MATHEMATICAL ENGINEERING TECHNICAL REPORTS

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METR 2009–01 January 2009

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Error Estimates with Explicit Constants for Sinc Approximation, Sinc Quadrature and Sinc Indefinite Integration

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January 2009

Abstract

Error estimates with explicit constants are given for approximations of functions, definite integrals and indefinite integrals by means of the Sinc approximation. Although in the literature various estimates have already been given for these approximations, they were basically for examining the rates of convergence, and several constants were left unevaluated. Giving more explicit estimates, i.e., evaluating these constants is of great practical importance, since by which we can reinforce the useful formulas with the concept of "verified numerical computations." We also improve some formulas themselves to decrease their computational costs. Numerical examples that confirm the theory are also given.

1 Introduction

The *Sinc approximation* on the whole real line is expressed as

$$F(x) \approx \sum_{j=-n}^{n} F(jh)S(j,h)(x), \quad x \in \mathbb{R}, \tag{1.1}$$

where $S(j,h)(x)$ is the so-called *Sinc function* defined by

$$S(j,h)(x) = \frac{\sin[\pi(x/h - j)]}{\pi(x/h - j)},$$

and h is a mesh size appropriately selected depending on n. A variety of approximation formulas are derived from the Sinc approximation. For example, the *Sinc quadrature* for the integral on

$(-\infty,\, \infty)$ is derived by integrating both sides of (1.1):

$$\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x \approx \int_{-\infty}^{\infty} \left\{ \sum_{j=-n}^{n} F(jh)S(j,h)(x) \right\} \mathrm{d}x = h \sum_{j=-n}^{n} F(jh), \tag{1.2}$$

which coincides with the (truncated) trapezoidal formula. Here the relation $\int_{-\infty}^{\infty} S(j,h)(x)\,\mathrm{d}x = h$ is used. Another example is the *Sinc indefinite integration* expressed as

$$\int_{-\infty}^{x} F(\sigma)\,\mathrm{d}\sigma \approx \int_{-\infty}^{x} \left\{ \sum_{j=-n}^{n} F(jh)S(j,h)(\sigma) \right\} \mathrm{d}\sigma = \sum_{j=-n}^{n} F(jh)J(j,h)(x), \tag{1.3}$$

where the basis function $J(j,h)(x)$ is computed via the *sine integral* $\mathrm{Si}(x) = \int_0^x \{\sin(\sigma)/\sigma\}\,\mathrm{d}\sigma$:

$$J(j,h)(x) = h\left\{ \frac{1}{2} + \frac{1}{\pi}\,\mathrm{Si}[\pi(x/h - j)] \right\}. \tag{1.4}$$

Other examples include *Sinc indefinite convolution*, *Harmonic-Sinc approximation*, approximation of derivatives, approximation of Hilbert and Cauchy transforms, and approximation of inversion of Fourier and Laplace transforms (see, for example, Stenger [20, 21]).

When the target interval is finite, say $(a,\, b)$, variable transformations are utilized. The most frequently-used transformation has been the *Single-Exponential* (*SE*) *transformation* [20, 21], while recently a stronger transformation, called the *Double-Exponential* (*DE*) *transformation* has been introduced [8, 25]. Under these transformations, some approximation formulas described above have been proved to enjoy exponential accuracy. For example, let us consider the Sinc approximation of a function f with the SE transformation, which we denote $f_{\text{SE-Sinc}}$ here. The error of the approximation can be estimated as (Stenger [20, 21]):

$$\sup_{t\in(a,\,b)} |f(t) - f_{\text{SE-Sinc}}(t)| \leq C\sqrt{n}\,\mathrm{e}^{-\sqrt{\pi d\mu n}}, \tag{1.5}$$

where d and μ are characteristic constants of f, and C is an "implicit" constant that does not depend on n, but can depend on other parameters in the scheme. If the DE transformation is used instead, the approximate function $f_{\text{DE-Sinc}}$ enjoys the faster convergence [27]:

$$\sup_{t\in(a,\,b)} |f(t) - f_{\text{DE-Sinc}}(t)| \leq C\,\mathrm{e}^{-\pi dn/\log(2dn/\mu)}. \tag{1.6}$$

Again, C is an implicit constant that at least does not depend on n. In either case, the fast convergence properties encouraged many authors to develop numerical schemes for a variety of problems, such as Fredholm integral equations [11, 15, 16], Volterra integral equations [11, 17], initial value problems of ordinary differential equations [3, 12], boundary value problems of second-order ordinary differential equations [2, 19, 23], and boundary value problems of fourth-order ordinary differential equations [9, 13, 18]. As a consequence, today such Sinc schemes (often grouped as the *Sinc methods*) are considered to be one of the most useful numerical methods that can apply to a wide range of problems.

The main aim of the present paper is to give more explicit error estimates than the existing estimates mentioned above, by clarifying the explicit forms of the constant C's. The reason of this is that in order to reinforce the promising schemes with the idea of *verified numerical computation*, which is a modern tool to design reliable and practical numerical libraries, estimates must be given without any ambiguity. See, for example, Corliss–Rall [4], Eiermann [5] and Petras [14] for numerical integration with guaranteed accuracy, where the errors of quadrature rules (such as Newton–Cotes or Gaussian formulas) have been estimated by strict inequalities with explicit constants. In other words, the existing estimates for the Sinc formulas are perfect in that they successfully reveal the convergence rates (note that the constant C's above are proved to be independent of n), but if we hope to know the quantity of the errors exactly, more explicit estimates are desired. Giving the sharp estimates of the constant C's is actually not an easy task, especially in the case of the DE transformation. In this project, we show estimates of the constant C's with explicit forms, concentrating on the approximation formulas for functions (1.1), definite integrals (1.2), and indefinite integrals (1.3), since they can be handled alike. In the case of indefinite integrals, we not only give the explicit constants, but also show that the existing estimates of the convergence rates can be improved.

As a second, subsidiary aim, we also improve some of the existing schemes themselves to reduce their computational cost. Recall that the original Sinc sampling formula is defined as an infinite sum, i.e.

$$F(x) \approx \sum_{j=-\infty}^{\infty} F(jh)S(j,h)(x), \quad x \in \mathbb{R}, \tag{1.7}$$

which is then truncated in the formula (1.1) assuming that $|F(x)|$ decays sufficiently fast as $x \to \pm\infty$. In the existing schemes, the truncation is always done symmetrically; i.e., $\sum_{-\infty}^{\infty}$ is approximated by $\sum_{-n}^{n}$ for some n. In this case the number of evaluation points of F is $2n+1$. It is, however, obviously not optimal when function's decay rates as $x \to -\infty$ and $x \to \infty$ are different. Suppose, for example, it is known that the function decays faster as $x \to -\infty$ than as $x \to \infty$. Then it should make sense to choose some $M < n$ and modify the formula (1.1) as

$$F(x) \approx \sum_{j=-M}^{n} F(jh)S(j,h)(x), \quad x \in \mathbb{R}, \tag{1.8}$$

which obviously reduces the cost to $M + n + 1$ ($< 2n + 1$). The usefulness of the form (1.8) has been pointed out by Stenger [20, 21] for some limited range of formulas. We extend this to other formulas covered in the present paper.

The organization of this paper is as follows. The main estimate results are stated in Section 2. For readers' convenience, we also briefly review conventional error analyses here. Then in Section 3 several numerical results are shown to confirm the theory. The proofs of the main theorems are left to Section 4. Section 5 is devoted to the conclusions.

2 Conventional and new error analyses for Sinc approximation, Sinc quadrature and Sinc indefinite integration

We first describe approximation formulas incorporated with variable transformations, and then summarize the conventional and new error analyses.

2.1 SE-Sinc approximation, SE-Sinc quadrature and SE-Sinc indefinite integration

The *SE transformation* is defined by

$$t = \psi_{\mathrm{SE}}(x) = \frac{b-a}{2}\tanh\left(\frac{x}{2}\right) + \frac{b+a}{2},$$

and its inverse is expressed as

$$x = \psi_{\mathrm{SE}}^{-1}(t) = \log\left(\frac{t-a}{b-t}\right).$$

The SE transformation maps $x \in \mathbb{R}$ into $t \in (a,\, b)$. Thus by considering $F(x) = f(\psi_{\mathrm{SE}}(x))$ in (1.8), we can apply the Sinc approximation (1.8) to a function $f(t)$ defined on a finite interval $(a,\, b)$:

$$f(\psi_{\mathrm{SE}}(x)) \approx \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))S(j,h)(x), \quad x \in \mathbb{R}.$$

Since $t = \psi_{\mathrm{SE}}(x)$, this approximation is equivalent to

$$f(t) \approx \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))S(j,h)(\psi_{\mathrm{SE}}^{-1}(t)), \quad t \in (a,\, b). \tag{2.1}$$

How the upper and lower bounds of the summation, N and M, are determined will be discussed later. We call this approximation the *SE-Sinc approximation*. Similarly, the SE transformation can be utilized for definite integration (1.2) and indefinite integration (1.3) as follows:

$$\int_a^b f(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(\psi_{\mathrm{SE}}(x))\psi_{\mathrm{SE}}'(x)\,\mathrm{d}x \approx h\sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh), \tag{2.2}$$

$$\int_a^t f(s)\,\mathrm{d}s = \int_{-\infty}^{\psi_{\mathrm{SE}}^{-1}(t)} f(\psi_{\mathrm{SE}}(\sigma))\psi_{\mathrm{SE}}'(\sigma)\,\mathrm{d}\sigma \approx \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh)J(j,h)(\psi_{\mathrm{SE}}^{-1}(t)). \tag{2.3}$$

We call these approximations the *SE-Sinc quadrature* and the *SE-Sinc indefinite integration*, respectively. Note that, as for indefinite integration, there is another type of formulas [6, 7, 20] where the Sinc function $S(j,h)$ is employed as a basis function, instead of $J(j,h)$. In the present paper, however, we will not get into them but focus solely on (2.3), since the other type is nothing but the combination of (2.1) and (2.3), and thus can be analyzed in a like manner.

2.2 DE-Sinc approximation, DE-Sinc quadrature and DE-Sinc indefinite integration

Recently it has turned out that replacing the SE transformation with the *DE transformation* accelerates the convergence rate of the Sinc schemes in many cases [8, 25]; in fact, certain optimality has been proved [22,24]. The DE transformation and its inverse are

$$t = \psi_{\mathrm{DE}}(x) = \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh(x)\right) + \frac{b+a}{2},$$
$$x = \psi_{\mathrm{DE}}^{-1}(t) = \log\left[\frac{1}{\pi}\log\left(\frac{z-a}{b-z}\right) + \sqrt{1+\left\{\frac{1}{\pi}\log\left(\frac{z-a}{b-z}\right)\right\}^2}\right].$$

The DE transformation maps the whole real line $\mathbb{R}$ onto a finite interval $(a,\, b)$, like as the SE transformation. Hence, in a similar manner, the following formulas can be obtained:

$$\text{DE-Sinc approximation:} \qquad f(t) \approx \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))S(j,h)(\psi_{\mathrm{DE}}^{-1}(t)), \tag{2.4}$$

$$\text{DE-Sinc quadrature:} \qquad \int_a^b f(t)\,\mathrm{d}t \approx h\sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi_{\mathrm{DE}}'(jh), \tag{2.5}$$

$$\text{DE-Sinc indefinite integration:} \qquad \int_a^t f(s)\,\mathrm{d}s \approx \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi_{\mathrm{DE}}'(jh)J(j,h)(\psi_{\mathrm{DE}}^{-1}(t)). \tag{2.6}$$

Like in the SE-Sinc case, there is another formula for indefinite integration [26]; but in the present paper (from the same reason as above) we focus on (2.6).

2.3 Function space

In order that the formulas above work accurately, the transformed function by the SE transformation or the DE transformation should be analytic and bounded on some strip domain,

$$\mathscr{D}_d = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < d\},$$

for a positive constant d. To be more specific, we introduce the following function space.

Definition 2.1. Let $\mathscr{D}$ be a simply-connected domain which satisfies $(a,\, b) \subset \mathscr{D}$, and let K, α, β be positive constants. Then $\mathbf{L}_{K,\alpha,\beta}(\mathscr{D})$ denotes the family of all functions f that are analytic on $\mathscr{D}$, and satisfy for all z in $\mathscr{D}$ the condition that

$$|f(z)| \leq K|Q_{\alpha,\beta}(z)|,$$

where $Q_{\alpha,\beta}(z) = (z-a)^{\alpha}(b-z)^{\beta}$. For simplicity, we write $Q_{1,1}(z)$ as $Q(z)$.

In what follows, $\mathscr{D}$ is either $\psi_{\mathrm{SE}}(\mathscr{D}_d)$ or $\psi_{\mathrm{DE}}(\mathscr{D}_d)$, where

$$\psi_{\mathrm{SE}}(\mathscr{D}_d) = \left\{z \in \mathbb{C} : \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d\right\},$$
$$\psi_{\mathrm{DE}}(\mathscr{D}_d) = \left\{z \in \mathbb{C} : \left|\arg\left[\frac{1}{\pi}\log\left(\frac{z-a}{b-z}\right) + \sqrt{1+\left\{\frac{1}{\pi}\log\left(\frac{z-a}{b-z}\right)\right\}^2}\right]\right| < d\right\}.$$

We here would like to emphasize that in the new theorems given below, estimates will be given explicitly using only the known parameters regarding the function space: i.e., K, α, β, d, and $b-a$.

2.4 Error analyses for the SE-Sinc case: existing and new results

As for the convergence rate of the SE-Sinc approximation (2.1), the next theorem has been known.

Theorem 2.2 (Stenger [20, Theorem 4.2.5])**.** Let $f \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$. Let $\mu = \min\{\alpha, \beta\}$, n be a positive integer, and h be selected by the formula

$$h = \sqrt{\frac{\pi d}{\mu n}}. \tag{2.7}$$

Furthermore, let M and N be positive integers defined by

$$\begin{cases} M = n, \quad N = \lceil \alpha n/\beta \rceil & (\text{if } \mu = \alpha) \\ N = n, \quad M = \lceil \beta n/\alpha \rceil & (\text{if } \mu = \beta) \end{cases} \tag{2.8}$$

respectively. Then there exists a constant C independent of n such that

$$\sup_{t\in(a,\,b)} \left| f(t) - \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh)) S(j,h)(\psi_{\mathrm{SE}}^{-1}(t)) \right| \leq C\sqrt{n}\,\mathrm{e}^{-\sqrt{\pi d \mu n}}. \tag{2.9}$$

Although it successfully reveals the fundamental convergence property, the constant C in (2.9) is left unestimated except the fact that it is independent of n. It is indispensable, however, to know its concrete form when we hope to guarantee the accuracy of the approximation. In this paper, we give the constant explicitly below.

Theorem 2.3. Assume that the assumptions of Theorem 2.2 are fulfilled. Then the inequality (2.9) holds with

$$C = \frac{2K(b-a)^{\alpha+\beta}}{\mu} \left[\frac{2}{\pi d(1-\mathrm{e}^{-2\sqrt{\pi d\mu}})\{\cos(d/2)\}^{\alpha+\beta}} + \sqrt{\frac{\mu}{\pi d}} \right].$$

Note that the constant C here depends only on K, α, β, d, and $b-a$, which are all known from the assumptions.

For the SE-Sinc quadrature (2.2), an estimate has been again given by Stenger [20].

Theorem 2.4 (Stenger [20, Theorem 4.2.6])**.** Let $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$. Let $\mu = \min\{\alpha, \beta\}$, n be a positive integer, and h be selected by the formula

$$h = \sqrt{\frac{2\pi d}{\mu n}}. \tag{2.10}$$

Furthermore, let M and N be positive integers defined by (2.8). Then there exists a constant C independent of n such that

$$\left| \int_a^b f(t)\,\mathrm{d}t - h \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh) \right| \leq C\,\mathrm{e}^{-\sqrt{2\pi d\mu n}}. \tag{2.11}$$

Again, the constant C in (2.11) is not given explicitly in the above theorem. We give its concrete form below.

Theorem 2.5. Assume that the assumptions of Theorem 2.4 are fulfilled. Then the inequality (2.11) holds with

$$C = \frac{2K(b-a)^{\alpha+\beta-1}}{\mu}\left[\frac{2}{(1-\mathrm{e}^{-\sqrt{2\pi d\mu}})\{\cos(d/2)\}^{\alpha+\beta}}+1\right].$$

Remark 1. Beighton–Noble [1] have given an error estimate for the (modified) SE-Sinc quadrature, but its convergence rate has been polynomial with respect to h. This is because the Euler–Maclaurin summation formula has been used in their analysis. In the present paper, the error is analyzed based on Stenger's [20] idea, and the exponential convergence rate is guaranteed.

The error analysis of the SE-Sinc indefinite integration (2.3) has been given by Haber [6], only in the case $M=N$.

Theorem 2.6 (Haber [6, Theorem 2])**.** Let $fQ \in \mathbf{L}_{K,\mu,\mu}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0<d<\pi$. Let n be a positive integer, and h be selected by the formula (2.7). Then there exists a constant C independent of n such that

$$\sup_{t\in(a,\,b)}\left|\int_a^t f(s)\,\mathrm{d}s - \sum_{j=-n}^{n} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh)J(j,h)(\psi_{\mathrm{SE}}^{-1}(t))\right| \leq C\sqrt{n}\,\mathrm{e}^{-\sqrt{\pi d\mu n}}.$$

We improve this analysis as follows.

Theorem 2.7. Let $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0<d<\pi$. Let $\mu=\min\{\alpha,\,\beta\}$, n be a positive integer, and h be selected by the formula (2.7). Furthermore, let M and N be positive integers defined by (2.8). Then it follows that

$$\sup_{t\in(a,\,b)}\left|\int_a^t f(s)\,\mathrm{d}s - \sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh)J(j,h)(\psi_{\mathrm{SE}}^{-1}(t))\right| \leq C\,\mathrm{e}^{-\sqrt{\pi d\mu n}},$$

where

$$C = \frac{2K(b-a)^{\alpha+\beta-1}}{\mu}\left[\frac{1}{d(1-\mathrm{e}^{-2\sqrt{\pi d\mu}})\{\cos(d/2)\}^{\alpha+\beta}}\sqrt{\frac{\pi d}{\mu}}+1.1\right].$$

Notice the differences between Theorem 2.6 and Theorem 2.7; the latter not only reveals the concrete form of the constant C, but also gives a sharper rate of convergence (notice $\sqrt{n}$ is now removed). It also adapts the optimal formula where the truncation is done at M and N with (generally) $M \neq N$.

2.5 Error analyses for the DE-Sinc case: existing and new results

Next we describe the results for the DE-Sinc cases. First, we would like to emphasize that all the existing schemes and analyses have been given only in the case $M=N$. In the new theorems below, however, we cover the optimal cases $M \neq N$.

The convergence rate of the DE-Sinc approximation (2.4) can be observed by the next theorem, which is faster than the SE-Sinc case.

Theorem 2.8 (Tanaka et al. [27, Theorem 3.1])**.** Let $f \in \mathbf{L}_{K,\mu,\mu}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let n be a positive integer with $n > \mu/(2d)$, and h be selected by the formula

$$h = \frac{\log(2dn/\mu)}{n}. \tag{2.12}$$

Then there exists a constant C independent of n such that

$$\sup_{t\in(a,\,b)} \left| f(t) - \sum_{j=-n}^{n} f(\psi_{\mathrm{DE}}(jh)) S(j,h)(\psi_{\mathrm{DE}}^{-1}(t)) \right| \leq C\,\mathrm{e}^{-\pi dn/\log(2dn/\mu)}.$$

We improve the estimate as follows:

Theorem 2.9. Let $f \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let $\mu = \min\{\alpha,\,\beta\}$, $\nu = \max\{\alpha,\,\beta\}$, n be a positive integer with $n \geq (\nu\,\mathrm{e})/(2d)$, and h be selected by the formula (2.12). Furthermore, let M and N be positive integers defined by

$$\begin{cases} M = n, \quad N = n - \lfloor \log(\beta/\alpha)/h \rfloor & (\text{if } \ \mu = \alpha) \\ N = n, \quad M = n - \lfloor \log(\alpha/\beta)/h \rfloor & (\text{if } \ \mu = \beta) \end{cases} \tag{2.13}$$

respectively. Then it follows that

$$\sup_{t\in(a,\,b)} \left| f(t) - \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh)) S(j,h)(\psi_{\mathrm{DE}}^{-1}(t)) \right| \leq C_1 \left[\frac{C_2}{1-\mathrm{e}^{-\pi\mu\,\mathrm{e}}} + \mu\,\mathrm{e}^{\frac{\pi}{2}\nu} \right] \mathrm{e}^{-\pi dn/\log(2dn/\mu)},$$

where the constants C_1 and C_2 are defined by

$$C_1 = \frac{2K(b-a)^{\alpha+\beta}}{\pi d\mu}, \tag{2.14}$$

$$C_2 = \frac{2}{\pi \cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)\cos d}. \tag{2.15}$$

Remark 2. One may notice that the conditions on n are different between Theorem 2.8 and Theorem 2.9. The condition $n > \mu/(2d)$ (in Theorem 2.8) is needed to assure the positiveness of the mesh size h. In Theorem 2.9, it is rewritten as $n \geq (\nu\,\mathrm{e})/(2d)$; this is because of the following reasons. Firstly, in order to modify the scheme itself so that it allows optimal truncation with M and N, we need $n > \nu/(2d)$ to assure the positiveness of M and N in (2.13). Secondly, with the condition $n > \nu/(2d)$, it is not possible to evaluate the maximum of $1/\log(2dn/\mu)$, which is included in the constant (see (4.4) in the proof). In order to establish explicit estimates, we further would like to relax the condition to $n \geq (\nu\,\mathrm{e})/(2d)$, which still seems reasonable, and then the term can be simply estimated as $1/\log(2dn/\mu) \leq 1/\log(2dn/\nu) \leq 1/\log(\mathrm{e}) = 1$.

For the DE-Sinc quadrature (2.5), the next error analysis has been given.

Theorem 2.10 (Tanaka et al. [28, Theorem 3.1])**.** Let $fQ \in \mathbf{L}_{K,\mu,\mu}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let n be a positive integer with $n > \mu/(4d)$, and h be selected by the formula

$$h = \frac{\log(4dn/\mu)}{n}. \tag{2.16}$$

Then there exists a constant C independent of n such that

$$\left|\int_a^b f(t)\,\mathrm{d}t - h\sum_{j=-n}^{n} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)\right| \leq C\,\mathrm{e}^{-2\pi dn/\log(4dn/\mu)}.$$

We refine the result as follows.

Theorem 2.11. Let $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let $\mu = \min\{\alpha,\,\beta\}$, $\nu = \max\{\alpha,\,\beta\}$, n be a positive integer with $n \geq (\nu\,\mathrm{e})/(4d)$, and h be selected by the formula (2.16). Furthermore, let M and N be positive integers defined by (2.13). Then it follows that

$$\left|\int_a^b f(t)\,\mathrm{d}t - h\sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)\right| \leq \tilde{C}_1\left[\frac{\tilde{C}_2}{1-\mathrm{e}^{-\frac{\pi}{2}\mu\,\mathrm{e}}} + \mathrm{e}^{\frac{\pi}{2}\nu}\right]\mathrm{e}^{-2\pi dn/\log(4dn/\mu)},$$

where the constants $\tilde{C}_1$ and $\tilde{C}_2$ are defined by

$$\tilde{C}_1 = \frac{2K(b-a)^{\alpha+\beta-1}}{\mu}, \qquad (2.17)$$

$$\tilde{C}_2 = \frac{2}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin d)\cos d}. \qquad (2.18)$$

The DE-Sinc indefinite integration (2.6) has been proposed by Muhammad–Mori [10], where a rough convergence analysis has also been discussed. We present here their results as a theorem by clarifying mathematical assumptions.

Theorem 2.12 (Muhammad–Mori [10])**.** Let $fQ \in \mathbf{L}_{K,\mu,\mu}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let $\mu' = \mu - \epsilon$ for ϵ with $0 < \epsilon < \mu$, n be a positive integer with $n > \mu'/(2d)$, and h be selected by the formula

$$h = \frac{\log(2dn/\mu')}{n}.$$

Then there exists a constant C independent of n such that

$$\sup_{t\in(a,\,b)}\left|\int_a^t f(s)\,\mathrm{d}s - \sum_{j=-n}^{n} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)J(j,h)(\psi_{\mathrm{DE}}^{-1}(t))\right| \leq C\,\mathrm{e}^{-\pi dn/\log(2dn/\mu')}.$$

By modifying the scheme and clarifying the concrete form of constants, we give a more explicit estimate, and additionally obtain a sharper rate of convergence as follows.

Theorem 2.13. Let $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let $\mu = \min\{\alpha,\,\beta\}$, $\nu = \max\{\alpha,\,\beta\}$, n be a positive integer with $n \geq (\nu\,\mathrm{e})/(2d)$, and h be selected by the formula (2.12). Furthermore, let M and N be positive integers defined by (2.13). Then it follows that

$$\begin{aligned}\sup_{t\in(a,\,b)}&\left|\int_a^t f(s)\,\mathrm{d}s - \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)J(j,h)(\psi_{\mathrm{DE}}^{-1}(t))\right|\\ &\leq \frac{\tilde{C}_1}{d}\left[\frac{\tilde{C}_2}{2}\frac{1}{1-\mathrm{e}^{-\pi\mu\,\mathrm{e}}} + \mathrm{e}^{\frac{\pi}{2}(\alpha+\beta)}\right]\frac{\log(2dn/\mu)}{n}\,\mathrm{e}^{-\pi dn/\log(2dn/\mu)},\end{aligned}$$

where the constants $\tilde{C}_1$ and $\tilde{C}_2$ are defined by (2.17) and (2.18), respectively.

3 Numerical examples

In this section, we present numerical results that confirm the estimates for the six approximations: (2.1)–(2.6). All programs are written in C with double-precision floating-point arithmetic, and GNU Scientific Library is used for computing the sine integral function in (1.4). We set the interval (a, b) to $(-1, 1)$, and consider test problems below.

Example 1 (approximation of a function)**.** Consider the function

$$f_1(t) = (1+t^2)^{1/2}(1+t)^{1/2}(1-t)^{3/4}.$$

This function is analytic on the domain $\psi_{\mathrm{SE}}(\mathscr{D}_{\pi/2})$ and $\psi_{\mathrm{DE}}(\mathscr{D}_{\pi/6})$, and satisfies

$$|f_1(z)| \leq 2|1+z|^{1/2}|1-z|^{3/4},$$

for all $z \in \psi_{\mathrm{SE}}(\mathscr{D}_{\pi/2})$ and $z \in \psi_{\mathrm{DE}}(\mathscr{D}_{\pi/6})$.

Example 2 (approximation of a definite integral)**.** Consider the function

$$f_2(t) = \frac{1}{2}(1+t^2)^{1/2} + \frac{1}{8}(1+t)^{-1/2}, \tag{3.1}$$

and its definite integral on $(-1, 1)$:

$$\int_{-1}^{1} f_2(t)\,\mathrm{d}t = \frac{1}{4}\left\{2\,\mathrm{arcsinh}(1) + 3\sqrt{2}\right\}.$$

The function f_2 is analytic on the domain $\psi_{\mathrm{SE}}(\mathscr{D}_{\pi/2})$ and $\psi_{\mathrm{DE}}(\mathscr{D}_{\pi/6})$, and satisfies

$$|f_2(z)Q(z)| \leq \left(2^{3/4} + \frac{1}{8}\right)|1+z|^{1/2}|1-z|^{1},$$

for all $z \in \psi_{\mathrm{SE}}(\mathscr{D}_{\pi/2})$ and $z \in \psi_{\mathrm{DE}}(\mathscr{D}_{\pi/6})$.

Example 3 (approximation of an indefinite integral)**.** Consider the function f_2 of (3.1) again, and its indefinite integral on $(-1, 1)$:

$$\int_{-1}^{t} f_2(s)\,\mathrm{d}s = \frac{1}{4}\left\{2^{1/2} + (1+t)^{1/2} + t(1+t^2)^{1/2} + \mathrm{arcsinh}(1) + \mathrm{arcsinh}(t)\right\}.$$

Figure 1 and 2 show the results for Example 1. Since $f_1 \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ with $K=2$, $\alpha = 1/2$, $\beta = 3/4$, $d = \pi/2$, and also $f_1 \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ with $K=2$, $\alpha = 1/2$, $\beta = 3/4$, $d = \pi/6$, we can apply Theorem 2.3 and Theorem 2.9 to estimate the approximation errors; the estimated maximum error is plotted as the dotted lines in the figures. The actual numerical error is checked on 1999 equally-spaced points, i.e. $t = -0.999, \ldots, 0.001, 0, 0.001, \ldots, 0.999$, and plotted as the solid line with + points. We can see that the estimate surely bounds the actual errors from above in both figures. Similarly, Figure 3 and 4 show the results for Example 2, and Figure 5 and 6 for Example 3. In both problems, the estimates are in fact sharp upper bounds of the actual error, when the effects of rounding errors are negligible.

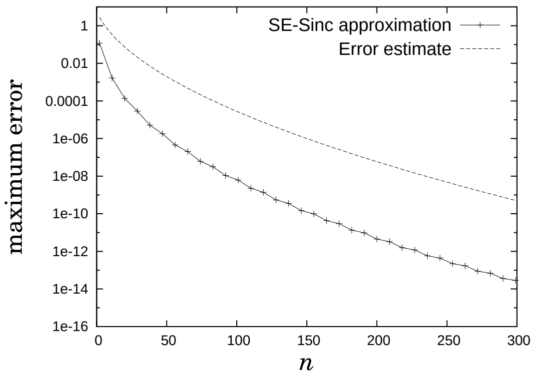

Figure 1. Error of the SE-Sinc approximation and its estimate.

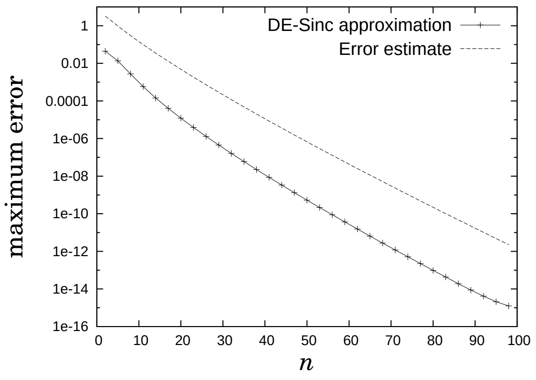

Figure 2. Error of the DE-Sinc approximation and its estimate.

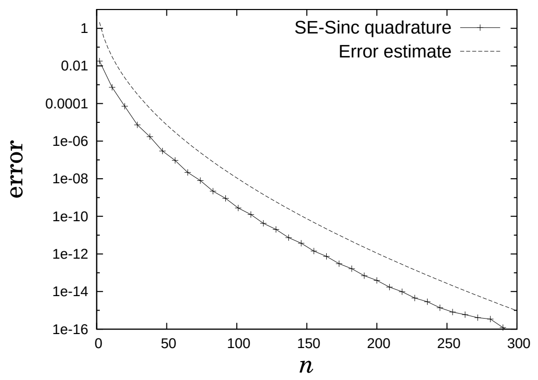

Figure 3. Error of the SE-Sinc quadrature and its estimate.

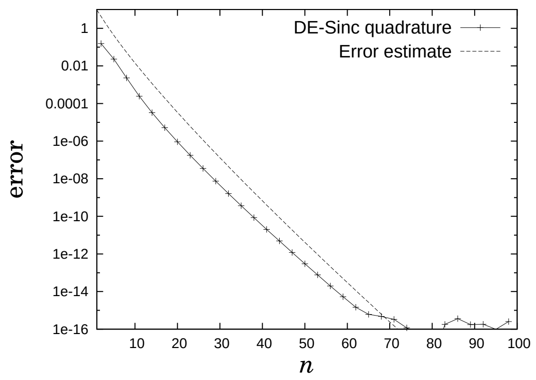

Figure 4. Error of the DE-Sinc quadrature and its estimate.

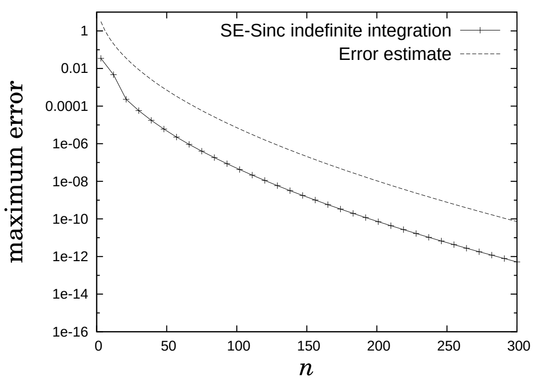

Figure 5. Error of the SE-Sinc indefinite integration and its estimate.

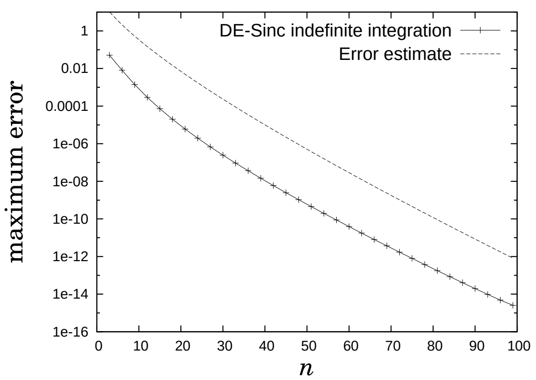

Figure 6. Error of the DE-Sinc indefinite integration and its estimate.

4 Proofs

In this section, we give proofs for the six new theorems above. Rough structures of the proofs are the same in all the cases, and thus in order to help readers understanding, we first outline the common structure in § 4.1, taking the SE-Sinc approximation (2.1) as an example. In particular, there we clarify which terms have been left unestimated. Then in § 4.2 detailed proofs for the SE-Sinc formulas are given, and in § 4.3 for the DE-Sinc formulas. Finally in § 4.4, we give the proof of supporting lemmas.

4.1 Sketch of the proofs

Proofs consist of evaluating two kinds of errors: *discretization error* and *truncation error*. In the SE-Sinc approximation (2.1), for example, the total error can be bounded as

$$\begin{aligned}
&\left|f(t)-\sum_{j=-M}^{N} f(\psi_{\mathrm{SE}}(jh))S(j,h)(\psi_{\mathrm{SE}}^{-1}(t))\right|\\
&\leq \left|f(t)-\sum_{j=-\infty}^{\infty} f(\psi_{\mathrm{SE}}(jh))S(j,h)(\psi_{\mathrm{SE}}^{-1}(t))\right|\\
&\quad+\left|\sum_{j=-\infty}^{-M-1} f(\psi_{\mathrm{SE}}(jh))S(j,h)(\psi_{\mathrm{SE}}^{-1}(t))+\sum_{j=N+1}^{\infty} f(\psi_{\mathrm{SE}}(jh))S(j,h)(\psi_{\mathrm{SE}}^{-1}(t))\right|.
\end{aligned}$$

The first term of the right hand side is the discretization error, and the second is the truncation error. Let us evaluate these terms in turn.

First, we consider the discretization error. To this end, it is indispensable to introduce the following function space.

Definition 4.1. Let $\mathscr{D}_d(\epsilon)$ be a rectangular domain defined for $0<\epsilon<1$ by

$$\mathscr{D}_d(\epsilon)=\{\zeta\in\mathbb{C}:|\operatorname{Re}\zeta|<1/\epsilon,\ |\operatorname{Im}\zeta|<d(1-\epsilon)\}.$$

Then $\mathbf{H}^1(\mathscr{D}_d)$ denotes the family of all functions F analytic on $\mathscr{D}_d$, and such that the norm $\mathcal{N}_1(F,d)$ is finite, where

$$\mathcal{N}_1(F,d)=\lim_{\epsilon\to 0}\oint_{\partial\mathscr{D}_d(\epsilon)}|F(\zeta)||\mathrm{d}\zeta|.$$

The discretization error of the Sinc approximation for a function F belonging to $\mathbf{H}^1(\mathscr{D}_d)$ has been estimated as follows.

Theorem 4.2 (Stenger [20, Theorem 3.1.3]). Let $F\in\mathbf{H}^1(\mathscr{D}_d)$. Then

$$\sup_{x\in\mathbb{R}}\left|F(x)-\sum_{j=-\infty}^{\infty}F(jh)S(j,h)(x)\right|\leq\frac{\mathcal{N}_1(F,d)}{\pi d(1-\mathrm{e}^{-2\pi d/h})}\,\mathrm{e}^{-\pi d/h}.$$

By setting $F(x)=f(\psi_{\mathrm{SE}}(x))$, we can apply the theorem to obtain an estimate for the SE-Sinc approximation.

Next, the truncation error can be evaluated by the next lemma.

Lemma 4.3 (Stenger [20, in the proof of Theorem 4.2.5])**.** Let $f \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$. Let $\mu = \min\{\alpha,\, \beta\}$, n be a positive integer, and M and N be positive integers defined by (2.8). Then it follows that

$$\left|\sum_{j=-\infty}^{-M-1} f(\psi_{\mathrm{SE}}(jh))S(j,h)(\psi_{\mathrm{SE}}^{-1}(t))\right| + \left|\sum_{j=N+1}^{\infty} f(\psi_{\mathrm{SE}}(jh))S(j,h)(\psi_{\mathrm{SE}}^{-1}(t))\right| \leq \frac{2K(b-a)^{\alpha+\beta}}{\mu h}\,\mathrm{e}^{-\mu n h}.$$

Notice that Theorem 4.2 and Lemma 4.3 refer to different function spaces $\mathbf{H}^1(\mathscr{D}_d)$ and $\mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$; the next lemma gives a link between these spaces.

Lemma 4.4 (Stenger [20, Theorem 4.2.4])**.** If $f \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, then $f(\psi_{\mathrm{SE}}(\cdot)) \in \mathbf{H}^1(\mathscr{D}_d)$.

By summing up the above results and by taking h according to (2.7), the constant C in (2.9) can be estimated as

$$\begin{aligned}\left\{\frac{\mathcal{N}_1(f(\psi_{\mathrm{SE}}(\cdot)),d)}{\pi d(1-\mathrm{e}^{-2\pi d/h})} + \frac{2K(b-a)^{\alpha+\beta}}{\mu h}\right\}\frac{1}{\sqrt{n}} &\leq \frac{\mathcal{N}_1(f(\psi_{\mathrm{SE}}(\cdot)),d)}{\pi d(1-\mathrm{e}^{-2\sqrt{\pi d\mu}})} + \frac{2K(b-a)^{\alpha+\beta}}{\sqrt{\pi d\mu}}\\ &\leq \frac{K\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\cdot)),d)}{\pi d(1-\mathrm{e}^{-2\sqrt{\pi d\mu}})} + \frac{2K(b-a)^{\alpha+\beta}}{\sqrt{\pi d\mu}}.\end{aligned} \tag{4.1}$$

There remains a term to be estimated: $\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\cdot)),d)$. To the authors' best knowledge, this term has never been explicitly evaluated, and one of the primal contributions of the present paper is that it is given for the first time. Its proof is left to § 4.4.

Lemma 4.5. Let α and β be positive constants, let $\mu = \min\{\alpha,\, \beta\}$, and let d be a constant with $0 < d < \pi$. Then

$$\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\cdot)),d) \leq \frac{4}{\mu}\left\{\frac{b-a}{\cos(d/2)}\right\}^{\alpha+\beta}.$$

This completes the desired explicit estimation. It turns out that the term $\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\cdot)),d)$ commonly appears in the other two formulas (2.2) and (2.3) as well, and proofs can be derived in like manner there.

The DE-Sinc cases can be handled in an analogous fashion. There are two terms to be evaluated: $\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{DE}}(\cdot)),d)$ for the DE-Sinc approximation (2.4), and $\mathcal{N}_1(\cosh(\cdot)Q_{\alpha,\beta}(\psi_{\mathrm{DE}}(\cdot)),d)$ for the DE-Sinc quadrature (2.5) and the DE-Sinc indefinite integration (2.6). The evaluation is given as follows, while its proof is left to § 4.4; we here like to mention that the proof gets far more complicated than the SE-Sinc case.

Lemma 4.6. Let α and β be positive constants, let $\mu = \min\{\alpha,\, \beta\}$, and let d be a constant with $0 < d < \pi/2$. Then

$$\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{DE}}(\cdot)),d) \leq \mathcal{N}_1(\cosh(\cdot)Q_{\alpha,\beta}(\psi_{\mathrm{DE}}(\cdot)),d) \leq \frac{4}{\pi\mu\cos d}\left\{\frac{b-a}{\cos(\frac{\pi}{2}\sin d)}\right\}^{\alpha+\beta}.$$

4.2 Proofs in the SE-Sinc case

4.2.1 Proof for the SE-Sinc approximation (2.1)

As described above, the proof of Theorem 2.3 can be immediately obtained by combining Lemma 4.5 with the inequality (4.1).

4.2.2 Proof for the SE-Sinc quadrature (2.2)

The discretization error of the Sinc quadrature has been analyzed in the literature as follows.

Theorem 4.7 (Stenger [20, Theorem 3.2.1])**.** Let $F \in \mathbf{H}^1(\mathscr{D}_d)$. Then

$$\left| \int_{-\infty}^{\infty} F(x)\,\mathrm{d}x - h \sum_{j=-\infty}^{\infty} F(jh) \right| \leq \frac{\mathcal{N}_1(F,d)}{1 - \,\mathrm{e}^{-2\pi d/h}} \,\mathrm{e}^{-2\pi d/h}.$$

Let us apply the theorem to the SE-Sinc quadrature. Since $Q(\psi_{\mathrm{SE}}(\zeta)) = (b-a)\psi_{\mathrm{SE}}'(\zeta)$, it follows that

$$|f(\psi_{\mathrm{SE}}(\zeta))\psi_{\mathrm{SE}}'(\zeta)| = \frac{1}{b-a}|f(\psi_{\mathrm{SE}}(\zeta))Q(\psi_{\mathrm{SE}}(\zeta))| \leq \frac{K}{b-a}|Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\zeta))|, \tag{4.2}$$

under the assumption that $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$. Thus it immediately follows $f(\psi_{\mathrm{SE}}(\cdot))\psi_{\mathrm{SE}}'(\cdot) \in \mathbf{H}^1(\mathscr{D}_d)$ because $\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\cdot)),d)$ is finite from Lemma 4.5. Therefore we can use Theorem 4.7 for $F(x) = f(\psi_{\mathrm{SE}}(x))\psi_{\mathrm{SE}}'(x)$ to bound the discretization error of the SE-Sinc quadrature as follows.

Lemma 4.8. Let $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, and let $\mu = \min\{\alpha,\, \beta\}$. Then

$$\left| \int_a^b f(t)\,\mathrm{d}t - h \sum_{j=-\infty}^{\infty} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh) \right| \leq \frac{4K(b-a)^{\alpha+\beta-1}}{\mu \cos^{\alpha+\beta}(d/2)} \frac{\mathrm{e}^{-2\pi d/h}}{1 - \,\mathrm{e}^{-2\pi d/h}}.$$

The truncation error is bounded as follows.

Lemma 4.9. Assume that the assumptions of Lemma 4.8 are fulfilled. Furthermore let n be a positive integer, and M and N be positive integers defined by (2.8). Then it follows that

$$\left| h \sum_{j=-\infty}^{-M-1} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh) \right| + \left| h \sum_{j=N+1}^{\infty} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh) \right| \leq \frac{2K(b-a)^{\alpha+\beta-1}}{\mu} \,\mathrm{e}^{-\mu n h}.$$

Proof. We can see that the same proof as Lemma 4.3 holds because $(b-a)f(\cdot)\psi_{\mathrm{SE}}'(\psi_{\mathrm{SE}}^{-1}(\cdot)) \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ from (4.2). ■

Combining Lemma 4.8 with Lemma 4.9, we obtain Theorem 2.5.

4.2.3 Proof for the SE-Sinc indefinite integration (2.3)

The discretization error of the Sinc indefinite integration has been analyzed as follows.

Theorem 4.10 (Stenger [20, Lemma 3.6.4])**.** Let $F \in \mathbf{H}^1(\mathscr{D}_d)$. Then

$$\sup_{x\in\mathbb{R}} \left| \int_{-\infty}^{x} F(\sigma)\,\mathrm{d}\sigma - \sum_{j=-\infty}^{\infty} F(jh)J(j,h)(x) \right| \leq \frac{\mathcal{N}_1(F,d)}{2d(1 - \,\mathrm{e}^{-2\pi d/h})} h \,\mathrm{e}^{-\pi d/h}.$$

From this we obtain an estimate for the SE-Sinc case in the same way as Lemma 4.8.

Lemma 4.11. Let $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, and let $\mu = \min\{\alpha, \beta\}$. Then

$$\sup_{t\in(a,\,b)} \left| \int_a^t f(s)\,\mathrm{d}s - \sum_{j=-\infty}^{\infty} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh)J(j,h)(\psi_{\mathrm{SE}}^{-1}(t)) \right| \leq \frac{2K(b-a)^{\alpha+\beta-1}}{d\mu\cos^{\alpha+\beta}(d/2)} \frac{h\,\mathrm{e}^{-\pi d/h}}{1-\,\mathrm{e}^{-2\pi d/h}}.$$

For the truncation error, it is necessary to bound the basis function $J(j, h)$. The next lemma gives the bound.

Lemma 4.12 (Stenger [20, Lemma 3.6.5])**.** For $x \in \mathbb{R}$, the function $J(j, h)(x)$ is bounded by

$$|J(j,h)(x)| \leq 1.1h.$$

Using this lemma and Lemma 4.9, we can bound the truncation error as follows.

Lemma 4.13. Assume that the assumptions of Lemma 4.9 are fulfilled. Then it follows that

$$\left| \sum_{j=-\infty}^{-M-1} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh)J(j,h)(\psi_{\mathrm{SE}}^{-1}(t)) \right| + \left| \sum_{j=N+1}^{\infty} f(\psi_{\mathrm{SE}}(jh))\psi_{\mathrm{SE}}'(jh)J(j,h)(\psi_{\mathrm{SE}}^{-1}(t)) \right|$$
$$\leq 1.1\frac{2K(b-a)^{\alpha+\beta-1}}{\mu}\,\mathrm{e}^{-\mu nh}.$$

Combining Lemma 4.11 with Lemma 4.13, we obtain Theorem 2.7.

4.3 Proofs in the DE-Sinc case

4.3.1 Error estimation for the DE-Sinc approximation (2.4)

From Lemma 4.6, it immediately follows that $f(\psi_{\mathrm{DE}}(\cdot)) \in \mathbf{H}^1(\mathscr{D}_d)$ if $f \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$, similar to Lemma 4.4. Therefore Theorem 4.2 can be used to bound the discretization error as follows.

Lemma 4.14. Let $f \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$, and let $\mu = \min\{\alpha, \beta\}$. Then

$$\sup_{t\in(a,\,b)} \left| f(t) - \sum_{j=-\infty}^{\infty} f(\psi_{\mathrm{DE}}(jh))S(j,h)(\psi_{\mathrm{DE}}^{-1}(t)) \right| \leq C_1 C_2 \frac{\mathrm{e}^{-\pi d/h}}{1-\,\mathrm{e}^{-2\pi d/h}},$$

where the constants C_1 and C_2 are defined by (2.14) and (2.15), respectively.

The truncation error is estimated in the next lemma.

Lemma 4.15. Assume that the assumptions of Lemma 4.14 are fulfilled. Furthermore let $\nu = \max\{\alpha, \beta\}$, n be a positive integer, and M and N be positive integers defined by (2.13). Then it follows that

$$\left| \sum_{j=-\infty}^{-M-1} f(\psi_{\mathrm{DE}}(jh))S(j,h)(\psi_{\mathrm{DE}}^{-1}(t)) \right| + \left| \sum_{j=N+1}^{\infty} f(\psi_{\mathrm{DE}}(jh))S(j,h)(\psi_{\mathrm{DE}}^{-1}(t)) \right| \leq 2d\,\mathrm{e}^{\frac{\pi}{2}\nu}C_1 \frac{\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)}}{h\,\mathrm{e}^{nh}},$$

where the constant C_1 is defined by (2.14).

Proof. Clearly $|S(j,h)(\psi_{\mathrm{DE}}^{-1}(t))| \leq 1$ for all $t \in (a,\, b)$. And since $f \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$, it follows for all $x \leq 0$ that

$$\begin{aligned}
|f(\psi_{\mathrm{DE}}(x))| &\leq K Q_{\alpha,\beta}(\psi_{\mathrm{DE}}(x)) \\
&= K \frac{(b-a)^{\alpha+\beta}}{(1+\,\mathrm{e}^{-\pi\sinh(x)})^{\alpha}(1+\,\mathrm{e}^{\pi\sinh(x)})^{\beta}} \\
&\leq K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\pi\alpha\sinh(x)} \\
&\leq K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\alpha}\,\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(-x)},
\end{aligned}$$

then the first sum is bounded as

$$\begin{aligned}
\left|\sum_{j=-\infty}^{-M-1} f(\psi_{\mathrm{DE}}(jh))S(j,h)(\psi_{\mathrm{DE}}^{-1}(t))\right| &\leq K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\alpha}\sum_{j=-\infty}^{-M-1}\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(-jh)} \\
&\leq K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\alpha}\int_{-\infty}^{-M}\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(-sh)}\,\mathrm{d}s \\
&\leq \frac{2K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\alpha}}{\pi\alpha h\,\mathrm{e}^{Mh}}\int_{-\infty}^{-M}\frac{\pi\alpha h}{2}\,\mathrm{e}^{-sh}\,\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(-sh)}\,\mathrm{d}s \\
&= \frac{2K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\alpha}}{\pi\alpha h\,\mathrm{e}^{Mh}}\,\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(Mh)}.
\end{aligned}$$

Furthermore using $\mu = \min\{\alpha,\, \beta\}$, $\nu = \max\{\alpha,\, \beta\}$, and the relations (2.13), we have

$$\frac{2K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\alpha}}{\pi\alpha h\,\mathrm{e}^{Mh}}\,\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(Mh)} \leq \frac{2K(b-a)^{\alpha+\beta}\,\mathrm{e}^{\frac{\pi}{2}\nu}}{\pi\mu h\,\mathrm{e}^{nh}}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)} = d\,\mathrm{e}^{\frac{\pi}{2}\nu}C_1\frac{\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)}}{h\,\mathrm{e}^{nh}}.$$

Similarly we can bound the second sum, thus the claim follows. ■

Then we can prove Theorem 2.9 as follows.

Proof. From Lemma 4.14 and Lemma 4.15, clearly it follows that

$$\sup_{t\in(a,\,b)}\left|f(t) - \sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))S(j,h)(\psi_{\mathrm{DE}}^{-1}(t))\right| \leq C_1\left[C_2\frac{\mathrm{e}^{-\pi d/h}}{1-\,\mathrm{e}^{-2\pi d/h}} + 2d\,\mathrm{e}^{\frac{\pi}{2}\nu}\frac{\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)}}{h\,\mathrm{e}^{nh}}\right].$$

Substituting (2.12) into the first term, we have

$$\frac{\mathrm{e}^{-\pi d/h}}{1-\,\mathrm{e}^{-2\pi d/h}} = \frac{\mathrm{e}^{-\pi dn/\log(2dn/\mu)}}{1-\,\mathrm{e}^{-\pi\mu(2dn/\mu)/\log(2dn/\mu)}} \leq \frac{\mathrm{e}^{-\pi dn/\log(2dn/\mu)}}{1-\,\mathrm{e}^{-\pi\mu\,\mathrm{e}}}, \tag{4.3}$$

since the function $\mathrm{e}^{\pi\mu x/\log x}$ has its minimum at $x = \mathrm{e}$. The second term can be evaluated as

$$\frac{\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)}}{h\,\mathrm{e}^{nh}} = \frac{\mu\,\mathrm{e}^{-\pi dn}}{2d\log(2dn/\mu)} = \frac{\exp\left\{-\frac{\pi\mu}{2}(2dn/\mu)\left(1-\frac{1}{\log(2dn/\mu)}\right)\right\}}{\log(2dn/\mu)}\frac{\mu}{2d}\,\mathrm{e}^{-\pi dn/\log(2dn/\mu)},$$

and using $n \geq (\nu\,\mathrm{e})/(2d) \geq (\mu\,\mathrm{e})/(2d)$, we have

$$\frac{\exp\left\{-\frac{\pi\mu}{2}(2dn/\mu)\left(1-\frac{1}{\log(2dn/\mu)}\right)\right\}}{\log(2dn/\mu)} \leq \frac{\exp\left\{-\frac{\pi\mu}{2}(\mathrm{e})\left(1-\frac{1}{\log(\mathrm{e})}\right)\right\}}{\log(\mathrm{e})} = 1, \tag{4.4}$$

since the left hand side is monotonically decreasing. Thus this theorem is established. ■

4.3.2 Proof for the DE-Sinc quadrature (2.5)

Noticing that

$$\psi_{\mathrm{DE}}'(\zeta) = \frac{\pi}{b-a}\cosh(\zeta)Q(\psi_{\mathrm{DE}}(\zeta)),$$

we easily obtain

$$|f(\psi_{\mathrm{DE}}(\zeta))\psi_{\mathrm{DE}}'(\zeta)| = \frac{\pi|\cosh(\zeta)|}{b-a}|f(\psi_{\mathrm{DE}}(x))Q(\psi_{\mathrm{DE}}(\zeta))| \leq \frac{\pi|\cosh(\zeta)|}{b-a}K|Q_{\alpha,\beta}(\zeta)|, \tag{4.5}$$

under the assumption that $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$. From this and Lemma 4.6, it immediately follows $f(\psi_{\mathrm{DE}}(\cdot))\psi_{\mathrm{DE}}'(\cdot) \in \mathbf{H}^1(\mathscr{D}_d)$. Therefore we can use Theorem 4.7 to bound the discretization error as follows.

Lemma 4.16. Let $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$, and let $\mu = \min\{\alpha, \beta\}$. Then

$$\left|\int_a^b f(t)\,\mathrm{d}t - h\sum_{j=-\infty}^{\infty} f(\psi_{\mathrm{DE}}(jh))\psi_{\mathrm{DE}}'(jh)\right| \leq \tilde{C}_1\tilde{C}_2\frac{\mathrm{e}^{-2\pi d/h}}{1-\mathrm{e}^{-2\pi d/h}},$$

where the constants $\tilde{C}_1$ and $\tilde{C}_2$ are defined by (2.17) and (2.18), respectively.

The truncation error is estimated by the next lemma.

Lemma 4.17. Assume that the assumptions of Lemma 4.16 are fulfilled. Furthermore let $\nu = \max\{\alpha, \beta\}$, n be a positive integer, and M and N be positive integers defined by (2.13). Then it follows that

$$\left|h\sum_{j=-\infty}^{-M-1} f(\psi_{\mathrm{DE}}(jh))\psi_{\mathrm{DE}}'(jh)\right| + \left|h\sum_{j=N+1}^{\infty} f(\psi_{\mathrm{DE}}(jh))\psi_{\mathrm{DE}}'(jh)\right| \leq \mathrm{e}^{\frac{\pi}{2}\nu}\tilde{C}_1\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)},$$

where the constant $\tilde{C}_1$ is defined by (2.17).

Proof. Since $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$ and (4.5), it follows for all $x \leq 0$ that

$$\begin{aligned}|f(\psi_{\mathrm{DE}}(x))\psi_{\mathrm{DE}}'(x)| &\leq \frac{\pi\cosh(x)}{b-a}\frac{K(b-a)^{\alpha+\beta}}{(1+\mathrm{e}^{-\pi\sinh(x)})^{\alpha}(1+\mathrm{e}^{\pi\sinh(x)})^{\beta}}\\ &\leq K(b-a)^{\alpha+\beta-1}\pi\cosh(x)\,\mathrm{e}^{\pi\alpha\sinh(x)},\end{aligned}$$

then the first sum is bounded as

$$\begin{aligned}\left|h\sum_{j=-\infty}^{-M-1} f(\psi_{\mathrm{DE}}(jh))\psi_{\mathrm{DE}}'(jh)\right| &\leq h\sum_{j=-\infty}^{-M-1} K(b-a)^{\alpha+\beta-1}\pi\cosh(jh)\,\mathrm{e}^{\pi\alpha\sinh(jh)}\\ &\leq K(b-a)^{\alpha+\beta-1}\int_{-\infty}^{-Mh}\pi\cosh(x)\,\mathrm{e}^{\pi\alpha\sinh(x)}\,\mathrm{d}x\\ &= \frac{K(b-a)^{\alpha+\beta-1}}{\alpha}\,\mathrm{e}^{-\pi\alpha\sinh(Mh)}\\ &\leq \frac{K(b-a)^{\alpha+\beta-1}\,\mathrm{e}^{\frac{\pi}{2}\alpha}}{\alpha}\,\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(Mh)}.\end{aligned}$$

Furthermore using $\mu = \min\{\alpha, \beta\}$, $\nu = \max\{\alpha, \beta\}$, and the relations (2.13), we have

$$\frac{K(b-a)^{\alpha+\beta-1}\,\mathrm{e}^{\frac{\pi}{2}\alpha}}{\alpha}\,\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(Mh)} \leq \frac{K(b-a)^{\alpha+\beta-1}\,\mathrm{e}^{\frac{\pi}{2}\nu}}{\mu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)} = \frac{\mathrm{e}^{\frac{\pi}{2}\nu}\tilde{C}_1}{2}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)}.$$

Similarly we can bound the second sum, thus the claim follows. ■

Now we are in a position to prove Theorem 2.11.

Proof. From Lemma 4.16 and Lemma 4.17, clearly it follows that

$$\left|\int_a^b f(t)\,\mathrm{d}t - h\sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)\right| \leq \tilde{C}_1\left[\tilde{C}_2\frac{\mathrm{e}^{-2\pi d/h}}{1-\,\mathrm{e}^{-2\pi d/h}} + \,\mathrm{e}^{\frac{\pi}{2}\nu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)}\right].$$

Substituting (2.16) into the first term, we have

$$\frac{\mathrm{e}^{-2\pi d/h}}{1-\,\mathrm{e}^{-2\pi d/h}} = \frac{\mathrm{e}^{-2\pi dn/\log(4dn/\mu)}}{1-\,\mathrm{e}^{-\frac{\pi}{2}\mu(4dn/\mu)/\log(4dn/\mu)}} \leq \frac{\mathrm{e}^{-2\pi dn/\log(4dn/\mu)}}{1-\,\mathrm{e}^{-\frac{\pi}{2}\mu\,\mathrm{e}}},$$

similar to (4.3). The second term can be evaluated as

$$\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)} = \,\mathrm{e}^{-2\pi dn} = \exp\left\{-\frac{\pi\mu}{2}(4dn/\mu)\left(1-\frac{1}{\log(4dn/\mu)}\right)\right\}\,\mathrm{e}^{-2\pi dn/\log(4dn/\mu)},$$

and using $n \geq (\nu\,\mathrm{e})/(4d) \geq (\mu\,\mathrm{e})/(4d)$, we have

$$\exp\left\{-\frac{\pi\mu}{2}(4dn/\mu)\left(1-\frac{1}{\log(4dn/\mu)}\right)\right\} \leq \exp\left\{-\frac{\pi\mu}{2}(\mathrm{e})\left(1-\frac{1}{\log(\mathrm{e})}\right)\right\} = 1,$$

since the left hand side is monotonically decreasing. It completes the proof. ■

4.3.3 Proof for the DE-Sinc indefinite integration (2.6)

Above we have already seen that $f(\psi_{\mathrm{DE}}(\cdot))\psi'_{\mathrm{DE}}(\cdot) \in \mathbf{H}^1(\mathscr{D}_d)$ if $fQ \in \mathbf{L}_{K,\alpha,\beta}(\psi_{\mathrm{DE}}(\mathscr{D}_d))$, thus the discretization error can be obtained as below by using Theorem 4.10 and Lemma 4.6.

Lemma 4.18. Assume that the assumptions of Lemma 4.16 are fulfilled. Then

$$\left|\int_a^t f(s)\,\mathrm{d}s - \sum_{j=-\infty}^{\infty} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)J(j,h)(\psi_{\mathrm{DE}}^{-1}(t))\right| \leq \frac{\tilde{C}_1\tilde{C}_2}{2d}\frac{h\,\mathrm{e}^{-\pi d/h}}{1-\,\mathrm{e}^{-2\pi d/h}},$$

where the constants $\tilde{C}_1$ and $\tilde{C}_2$ are defined by (2.17) and (2.18), respectively.

The truncation error is estimated as follows; since it can be easily obtained from Lemma 4.12 and Lemma 4.17, we omit the proof.

Lemma 4.19. Assume that the assumptions of Lemma 4.17 are fulfilled. Then it follows that

$$\left|\sum_{j=-\infty}^{-M-1} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)J(j,h)(\psi_{\mathrm{DE}}^{-1}(t))\right| + \left|\sum_{j=N+1}^{\infty} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)J(j,h)(\psi_{\mathrm{DE}}^{-1}(t))\right|$$
$$\leq 1.1\,\mathrm{e}^{\frac{\pi}{2}\nu}\tilde{C}_1\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)},$$

where the constant $\tilde{C}_1$ is defined by (2.17).

Finally, we prove Theorem 2.13.

Proof. From Lemma 4.18 and Lemma 4.19, clearly it follows that

$$\begin{aligned}&\sup_{t\in(a,\,b)}\left|\int_a^t f(s)\,\mathrm{d}s-\sum_{j=-M}^{N} f(\psi_{\mathrm{DE}}(jh))\psi'_{\mathrm{DE}}(jh)J(j,h)(\psi_{\mathrm{DE}}^{-1}(t))\right|\\&\leq \tilde{C}_1\left[\frac{\tilde{C}_2}{2d}\frac{\mathrm{e}^{-\pi d/h}}{1-\mathrm{e}^{-2\pi d/h}}+\frac{1.1}{h}\,\mathrm{e}^{\frac{\pi}{2}\nu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)}\right]h.\end{aligned}$$

We can use (4.3) for the first term. For the second term, we have

$$\begin{aligned}\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)}&\leq \mathrm{e}^{\frac{\pi}{2}\mu}\,\mathrm{e}^{-\pi\mu\cosh(nh)}\\&=\mathrm{e}^{\frac{\pi}{2}\mu}\exp\left[-\frac{\pi}{2}\mu\left\{(2dn/\mu)+\frac{1}{(2dn/\mu)}\right\}\right]\\&=\mathrm{e}^{\frac{\pi}{2}\mu}\exp\left[-\frac{\pi}{2}\mu\left\{(2dn/\mu)+\frac{1}{(2dn/\mu)}-\frac{(2dn/\mu)}{\log(2dn/\mu)}\right\}\right]\,\mathrm{e}^{-\pi dn/\log(2dn/\mu)}.\end{aligned}$$

Furthermore, since $n\geq(\nu\,\mathrm{e})/(2d)\geq(\mu\,\mathrm{e})/(2d)$, it follows that

$$\frac{1.1}{h}=\frac{1.1\mu}{2d}\times\frac{(2dn/\mu)}{\log(2dn/\mu)}\leq\frac{1.1\mu}{2d}\times\mathrm{e}^2\left\{(2dn/\mu)+\frac{1}{(2dn/\mu)}-\frac{(2dn/\mu)}{\log(2dn/\mu)}\right\}.$$

If we set a function g as $g(x)=x\,\mathrm{e}^{-\frac{\pi}{2}\mu x}$, which has its maximum at $x=2/(\pi\mu)$, we have

$$\begin{aligned}\frac{1.1}{h}\,\mathrm{e}^{\frac{\pi}{2}\nu}\,\mathrm{e}^{-\frac{\pi}{2}\mu\exp(nh)}&\leq \mathrm{e}^{\frac{\pi}{2}(\mu+\nu)}\frac{1.1\mu}{2d}\,\mathrm{e}^2\left[g\left((2dn/\mu)+\frac{1}{(2dn/\mu)}-\frac{(2dn/\mu)}{\log(2dn/\mu)}\right)\right]\,\mathrm{e}^{-\pi dn/\log(2dn/\mu)}\\&\leq \mathrm{e}^{\frac{\pi}{2}(\mu+\nu)}\frac{1.1\mu}{2d}\,\mathrm{e}^2\left[\frac{2}{\pi\mu\,\mathrm{e}}\right]\,\mathrm{e}^{-\pi dn/\log(2dn/\mu)}\\&=\frac{1.1\,\mathrm{e}}{\pi}\frac{\mathrm{e}^{\frac{\pi}{2}(\mu+\nu)}}{d}\,\mathrm{e}^{-\pi dn/\log(2dn/\mu)}\\&<\frac{\mathrm{e}^{\frac{\pi}{2}(\mu+\nu)}}{d}\,\mathrm{e}^{-\pi dn/\log(2dn/\mu)}.\end{aligned}$$

Furthermore using $\mathrm{e}^{\frac{\pi}{2}(\mu+\nu)}=\mathrm{e}^{\frac{\pi}{2}(\alpha+\beta)}$, we obtain the desired inequality. ∎

4.4 Proofs of Lemma 4.5 and Lemma 4.6

Here we prove Lemma 4.5 and Lemma 4.6.

First we consider Lemma 4.5. Recall that $Q_{\alpha,\beta}(z)=(z-a)^{\alpha}(b-z)^{\beta}$. If we apply a variable transformation $z=\psi_{\mathrm{SE}}(\zeta)$, we get

$$Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\zeta))=\frac{(b-a)^{\alpha+\beta}}{(1+\mathrm{e}^{-\zeta})^{\alpha}(1+\mathrm{e}^{\zeta})^{\beta}}.$$

In view of this, we can see that the next lemma is essential to estimate $\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\cdot)),d)$.

Lemma 4.20. Let x and y be real numbers with $|y| < \pi$, and let $\zeta = x + \mathrm{i}\, y$. Then

$$\left|\frac{1}{1+\mathrm{e}^{\zeta}}\right| \leq \frac{1}{(1+\mathrm{e}^{x})\cos(y/2)}, \tag{4.6}$$

$$\left|\frac{1}{1+\mathrm{e}^{-\zeta}}\right| \leq \frac{1}{(1+\mathrm{e}^{-x})\cos(y/2)}. \tag{4.7}$$

Proof. We only prove the first inequality (4.6), because the second one (4.7) can be easily derived by replacing ζ with $-\zeta$. Using

$$\cosh^2(x/2) - \sin^2(y/2) \geq \cosh^2(x/2)\{1 - \sin^2(y/2)\} = \cosh^2(x/2)\cos^2(y/2),$$

we have

$$\left|\frac{1}{1+\mathrm{e}^{\zeta}}\right| = \frac{\mathrm{e}^{-x/2}}{2\sqrt{\cosh^2(x/2) - \sin^2(y/2)}} \leq \frac{\mathrm{e}^{-x/2}}{2\cosh(x/2)\cos(y/2)} = \frac{1}{(1+\mathrm{e}^{x})\cos(y/2)},$$

which establishes the lemma. ∎

Using this lemma, we can estimate $\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\cdot)), d)$ (Lemma 4.5).

Proof. For all ϵ with $0 < \epsilon < 1$, we have

$$\lim_{x\to\pm\infty}\int_{-d(1-\epsilon)}^{d(1-\epsilon)} |Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(x+\mathrm{i}\, y))|\,\mathrm{d}y \leq \lim_{x\to\pm\infty}\frac{(b-a)^{\alpha+\beta}}{(1+\mathrm{e}^{-x})^{\alpha}(1+\mathrm{e}^{x})^{\beta}}\int_{-d(1-\epsilon)}^{d(1-\epsilon)}\frac{\mathrm{d}y}{\cos^{\alpha+\beta}(y/2)} = 0,$$

if we note Lemma 4.20. Therefore $\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\cdot)), d)$ can be written as

$$\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\cdot)), d) = \lim_{y\to d}\int_{-\infty}^{\infty}|Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(x+\mathrm{i}\, y))|\,\mathrm{d}x + \lim_{y\to -d}\int_{-\infty}^{\infty}|Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(x+\mathrm{i}\, y))|\,\mathrm{d}x. \tag{4.8}$$

Again using Lemma 4.20, we can estimate it as

$$\begin{aligned}
\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{SE}}(\cdot)), d) &\leq 2\left\{\frac{b-a}{\cos(d/2)}\right\}^{\alpha+\beta}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(1+\mathrm{e}^{-x})^{\alpha}(1+\mathrm{e}^{x})^{\beta}}\\
&\leq 2\left\{\frac{b-a}{\cos(d/2)}\right\}^{\alpha+\beta}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{(1+\mathrm{e}^{-x})^{\mu}(1+\mathrm{e}^{x})^{\mu}}\\
&= 4\left\{\frac{b-a}{\cos(d/2)}\right\}^{\alpha+\beta}\int_{0}^{\infty}\left\{\frac{\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}\right\}^{\mu}\mathrm{d}x\\
&\leq 4\left\{\frac{b-a}{\cos(d/2)}\right\}^{\alpha+\beta}\int_{0}^{\infty}\mathrm{e}^{-\mu x}\,\mathrm{d}x\\
&= 4\left\{\frac{b-a}{\cos(d/2)}\right\}^{\alpha+\beta}\frac{1}{\mu}.
\end{aligned}$$

∎

Now we switch to Lemma 4.6. The function $\cosh(\zeta)$ is bounded as

$$|\cosh(x+\mathrm{i}\, y)| = \sqrt{\cosh^2(x) - \sin^2(y)} \leq \cosh(x), \tag{4.9}$$

for all x, $y \in \mathbb{R}$. The main difficulty in proving Lemma 4.6 is that bounding the function $Q_{\alpha,\beta}(\psi_{\mathrm{DE}}(\zeta))$ is quite a complicated task. Since

$$Q_{\alpha,\beta}(\psi_{\mathrm{DE}}(\zeta)) = \frac{(b-a)^{\alpha+\beta}}{(1+\mathrm{e}^{-\pi\sinh\zeta})^{\alpha}(1+\mathrm{e}^{\pi\sinh\zeta})^{\beta}},$$

the next lemma is essential in the project.

Lemma 4.21. Let x and y be real numbers with $|y| < \pi/2$, and let $\zeta = x + \mathrm{i}\,y$. Then

$$\left|\frac{1}{1+\mathrm{e}^{\pi\sinh\zeta}}\right| \leq \frac{1}{(1+\mathrm{e}^{\pi\sinh(x)\cos y})\cos(\frac{\pi}{2}\sin y)}, \tag{4.10}$$

$$\left|\frac{1}{1+\mathrm{e}^{-\pi\sinh\zeta}}\right| \leq \frac{1}{(1+\mathrm{e}^{-\pi\sinh(x)\cos y})\cos(\frac{\pi}{2}\sin y)}.$$

We like to leave its long proof to the end of this section. If we accept this lemma, we can derive Lemma 4.6 as follows.

Proof. We write $\varOmega(\zeta) = \cosh(\zeta)Q_{\alpha,\beta}(\psi_{\mathrm{DE}}(\zeta))$ for simplicity. It is sufficient to bound $\mathcal{N}_1(\varOmega, d)$ since clearly $\mathcal{N}_1(Q_{\alpha,\beta}(\psi_{\mathrm{DE}}(\cdot)), d) \leq \mathcal{N}_1(\varOmega, d)$ holds. Using (4.9) and Lemma 4.21, we have

$$\begin{aligned}
|\varOmega(x+\mathrm{i}\,y)| &\leq \cosh(x)\left\{\frac{b-a}{\cos(\frac{\pi}{2}\sin y)}\right\}^{\alpha+\beta}\frac{1}{(1+\mathrm{e}^{-\pi\sinh(x)\cos y})^{\alpha}(1+\mathrm{e}^{\pi\sinh(x)\cos y})^{\beta}}\\
&\leq \cosh(x)\left\{\frac{b-a}{\cos(\frac{\pi}{2}\sin y)}\right\}^{\alpha+\beta}\frac{1}{(1+\mathrm{e}^{-\pi\sinh(x)\cos y})^{\mu}(1+\mathrm{e}^{\pi\sinh(x)\cos y})^{\mu}}\\
&\leq \cosh(x)\left\{\frac{b-a}{\cos(\frac{\pi}{2}\sin y)}\right\}^{\alpha+\beta}\mathrm{e}^{-\pi\mu\sinh(|x|)\cos y}
\end{aligned} \tag{4.11}$$

for all $x \in \mathbb{R}$ and $y \in [-d,\, d]$. Then it follows for all ϵ with $0 < \epsilon < 1$ that

$$\lim_{x\to\pm\infty}\int_{-d(1-\epsilon)}^{d(1-\epsilon)}|\varOmega(x+\mathrm{i}\,y)|\,\mathrm{d}y \leq \lim_{x\to\pm\infty}\frac{(b-a)^{\alpha+\beta}\cosh(x)}{\mathrm{e}^{\pi\mu\sinh(|x|)\cos d(1-\epsilon)}}\int_{-d(1-\epsilon)}^{d(1-\epsilon)}\frac{\mathrm{d}y}{\cos^{\alpha+\beta}(\frac{\pi}{2}\sin y)} = 0.$$

Therefore, similar to (4.8), we can see that

$$\mathcal{N}_1(\varOmega, d) = \lim_{y\to d}\int_{-\infty}^{\infty}|\varOmega(x+\mathrm{i}\,y)|\,\mathrm{d}x + \lim_{y\to -d}\int_{-\infty}^{\infty}|\varOmega(x+\mathrm{i}\,y)|\,\mathrm{d}x,$$

and by (4.11), it is estimated as

$$\begin{aligned}
\mathcal{N}_1(\varOmega, d) &\leq 2\left\{\frac{b-a}{\cos(\frac{\pi}{2}\sin d)}\right\}^{\alpha+\beta}\int_{-\infty}^{\infty}\cosh(x)\,\mathrm{e}^{-\pi\mu\sinh(|x|)\cos d}\,\mathrm{d}x\\
&= 4\left\{\frac{b-a}{\cos(\frac{\pi}{2}\sin d)}\right\}^{\alpha+\beta}\int_{0}^{\infty}\cosh(x)\,\mathrm{e}^{-\pi\mu\sinh(x)\cos d}\,\mathrm{d}x\\
&= 4\left\{\frac{b-a}{\cos(\frac{\pi}{2}\sin d)}\right\}^{\alpha+\beta}\frac{1}{\pi\mu\cos d}.
\end{aligned}$$

This is the desired inequality. ∎

Finally we finish this section by proving Lemma 4.21.

Proof. We only need to show (4.10) for the same reason as in Lemma 4.20. Clearly (4.10) holds if $y = 0$, thus we assume $y \neq 0$ below. The left hand side of (4.10) is equal to

$$\left| \frac{1}{1 + \mathrm{e}^{\pi \sinh(x + \mathrm{i}\, y)}} \right| = \frac{1}{(1 + \mathrm{e}^{\pi \sinh(x) \cos y}) \sqrt{g(x,y)}},$$

where

$$g(x,y) = 1 - \frac{\sin^2(\frac{\pi}{2} \cosh(x) \sin y)}{\cosh^2(\frac{\pi}{2} \sinh(x) \cos y)}.$$

Then for (4.10) it is sufficient to prove the inequality $g(x,y) \geq g(0,y)$, because

$$g(0,y) = 1 - \sin^2(\tfrac{\pi}{2} \sin y) = \cos^2(\tfrac{\pi}{2} \sin y).$$

Since g is an even function, we can assume $x \geq 0$ and $0 < y < \pi/2$ without loss of generality. In what follows, we prove $g(x,y) \geq g(0,y)$ holds for each y; this is done in the following two steps:

1) Show $\frac{\partial}{\partial x} g(x,y) \geq 0$ for all x with $0 \leq x \leq x_0$,

2) Show $g(x,y) \geq g(x_0, y)$ for all x with $x_0 < x$,

where $x_0 = \log((1 + \cos y)/\sin y)$.

Let us first consider the second one, which is relatively easy. Clearly it holds that

$$g(x,y) = 1 - \frac{\sin^2(\frac{\pi}{2} \cosh(x) \sin y)}{\cosh^2(\frac{\pi}{2} \sinh(x) \cos y)} \geq 1 - \frac{1}{\cosh^2(\frac{\pi}{2} \sinh(x) \cos y)} = \tanh^2(\tfrac{\pi}{2} \sinh(x) \cos y).$$

The equality holds when $x = x_0$. Since the function $\tanh^2(\frac{\pi}{2} \sinh(x) \cos y)$ is monotonically increasing with respect to x, it follows for all x with $x > x_0$ that

$$g(x,y) \geq \tanh^2(\tfrac{\pi}{2} \sinh(x) \cos y) \geq \tanh^2(\tfrac{\pi}{2} \sinh(x_0) \cos y) = g(x_0, y).$$

This completes the second step.

Next we consider the first step, $0 \leq x \leq x_0$. Notice that $1 \leq \cosh(x) \leq 1/\sin(y)$ in this range of x. Considering the derivative of $g(x,y)$, we have

$$\begin{aligned} &\frac{\partial}{\partial x} g(x,y) \\ &= \frac{\pi \sin(\frac{\pi}{2} \cosh(x) \sin y) \sin(\frac{\pi}{2} \cosh(x) \sin(y) + y) \sinh(\frac{\pi}{2} \sinh(x) \cos(y) + x)}{2 \cosh^3(\frac{\pi}{2} \sinh(x) \cos y)} \left\{ g_1(x,y) + g_2(x,y) \right\}, \end{aligned}$$

where

$$\begin{aligned} g_1(x,y) &= \frac{\sin(\frac{\pi}{2} \cosh(x) \sin(y) - y)}{\sin(\frac{\pi}{2} \cosh(x) \sin(y) + y)}, \\ g_2(x,y) &= \frac{\sinh(\frac{\pi}{2} \sinh(x) \cos(y) - x)}{\sinh(\frac{\pi}{2} \sinh(x) \cos(y) + x)}. \end{aligned}$$

Let us prove $g_1(x,y) + g_2(x,y) \geq 0$, which then proves $\frac{\partial}{\partial x} g(x,y) \geq 0$. We readily see

$$\begin{aligned} \sin(\tfrac{\pi}{2}\cosh(x)\sin(y) + y) &\geq 0, \\ \sin(\tfrac{\pi}{2}\cosh(x)\sin(y) - y) &\geq 0, \\ \sinh(\tfrac{\pi}{2}\sinh(x)\cos(y) + x) &\geq 0, \end{aligned}$$

since $1 \leq \cosh(x) \leq 1/\sin(y)$. Moreover, if $\frac{\pi}{2}\cos y \geq 1$,

$$\sinh(\tfrac{\pi}{2}\sinh(x)\cos(y) - x) \geq 0.$$

Hence we can conclude $g_1(x,y) + g_2(x,y) \geq 0$ if $0 < y \leq \arccos(2/\pi)$.

Assume $\arccos(2/\pi) < y < \pi/2$ below. In this range of y, we prove $g_1(x,y) + g_2(x,y) \geq 0$ by showing:

1a) $\frac{\partial}{\partial x} g_1(x,y) \geq 0$ and $\frac{\partial}{\partial x} g_2(x,y) \geq 0$,

1b) $g_1(0,y) + g_2(0,y) \geq 0$.

The claim 1a) can be shown as follows. For the derivative of $g_1(x,y)$, it immediately holds that

$$\frac{\partial}{\partial x} g_1(x,y) = \frac{\pi \sinh(x)\sin(2y)\sin y}{2\sin^2(\frac{\pi}{2}\cosh(x)\sin(y) + y)} \geq 0.$$

For the derivative of $g_2(x,y)$, we have

$$\begin{aligned} \frac{\partial}{\partial x} g_2(x,y) &= \frac{(\frac{\pi}{2}\cos y)\cosh(x)\sinh(2x) - \sinh(2(\frac{\pi}{2}\cos y)\sinh x)}{\sinh^2((\frac{\pi}{2}\cos y)\sinh(x) + x)} \\ &\geq (\tfrac{\pi}{2}\cos y)\frac{\{\cosh(x)\sinh(2x) - \sinh(2\sinh x)\}}{\sinh^2((\frac{\pi}{2}\cos y)\sinh(x) + x)}, \end{aligned}$$

since $0 < \frac{\pi}{2}\cos y < 1$. Furthermore differentiating the numerator of the right hand side, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\cosh(x)\sinh(2x) - \sinh(2\sinh x)\} = \cosh(x)\{3\cosh(2x) - 2\cosh(2\sinh x) - 1\}.$$

Recalling $0 \leq x \leq x_0 = \log((1+\cos y)/\sin y)$ and $\arccos(2/\pi) < y < \pi/2$, we can see

$$0 \leq x \leq \log\left(\frac{1+\cos y}{\sin y}\right) < \frac{1}{2}\log\left(\frac{\frac{\pi}{2}+1}{\frac{\pi}{2}-1}\right),$$

and in this range of x, it follows that

$$3\cosh(2x) - 2\cosh(2\sinh x) - 1 \geq 0.$$

Thus $\frac{\partial}{\partial x} g_2(x,y) \geq 0$ holds.

The proof is completed by showing 1b):

$$g_1(0,y) + g_2(0,y) = \frac{\sin(\frac{\pi}{2}\sin(y) - y)}{\sin(\frac{\pi}{2}\sin(y) + y)} + \frac{\frac{\pi}{2}\cos(y) - 1}{\frac{\pi}{2}\cos(y) + 1} \geq 0.$$

Let us start with the obvious inequality for $0 \leq s \leq \pi/2$:

$$\left(\frac{\pi}{2} + s\right)\sin(s) \geq \frac{\pi}{2}\sin(s) \geq s,$$

from which we have

$$\left(\frac{\pi}{2} - s\right)\left(\frac{\pi}{2} + s\right)\sin(s) \geq \left(\frac{\pi}{2} - s\right)s \geq \cos(s)s,$$

and it follows for $0 \leq s \leq \pi/2$ that

$$2\sin(s)\left\{\left(\frac{\pi}{2}\right)^2 - s^2\right\} \geq 2s\cos s.$$

Putting $s = \frac{\pi}{2}\sin(y)$ here, we have

$$2\sin(\tfrac{\pi}{2}\sin y)\left\{\tfrac{\pi}{2}\cos y\right\}^2 \geq 2(\tfrac{\pi}{2}\sin y)\cos(\tfrac{\pi}{2}\sin y).$$

This inequality is equivalent to:

$$\begin{aligned}&\left\{\sin(\tfrac{\pi}{2}\sin y)(\tfrac{\pi}{2}\cos y) - (\tfrac{\pi}{2}\sin y)\cos(\tfrac{\pi}{2}\sin y)\right\}\left\{(\tfrac{\pi}{2}\cos y) + 1\right\}\\ &\geq -\left\{\sin(\tfrac{\pi}{2}\sin y)(\tfrac{\pi}{2}\cos y) + (\tfrac{\pi}{2}\sin y)\cos(\tfrac{\pi}{2}\sin y)\right\}\left\{(\tfrac{\pi}{2}\cos y) - 1\right\},\end{aligned}$$

which then is equal to:

$$\frac{\sin(\frac{\pi}{2}\sin y)(\frac{\pi}{2}\cos y) - (\frac{\pi}{2}\sin y)\cos(\frac{\pi}{2}\sin y)}{\sin(\frac{\pi}{2}\sin y)(\frac{\pi}{2}\cos y) + (\frac{\pi}{2}\sin y)\cos(\frac{\pi}{2}\sin y)} + \frac{(\frac{\pi}{2}\cos y) - 1}{(\frac{\pi}{2}\cos y) + 1} \geq 0.$$

The left hand side is nothing but $g_1(0, y) + g_2(0, y)$, which completes the proof. ■

5 Concluding remarks

In this paper, explicit error estimates have been given for the SE/DE-Sinc approximation, the SE/DE-Sinc quadrature, and the SE/DE-Sinc indefinite integration, i.e. (2.1)–(2.6). By "explicit" we mean that the estimates are given with all the constants explicitly clarified; this is in contrast to the existing convergence analyses by several authors [6, 10, 20, 27, 28] where the convergence rates have been successfully revealed, but the constants have been left unevaluated. Giving explicit estimates is quite important from practical perspective, since it enables us to guarantee the accuracy of approximations in actual computations, and make the numerical formulas more reliable and practical. We have also improved some formulas themselves so that the computational costs are decreased. This is done by replacing the symmetric truncation, like (1.1), with the optimal truncation, like (1.8). The numerical results have been also shown, which confirm the theory.

Future works include the followings. First, we are now in fact constructing libraries with guaranteed accuracy based on the new explicit estimates. This will be reported soon elsewhere. Second, similar explicit estimates are desired for other Sinc formulas, such as approximations of derivatives and indefinite convolutions.

Acknowledgement

The authors are greatly indebted to Ken'ichiro Tanaka of Tokio Marine Financial Solutions LTD., for several helpful comments concerning Lemma 4.21. This work was supported by Global COE Program "The research and training center for new development in mathematics," MEXT, Japan.

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