

Figure 11: Uncertain external loads in the worst scenarios for the robust optimal solution of the grid truss: (i) the displacement constraint of the node (g) is active; (ii) the displacement constraint of the node (h) is active.

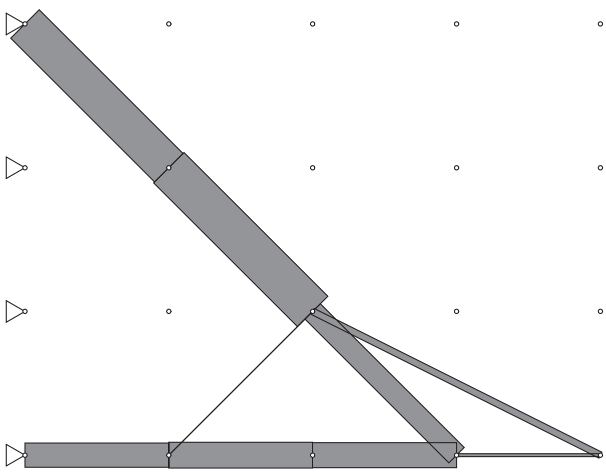


Figure 12: Optimal solution of the grid truss without uncertainty.

## 6 Conclusions

In this paper we have considered some formulations of robust optimization of structures under the non-probabilistic uncertainty of static load. Particularly, it has been shown that the robust structural optimization problem is formulated as an MPEC (mathematical program with complementarity constraints) problem. For applying a standard nonlinear programming approach such as the SQP method, we have proposed an implicit reformulation of the MPEC based on the smoothed Fischer–Burmeister function. In our implicit formulation the parameter $\rho$ for smoothing the Fischer–Burmeister function is treated as an independent variable, and we add an equality constraint with which $\rho$ vanishes at the optimal solution automatically. Consequently, $\rho$ decreases automatically as the optimization algorithm approaches to convergence.

Through the numerical experiments it has been shown that the presented implicit formulation of the robust structural optimization can be solved without any difficulties by using a standard SQP algorithm. The characteristics of the obtained robust optimal designs have been discussed by comparing the conventional optimal designs without considering uncertainty.

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