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An Approximation Algorithm for the Traveling Tournament Problem*

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Abstract. This paper deals with the traveling tournament problem, which is a well-known benchmark problem in the field of tournament timetabling. We propose a new lower bound for the traveling tournament problem, and construct a randomized approximation algorithm yielding a feasible solution whose approximation ratio is less than $2+(9/4)/(n-1)$, where n is the number of teams. For the traveling tournament problem, this is the first approximation algorithm with a constant approximation ratio, which is less than 2+3/4.
Key words: traveling tournament problem, lower bound, approximation algorithm, randomized algorithm, tournament, timetabling, scheduling

1 Introduction

In the field of tournament timetabling, the traveling tournament problem (TTP) is a well-known benchmark problem established by Easton, Nemhauser and Trick [2]. The objective of TTP is to make a round-robin tournament that minimizes total traveling distance. The problem TTP includes an optimization aspect like the traveling salesman problem (TSP) and the vehicle routing problem. However, TTP is surprisingly harder than TSP; there is a 10-teams TTP instance not yet solved exactly [10]. This is very contrastive to TSP, where a 10-cities instance of TSP is obviously easy.

Various studies on TTP have been appeared in recent years [2, 9]. Most of the best upper bounds of TTP instances are obtained by metaheuristic algorithms [1, 10, 11]. On the other hand, there are a few researches about lower bound and exact method for TTP [2], and no approximation algorithm for TTP is proposed so far.

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In this paper, we propose a new lower bound for TTP, and construct a randomized approximation algorithm yielding a feasible solution whose approximation ratio is less than $2 + (9/4)/(n-1)$ where n is the number of teams. Our result is the first approximation algorithm with a constant approximation ratio, which is less than 2+3/4.

2 Traveling Tournament Problem

In this section, we introduce some terminology and then define TTP. For more discussions on TTP and its variations, see [7, 8].

We are given a set of teams $T = \{1, 2, \ldots, n\}$ where $n \geq 4$ and even, and each team has its home venue. A game is specified by an ordered pair of teams. A double round-robin tournament is a set of games in which every team plays every other team once at its home venue and once at away (i.e., at the venue of the opponent); hence, exactly $2(n-1)$ slots are required to complete a double round-robin tournament.

Each team stays its home venue before a tournament, and then travels to play its games at the chosen venues. After a tournament, each team goes back to its home venue (if necessary). We note that, when a team plays two consecutive away games, the team goes directly from the venue of the first opponent to the other without returning to its home venue.

For any pair of teams $i, j \in T$, $d_{ij} \geq 0$ denotes the distance between i's and j's venues. Throughout this paper we assume that $d_{ii} = 0$ and triangle inequalities ($d_{ij} + d_{jk} \geq d_{ik}$) hold for any $i, j, k \in T$.

The traveling tournament problem is defined as follows.

Traveling Tournament Problem (TTP) [2]
Input: a set of teams T and a distance matrix $D = (d_{ij})$, indexed by T.
Output: a double round-robin tournament S of n teams such that

1. no team plays more than three consecutive away games;
2. no team plays more than three consecutive home games;
3. teams i at j immediately followed by j at i is prohibited (no repeaters);
4. the total distance traveled by the teams is minimized.

In the rest of this paper, a double round-robin tournament satisfying the above conditions 1–3 (1–4) is called a *feasible* (*optimal*, respectively) tournament.

Most of the best upper bounds for TTP are obtained by metaheuristic algorithms [1, 10, 11]. However, to prove the optimality of TTP is very difficult; for instance, there still exists an open instance of $n = 10$ [10].

3 Lower Bound

In this section, we consider lower bounds for TTP. The *independent lower bound*, proposed in [2], is a well-known lower bound for TTP. This bound is obtained by summing up of minimum traveling distance of each team; in other words, one has to solve n instances of 3-customer vehicle routing problem (see [5] for 3-customer vehicle routing problem). Although the performance of the independent lower bound is experimentally good (about 5% smaller than the optimal value of an instance), to get this bound is NP-hard in general because the 3-customer vehicle routing problem is NP-hard [6].

Here we propose a new lower bound, which is less tight than the independent lower bound but easy to compute. We denote the sum total of ordered pair of distances between venues by Δ, i.e., $\Delta \stackrel{\text{def.}}{=} \sum_{(i,j)\in T^2} d_{ij}$. In addition, to simply show the status of before and after a tournament, we introduce two artificial slots 0 and $2n-1$, and assume that each team is at home in these slots.

Theorem 1 provides a lower bound of the optimal value of TTP. (The idea of the following method is similar to the "*radial cost*" of vehicle routing problems [4].)

Theorem 1. *The optimal value z^* of TTP satisfies that $z^* \geq (2/3)\Delta$.*

Proof.

Let S^* be an optimal tournament of a given instance of TTP.

Suppose that in S^* team i plays three consecutive away games at teams j_1, j_2, then j_3. The total traveling distance of team i corresponding to these games, denoted by $d(i; j_1, j_2, j_3)$, is $d_{ij_1} + d_{j_1 j_2} + d_{j_2 j_3} + d_{j_3 i}$. From the triangle inequality, we have $d(i; j_1, j_2, j_3) = d_{ij_1} + d_{j_1 j_2} + d_{j_2 j_3} + d_{j_3 i} \geq d_{ij_1} + d_{j_1 j_2} + d_{j_2 i} \geq d_{ij_1} + d_{j_1 i}$. Similarly, the following inequalities hold: $d(i; j_1, j_2, j_3) \geq d_{ij_2} + d_{j_2 i}$ and $d(i; j_1, j_2, j_3) \geq d_{ij_3} + d_{j_3 i}$. Hence, $d(i; j_1, j_2, j_3) \geq (d_{ij_1} + d_{j_1 i} + d_{ij_2} + d_{j_2 i} + d_{ij_3} + d_{j_3 i})/3$.

Next, consider the case that team i is at home in a particular slot, plays two consecutive away games at teams j_4 and j_5, then goes back to the home of i. The corresponding distance $d(i; j_4, j_5)$ is $d_{ij_4} + d_{j_4 j_5} + d_{j_5 i}$. We can easily show that $d(i; j_4, j_5) \geq (d_{ij_4} + d_{j_4 i} + d_{ij_5} + d_{j_5 i})/2 \geq (d_{ij_4} + d_{j_4 i} + d_{ij_5} + d_{j_5 i})/3$.

Finally, consider the case that team i is at home in a particular slot, plays an away game at team j_6, then goes back to its home. For the corresponding distance $d(i; j_6)$, we have $d(i; j_6) = d_{ij_6} + d_{j_6 i} \geq (d_{ij_6} + d_{j_6 i})/3$.

From the above, in the tournament S^* the traveling distance of team i is at least $(1/3)\sum_{j\in T\setminus\{i\}}(d_{ij} + d_{ji})$. Therefore we have $z^* \geq (1/3)\sum_{i\in T}\sum_{j\in T\setminus\{i\}}(d_{ij} + d_{ji}) = (2/3)\Delta$. □

Corollary 1. *Let z^* be the optimal value of an instance of TTP. The traveling distance of every feasible tournament of the instance is at most $3z^*$.*

Proof. When team i has no consecutive away games, the distance traveled by i becomes the longest, i.e., $\sum_{j\in T\setminus\{i\}}(d_{ij} + d_{ji})$. Thus, the traveling distance of every tournament is at most $\sum_{i\in T}\sum_{j\in T\setminus\{i\}}(d_{ij} + d_{ji}) = 2\Delta$, which is three times the lower bound obtained by Theorem 1. □

Corollary 1 says that every algorithm yielding a feasible tournament of TTP is a 3-approximation algorithm for TTP. However, in the next section we propose a better approximation algorithm.

4 Approximation Algorithm

In this section, for constructing good feasible tournaments, we propose a randomized algorithm based on the Modified Circle Method (MCM), which was proposed in [3] by the authors for the constant distance traveling tournament problem. See the appendix for the detail of MCM.

When a team plays at away in at least one of slots s and $s+1$, we say that the team has a *move* between these slots. On the number of moves in a feasible tournament produced by MCM, we have the following result.

Theorem 2. [3] *The algorithm MCM produces a feasible tournament S in which the number of moves, denoted by $M(S)$, satisfies that*

$$M(S) = \begin{cases} (4/3)n^2 - (2/3)n - 1 & (n \equiv 0 \mod 3), \\ (4/3)n^2 - (1/2)n - 4/3 & (n \equiv 1 \mod 3), \\ (4/3)n^2 + (1/6)n - 5/3 & (n \equiv 2 \mod 3). \end{cases}$$

Here we note that MCM runs in $\mathrm{O}(n^2)$. Now we propose a simple randomized algorithm.

Algorithm 1.
Step 1: Construct a feasible tournament S by MCM.
Step 2: Randomly permute the names of teams in S.

Theorem 3. *The approximation ratio of Algorithm 1 is bounded by*

$$2 + (9/4)/(n-1).$$

Proof. We denote the distance of a tournament obtained by Algorithm 1 by a random variable Z. The random permutation of names of teams implies that each move in tournament S obtained by MCM is assigned to a fixed pair of (mutually distinct) team venues with probability $1/(n^2-n)$. Thus, the expectation of the distance of a move is $\Delta/(n^2-n)$. Since MCM outputs a feasible tournament whose number of moves is less than $(4/3)n^2 + (1/6)n$, we have

$$\mathrm{E}[Z] < \left((4/3)n^2 + (1/6)n\right) \Delta/(n^2-n).$$

From the above, the approximation ratio of Algorithm 1 is strictly bounded by

$$\frac{\left((4/3)n^2 + (1/6)n\right) \Delta/(n^2-n)}{z^*} \leq \frac{\left((4/3)n^2 + (1/6)n\right) \Delta/(n^2-n)}{(2/3)\Delta} = (3/2)\frac{(4/3)n^2 + (1/6)n}{n^2-n} = 2 + \frac{9/4}{n-1}. \quad \square$$

Theorem 3 says that when we generate a number of random permutations, the probability to find a tournament whose approximation ratio is less than $2 + (9/4)/(n-1)$ converges to 1. Since $n \geq 4$, Algorithm 1 is a randomized approximation algorithm with a constant approximation ratio, which is less than 2+3/4.

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Appendix: Modified Circle Method

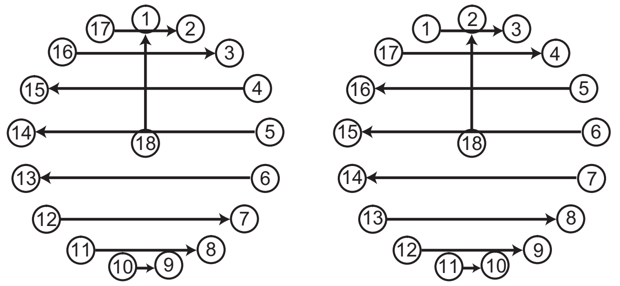
Here we describe the procedure of the Modified Circle Method (MCM), which was proposed in [3] by the authors.

Denote the set of teams by $T = \{1, 2, \ldots, n\}$. We introduce a directed graph $G^{\mathrm{e}} = (T, A^{\mathrm{e}})$ with a vertex set T and a set of mutually disjoint directed edges

$$\begin{aligned} A^{\mathrm{e}} \stackrel{\text{def.}}{=} &\{(j, n+1-j) \ : \ \lceil j/3 \rceil \text{ is even}, \ 1 \leq j \leq n/2\} \\ &\cup \{(n+1-j, j) \ : \ \lceil j/3 \rceil \text{ is odd}, \ 1 \leq j \leq n/2\}. \end{aligned}$$

Let $G^{\mathrm{o}} = (T, A^{\mathrm{o}})$ be a directed graph obtained from G^{e} by reversing the direction of the edge between 1 and n. For each $s \in \{1, 2, \ldots, n-1\}$, we define a permutation π^s by $(\pi^s(1), \pi^s(2), \ldots, \pi^s(n)) = (s, s+1, \ldots, n-1, 1, 2, \ldots, s-1, n)$. For any permutation π on T, $G^{\mathrm{e}}(\pi)$ (resp., $G^{\mathrm{o}}(\pi)$) denotes the set of $n/2$ games satisfying that every directed edge $(u, v) \in A^{\mathrm{e}}$ (resp., A^{o}) corresponds to a game between $\pi(u)$ and $\pi(v)$ held at the home venue of $\pi(v)$.

Consider the case that $n \equiv 0 \bmod 3$. Let X be a single round-robin tournament satisfying that games in slot s are defined by $G^{\mathrm{o}}(\pi^s)$ if $s \in \{1, 2, 3\} \bmod 6$, and by $G^{\mathrm{e}}(\pi^s)$ if $s \in \{4, 5, 0\} \bmod 6$. Figure 1 shows the games of the first four



(a) $s = 1$ (b) $s = 2$ (c) $s = 3$ (d) $s = 4$

Fig. 1. Games of the first four slots in X, where $n = 18$.

slots in X when $n = 18$. For each $i \in \{1, 2, \ldots, n/3 - 1\}$, we denote a partial schedule of X consisting of a sequence of three slots $(3i-2, 3i-1, 3i)$ by X_i. In addition, we denote the partial schedule of X consisting of two slots $(n-2, n-1)$ by $X_{n/3}$. Now we construct a double round-robin tournament Y by concatenating these partial schedules as follows: $Y = (X_1, \overline{X_1}, \overline{X_2}, X_2, X_3, \overline{X_3}, \overline{X_4}, X_4, X_5, \ldots, \overline{X_{\frac{n}{3}-1}}, \overline{X_{\frac{n}{3}}}, X_{\frac{n}{3}})$, where $\overline{X_i}$ is the partial schedule obtained from X_i by reversing all venues.

Consider the case that $n \equiv 1 \bmod 3$. We construct a single round-robin tournament X with the same method as above. For each $i \in \{1, 2, \ldots, (n-1)/3\}$, we denote a partial schedule of X consisting of a sequence of three slots $(3i-2, 3i-1, 3i)$ by X_i. We construct a double round-robin tournament Y by concatenating the partial schedules as follows: $Y = (X_1, \overline{X_1}, \overline{X_2}, X_2, X_3, \ldots, X_{\frac{n-1}{3}}, \overline{X_{\frac{n-1}{3}}})$.

Consider the case that $n \equiv 2 \bmod 3$. Let $\widetilde{G}^{\mathrm{e}}$ (resp., $\widetilde{G}^{\mathrm{o}}$) be a directed graph obtained from G^{e} (resp., G^{o}) by reversing the direction of the edge between $n/2-1$ and $n/2+2$. We construct a single round-robin tournament X as well as the above cases using directed graphs $\widetilde{G}^{\mathrm{e}}$ and $\widetilde{G}^{\mathrm{o}}$. For each $i \in \{2, 3, \ldots, (n-2)/3\}$, we denote a partial schedule of X consisting of a sequence of three slots $(3i-3, 3i-2, 3i-1)$ by X_i. We denote the partial schedules of X consisting of two slots $(1, 2)$ by X_1 and two slots $(n-2, n-1)$ by $X_{(n+1)/3}$. We construct a double round-robin tournament Y by concatenating these partial schedules as follows: $Y = (X_1, \overline{X_1}, \overline{X_2}, X_2, X_3, \ldots, X_{\frac{n+1}{3}}, \overline{X_{\frac{n+1}{3}}})$.

For all the cases, the single round-robin tournament X has neither partial home-away patterns HHHH, AAAA, HAH nor AHA; that is, each team has at most three consecutive home/away games in a single round-robin tournament X. The time complexity to construct X and Y is $\mathrm{O}(n^2)$, i.e., MCM runs in linear time in the output size.

On the number of moves of the tournament produced by MCM, Theorem 2 holds. For the proof of Theorem 2, see [3].