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Hybrid Analysis of Nonlinear Time-Varying Circuits Providing DAEs with Index at Most One

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Abstract

Commercial packages for transient circuit simulation are often based on the modified nodal analysis (MNA) which allows an automatic setup of model equations and requires a nearly minimal number of variables. However, it may lead to differential-algebraic equations (DAEs) with higher index. Here, we present a hybrid analysis for nonlinear time-varying circuits leading to DAEs with index at most one. This hybrid analysis is based merely on the network topology, which possibly leads to an automatic setup of the hybrid equations from netlists. Moreover, we prove that the minimum index of the DAE arising from the hybrid analysis never exceeds the index from MNA. As a positive side effect, the number of equations from the hybrid analysis is always no greater than that one from the MNA. This suggests that the hybrid analysis is superior to MNA in numerical accuracy and computational effort.

1 Introduction

When modelling electric circuits for transient simulation, one has to regard Kirchhoff's laws for the network and the constitutive equations for the different types of network elements. They are originally based on the branch voltages and the branch currents existing in the network. They form the basis for all modelling approaches as for instance the popular modified nodal analysis (MNA).

Concerning the huge number of variables involved (all branch voltages and branch currents), one is interested in a reduced system reflecting the complete circuit behaviour that can be generated automatically. Whereas MNA focuses on a description depending mainly on nodal potentials, the hybrid analysis approach [1] here employs certain branch voltages and branch currents obtained from a construction of a particular *normal tree*.

A normal tree is a tree containing all independent voltage sources, no independent current sources, a maximal number of capacitive branches, and a minimal number of inductive

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branches. Normal trees have already been used in [2] for state approaches for linear RLC networks. The results have been extended in [3] for linear circuits containing ideal transformers, nullors, independent/dependent sources, resistors, inductors, capacitors, and, under a topological restriction, gyrators.

The hybrid analysis is a common generalization of the loop analysis and the cutset analysis, which are classical circuit analysis methods. Kron [4] proposed the hybrid analysis in 1939, and Amari [5] and Branin [6] developed it further in 1960s. In contrast to MNA, the hybrid analysis retains flexibility in the selection of a normal tree, which can be exploited to find a model description that reduces the numerical difficulties.

The differential-algebraic equations (DAEs) arising from the hybrid analysis are called the *hybrid equations.* Recently, the analysis of the *index* of the hybrid equations has been developed. For linear time-invariant electric circuits which are composed of resistors, inductors, capacitors, independent voltage/current sources, and dependent voltage/current sources, an algorithm for finding an optimal hybrid analysis which minimizes the index of the hybrid equations was proposed in [7]. For linear time-invariant RLC circuits, it is shown in [8] that the index of the hybrid equations never exceeds one, while MNA often results in a DAE with index two. Moreover, [8] gives a structural characterization of circuits with index zero.

For nonlinear time-varying circuits, this paper shows that the index of the hybrid equations is at most one, and gives a structural characterization for the index being zero, which is an extension of the results in [8]. By this structural characterization, we prove that the minimum index of the hybrid equations does not exceed the index of the DAE arising from MNA (cf. [9, 10, 11, 12]). Here, we follow the hybrid analysis approach in [7] but use projection techniques (cf. [11]) in order to prove the index results for general nonlinear time-varying circuit systems.

The organization of this paper is as follows. In Section 2, we describe nonlinear time-varying circuits. We present the procedure of the hybrid analysis in Section 3. Section 4 is devoted to the definition of the tractability index of DAEs. We analyze the hybrid equation system in Section 5, and characterize its index in Section 6. In Section 7, we make comparisons between the hybrid analysis and MNA. Finally, Section 8 concludes this paper.

2 Nonlinear Time-Varying Circuits

Here, we consider nonlinear time-varying circuits composed of resistors, conductors, inductors, capacitors, and voltage/current sources.

We denote the vector of currents through all branches of the circuit by $\boldsymbol{i}$, and the vector of voltages across all branches by $\boldsymbol{u}$. Let V, J, C, L, R, G, S_V, and S_J denote the sets of independent voltage sources, independent current sources, capacitors, inductors, resistors, conductors, controlled voltage sources, and controlled current sources, respectively. The vector of currents through independent voltage sources, independent current sources, capacitors, inductors, resistors, conductors, controlled voltage sources, and controlled current sources are denoted by $\boldsymbol{i}_V$, $\boldsymbol{i}_J$, $\boldsymbol{i}_C$, $\boldsymbol{i}_L$, $\boldsymbol{i}_R$, $\boldsymbol{i}_G$, $\boldsymbol{i}_{S_V}$, and $\boldsymbol{i}_{S_J}$. The vector of voltages across independent voltage sources, independent current sources, capacitors, inductors, resistors, conductors, controlled voltage sources, and controlled current sources are denoted by $\boldsymbol{u}_V$, $\boldsymbol{u}_J$, $\boldsymbol{u}_C$, $\boldsymbol{u}_L$, $\boldsymbol{u}_R$, $\boldsymbol{u}_G$, $\boldsymbol{u}_{S_V}$, and $\boldsymbol{u}_{S_J}$. The physical characteristics of elements determine *constitutive equations.*

Independent voltage and current sources simply read as

$$\boldsymbol{u}_V = \boldsymbol{v}_s(t) \quad \text{and} \quad \boldsymbol{i}_J = \boldsymbol{j}_s(t). \tag{1}$$

We assume that the constitutive equations of capacitors and inductors are described by

$$\boldsymbol{i}_C = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}(\boldsymbol{u}_C, t) \quad \text{and} \quad \boldsymbol{u}_L = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}(\boldsymbol{i}_L, t). \tag{2}$$

Moreover, we assume that conductors and resistors are described by

$$\boldsymbol{i}_G = \boldsymbol{g}(\boldsymbol{u}_G, t) \quad \text{and} \quad \boldsymbol{u}_R = \boldsymbol{r}(\boldsymbol{i}_R, t).$$

Finally, let the controlled sources be given in the form of

$$\boldsymbol{i}_{S_J} = \boldsymbol{\gamma}(\boldsymbol{i}_{S_V}, \boldsymbol{u}_{S_J}, t) \quad \text{and} \quad \boldsymbol{u}_{S_V} = \boldsymbol{\rho}(\boldsymbol{i}_{S_V}, \boldsymbol{u}_{S_J}, t).$$

A square matrix U is called *positive definite* if $\boldsymbol{x}^\top U \boldsymbol{x} > 0$ for all $\boldsymbol{x} \neq \mathbf{0}$. In this paper, we assume the following conditions.

Assumption 2.1. *The capacitance matrix C, the conductance matrix G, the resistance matrix R, the inductance matrix L, and the controlled source matrix S given by*

$$C = \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{u}_C}, \quad G = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}_G}, \quad R = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{i}_R}, \quad L = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{i}_L}, \quad \textit{and} \quad S = \begin{pmatrix} \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{i}_{S_V}} & \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{u}_{S_J}} \\ \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{i}_{S_V}} & \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{u}_{S_J}} \end{pmatrix}$$

are all positive definite.[1]

Introducing

$$\boldsymbol{u}_Y := \begin{pmatrix} \boldsymbol{u}_G \\ \boldsymbol{u}_{S_J} \end{pmatrix}, \quad \boldsymbol{u}_Z := \begin{pmatrix} \boldsymbol{u}_R \\ \boldsymbol{u}_{S_V} \end{pmatrix}, \quad \boldsymbol{i}_Y := \begin{pmatrix} \boldsymbol{i}_G \\ \boldsymbol{i}_{S_J} \end{pmatrix}, \quad \boldsymbol{i}_Z := \begin{pmatrix} \boldsymbol{i}_R \\ \boldsymbol{i}_{S_V} \end{pmatrix}$$

and

$$\boldsymbol{f}(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t) := \begin{pmatrix} \boldsymbol{g}(\boldsymbol{u}_G, t) \\ \boldsymbol{\gamma}(\boldsymbol{i}_{S_V}, \boldsymbol{u}_{S_J}, t) \end{pmatrix}, \quad \boldsymbol{h}(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t) := \begin{pmatrix} \boldsymbol{r}(\boldsymbol{i}_R, t) \\ \boldsymbol{\rho}(\boldsymbol{i}_{S_V}, \boldsymbol{u}_{S_J}, t) \end{pmatrix},$$

we find

$$\boldsymbol{i}_Y = \boldsymbol{f}(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t), \quad \boldsymbol{u}_Z = \boldsymbol{h}(\boldsymbol{i}_Z, \boldsymbol{u}_Y, t) \tag{3}$$

and

$$\begin{pmatrix} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{i}_Z} & \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}_Y} \\ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{i}_Z} & \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}_Y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{i}_R} & 0 & 0 & 0 \\ 0 & \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{i}_{S_V}} & 0 & \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{u}_{S_J}} \\ 0 & 0 & \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}_G} & 0 \\ 0 & \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{i}_{S_V}} & 0 & \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{u}_{S_J}} \end{pmatrix}. \tag{4}$$

[1] Assuming the controlled source matrix S to be positive definite is very restrictive and usually not fulfilled when controlled sources are considered alone. However, controlled sources are often used to describe certain transistor behaviour. Considering the whole static behavior of a transistor (e.g. including bulk resistances) as a controlled source may lead to a positive definite matrix S.

Let $\Gamma = (W, E)$ be the network graph with vertex set W and edge set E. An edge in Γ corresponds to a branch that contains one element in the circuit. For a consistent model description, Γ contains no cycles consisting of independent voltage sources only and no cutsets consisting of independent current sources only. We split E into E_y and E_z, i.e., $E_y \cup E_z = E$ and $E_y \cap E_z = \emptyset$. A partition (E_y, E_z) is called an *admissible partition*, if E_y includes all the independent voltage sources, all the capacitors, all the conductors as well as all the controlled current sources, and E_z includes all the independent current sources, all the inductors, all the resistors as well as all the controlled voltage sources.

We call a spanning tree T of Γ a *reference tree* if T contains all the edges of the independent voltage sources, no edges of the independent current sources, and as many edges in E_y as possible. Note that a reference tree T may contain some edges in E_z. A reference tree is called *normal* if it contains as many edges corresponding to capacitors and as few edges corresponding to inductors as possible. The cotree of T is denoted by $\overline{T} = E \setminus T$. The hybrid equations are determined by an admissible partition (E_y, E_z) and a reference tree T, which is not necessarily normal. For the sake of simplicity, we adopt a normal reference tree throughout this paper.

With respect to a normal reference tree T, we further split $\boldsymbol{i}$ and $\boldsymbol{u}$ into

$$\boldsymbol{i} = (\boldsymbol{i}_V, \boldsymbol{i}_C^\tau, \boldsymbol{i}_Y^\tau, \boldsymbol{i}_Z^\tau, \boldsymbol{i}_L^\tau, \boldsymbol{i}_C^\lambda, \boldsymbol{i}_Y^\lambda, \boldsymbol{i}_Z^\lambda, \boldsymbol{i}_L^\lambda, \boldsymbol{i}_J)^\top$$

and

$$\boldsymbol{u} = (\boldsymbol{u}_V, \boldsymbol{u}_C^\tau, \boldsymbol{u}_Y^\tau, \boldsymbol{u}_Z^\tau, \boldsymbol{u}_L^\tau, \boldsymbol{u}_C^\lambda, \boldsymbol{u}_Y^\lambda, \boldsymbol{u}_Z^\lambda, \boldsymbol{u}_L^\lambda, \boldsymbol{u}_J)^\top,$$

where the superscripts τ and λ designate the tree T and the cotree $\overline{T}$. With respect to a normal reference tree T, the vector valued function $\boldsymbol{f}$ is also split into $\boldsymbol{f}^\tau$ and $\boldsymbol{f}^\lambda$. Similarly, we split $\boldsymbol{h}$, $\boldsymbol{q}$, and $\boldsymbol{\phi}$. The matrix C and L are written in the form of

$$\begin{pmatrix} C_\tau^\tau & C_\lambda^\tau \\ C_\tau^\lambda & C_\lambda^\lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L_\tau^\tau & L_\lambda^\tau \\ L_\tau^\lambda & L_\lambda^\lambda \end{pmatrix},$$

where

$$C_\tau^\tau = \frac{\partial \boldsymbol{q}^\tau}{\partial \boldsymbol{u}_C^\tau}, \quad C_\lambda^\tau = \frac{\partial \boldsymbol{q}^\tau}{\partial \boldsymbol{u}_C^\lambda}, \quad C_\tau^\lambda = \frac{\partial \boldsymbol{q}^\lambda}{\partial \boldsymbol{u}_C^\tau}, \quad C_\lambda^\lambda = \frac{\partial \boldsymbol{q}^\lambda}{\partial \boldsymbol{u}_C^\lambda},$$
$$L_\tau^\tau = \frac{\partial \boldsymbol{\phi}^\tau}{\partial \boldsymbol{i}_L^\tau}, \quad L_\lambda^\tau = \frac{\partial \boldsymbol{\phi}^\tau}{\partial \boldsymbol{i}_L^\lambda}, \quad L_\tau^\lambda = \frac{\partial \boldsymbol{\phi}^\lambda}{\partial \boldsymbol{i}_L^\tau}, \quad L_\lambda^\lambda = \frac{\partial \boldsymbol{\phi}^\lambda}{\partial \boldsymbol{i}_L^\lambda}.$$

Let us define

$$Z = \begin{pmatrix} Z_\tau^\tau & Z_\lambda^\tau \\ Z_\tau^\lambda & Z_\lambda^\lambda \end{pmatrix}, \quad H = \begin{pmatrix} H_\tau^\tau & H_\lambda^\tau \\ H_\tau^\lambda & H_\lambda^\lambda \end{pmatrix}, \quad F = \begin{pmatrix} F_\tau^\tau & F_\lambda^\tau \\ F_\tau^\lambda & F_\lambda^\lambda \end{pmatrix}, \quad Y = \begin{pmatrix} Y_\tau^\tau & Y_\lambda^\tau \\ Y_\tau^\lambda & Y_\lambda^\lambda \end{pmatrix},$$

where

$$
\begin{aligned}
&Z_\tau^\tau = \frac{\partial \boldsymbol{h}^\tau}{\partial \boldsymbol{i}_Z^\tau}, \quad Z_\lambda^\tau = \frac{\partial \boldsymbol{h}^\tau}{\partial \boldsymbol{i}_Z^\lambda}, \quad Z_\tau^\lambda = \frac{\partial \boldsymbol{h}^\lambda}{\partial \boldsymbol{i}_Z^\tau}, \quad Z_\lambda^\lambda = \frac{\partial \boldsymbol{h}^\lambda}{\partial \boldsymbol{i}_Z^\lambda},\\
&H_\tau^\tau = \frac{\partial \boldsymbol{h}^\tau}{\partial \boldsymbol{u}_Y^\tau}, \quad H_\lambda^\tau = \frac{\partial \boldsymbol{h}^\tau}{\partial \boldsymbol{u}_Y^\lambda}, \quad H_\tau^\lambda = \frac{\partial \boldsymbol{h}^\lambda}{\partial \boldsymbol{u}_Y^\tau}, \quad H_\lambda^\lambda = \frac{\partial \boldsymbol{h}^\lambda}{\partial \boldsymbol{u}_Y^\lambda},\\
&F_\tau^\tau = \frac{\partial \boldsymbol{f}^\tau}{\partial \boldsymbol{i}_Z^\tau}, \quad F_\lambda^\tau = \frac{\partial \boldsymbol{f}^\tau}{\partial \boldsymbol{i}_Z^\lambda}, \quad F_\tau^\lambda = \frac{\partial \boldsymbol{f}^\lambda}{\partial \boldsymbol{i}_Z^\tau}, \quad F_\lambda^\lambda = \frac{\partial \boldsymbol{f}^\lambda}{\partial \boldsymbol{i}_Z^\lambda},\\
&Y_\tau^\tau = \frac{\partial \boldsymbol{f}^\tau}{\partial \boldsymbol{u}_Y^\tau}, \quad Y_\lambda^\tau = \frac{\partial \boldsymbol{f}^\tau}{\partial \boldsymbol{u}_Y^\lambda}, \quad Y_\tau^\lambda = \frac{\partial \boldsymbol{f}^\lambda}{\partial \boldsymbol{u}_Y^\tau}, \quad Y_\lambda^\lambda = \frac{\partial \boldsymbol{f}^\lambda}{\partial \boldsymbol{u}_Y^\lambda}.
\end{aligned}
$$

Then $\begin{pmatrix} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{i}_Z} & \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{u}_Y} \\ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{i}_Z} & \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}_Y} \end{pmatrix}$ is written in the form of $\begin{pmatrix} Z & H \\ F & Y \end{pmatrix}$, which is positive definite by Assumption 2.1 and (4).

By the definition of a normal reference tree, the *fundamental cutset matrix* K is given by

$$
K = \begin{array}{c} \begin{matrix} \boldsymbol{i}_V & \boldsymbol{i}_C^\tau & \boldsymbol{i}_Y^\tau & \boldsymbol{i}_Z^\tau & \boldsymbol{i}_L^\tau & \boldsymbol{i}_C^\lambda & \boldsymbol{i}_Y^\lambda & \boldsymbol{i}_Z^\lambda & \boldsymbol{i}_L^\lambda & \boldsymbol{i}_J \end{matrix} \\ \begin{pmatrix} I & 0 & 0 & 0 & 0 & A_{VC} & A_{VY} & A_{VZ} & A_{VL} & A_{VJ} \\ 0 & I & 0 & 0 & 0 & A_{CC} & A_{CY} & A_{CZ} & A_{CL} & A_{CJ} \\ 0 & 0 & I & 0 & 0 & 0 & A_{YY} & A_{YZ} & A_{YL} & A_{YJ} \\ 0 & 0 & 0 & I & 0 & 0 & 0 & A_{ZZ} & A_{ZL} & A_{ZJ} \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & A_{LL} & A_{LJ} \end{pmatrix} \end{array}.
$$

Then *Kirchhoff's current law* (KCL), which states that the sum of currents entering each node is equal to zero, may be written as

$$K\boldsymbol{i} = \boldsymbol{0}.$$

This is rewritten as

$$\boldsymbol{i}_V = -A_{VC}\boldsymbol{i}_C^\lambda - A_{VY}\boldsymbol{i}_Y^\lambda - A_{VZ}\boldsymbol{i}_Z^\lambda - A_{VL}\boldsymbol{i}_L^\lambda - A_{VJ}\boldsymbol{i}_J, \tag{5}$$

$$\boldsymbol{i}_C^\tau + A_{CC}\boldsymbol{i}_C^\lambda + A_{CY}\boldsymbol{i}_Y^\lambda + A_{CZ}\boldsymbol{i}_Z^\lambda + A_{CL}\boldsymbol{i}_L^\lambda + A_{CJ}\boldsymbol{i}_J = \boldsymbol{0}, \tag{6}$$

$$\boldsymbol{i}_Y^\tau + A_{YY}\boldsymbol{i}_Y^\lambda + A_{YZ}\boldsymbol{i}_Z^\lambda + A_{YL}\boldsymbol{i}_L^\lambda + A_{YJ}\boldsymbol{i}_J = \boldsymbol{0}, \tag{7}$$

$$\boldsymbol{i}_Z^\tau = -A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{i}_J, \tag{8}$$

$$\boldsymbol{i}_L^\tau = -A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{i}_J. \tag{9}$$

Kirchhoff's voltage law (KVL), which states that the sum of voltages in each loop of the network is equal to zero, provides

$$K^\perp \boldsymbol{u} = \boldsymbol{0}$$

with $K^\perp$ being the *fundamental loop matrix*

$$
K^\perp = \begin{array}{c} \begin{matrix} \boldsymbol{u}_V & \boldsymbol{u}_C^\tau & \boldsymbol{u}_Y^\tau & \boldsymbol{u}_Z^\tau & \boldsymbol{u}_L^\tau & \boldsymbol{u}_C^\lambda & \boldsymbol{u}_Y^\lambda & \boldsymbol{u}_Z^\lambda & \boldsymbol{u}_L^\lambda & \boldsymbol{u}_J \end{matrix} \\ \begin{pmatrix} -A_{VC}^\top & -A_{CC}^\top & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ -A_{VY}^\top & -A_{CY}^\top & -A_{YY}^\top & 0 & 0 & 0 & I & 0 & 0 & 0 \\ -A_{VZ}^\top & -A_{CZ}^\top & -A_{YZ}^\top & -A_{ZZ}^\top & 0 & 0 & 0 & I & 0 & 0 \\ -A_{VL}^\top & -A_{CL}^\top & -A_{YL}^\top & -A_{ZL}^\top & -A_{LL}^\top & 0 & 0 & 0 & I & 0 \\ -A_{VJ}^\top & -A_{CJ}^\top & -A_{YJ}^\top & -A_{ZJ}^\top & -A_{LJ}^\top & 0 & 0 & 0 & 0 & I \end{pmatrix} \end{array}.
$$

This is rewritten as

$$\boldsymbol{u}_C^\lambda = A_{VC}^\top \boldsymbol{u}_V + A_{CC}^\top \boldsymbol{u}_C^\tau, \tag{10}$$
$$\boldsymbol{u}_Y^\lambda = A_{VY}^\top \boldsymbol{u}_V + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau, \tag{11}$$
$$\boldsymbol{u}_Z^\lambda - A_{VZ}^\top \boldsymbol{u}_V - A_{CZ}^\top \boldsymbol{u}_C^\tau - A_{YZ}^\top \boldsymbol{u}_Y^\tau - A_{ZZ}^\top \boldsymbol{u}_Z^\tau = \boldsymbol{0}, \tag{12}$$
$$\boldsymbol{u}_L^\lambda - A_{VL}^\top \boldsymbol{u}_V - A_{CL}^\top \boldsymbol{u}_C^\tau - A_{YL}^\top \boldsymbol{u}_Y^\tau - A_{ZL}^\top \boldsymbol{u}_Z^\tau - A_{LL}^\top \boldsymbol{u}_L^\tau = \boldsymbol{0}, \tag{13}$$
$$\boldsymbol{u}_J = A_{VJ}^\top \boldsymbol{u}_V + A_{CJ}^\top \boldsymbol{u}_C^\tau + A_{YJ}^\top \boldsymbol{u}_Y^\tau + A_{ZJ}^\top \boldsymbol{u}_Z^\tau + A_{LJ}^\top \boldsymbol{u}_L^\tau. \tag{14}$$

3 Hybrid Analysis

In this section, we describe the procedure of the hybrid analysis. The idea is to use all constitutive equations such that the equations $K\boldsymbol{i} = \boldsymbol{0}$ and $K^\perp \boldsymbol{u} = \boldsymbol{0}$ provide a system depending on $\boldsymbol{u}_C^\tau$, $\boldsymbol{u}_Y^\tau$, $\boldsymbol{i}_Z^\lambda$, and $\boldsymbol{i}_L^\lambda$ only. The second and third line of $K\boldsymbol{i} = \boldsymbol{0}$ as well as the third and fourth line of $K^\perp \boldsymbol{u} = \boldsymbol{0}$ provide us the *hybrid equations* (or *hybrid equation system*)

$$\begin{aligned}
-A_{CZ}^\top \boldsymbol{u}_C^\tau - A_{YZ}^\top \boldsymbol{u}_Y^\tau - A_{ZZ}^\top \boldsymbol{h}^\tau + \boldsymbol{h}^\lambda &= A_{VZ}^\top \boldsymbol{v}_s(t), \\
-A_{CL}^\top \boldsymbol{u}_C^\tau - A_{YL}^\top \boldsymbol{u}_Y^\tau - A_{ZL}^\top \boldsymbol{h}^\tau - A_{LL}^\top \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\tau + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\lambda &= A_{VL}^\top \boldsymbol{v}_s(t), \\
A_{CY}\boldsymbol{f}^\lambda + A_{CZ}\boldsymbol{i}_Z^\lambda + A_{CL}\boldsymbol{i}_L^\lambda + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\tau + A_{CC}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\lambda &= -A_{CJ}\boldsymbol{j}_s(t), \\
\boldsymbol{f}^\tau + A_{YY}\boldsymbol{f}^\lambda + A_{YZ}\boldsymbol{i}_Z^\lambda + A_{YL}\boldsymbol{i}_L^\lambda &= -A_{YJ}\boldsymbol{j}_s(t),
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{q}^\tau &= \boldsymbol{q}^\tau(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, t), \\
\boldsymbol{q}^\lambda &= \boldsymbol{q}^\lambda(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, t), \\
\boldsymbol{f}^\tau &= \boldsymbol{f}^\tau(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau, t), \\
\boldsymbol{f}^\lambda &= \boldsymbol{f}^\lambda(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau, t), \\
\boldsymbol{h}^\tau &= \boldsymbol{h}^\tau(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau, t), \\
\boldsymbol{h}^\lambda &= \boldsymbol{h}^\lambda(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau, t), \\
\boldsymbol{\phi}^\tau &= \boldsymbol{\phi}^\tau(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, t), \\
\boldsymbol{\phi}^\lambda &= \boldsymbol{\phi}^\lambda(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, t).
\end{aligned}$$

The derivation of the hybrid equations is given in Appendix A. The procedure of the hybrid analysis is as follows.

1. The values of $\boldsymbol{u}_V$ and $\boldsymbol{i}_J$ are obvious from (1).

2. Compute the values of $\boldsymbol{i}_Z^\lambda$, $\boldsymbol{i}_L^\lambda$ and $\boldsymbol{u}_C^\tau$, $\boldsymbol{u}_Y^\tau$ by solving the hybrid equations.

3. Compute the values of $\boldsymbol{i}_Z^\tau$, $\boldsymbol{i}_L^\tau$ from (8) and (9), and $\boldsymbol{u}_C^\lambda$, $\boldsymbol{u}_Y^\lambda$ from (10) and (11).

4. Compute the values of $\boldsymbol{u}_Z^\tau$, $\boldsymbol{u}_Z^\lambda$, $\boldsymbol{u}_L^\tau$, $\boldsymbol{u}_L^\lambda$, and $\boldsymbol{i}_C^\tau$, $\boldsymbol{i}_C^\lambda$, $\boldsymbol{i}_Y^\tau$, $\boldsymbol{i}_Y^\lambda$ by substituting the values obtained in Steps 1–3 into (2) and (3).

5. Compute the values of $\boldsymbol{i}_V$ and $\boldsymbol{u}_J$ by substituting the values obtained in Steps 1–4 into (5) and (14).

In the case of $E_y = \emptyset$, the above procedure is called the *loop analysis* or the *tieset analysis*. In the case of $E_z = \emptyset$, it is called the *cutset analysis*.

All operations in Steps 3–5 are substitutions and differentiations of the obtained solutions. Consequently, the numerical difficulty is determined by the index of the hybrid equation system. This hybrid equation system depends on the branch voltages $\boldsymbol{u}_C^{\mathcal{T}}$ of capacitors in the tree T, the branch voltages $\boldsymbol{u}_Y^{\mathcal{T}}$ of conductors and controlled current sources in T, the branch currents $\boldsymbol{i}_L^{\lambda}$ of inductors in the cotree $\overline{T}$, and the branch currents $\boldsymbol{i}_Z^{\lambda}$ of resistors and controlled voltage sources in $\overline{T}$. Higher index variables as known from MNA do not appear in the hybrid equation system. In this paper, we prove that the hybrid equation system has index at most one. The proof relies on the *tractability index* concept with the use of projector based analysis.

4 DAEs with Properly Stated Leading Term

Consider a DAE in the form of

$$A \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{d}(\boldsymbol{x}(t), t) + \boldsymbol{b}(\boldsymbol{x}(t), t) = \mathbf{0}. \tag{15}$$

Let A be an $m \times n$ matrix. We define

$$D(\boldsymbol{x}, t) := \frac{\partial \boldsymbol{d}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}, \quad B(\boldsymbol{x}, t) := \frac{\partial \boldsymbol{b}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}, \quad \text{and} \quad M(\boldsymbol{x}, t) := AD(\boldsymbol{x}, t).$$

A matrix P satisfying $P^2 = P$ is called a *projector*. Moreover, a projector P is called a *projector onto* a subspace Σ if $\operatorname{im} P = \Sigma$.

Definition 4.1 ([13, Definition 2.1])**.** *The equation* (15) *is a DAE with properly stated leading term if the size of* $D(\boldsymbol{x}, t)$ *is* $n \times m$*,*

$$\ker A \oplus \operatorname{im} D(\boldsymbol{x}, t) = \mathbb{R}^n \tag{16}$$

holds for all $\boldsymbol{x}$ *and* t *from the definition domain, and there is an* $n \times n$ *projector function* $P(t)$ *continuously differentiable with respect to* t *such that* $\ker P(t) = \ker A$*,* $\operatorname{im} P(t) = \operatorname{im} D(\boldsymbol{x}, t)$*, and* $\boldsymbol{d}(\boldsymbol{x}, t) = P(t)\boldsymbol{d}(\boldsymbol{x}, t)$*.*

A DAE with properly stated leading term (15) arises in circuit simulation via circuit analysis methods such as MNA [14]. A DAE with properly stated leading term was first introduced in [15]. The analysis of such DAEs has been developed in [14, 16, 17, 18, 19].

Lemma 4.2 ([14, Lemma A.1])**.** *Let* A *be an* $m \times n$ *matrix and* $D(\boldsymbol{x}, t)$ *be an* $n \times m$ *matrix. Then, the relation* $\ker A \oplus \operatorname{im} D(\boldsymbol{x}, t) = \mathcal{R}^n$ *is equivalent to the following three conditions:*

$$\operatorname{im} M(\boldsymbol{x}, t) = \operatorname{im} A, \ \ker M(\boldsymbol{x}, t) = \ker D(\boldsymbol{x}, t), \ \ker A \cap \operatorname{im} D(\boldsymbol{x}, t) = \{\mathbf{0}\},$$

where $M(\boldsymbol{x}, t) = AD(\boldsymbol{x}, t)$*.*

Obviously, the DAE (15) represents a regular ODE if and only if the matrix $M(\boldsymbol{x},t)$ is nonsingular for all $\boldsymbol{x}$ and t of the definition domain. In this case we say that the DAE (15) has index 0. In the case of a singular matrix $M(\boldsymbol{x},t)$ for all $\boldsymbol{x}$ and t, the DAE (15) contains algebraic equations. Furthermore, one may have to differentiate certain part of the system to get a solution. A simple criteria for the absence of this problem is given by the tractability index 1 condition (see [13], Theorem 4.3).

Definition 4.3 ([13, Definition 3.3])**.** *The DAE* (15) *is regular with index 1 on their definition domain if* $M(\boldsymbol{x},t)$ *is singular and*

$$\ker D(\boldsymbol{x},t) \cap \{\boldsymbol{z} \in \mathbb{R}^m \mid B(\boldsymbol{x},t)\boldsymbol{z} \in \operatorname{im} M(\boldsymbol{x},t)\} = \{\boldsymbol{0}\}$$

for all $(\boldsymbol{x},t)$ *of the definition domain.*

Remark 4.4 ([20, Remark 4.6])**.** *A DAE* (15) *is regular with index 1 if and only if the matrix* $M(\boldsymbol{x},t)+B(\boldsymbol{x},t)Q(\boldsymbol{x},t)$ *is nonsingular for all* $\boldsymbol{x}$ *and* t *with a projector* $Q(\boldsymbol{x},t)$ *onto* $\ker M(\boldsymbol{x},t)$.

We now adduce a useful lemma.

Lemma 4.5 ([10, Lemma 3])**.** *Let* U *be a* $\mu \times \mu$ *positive definite matrix and* N *be a* $k \times \mu$ *rectangular matrix. Then* $\ker NUN^\top = \ker N^\top$ *and* $\operatorname{im} NUN^\top = \operatorname{im} N$ *hold.*

5 Hybrid Equations with Properly Stated Leading Term

In this section, we rewrite the hybrid equation system as a DAE with properly stated leading term. We first define a reflexive generalized inverse.

Definition 5.1. *A reflexive generalized inverse of a matrix* A *is a matrix* A^- *which satisfies* $AA^-A = A$ *and* $A^-AA^- = A^-$.

A reflexive generalized inverse A^- satisfies

$$\dim \operatorname{im} A^-A = \dim \operatorname{im} A. \tag{17}$$

We now define

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -A_{LL}^\top & I & 0 & 0 \\ 0 & 0 & I & A_{CC} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{d}(\boldsymbol{x},t) = A^-A \begin{pmatrix} \boldsymbol{\phi}^\tau(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, t) \\ \boldsymbol{\phi}^\lambda(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, t) \\ \boldsymbol{q}^\tau(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, t) \\ \boldsymbol{q}^\lambda(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, t) \end{pmatrix},$$

$$\boldsymbol{x}(t) = \begin{pmatrix} \boldsymbol{i}_Z^\lambda \\ \boldsymbol{i}_L^\lambda \\ \boldsymbol{u}_C^\tau \\ \boldsymbol{u}_Y^\tau \end{pmatrix}, \quad \boldsymbol{b}(\boldsymbol{x},t) = \begin{pmatrix} -A_{VZ}^\top \boldsymbol{v}_s(t) - A_{CZ}^\top \boldsymbol{u}_C^\tau - A_{YZ}^\top \boldsymbol{u}_Y^\tau - A_{ZZ}^\top \boldsymbol{h}^\tau + \boldsymbol{h}^\lambda \\ -A_{VL}^\top \boldsymbol{v}_s(t) - A_{CL}^\top \boldsymbol{u}_C^\tau - A_{YL}^\top \boldsymbol{u}_Y^\tau - A_{ZL}^\top \boldsymbol{h}^\tau \\ A_{CY}\boldsymbol{f}^\lambda + A_{CZ}\boldsymbol{i}_Z^\lambda + A_{CL}\boldsymbol{i}_L^\lambda + A_{CJ}\boldsymbol{j}_s(t) \\ \boldsymbol{f}^\tau + A_{YY}\boldsymbol{f}^\lambda + A_{YZ}\boldsymbol{i}_Z^\lambda + A_{YL}\boldsymbol{i}_L^\lambda + A_{YJ}\boldsymbol{j}_s(t) \end{pmatrix}.$$

By $A = AA^{-}A$, this gives the hybrid equation system in the form of (15). The Jacobian matrices $D(\boldsymbol{x}, t)$ and $B(\boldsymbol{x}, t)$ are given by

$$D(\boldsymbol{x}, t) = A^{-}A \begin{pmatrix} 0 & -L_{\tau}^{\tau} A_{LL} + L_{\lambda}^{\tau} & 0 & 0 \\ 0 & -L_{\tau}^{\lambda} A_{LL} + L_{\lambda}^{\lambda} & 0 & 0 \\ 0 & 0 & C_{\tau}^{\tau} + C_{\lambda}^{\tau} A_{CC}^{\top} & 0 \\ 0 & 0 & C_{\tau}^{\lambda} + C_{\lambda}^{\lambda} A_{CC}^{\top} & 0 \end{pmatrix}$$

and

$$B(\boldsymbol{x}, t) = \begin{pmatrix} B_Z(\boldsymbol{x}, t) & A_{ZZ}^{\top} Z_{\tau}^{\tau} A_{ZL} - Z_{\tau}^{\lambda} A_{ZL} & -A_{CZ}^{\top} & -A_{YZ}^{\top} + B_H(\boldsymbol{x}, t) \\ A_{ZL}^{\top} Z_{\tau}^{\tau} A_{ZZ} - A_{ZL}^{\top} Z_{\lambda}^{\tau} & A_{ZL}^{\top} Z_{\tau}^{\tau} A_{ZL} & -A_{CL}^{\top} & -A_{YL}^{\top} - A_{ZL}^{\top} H_{\tau}^{\tau} - A_{ZL}^{\top} H_{\lambda}^{\tau} A_{YY}^{\top} \\ A_{CZ} - A_{CY} F_{\tau}^{\lambda} A_{ZZ} + A_{CY} F_{\lambda}^{\lambda} & A_{CL} & A_{CY} Y_{\lambda}^{\lambda} A_{CY}^{\top} & A_{CY} Y_{\tau}^{\lambda} + A_{CY} Y_{\lambda}^{\lambda} A_{YY}^{\top} \\ A_{YZ} + B_F(\boldsymbol{x}, t) & A_{YL} & Y_{\lambda}^{\tau} A_{CY}^{\top} + A_{YY} Y_{\lambda}^{\lambda} A_{CY}^{\top} & B_Y(\boldsymbol{x}, t) \end{pmatrix},$$

where

$$\begin{aligned} B_Z(\boldsymbol{x}, t) &= A_{ZZ}^{\top} Z_{\tau}^{\tau} A_{ZZ} - A_{ZZ}^{\top} Z_{\lambda}^{\tau} - Z_{\tau}^{\lambda} A_{ZZ} + Z_{\lambda}^{\lambda}, \\ B_H(\boldsymbol{x}, t) &= -A_{ZZ}^{\top} H_{\tau}^{\tau} - A_{ZZ}^{\top} H_{\lambda}^{\tau} A_{YY}^{\top} + H_{\tau}^{\lambda} + H_{\lambda}^{\lambda} A_{YY}^{\top}, \\ B_Y(\boldsymbol{x}, t) &= Y_{\tau}^{\tau} + Y_{\lambda}^{\tau} A_{YY}^{\top} + A_{YY} Y_{\tau}^{\lambda} + A_{YY} Y_{\lambda}^{\lambda} A_{YY}^{\top}, \\ B_F(\boldsymbol{x}, t) &= -F_{\tau}^{\tau} A_{ZZ} + F_{\lambda}^{\tau} - A_{YY} F_{\tau}^{\lambda} A_{ZZ} + A_{YY} F_{\lambda}^{\lambda}. \end{aligned}$$

Then these matrices have the following property.

Lemma 5.2. *If* $\begin{pmatrix} Z & H \\ F & Y \end{pmatrix}$ *is positive definite, then* $\begin{pmatrix} B_Z(\boldsymbol{x}, t) & B_H(\boldsymbol{x}, t) \\ B_F(\boldsymbol{x}, t) & B_Y(\boldsymbol{x}, t) \end{pmatrix}$ *is positive definite.*

Proof. Using $A_Z = \begin{pmatrix} -A_{ZZ}^{\top} & I \end{pmatrix}$ and $A_Y = \begin{pmatrix} I & A_{YY} \end{pmatrix}$, we have

$$\begin{pmatrix} B_Z(\boldsymbol{x}, t) & B_H(\boldsymbol{x}, t) \\ B_F(\boldsymbol{x}, t) & B_Y(\boldsymbol{x}, t) \end{pmatrix} = \begin{pmatrix} A_Z & 0 \\ 0 & A_Y \end{pmatrix} \begin{pmatrix} Z & H \\ F & Y \end{pmatrix} \begin{pmatrix} A_Z & 0 \\ 0 & A_Y \end{pmatrix}^{\top}.$$

Since $\begin{pmatrix} Z & H \\ F & Y \end{pmatrix}$ is positive definite and A_Y and A_Z are of full row rank, $\begin{pmatrix} B_Z(\boldsymbol{x}, t) & B_H(\boldsymbol{x}, t) \\ B_F(\boldsymbol{x}, t) & B_Y(\boldsymbol{x}, t) \end{pmatrix}$ is also positive definite. □

Let us define

$$\Omega(\boldsymbol{x}, t) = \begin{pmatrix} L_{\tau}^{\tau} & L_{\lambda}^{\tau} & 0 & 0 \\ L_{\tau}^{\lambda} & L_{\lambda}^{\lambda} & 0 & 0 \\ 0 & 0 & C_{\tau}^{\tau} & C_{\lambda}^{\tau} \\ 0 & 0 & C_{\tau}^{\lambda} & C_{\lambda}^{\lambda} \end{pmatrix}.$$

Now we have

$$D(\boldsymbol{x}, t) = A^{-}A \begin{pmatrix} L_{\tau}^{\tau} & L_{\lambda}^{\tau} & 0 & 0 \\ L_{\tau}^{\lambda} & L_{\lambda}^{\lambda} & 0 & 0 \\ 0 & 0 & C_{\tau}^{\tau} & C_{\lambda}^{\tau} \\ 0 & 0 & C_{\tau}^{\lambda} & C_{\lambda}^{\lambda} \end{pmatrix} \begin{pmatrix} 0 & -A_{LL} & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & A_{CC}^{\top} & 0 \end{pmatrix} = A^{-}A\Omega(\boldsymbol{x}, t)A^{\top}.$$

Hence $M(\boldsymbol{x}, t) = AA^{-}A\Omega(\boldsymbol{x}, t)A^{\top} = A\Omega(\boldsymbol{x}, t)A^{\top}$ holds. By using this equation, we obtain the following two lemmas.

Lemma 5.3. *Suppose that* $\Omega(\boldsymbol{x}, t)$ *is positive definite. Then* $\operatorname{im} M(\boldsymbol{x}, t) = \operatorname{im} A$ *and* $\ker M(\boldsymbol{x}, t) = \ker D(\boldsymbol{x}, t)$ *hold.*

Proof. Since $\Omega(\boldsymbol{x}, t)$ is positive definite, it follows from Lemma 4.5 that $\ker M(\boldsymbol{x}, t) = \ker A^\top$ and $\operatorname{im} M(\boldsymbol{x}, t) = \operatorname{im} A$. We now have

$$\ker D(\boldsymbol{x}, t) = \ker A^- A\Omega(\boldsymbol{x}, t)A^\top \supseteq \ker A^\top.$$

Let $\boldsymbol{z}$ be an element in $\ker D(\boldsymbol{x}, t)$. Then, by $D(\boldsymbol{x}, t)\boldsymbol{z} = \boldsymbol{0}$, we have $\boldsymbol{z}^\top AA^- A\Omega(\boldsymbol{x}, t)A^\top \boldsymbol{z} = 0$, which is $\boldsymbol{z}^\top A\Omega(\boldsymbol{x}, t)A^\top \boldsymbol{z} = 0$. Since $\Omega(\boldsymbol{x}, t)$ is positive definite, we have $A^\top \boldsymbol{z} = \boldsymbol{0}$. Hence $\boldsymbol{z} \in \ker A^\top$ holds. Thus we obtain $\ker D(\boldsymbol{x}, t) = \ker A^\top = \ker M(\boldsymbol{x}, t)$. ☐

Lemma 5.4. *Suppose that* $\Omega(\boldsymbol{x}, t)$ *is positive definite. Then* $\ker A \cap \operatorname{im} D(\boldsymbol{x}, t) = \{\boldsymbol{0}\}$ *holds.*

Proof. Let $\boldsymbol{z}$ be an element in $\ker A \cap \operatorname{im} D(\boldsymbol{x}, t)$. Then we have $A\boldsymbol{z} = \boldsymbol{0}$ and $\boldsymbol{z} = D(\boldsymbol{x}, t)\boldsymbol{y}$ for some $\boldsymbol{y}$. Hence $AD(\boldsymbol{x}, t)\boldsymbol{y} = \boldsymbol{0}$ holds, which implies that $\boldsymbol{y} \in \ker AD(\boldsymbol{x}, t) = \ker D(\boldsymbol{x}, t)$ by Lemma 5.3. Thus we obtain $\boldsymbol{z} = D(\boldsymbol{x}, t)\boldsymbol{y} = \boldsymbol{0}$. ☐

With the use of a reflexive generalized inverse, we define a constant projector $P = A^- A$. Then the projector P has the following property.

Lemma 5.5. *Suppose that* $\Omega(\boldsymbol{x}, t)$ *is positive definite. For a projector* $P = A^- A$*, we have* $\ker P = \ker A$ *and* $\operatorname{im} P = \operatorname{im} D(\boldsymbol{x}, t)$*.*

Proof. We first prove $\ker P = \ker A$. It clearly holds that $\ker P = \ker A^- A \supseteq \ker A$. For any $\boldsymbol{z} \in \ker P$, we have $A^- A\boldsymbol{z} = \boldsymbol{0}$. Hence $AA^- A\boldsymbol{z} = \boldsymbol{0}$ holds, which implies $\boldsymbol{z} \in \ker A$. Thus we obtain $\ker P \subseteq \ker A$.

Secondly, we prove $\operatorname{im} P = \operatorname{im} D(\boldsymbol{x}, t)$. It clearly holds that $\operatorname{im} D(\boldsymbol{x}, t) = \operatorname{im} A^- A\Omega(\boldsymbol{x}, t)A^\top \subseteq \operatorname{im} P$. By the proof of Lemma 5.3, $\ker D(\boldsymbol{x}, t) = \ker A^\top$ holds. Hence we have

$$\dim \operatorname{im} D(\boldsymbol{x}, t) = m - \dim \ker D(\boldsymbol{x}, t) = m - \dim \ker A^\top = \dim \operatorname{im} A^\top.$$

It follows from (17) that

$$\dim \operatorname{im} A^\top = \dim \operatorname{im} A = \dim \operatorname{im} A^- A,$$

which implies $\dim \operatorname{im} D(\boldsymbol{x}, t) = \dim \operatorname{im} P$. Thus we obtain $\operatorname{im} P = \operatorname{im} D(\boldsymbol{x}, t)$. ☐

By Lemmas 5.3–5.5, we obtain the following proposition.

Proposition 5.6. *Suppose that* $\Omega(\boldsymbol{x}, t)$ *is positive definite. Then the hybrid equation system in the form of* (15) *is a DAE with properly stated leading term.*

Proof. We obtain (16) by Lemmas 4.2, 5.3, and 5.4. Moreover, $P\boldsymbol{d}(\boldsymbol{x}, t) = A^- A\boldsymbol{d}(\boldsymbol{x}, t) = \boldsymbol{d}(\boldsymbol{x}, t)$ holds. Thus, by Lemma 5.5, P is a projector satisfying Definition 4.1. ☐

Let us define

$$Q = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

In fact, Q is a projector satisfying the condition in Remark 4.4 as follows.

Lemma 5.7. *Suppose that* $\Omega(\boldsymbol{x},t)$ *is positive definite. Then* $\operatorname{im} Q = \ker M(\boldsymbol{x},t)$ *holds.*

Proof. Since $\Omega(\boldsymbol{x},t)$ is positive definite, it follows from Lemma 4.5 that $\ker M(\boldsymbol{x},t) = \ker A^\top$. Hence we obtain

$$\ker M(\boldsymbol{x},t) = \{\boldsymbol{z} \mid A^\top \boldsymbol{z} = \boldsymbol{0}\} = \operatorname{im} Q$$

by the definition of A. □

6 Index of Hybrid Equations

In this section, we prove that the index of the hybrid equations is at most one, and give a structural criteria for hybrid equations with index zero.

We now introduce the *Resistor-Acyclic condition* for admissible partition (E_y, E_z), which is proved in Theorem 6.2 to be a necessary and sufficient condition for hybrid equations with index zero.

[**Resistor-Acyclic condition**]

- Each conductor and controlled current source in E_y belongs to a cycle consisting of independent voltage sources, capacitors, and itself.
- Each resistor and controlled voltage source in E_z belongs to a cutset consisting of inductors, independent current sources, and itself.

The Resistor-Acyclic condition can be expressed as follows.

Lemma 6.1. *An admissible partition* (E_y, E_z) *satisfies the Resistor-Acyclic condition if and only if there exists a normal reference tree* T *such that* $Y \subseteq \overline{T}$ *and* $Z \subseteq T$. □

We obtain the following theorem concerning the index.

Theorem 6.2. *Under Assumption 2.1, the index of the hybrid equations is at most one for any admissible partition* (E_y, E_z) *and normal reference tree* T*. Moreover, the index is zero if and only if an admissible partition* (E_y, E_z) *satisfies the Resistor-Acyclic condition.*

Proof. The index of the hybrid equations is zero if and only if $M(\boldsymbol{x},t) = A\Omega(\boldsymbol{x},t)A^\top$ is nonsingular. Since $\Omega(\boldsymbol{x},t)$ is positive definite, this is equivalent to the condition that A is full row rank, which means that we have no variables $\boldsymbol{i}_Z^\lambda$ and $\boldsymbol{u}_Y^\tau$. In other words, $Y \subseteq \overline{T}$ and $Z \subseteq T$ hold. This is the Resistor-Acyclic condition by Lemma 6.1.

In order to prove that the index of the hybrid equations is at most one, we show that $M(\boldsymbol{x},t) + B(\boldsymbol{x},t)Q$ is nonsingular. Now we have

$$\begin{aligned} M(\boldsymbol{x},t) + B(\boldsymbol{x},t)Q &= M(\boldsymbol{x},t) + \begin{pmatrix} B_Z(\boldsymbol{x},t) & 0 & 0 & -A_{YZ}^\top + B_H(\boldsymbol{x},t) \\ A_{ZL}^\top Z_\tau^\tau A_{ZZ} - A_{ZL}^\top Z_\lambda^\tau & 0 & 0 & -A_{YL}^\top - A_{ZL}^\top H_\tau^\tau - A_{ZL}^\top H_\lambda^\tau A_{YY}^\top \\ A_{CZ} - A_{CY} F_\tau^\lambda A_{ZZ} + A_{CY} F_\lambda^\lambda & 0 & 0 & A_{CY} Y_\tau^\lambda + A_{CY} Y_\lambda^\lambda A_{YY}^\top \\ A_{YZ} + B_F(\boldsymbol{x},t) & 0 & 0 & B_Y(\boldsymbol{x},t) \end{pmatrix} \\ &= \begin{pmatrix} B_Z(\boldsymbol{x},t) & 0 & 0 & -A_{YZ}^\top + B_H(\boldsymbol{x},t) \\ A_{ZL}^\top Z_\tau^\tau A_{ZZ} - A_{ZL}^\top Z_\lambda^\tau & M_L(\boldsymbol{x},t) & 0 & -A_{YL}^\top - A_{ZL}^\top H_\tau^\tau - A_{ZL}^\top H_\lambda^\tau A_{YY}^\top \\ A_{CZ} - A_{CY} F_\tau^\lambda A_{ZZ} + A_{CY} F_\lambda^\lambda & 0 & M_C(\boldsymbol{x},t) & A_{CY} Y_\tau^\lambda + A_{CY} Y_\lambda^\lambda A_{YY}^\top \\ A_{YZ} + B_F(\boldsymbol{x},t) & 0 & 0 & B_Y(\boldsymbol{x},t) \end{pmatrix}, \end{aligned}$$

where

$$M_L(\boldsymbol{x},t) = A_{LL}^\top L_\tau^\tau A_{LL} - A_{LL}^\top L_\lambda^\tau - L_\tau^\lambda A_{LL} + L_\lambda^\lambda,$$
$$M_C(\boldsymbol{x},t) = C_\tau^\tau + C_\lambda^\tau A_{CC}^\top + A_{CC} C_\tau^\lambda + A_{CC} C_\lambda^\lambda A_{CC}^\top.$$

Then $M_L(\boldsymbol{x},t)$ and $M_C(\boldsymbol{x},t)$ are nonsingular, because these are expressed by

$$M_L(\boldsymbol{x},t) = \begin{pmatrix} -A_{LL}^\top & I \end{pmatrix} \begin{pmatrix} L_\tau^\tau & L_\lambda^\tau \\ L_\tau^\lambda & L_\lambda^\lambda \end{pmatrix} \begin{pmatrix} -A_{LL} \\ I \end{pmatrix},$$
$$M_C(\boldsymbol{x},t) = \begin{pmatrix} I & A_{CC} \end{pmatrix} \begin{pmatrix} C_\tau^\tau & C_\lambda^\tau \\ C_\tau^\lambda & C_\lambda^\lambda \end{pmatrix} \begin{pmatrix} I \\ A_{CC}^\top \end{pmatrix}.$$

Finally, we prove that the determinant of $M(\boldsymbol{x},t) + B(\boldsymbol{x},t)Q$ is nonzero. The determinant is given by

$$\det(M(\boldsymbol{x},t) + B(\boldsymbol{x},t)Q) = \det M_L(\boldsymbol{x},t) \cdot \det M_C(\boldsymbol{x},t) \cdot \det \begin{pmatrix} B_Z(\boldsymbol{x},t) & -A_{YZ}^\top + B_H(\boldsymbol{x},t) \\ A_{YZ} + B_F(\boldsymbol{x},t) & B_Y(\boldsymbol{x},t) \end{pmatrix}.$$

We now have

$$\begin{pmatrix} B_Z(\boldsymbol{x},t) & -A_{YZ}^\top + B_H(\boldsymbol{x},t) \\ A_{YZ} + B_F(\boldsymbol{x},t) & B_Y(\boldsymbol{x},t) \end{pmatrix} = \begin{pmatrix} 0 & -A_{YZ}^\top \\ A_{YZ} & 0 \end{pmatrix} + \begin{pmatrix} B_Z(\boldsymbol{x},t) & B_H(\boldsymbol{x},t) \\ B_F(\boldsymbol{x},t) & B_Y(\boldsymbol{x},t) \end{pmatrix},$$

which is the sum of a positive semidefinite matrix and a positive definite matrix by Lemma 5.2. Hence this matrix is positive definite, which implies that its determinant is nonzero. Since $\det M_L(\boldsymbol{x},t) \neq 0$ and $\det M_C(\boldsymbol{x},t) \neq 0$, we obtain $\det(M(\boldsymbol{x},t) + B(\boldsymbol{x},t)Q) \neq 0$. □

Remark 6.3. *For nonlinear time-varying circuits composed of resistors (all modelled as conductances), inductors, capacitors, and voltage/current sources, the dimension of the hybrid equation system is no greater than that one for the MNA system. This is because* $\dim(\boldsymbol{u}_C^\tau, \boldsymbol{u}_Y^\tau) < n$ *for* n *being the number of nodes of the circuit,* $\dim \boldsymbol{i}_L^\lambda$ *is not greater than the number of inductors in the system, and* $\dim \boldsymbol{i}_Z^\lambda$ *is not greater than the number of (controlled) voltage sources of the system.*

7 Comparison of Hybrid Analysis with MNA

We consider nonlinear time-varying circuits without controlled voltage/current sources. Let us assume that the constitutive equations of resistors can be rewritten as those of conductors, and vice versa.[2] Then we can choose an admissible partition among the multitude of possibilities and so do the hybrid equations. In this section, we prove that the index of a DAE arising from the optimal hybrid analysis does not exceed that from MNA.

[2]For example, the function $\boldsymbol{r}$ of a resistor described by Ohm's law is a linear time-invariant function, and $\boldsymbol{r}$ of a diode is an exponential function with respect to its voltage variable. Thus we can regard resistors as conductors, and vice versa.

MNA is the most commonly used analysis method. We call the DAE arising from MNA the *MNA equations*. The index of the MNA equations is characterized by the circuit structure. We now define an *L-I cutset* and a *C-V loop*.

Definition 7.1. *An L-I cutset is a cutset consisting of inductors only, or a cutset consisting of inductors and current sources only. A C-V loop is a cycle consisting of capacitors and voltage sources only.*

Note that a cycle consisting of capacitors only is not a C-V loop. For nonlinear time-varying circuits, the index has the following property.

Theorem 7.2 ([11, Theorem 4.1],[21, Theorem 1.5])**.** *For nonlinear time-varying circuits composed of resistors, inductors, capacitors, and independent voltage/current sources, MNA leads to a DAE with index at most one if and only if the network contains neither L-I cutsets nor C-V loops. Otherwise, MNA leads to a DAE with index two.*

This theorem is generalized for nonlinear time-varying electric circuits containing controlled sources which satisfy certain conditions [11]. Moreover, the following theorem gives a necessary and sufficient condition for the index being zero.

Theorem 7.3 ([21, Theorem 1.5],[12, Theorem 2])**.** *For nonlinear time-varying circuits composed of resistors, inductors, capacitors, and independent voltage/current sources, the index of the MNA equations is zero if and only if the network does not contain independent voltage sources and has a spanning tree consisting of capacitors only.*

Theorems 6.2 and 7.3 imply that the minimum index of the hybrid equations does not exceed that of the MNA equations as follows.

Corollary 7.4. *For nonlinear time-varying circuits composed of resistors, conductors, inductors, capacitors, and independent voltage/current sources, the minimum index of the hybrid equations never exceeds the index of the MNA equations.*

Proof. Let (E_y, E_z) be an admissible partition such that E_z includes no resistors. We prove that the index of the hybrid equations with this admissible partition does not exceed the index of the MNA equations, which completes the proof.

By Theorem 6.2, the index of the hybrid equations is at most one. This implies that if the index of the MNA equations is more than zero, then that of the hybrid equations does not exceed it. Therefore, it suffices to prove that if the index of the MNA equations is zero, then that of the hybrid equations is also zero. Let us assume that the index of the MNA equations is zero. Then, the network has a spanning tree consisting of capacitors only by Theorem 7.3. Hence, each conductor in E_y belongs to a cycle consisting of capacitors and itself. Since E_z includes no resistors, (E_y, E_z) satisfies the Resistor-Acyclic condition. Therefore, it follows from Theorem 6.2 that the index of the hybrid equations is zero with admissible partition (E_y, E_z) such that E_z includes no resistors. □

8 Conclusion

For nonlinear time-varying circuits composed of resistors, conductors, inductors, capacitors, independent voltage/current sources, and controlled voltage/current sources, we have proved that the index of the hybrid equations never exceeds one, and given a structural characterization of circuits with index zero under the assumption that the capacitance matrix, the conductance matrix, the resistance matrix, the inductance matrix, and the controlled source matrix are all positive definite. The proof relies on the tractability index concept with the use of projector based analysis. Moreover, for nonlinear time-varying circuits without controlled sources, we have shown that the minimum index of hybrid equations does not exceed the index of MNA equations, which suggests that the hybrid analysis is superior to MNA in numerical accuracy.

The case of nonlinear time-varying circuits which may contain a wide class of controlled sources is under current research but seems to have no limit of the approach when certain topological criteria are satisfied.

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A Derivation of Hybrid Equations

Briefly expressed, we derive the hybrid equations by reducing the network equations to the cutset equations (6), (7) and the loop equations (12), (13) depending on the tree variables $\boldsymbol{u}_C^\tau$, $\boldsymbol{u}_Y^\tau$ and the cotree variables $\boldsymbol{i}_Z^\lambda$, $\boldsymbol{i}_L^\lambda$ only. For ease in notation, we omit the time argument t.

First, we transform the cutset equation (6) for capacitive tree branches. Using the constitutive equations (2) and (3), we may substitute $\boldsymbol{i}_C$ as well as $\boldsymbol{i}_Y$ and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\tau(\boldsymbol{u}_C^\tau,\boldsymbol{u}_C^\lambda) + A_{CC}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\lambda(\boldsymbol{u}_C^\tau,\boldsymbol{u}_C^\lambda) + A_{CY}\boldsymbol{f}^\lambda(\boldsymbol{i}_Z^\tau,\boldsymbol{i}_Z^\lambda,\boldsymbol{u}_Y^\tau,\boldsymbol{u}_Y^\lambda) + A_{CZ}\boldsymbol{i}_Z^\lambda + A_{CL}\boldsymbol{i}_L^\lambda + A_{CJ}\boldsymbol{i}_J = \mathbf{0}.$$

The cutset equations (8) for resistive tree branches, the loop equations (10) for capacitive cotree branches, and the loop equations (11) for conductive cotree branches provide substitutions for $\boldsymbol{i}_Z^\tau$, $\boldsymbol{u}_C^\lambda$, and $\boldsymbol{u}_Y^\lambda$. Hence, the cutset equations (6) for capacitive tree branches result in

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\tau(\boldsymbol{u}_C^\tau, A_{VC}^\top\boldsymbol{u}_V + A_{CC}^\top\boldsymbol{u}_C^\tau) + A_{CC}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\lambda(\boldsymbol{u}_C^\tau, A_{VC}^\top\boldsymbol{u}_V + A_{CC}^\top\boldsymbol{u}_C^\tau)\\
&\qquad + A_{CY}\boldsymbol{f}^\lambda(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{i}_J, \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top\boldsymbol{u}_V + A_{CY}^\top\boldsymbol{u}_C^\tau + A_{YY}^\top\boldsymbol{u}_Y^\tau)\\
&\qquad\qquad + A_{CZ}\boldsymbol{i}_Z^\lambda + A_{CL}\boldsymbol{i}_L^\lambda + A_{CJ}\boldsymbol{i}_J = \mathbf{0}.
\end{aligned}$$

Regarding the constitutive equations (1) for independent sources, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\tau(\boldsymbol{u}_C^\tau, A_{VC}^\top\boldsymbol{v}_s(t) + A_{CC}^\top\boldsymbol{u}_C^\tau) + A_{CC}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\lambda(\boldsymbol{u}_C^\tau, A_{VC}^\top\boldsymbol{v}_s(t) + A_{CC}^\top\boldsymbol{u}_C^\tau)\\
&\qquad + A_{CY}\boldsymbol{f}^\lambda(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top\boldsymbol{v}_s(t) + A_{CY}^\top\boldsymbol{u}_C^\tau + A_{YY}^\top\boldsymbol{u}_Y^\tau)\\
&\qquad\qquad + A_{CZ}\boldsymbol{i}_Z^\lambda + A_{CL}\boldsymbol{i}_L^\lambda + A_{CJ}\boldsymbol{j}_s(t) = \mathbf{0}. \qquad (18)
\end{aligned}$$

In a similar way, we can transform the cutset equations (7) for conductive tree branches to obtain

$$\begin{aligned}
&\boldsymbol{f}^\tau(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top\boldsymbol{v}_s(t) + A_{CY}^\top\boldsymbol{u}_C^\tau + A_{YY}^\top\boldsymbol{u}_Y^\tau)\\
&\qquad + A_{YY}\boldsymbol{f}^\lambda(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top\boldsymbol{v}_s(t) + A_{CY}^\top\boldsymbol{u}_C^\tau + A_{YY}^\top\boldsymbol{u}_Y^\tau)\\
&\qquad\qquad + A_{YZ}\boldsymbol{i}_Z^\lambda + A_{YL}\boldsymbol{i}_L^\lambda + A_{YJ}\boldsymbol{j}_s(t) = \mathbf{0}. \qquad (19)
\end{aligned}$$

Next, we transform the loop equations (13) for inductive cotree branches. Applying the constitutive equations (2) and (3), we may substitute $\boldsymbol{u}_L$ as well as $\boldsymbol{u}_Z$ and we obtain

$$-A_{VL}^\top\boldsymbol{u}_V - A_{CL}^\top\boldsymbol{u}_C^\tau - A_{YL}^\top\boldsymbol{u}_Y^\tau - A_{ZL}^\top\boldsymbol{h}^\tau(\boldsymbol{i}_Z^\tau,\boldsymbol{i}_Z^\lambda,\boldsymbol{u}_Y^\tau,\boldsymbol{u}_Y^\lambda) - A_{LL}^\top\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\tau(\boldsymbol{i}_L^\tau,\boldsymbol{i}_L^\lambda) + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\lambda(\boldsymbol{i}_L^\tau,\boldsymbol{i}_L^\lambda) = \mathbf{0}.$$

The cutset equations (8) for resistive tree branches, the cutset equations (9) for inductive tree branches, and the loop equations (11) for conductive cotree branches provide substitutions for $\boldsymbol{i}_Z^\tau$, $\boldsymbol{i}_L^\tau$ and $\boldsymbol{u}_Y^\lambda$. Hence, the loop equations (13) for inductive cotree branches result in

$$\begin{aligned}
&-A_{VL}^\top\boldsymbol{u}_V - A_{CL}^\top\boldsymbol{u}_C^\tau - A_{YL}^\top\boldsymbol{u}_Y^\tau\\
&\qquad - A_{ZL}^\top\boldsymbol{h}^\tau(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{i}_J, \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top\boldsymbol{u}_V + A_{CY}^\top\boldsymbol{u}_C^\tau + A_{YY}^\top\boldsymbol{u}_Y^\tau)\\
&\qquad\qquad - A_{LL}^\top\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\tau(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{i}_J, \boldsymbol{i}_L^\lambda) + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\lambda(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{i}_J, \boldsymbol{i}_L^\lambda) = \mathbf{0}.
\end{aligned}$$

Regarding the constitutive equations (1) for independent sources, we obtain

$$\begin{aligned}
&-A_{VL}^\top \boldsymbol{v}_s(t) - A_{CL}^\top \boldsymbol{u}_C^\tau - A_{YL}^\top \boldsymbol{u}_Y^\tau \\
&\quad - A_{ZL}^\top \boldsymbol{h}^\tau(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau) \\
&\qquad - A_{LL}^\top \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\tau(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda) + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\lambda(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda) = \mathbf{0}. \qquad (20)
\end{aligned}$$

Finally, we transform (12) similarly and get

$$\begin{aligned}
&-A_{VZ}^\top \boldsymbol{v}_s(t) - A_{CZ}^\top \boldsymbol{u}_C^\tau - A_{YZ}^\top \boldsymbol{u}_Y^\tau \\
&\quad - A_{ZZ}^\top \boldsymbol{h}^\tau(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau) \\
&\quad + \boldsymbol{h}^\lambda(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau) = \mathbf{0}. \qquad (21)
\end{aligned}$$

Thus, by (18)–(21), we obtain the hybrid equations

$$\begin{aligned}
-A_{VZ}^\top \boldsymbol{v}_s(t) - A_{CZ}^\top \boldsymbol{u}_C^\tau - A_{YZ}^\top \boldsymbol{u}_Y^\tau - A_{ZZ}^\top \boldsymbol{h}^\tau + \boldsymbol{h}^\lambda &= \mathbf{0}, \\
-A_{VL}^\top \boldsymbol{v}_s(t) - A_{CL}^\top \boldsymbol{u}_C^\tau - A_{YL}^\top \boldsymbol{u}_Y^\tau - A_{ZL}^\top \boldsymbol{h}^\tau - A_{LL}^\top \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\tau + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}^\lambda &= \mathbf{0}, \\
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\tau + A_{CC}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{q}^\lambda + A_{CY}\boldsymbol{f}^\lambda + A_{CZ}\boldsymbol{i}_Z^\lambda + A_{CL}\boldsymbol{i}_L^\lambda + A_{CJ}\boldsymbol{j}_s(t) &= \mathbf{0}, \\
\boldsymbol{f}^\tau + A_{YY}\boldsymbol{f}^\lambda + A_{YZ}\boldsymbol{i}_Z^\lambda + A_{YL}\boldsymbol{i}_L^\lambda + A_{YJ}\boldsymbol{j}_s(t) &= \mathbf{0},
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{q}^\tau &= \boldsymbol{q}^\tau(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, t), \\
\boldsymbol{q}^\lambda &= \boldsymbol{q}^\lambda(\boldsymbol{u}_C^\tau, A_{VC}^\top \boldsymbol{v}_s(t) + A_{CC}^\top \boldsymbol{u}_C^\tau, t), \\
\boldsymbol{f}^\tau &= \boldsymbol{f}^\tau(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau, t), \\
\boldsymbol{f}^\lambda &= \boldsymbol{f}^\lambda(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau, t), \\
\boldsymbol{h}^\tau &= \boldsymbol{h}^\tau(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau, t), \\
\boldsymbol{h}^\lambda &= \boldsymbol{h}^\lambda(-A_{ZZ}\boldsymbol{i}_Z^\lambda - A_{ZL}\boldsymbol{i}_L^\lambda - A_{ZJ}\boldsymbol{j}_s(t), \boldsymbol{i}_Z^\lambda, \boldsymbol{u}_Y^\tau, A_{VY}^\top \boldsymbol{v}_s(t) + A_{CY}^\top \boldsymbol{u}_C^\tau + A_{YY}^\top \boldsymbol{u}_Y^\tau, t), \\
\boldsymbol{\phi}^\tau &= \boldsymbol{\phi}^\tau(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, t), \\
\boldsymbol{\phi}^\lambda &= \boldsymbol{\phi}^\lambda(-A_{LL}\boldsymbol{i}_L^\lambda - A_{LJ}\boldsymbol{j}_s(t), \boldsymbol{i}_L^\lambda, t).
\end{aligned}$$