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Tomoaki OKAYAMA, Takayasu MATSUO,
Masaaki SUGIHARA

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DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.keisu.t-u-tokyo.ac.jp/research/techrep/index.html

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Tomoaki OKAYAMA*, Takayasu MATSUO†, Masaaki SUGIHARA‡

Department of Mathematical Informatics
Graduate School of Information Science and Technology
The University of Tokyo

*Tomoaki_Okayama@mist.i.u-tokyo.ac.jp,
†matsuo@mist.i.u-tokyo.ac.jp,
‡m_sugihara@mist.i.u-tokyo.ac.jp

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Abstract

In this paper we propose new numerical methods for Fredholm integral equations of the second kind with weakly singular kernels. The methods are developed by means of the Sinc approximation with smoothing transformations, which is an effective technique against the singularities of the equations. Numerical examples show that the methods achieve exponential convergence, and in this sense the methods improve conventional results where only polynomial convergence have been reported so far.

1 Introduction

The purpose of this paper is to develop high order numerical methods for Fredholm integral equations of the form

$$\lambda u(t) - \int_a^b |t-s|^{p-1} k(t,s) u(s)\,\mathrm{d}s = g(t), \quad a \leq t \leq b, \tag{1.1}$$

where $\lambda \neq 0$, $0 < p < 1$, k and g are given smooth functions, and u is the solution to be determined. Equations of this form often arise in practical applications such as Dirichlet problems, mathematical problems of radiative equilibrium, radiative heat transfer problems [11, 15, 16].

The construction of high order methods for the equations is, however, not an easy task because of the singularity in the "weakly singular" kernel $|t-s|^{p-1}k(t,s)$ (note that $p < 1$); in fact, in this case the solution u is generally not differentiable at the endpoints (i.e. $t = a$ and $t = b$) [6, 12, 15, 21], and due to this, to the best of the authors' knowledge the best convergence rate ever achieved remains only at polynomial order. For example, if we set uniform meshes with $n+1$ grid points and apply the spline methods of order m, then the convergence rate is only $\mathrm{O}(n^{-2p})$ at most [5, 16], and it can not be improved by increasing m. One way of remedying this is to introduce graded meshes [5, 16, 22]. Then the rate is improved to $\mathrm{O}(n^{-m})$ [2, 22] which now depends on m, but still at polynomial order. Furthermore, as pointed in Monegato–Scuderi [8], this idea contains several substantial drawbacks such as that the implementation is complicated compared to the case of uniform meshes, and that the system of linear equations generated in this way becomes rapidly ill-conditioned as m increases. To counter these issues, Monegato–Scuderi [8] have proposed to introduce a smoothing transformation, instead of graded meshes, with which the solution can be made arbitrarily smooth. Then with the standard spline method on uniform meshes the better rate $\mathrm{O}(n^{-m})$ can be obtained without the drawbacks above (from the same reason, the concept of a smoothing transformation has recently been used by several authors [4, 10]). Other methods for the equations (1.1) include [1, 7], whose convergence rates are all of polynomial.

On the other hand, a method with *exponential* order convergence rate has been developed by Riley [13] for Volterra integral equations of the form

$$u(t) - \int_a^t (t-s)^{p-1} k(t,s) u(s)\,\mathrm{d}s = g(t), \quad a \leq t \leq b,$$

where the kernel is also assumed to be weakly singular, and the solution u is generally not differentiable at $t = a$ (cf. Brunner [3]). The key here is to utilize not only the concept of a smoothing transformation described above but this time also the so-called Sinc approximation; this is motivated by the fact that the combination is generally an effective tool for functions with derivative singularity at endpoints (cf. Stenger [17]). Riley then confirmed numerically that his method in fact achieves exponential convergence $\mathrm{O}(\exp(-c_1\sqrt{n}))$ despite the singularity. Furthermore, it can be examined numerically that in his method the system of linear equations is very well-conditioned.

With these backgrounds, we propose two new numerical methods for the equations (1.1). The first method is given by simply extending Riley's idea to the Fredholm case. It is then shown by numerical experiments that the new method enjoys the same convergence rate $\mathrm{O}(\exp(-c_1\sqrt{n}))$ as in the Volterra case. The second method is derived by replacing the smoothing transformation employed in the first method, the standard *tanh transformation*, with the so-called *double exponential transformation*. This modification is motivated by an observation that in various cases [9, 18] such replacement drastically improves the order of convergence. In fact, it turns out that the modification works well also in our case, and numerical experiments suggest that the convergence rate is improved to $\mathrm{O}(\exp(-c_2 n/\log n))$. In both of the new methods the linear equations are very well-conditioned. Finally, we also give a way of estimating a tuning parameter d which is the most essential parameter in the methods and substantially affects their performance. We like to emphasize that this point has been left unanswered in Riley [13].

The organization of this paper is as follows. In Section 2, we state basic theorems of the Sinc methods, which are referred to in the subsequent sections. In Section 3, two numerical methods are derived by means of the Sinc approximation. In Section 4, we analyze the regularity of the solution u of the equation (1.1). In Section 5, we give error bounds of the proposed methods. In Section 6 we show numerical results. Finally in Section 7 we conclude this paper.

2 Basic definitions and theorems of Sinc methods

2.1 Sinc approximation

The original Sinc approximation is expressed as

$$f(x) \approx \sum_{j=-N}^{N} f(jh)S(j,h)(x), \quad x \in \mathbb{R}, \tag{2.1}$$

where the basis function $S(j,h)(x)$ (the so-called *Sinc function*) is defined by

$$S(j,h)(x) = \frac{\sin \pi(x/h - j)}{\pi(x/h - j)},$$

and h is a step size appropriately chosen depending on a given positive integer N. Note that the approximation formula (2.1) is defined on $x \in \mathbb{R}$, whereas the target equation (1.1) is defined on the finite interval $(a,\, b)$. In order to relate these two intervals, $\mathbb{R}$ and $(a,\, b)$, the *tanh transformation* (and its inverse)

$$\begin{aligned} t &= \phi_{a,b}^{\mathrm{SE}}(x) = \frac{b-a}{2}\tanh\left(\frac{x}{2}\right) + \frac{b+a}{2}, \\ x &= \{\phi_{a,b}^{\mathrm{SE}}\}^{-1}(t) = \log\left(\frac{t-a}{b-t}\right) \end{aligned}$$

can be introduced [17]. Throughout this paper, we call this the *single exponential* (*SE*) *transformation*, and the combination of (2.1) and the SE transformation the *SE-Sinc approximation.*

In order that the formula (2.1) on $\mathbb{R}$ works accurately, a function to be approximated should be analytic on a strip domain, $\mathscr{D}_d = \{z \in \mathbb{C} : |\operatorname{Im} z| < d\}$ for some $d > 0$, and also should be bounded in some sense. When incorporated with the SE transformation, the conditions should be considered on the translated domain

$$\phi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d) = \left\{ z \in \mathbb{C} : \left| \arg\left(\frac{z-a}{b-z}\right) \right| < d \right\}.$$

In order to clarify the conditions more precisely, it is convenient to introduce the following function space.

Definition 2.1. Let $\mathscr{D}$ be a simply-connected domain which satisfies $(a,\, b) \subset \mathscr{D}$, and let β, γ be positive constants. Then $\mathbf{L}_{\beta,\gamma}(\mathscr{D})$ denotes the family of all functions f that satisfy the following conditions: (i) f is analytic in $\mathscr{D}$; (ii) there exists a constant C such that

$$|f(z)| \leq C|z-a|^{\beta}|b-z|^{\gamma} \tag{2.2}$$

holds for all z in $\mathscr{D}$. For later convenience, let us denote $\mathbf{L}_{\beta}(\mathscr{D}) = \mathbf{L}_{\beta,\beta}(\mathscr{D})$ and introduce a function $Q(z) = (z-a)(b-z)$.

When $f \in \mathbf{L}_{\alpha}(\phi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d))$ for some positive constants d and α, the next theorem guarantees the exponential convergence of the SE-Sinc approximation.

Theorem 2.2 (Stenger [17, Theorem 4.2.5])**.** Let $f \in \mathbf{L}_\alpha(\phi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let also N be a positive integer, and h be given by the formula

$$h = \sqrt{\frac{\pi d}{\alpha N}}. \tag{2.3}$$

Then there exists a constant C independent of N, such that

$$\max_{a \le t \le b} \left| f(t) - \sum_{j=-N}^{N} f(\phi_{a,b}^{\mathrm{SE}}(jh)) S(j,h)(\{\phi_{a,b}^{\mathrm{SE}}\}^{-1}(t)) \right| \le C\sqrt{N} \exp\left(-\sqrt{\pi d \alpha N}\right).$$

It is also possible to employ the *double exponential* (*DE*) *transformation* (cf. [9,18]) in place of the SE transformation. The transformation and its inverse are

$$t = \phi_{a,b}^{\mathrm{DE}}(x) = \frac{b-a}{2} \tanh\left(\frac{\pi}{2}\sinh(x)\right) + \frac{b+a}{2},$$
$$x = \{\phi_{a,b}^{\mathrm{DE}}\}^{-1}(t) = \log\left[\frac{1}{\pi}\log\left(\frac{t-a}{b-t}\right) + \sqrt{1 + \left\{\frac{1}{\pi}\log\left(\frac{t-a}{b-t}\right)\right\}^2}\right].$$

This transformation maps $\mathbb{R}$ onto (a, b), and maps $\mathscr{D}_d$ onto the domain:

$$\phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d) = \left\{ z \in \mathbb{C} : \left| \arg\left[\frac{1}{\pi}\log\left(\frac{z-a}{b-z}\right) + \sqrt{1 + \left\{\frac{1}{\pi}\log\left(\frac{z-a}{b-z}\right)\right\}^2}\right] \right| < d \right\}.$$

If $f \in \mathbf{L}_\alpha(\phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d))$, the Sinc approximation with the DE transformation is extremely accurate as stated below. We call this approximation the *DE-Sinc approximation*.

Theorem 2.3 (Tanaka et al. [19, Theorem 3.1])**.** Let $f \in \mathbf{L}_\alpha(\phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$, let N be a positive integer, and let h be selected by the formula

$$h = \frac{\log(2dN/\alpha)}{N}. \tag{2.4}$$

Then there exists a constant C which is independent of N, such that

$$\max_{a \le t \le b} \left| f(t) - \sum_{j=-N}^{N} f(\phi_{a,b}^{\mathrm{DE}}(jh)) S(j,h)(\{\phi_{a,b}^{\mathrm{DE}}\}^{-1}(t)) \right| \le C \exp\left\{\frac{-\pi d N}{\log(2dN/\alpha)}\right\}.$$

The common assumption $f \in \mathbf{L}_\alpha(\mathscr{D})$ in Theorem 2.2 and 2.3 implies that the approximated function must tend to 0 as $t \to a$ and $t \to b$ in view of the condition (2.2). In order to handle more general cases, it is convenient to introduce the translation:

$$\mathcal{T}[f](t) = f(t) - \frac{(b-t)f(a) + (t-a)f(b)}{b-a}, \tag{2.5}$$

which maps a function with general boundary values to the one with 0 boundary values. With this notion, let us introduce another function space $\mathbf{M}_\alpha(\mathscr{D})$ in the following definitions as a family of functions such that $\mathcal{T}f$ belongs to $\mathbf{L}_\alpha(\mathscr{D})$.

Definition 2.4. Let $\mathscr{D}$ be a bounded and simply-connected domain. We denote by $\mathbf{HC}(\mathscr{D})$ the family of all functions that are analytic in $\mathscr{D}$ and continuous on $\overline{\mathscr{D}}$. This function space is complete with the norm $\|\cdot\|_{\mathbf{HC}(\mathscr{D})}$ defined by

$$\|f\|_{\mathbf{HC}(\mathscr{D})} = \max_{z\in\overline{\mathscr{D}}} |f(z)|.$$

Definition 2.5. Let α be a constant which satisfies $0 < \alpha \leq 1$ and let $\mathscr{D}$ be a bounded and simply-connected domain such that $(a, b) \subset \mathscr{D}$. The space $\mathbf{M}_\alpha(\mathscr{D})$ consists of all functions f that satisfy the following conditions: (i) $f \in \mathbf{HC}(\mathscr{D})$; (ii) there exists a constant C for all z in $\mathscr{D}$ such that

$$\begin{aligned} |f(z) - f(a)| &\leq C|z - a|^\alpha, \\ |f(b) - f(z)| &\leq C|b - z|^\alpha. \end{aligned}$$

Remark 1. Functions in $\mathbf{L}_\alpha(\mathscr{D})$ or $\mathbf{M}_\alpha(\mathscr{D})$ are analytic in $\mathscr{D}$, but may have a singularity on the boundary of $\mathscr{D}$. In particular, the function spaces contain a function that is not differentiable at the endpoints, like the solution of the equations (1.1).

2.2 Sinc quadrature

The Sinc approximation can be applied to definite integration based on the function approximation described above; it is called the *Sinc quadrature*. When incorporated with the SE transformation, the quadrature rule is designated as the *SE-Sinc quadrature*, and shows exponential convergence as the next theorem states.

Theorem 2.6 (Stenger [17, Theorem 4.2.6])**.** Assume that f satisfies $(fQ) \in \mathbf{L}_{\beta,\gamma}(\phi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d))$ for d with $0 < d < \pi$. Let $\alpha = \min\{\beta, \gamma\}$, N be a positive integer, and h be selected by the formula

$$h = \sqrt{\frac{2\pi d}{\alpha N}}. \tag{2.6}$$

Furthermore, let m and n be positive integers defined by

$$\begin{cases} m = N, \quad n = \lceil \beta N/\gamma \rceil, & (\text{if } \alpha = \beta) \\ n = N, \quad m = \lceil \gamma N/\beta \rceil, & (\text{if } \alpha = \gamma) \end{cases} \tag{2.7}$$

respectively. Then there exists a constant C which is independent of N, such that

$$\left| \int_a^b f(s)\,\mathrm{d}s - h \sum_{j=-m}^{n} f(\phi^{\mathrm{SE}}_{a,b}(jh))\{\phi^{\mathrm{SE}}_{a,b}\}'(jh) \right| \leq C \exp\left(-\sqrt{2\pi d\alpha N}\right).$$

It is also possible to employ the DE transformation in place of the SE transformation, and it further accelerates the convergence as the next theorem shows. We call the quadrature rule the *DE-Sinc quadrature*. The theorem is a straightforward extension of the error analysis in Tanaka et al. [20, Theorem 3.1]

Theorem 2.7. Assume that f satisfies $(fQ) \in \mathbf{L}_{\beta,\gamma}(\phi^{\mathrm{DE}}_{a,b}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let $\alpha = \min\{\beta, \gamma\}$, N be a positive integer, and h be selected by the formula

$$h = \frac{\log(4dN/\alpha)}{N}. \tag{2.8}$$

Furthermore, let m and n be positive integers defined by

$$\begin{cases} m = N, \quad n = N + \lceil \log(\beta/\gamma)/h \rceil, & (\text{if } \alpha = \beta) \\ n = N, \quad m = N + \lceil \log(\gamma/\beta)/h \rceil, & (\text{if } \alpha = \gamma) \end{cases} \tag{2.9}$$

respectively. Then there exists a constant C which is independent of N, such that

$$\left| \int_a^b f(s)\,\mathrm{d}s - h \sum_{j=-m}^{n} f(\phi_{a,b}^{\mathrm{DE}}(jh))\{\phi_{a,b}^{\mathrm{DE}}\}'(jh) \right| \leq C \exp\left\{ \frac{-2\pi d N}{\log(4dN/\alpha)} \right\}.$$

Note that the step sizes h in the formula (2.6) and (2.8) (quadrature case) are different from those in the formula (2.3) and (2.4) (approximation case), if all α's are identical. This might complicate the implementation task. One simple way of working around this is to choose the same step sizes in the quadrature rules as those in the approximations. In this case, Theorem 2.6 and 2.7 should be modified as follows. We omit the proof because it is similar to those given in Stenger [17, Corollary 4.2.7]. For later analysis, the term $(b-a)$ is stated separately from a constant C here.

Corollary 2.8. Assume that f satisfies $(fQ) \in \mathbf{L}_{\beta,\gamma}(\phi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$. Let $\alpha = \min\{\beta, \gamma\}$, N be a positive integer, and h be selected by the formula (2.3). Furthermore, let m and n be positive integers defined by the formula (2.7). Then there exists a constant C which is independent of a, b and N, such that

$$\left| \int_a^b f(s)\,\mathrm{d}s - h \sum_{j=-m}^{n} f(\phi_{a,b}^{\mathrm{SE}}(jh))\{\phi_{a,b}^{\mathrm{SE}}\}'(jh) \right| \leq C(b-a)^{\beta+\gamma-1} \exp\left(-\sqrt{\pi d \alpha N} \right).$$

Corollary 2.9. Assume that f satisfies $(fQ) \in \mathbf{L}_{\beta,\gamma}(\phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let $\alpha = \min\{\beta, \gamma\}$, N be a positive integer, and h be selected by the formula (2.4). Furthermore, let m and n be positive integers defined by the formula (2.9). Then there exists a constant C which is independent of a, b and N, such that

$$\left| \int_a^b f(s)\,\mathrm{d}s - h \sum_{j=-m}^{n} f(\phi_{a,b}^{\mathrm{DE}}(jh))\{\phi_{a,b}^{\mathrm{DE}}\}'(jh) \right| \leq C(b-a)^{\beta+\gamma-1} \exp\left\{ \frac{-2\pi d N}{\log(2dN/\alpha)} \right\}.$$

3 Sinc-collocation methods

In this section, we describe two collocation schemes by means of the Sinc approximation. In the first scheme the SE transformation is utilized, and in the second one the DE transformation is employed.

3.1 SE-Sinc scheme

The solution u is assumed to belong to $\mathbf{M}_{\alpha}(\phi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d))$ here. We need the values of d and α, which depend on the *unknown* solution u. This point will be discussed in Section 4, and at the moment we simply assume the parameter d is somehow known, and $\alpha = p$. Then the translated function $\mathcal{T}u$ belongs to $\mathbf{L}_p(\phi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d))$, with $\mathcal{T}$ defined as in (2.5). According to Theorem 2.2 the function can be accurately approximated as

$$\mathcal{T}[u](t) \approx \sum_{j=-N}^{N} \mathcal{T}[u](\phi^{\mathrm{SE}}_{a,b}(jh))S(j,h)(\{\phi^{\mathrm{SE}}_{a,b}\}^{-1}(t)).$$

Based on this, the original solution u is approximated by the function

$$\mathcal{P}^{\mathrm{SE}}_N[u](t) = u(a)w_a(t) + \sum_{j=-N}^{N} \mathcal{T}[u](\phi^{\mathrm{SE}}_{a,b}(jh))S(j,h)(\{\phi^{\mathrm{SE}}_{a,b}\}^{-1}(t)) + u(b)w_b(t), \tag{3.1}$$

where w_a and w_b are auxiliary basis functions defined by $w_a(t) = (b-t)/(b-a)$, $w_b(t) = (t-a)/(b-a)$. The step size h is given by (2.3) with $\alpha = p$. Note that for a given general continuous function f, $\mathcal{P}^{\mathrm{SE}}_N f$ is nothing but its interpolation by Sinc functions with support abscissas:

$$t^{\mathrm{SE}}_i = \begin{cases} a & (i = -N-1), \\ \phi^{\mathrm{SE}}_{a,b}(ih) & (i = -N, \ldots, N), \\ b & (i = N+1). \end{cases}$$

In order to solve the problem, the unknown coefficients in the right hand side of (3.1) should be determined. For the purpose, let us set an approximate solution u^{SE}_N as

$$u^{\mathrm{SE}}_N(t) = c_{-N-1}w_a(t) + \sum_{j=-N}^{N} c_j S(j,h)(\{\phi^{\mathrm{SE}}_{a,b}\}^{-1}(t)) + c_{N+1}w_b(t),$$

and substitute this into the equation (1.1). Then consider its collocation on $n = 2N+3$ sampling points at $t = t^{\mathrm{SE}}_i$. This results in the following system of linear equations:

$$\lambda u^{\mathrm{SE}}_N(t^{\mathrm{SE}}_i) - \int_a^b |t^{\mathrm{SE}}_i - s|^{p-1} k(t^{\mathrm{SE}}_i, s) u^{\mathrm{SE}}_N(s)\,\mathrm{d}s = g(t^{\mathrm{SE}}_i), \quad i = -N-1, -N, \ldots, N, N+1. \tag{3.2}$$

Next, we proceed to the approximation of integrals in (3.2). Since the SE-Sinc quadrature (Corollary 2.8) does not allow any singularity in the target interval, we split the integral into two at $s = t^{\mathrm{SE}}_i$:

$$\begin{aligned} &\int_a^b |t^{\mathrm{SE}}_i - s|^{p-1} k(t^{\mathrm{SE}}_i, s) u^{\mathrm{SE}}_N(s)\,\mathrm{d}s \\ &= \int_a^{t^{\mathrm{SE}}_i} (t^{\mathrm{SE}}_i - s)^{p-1} k(t^{\mathrm{SE}}_i, s) u^{\mathrm{SE}}_N(s)\,\mathrm{d}s + \int_{t^{\mathrm{SE}}_i}^b (s - t^{\mathrm{SE}}_i)^{p-1} k(t^{\mathrm{SE}}_i, s) u^{\mathrm{SE}}_N(s)\,\mathrm{d}s, \end{aligned} \tag{3.3}$$

so that the singular point only appears as the endpoints. Suppose that $k(t,\cdot) \in \mathbf{HC}(\phi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d))$ uniformly for all $t \in [a, b]$, and the integrand of the first integral satisfies the assumptions in Corollary 2.8 with $\beta = 1$ and $\gamma = p$. Then the first integral can be accurately approximated by $\mathcal{A}^{\mathrm{SE}}_N[u^{\mathrm{SE}}_N](t^{\mathrm{SE}}_i)$. Here the operator $\mathcal{A}^{\mathrm{SE}}_N$ is defined by

$$\begin{aligned} \mathcal{A}^{\mathrm{SE}}_N[f](t) &= h \sum_{m=-M}^{N} (t - \phi^{\mathrm{SE}}_{a,t}(mh))^{p-1} k(t, \phi^{\mathrm{SE}}_{a,t}(mh)) f(\phi^{\mathrm{SE}}_{a,t}(mh)) \{\phi^{\mathrm{SE}}_{a,t}\}'(mh) \\ &= (t-a)^p h \sum_{m=-M}^{N} \frac{k(t, \phi^{\mathrm{SE}}_{a,t}(mh)) f(\phi^{\mathrm{SE}}_{a,t}(mh))}{(1+\mathrm{e}^{-mh})(1+\mathrm{e}^{mh})^p}, \end{aligned}$$

where N and h are the same ones in (3.1), and M is set by $M = \lceil pN \rceil$. Note that the variable transformation is not $\phi^{\mathrm{SE}}_{a,b}(\cdot)$, but $\phi^{\mathrm{SE}}_{a,t}(\cdot)$. The second integral in (3.3) can be handled in a similar manner with the operator $\mathcal{B}^{\mathrm{SE}}_N$ defined by

$$\mathcal{B}^{\mathrm{SE}}_N[f](t) = (b-t)^p h \sum_{m=-N}^{M} \frac{k(t, \phi^{\mathrm{SE}}_{t,b}(mh)) f(\phi^{\mathrm{SE}}_{t,b}(mh))}{(1+\mathrm{e}^{-mh})^p (1+\mathrm{e}^{mh})}.$$

Note the differences from $\mathcal{A}^{\mathrm{SE}}_N$. Then if we introduce $\mathcal{K}^{\mathrm{SE}}_N = \mathcal{A}^{\mathrm{SE}}_N + \mathcal{B}^{\mathrm{SE}}_N$, we reach the final linear system to be solved in matrix-vector form:

$$(E^{\mathrm{SE}}_n - K^{\mathrm{SE}}_n)\boldsymbol{c}_n = \boldsymbol{g}^{\mathrm{SE}}_n, \tag{3.4}$$

where $\boldsymbol{c}_n = [c_{-N-1},\, c_{-N},\, \ldots,\, c_N,\, c_{N+1}]^{\mathrm{T}}$, $\boldsymbol{g}^{\mathrm{SE}}_n = [g(a),\, g(t^{\mathrm{SE}}_{-N}),\, \ldots,\, g(t^{\mathrm{SE}}_N),\, g(b)]^{\mathrm{T}}$, and E^{SE}_n, K^{SE}_n are the $n \times n$ matrices:

$$E^{\mathrm{SE}}_n = \lambda \left[\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \hline w_a(t^{\mathrm{SE}}_{-N}) & 1 & & \mathrm{O} & w_b(t^{\mathrm{SE}}_{-N}) \\ \vdots & & \ddots & & \vdots \\ w_a(t^{\mathrm{SE}}_N) & \mathrm{O} & & 1 & w_b(t^{\mathrm{SE}}_N) \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right],$$

$$K^{\mathrm{SE}}_n = \left[\begin{array}{c|ccc|c} \mathcal{B}^{\mathrm{SE}}_N[w_a](a) & \cdots & \mathcal{B}^{\mathrm{SE}}_N[S(j,h)(\{\phi^{\mathrm{SE}}_{a,b}\}^{-1}(\cdot))](a) & \cdots & \mathcal{B}^{\mathrm{SE}}_N[w_b](a) \\ \hline \mathcal{K}^{\mathrm{SE}}_N[w_a](t^{\mathrm{SE}}_{-N}) & \cdots & \mathcal{K}^{\mathrm{SE}}_N[S(j,h)(\{\phi^{\mathrm{SE}}_{a,b}\}^{-1}(\cdot))](t^{\mathrm{SE}}_{-N}) & \cdots & \mathcal{K}^{\mathrm{SE}}_N[w_b](t^{\mathrm{SE}}_{-N}) \\ \vdots & & \vdots & & \vdots \\ \mathcal{K}^{\mathrm{SE}}_N[w_a](t^{\mathrm{SE}}_N) & \cdots & \mathcal{K}^{\mathrm{SE}}_N[S(j,h)(\{\phi^{\mathrm{SE}}_{a,b}\}^{-1}(\cdot))](t^{\mathrm{SE}}_N) & \cdots & \mathcal{K}^{\mathrm{SE}}_N[w_b](t^{\mathrm{SE}}_N) \\ \hline \mathcal{A}^{\mathrm{SE}}_N[w_a](b) & \cdots & \mathcal{A}^{\mathrm{SE}}_N[S(j,h)(\{\phi^{\mathrm{SE}}_{a,b}\}^{-1}(\cdot))](b) & \cdots & \mathcal{A}^{\mathrm{SE}}_N[w_b](b) \end{array}\right].$$

By obtaining the coefficients vector $\boldsymbol{c}_n$, we get the approximate solution u^{SE}_N.

3.2 DE-Sinc scheme

Now we switch the focus to the DE-Sinc case. Throughout this subsection, the solution u is assumed to belong to $\mathbf{M}_p(\phi^{\mathrm{DE}}_{a,b}(\mathscr{D}_d))$. Since $\mathcal{T}u$ belongs to $\mathbf{L}_p(\phi^{\mathrm{DE}}_{a,b}(\mathscr{D}_d))$, the solution u can be accurately approximated by

$$\mathcal{P}^{\mathrm{DE}}_N[u](t) = u(a)w_a(t) + \sum_{j=-N}^{N} \mathcal{T}[u](\phi^{\mathrm{DE}}_{a,b}(jh)) S(j,h)(\{\phi^{\mathrm{DE}}_{a,b}\}^{-1}(t)) + u(b)w_b(t), \tag{3.5}$$

where the step size h is selected by (2.4) with $\alpha = p$ in view of Theorem 2.3. Accordingly we set the approximate solution u^{DE}_N as

$$u^{\mathrm{DE}}_N(t) = c_{-N-1}w_a(t) + \sum_{j=-N}^{N} c_j S(j,h)(\{\phi^{\mathrm{DE}}_{a,b}\}^{-1}(t)) + c_{N+1}w_b(t),$$

and substitute this u^{DE}_N into the equation (1.1). Then setting n sampling points:

$$t^{\mathrm{DE}}_i = \begin{cases} a & (i = -N-1), \\ \phi^{\mathrm{DE}}_{a,b}(ih) & (i = -N,\, \ldots,\, N), \\ b & (i = N+1), \end{cases}$$

which are the interpolation points of the approximate function (3.5), we obtain the system of linear equations like (3.2). The approximation of the integral in the system is done in like manner as the SE case; the integrals are split at $s = t_i^{\mathrm{DE}}$ like (3.3), and the resulting integrals are approximated by $\mathcal{A}_N^{\mathrm{DE}}[u_N^{\mathrm{DE}}](t_i^{\mathrm{DE}})$ and $\mathcal{B}_N^{\mathrm{DE}}[u_N^{\mathrm{DE}}](t_i^{\mathrm{DE}})$ according to Corollary 2.9 under the assumption that $k(t,\cdot) \in \mathbf{HC}(\phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d))$ uniformly for all $t \in [a,\, b]$. Here $\mathcal{A}_N^{\mathrm{DE}}$ and $\mathcal{B}_N^{\mathrm{DE}}$ are defined by

$$\mathcal{A}_N^{\mathrm{DE}}[f](t) = (t-a)^p h \sum_{m=-M}^{N} \frac{k(t,\phi_{a,t}^{\mathrm{DE}}(mh)) f(\phi_{a,t}^{\mathrm{DE}}(mh)) \pi \cosh(mh)}{(1+\mathrm{e}^{-\pi\sinh(mh)})(1+\mathrm{e}^{\pi\sinh(mh)})^p},$$

$$\mathcal{B}_N^{\mathrm{DE}}[f](t) = (b-t)^p h \sum_{m=-N}^{M} \frac{k(t,\phi_{t,b}^{\mathrm{DE}}(mh)) f(\phi_{t,b}^{\mathrm{DE}}(mh)) \pi \cosh(mh)}{(1+\mathrm{e}^{-\pi\sinh(mh)})^p(1+\mathrm{e}^{\pi\sinh(mh)})},$$

where N and h are the same ones in (3.5), and M is defined by $M = N + \lceil \log(p)/h \rceil$. We also introduce $\mathcal{K}_N^{\mathrm{DE}}$ as $\mathcal{K}_N^{\mathrm{DE}} = \mathcal{A}_N^{\mathrm{DE}} + \mathcal{B}_N^{\mathrm{DE}}$. With these notions, the final system of linear equations is expressed as

$$(E_n^{\mathrm{DE}} - K_n^{\mathrm{DE}})\boldsymbol{c}_n = \boldsymbol{g}_n^{\mathrm{DE}}, \tag{3.6}$$

where $\boldsymbol{g}_n^{\mathrm{DE}} = [g(a),\, g(t_{-N}^{\mathrm{DE}}),\, \ldots,\, g(t_N^{\mathrm{DE}}),\, g(b)]^{\mathrm{T}}$ and E_n^{DE}, K_n^{DE} are the $n \times n$ matrices:

$$E_n^{\mathrm{DE}} = \lambda \left[\begin{array}{c|ccc|c} 1 & 0 & \cdots & 0 & 0 \\ \hline w_a(t_{-N}^{\mathrm{DE}}) & 1 & & \mathrm{O} & w_b(t_{-N}^{\mathrm{DE}}) \\ \vdots & & \ddots & & \vdots \\ w_a(t_N^{\mathrm{DE}}) & \mathrm{O} & & 1 & w_b(t_N^{\mathrm{DE}}) \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array}\right],$$

$$K_n^{\mathrm{DE}} = \left[\begin{array}{c|ccc|c} \mathcal{B}_N^{\mathrm{DE}}[w_a](a) & \cdots & \mathcal{B}_N^{\mathrm{DE}}[S(j,h)(\{\phi_{a,b}^{\mathrm{DE}}\}^{-1}(\cdot))](a) & \cdots & \mathcal{B}_N^{\mathrm{DE}}[w_b](a) \\ \hline \mathcal{K}_N^{\mathrm{DE}}[w_a](t_{-N}^{\mathrm{DE}}) & \cdots & \mathcal{K}_N^{\mathrm{DE}}[S(j,h)(\{\phi_{a,b}^{\mathrm{DE}}\}^{-1}(\cdot))](t_{-N}^{\mathrm{DE}}) & \cdots & \mathcal{K}_N^{\mathrm{DE}}[w_b](t_{-N}^{\mathrm{DE}}) \\ \vdots & & \vdots & & \vdots \\ \mathcal{K}_N^{\mathrm{DE}}[w_a](t_N^{\mathrm{DE}}) & \cdots & \mathcal{K}_N^{\mathrm{DE}}[S(j,h)(\{\phi_{a,b}^{\mathrm{DE}}\}^{-1}(\cdot))](t_N^{\mathrm{DE}}) & \cdots & \mathcal{K}_N^{\mathrm{DE}}[w_b](t_N^{\mathrm{DE}}) \\ \hline \mathcal{A}_N^{\mathrm{DE}}[w_a](b) & \cdots & \mathcal{A}_N^{\mathrm{DE}}[S(j,h)(\{\phi_{a,b}^{\mathrm{DE}}\}^{-1}(\cdot))](b) & \cdots & \mathcal{A}_N^{\mathrm{DE}}[w_b](b) \end{array}\right].$$

By obtaining the coefficients vector $\boldsymbol{c}_n$, we get the approximate solution u_N^{DE}.

4 How to determine the parameters d and α

As remarked in the previous section, it is mandatory to choose parameters d and α for setting the step size h by (2.3) or (2.4). These parameters, however, should depend on the *unknown* solution u, and it is hard to know *before* solving the problem; in fact, the parameter d indicates the size of the holomorphic domain of u, and α the order of Hölder continuous of u (recall that in the previous section it is assumed that $u \in \mathbf{M}_\alpha(\phi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d))$ or $u \in \mathbf{M}_\alpha(\phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d))$, and $\alpha = p$). Although the choice substantially affects the performance of the Sinc methods (see Section 2), this point has not been fully answered in Riley [13]. In the present paper, we give an answer to the issue for the Fredholm problem.

Let us introduce the integral operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{K} : \mathbf{HC}(\mathscr{D}) \to \mathbf{HC}(\mathscr{D})$ defined by

$$\mathcal{A}[f](z) = \int_a^z (z-w)^{p-1} k(z,w) f(w)\,\mathrm{d}w,$$
$$\mathcal{B}[f](z) = \int_z^b (w-z)^{p-1} k(z,w) f(w)\,\mathrm{d}w,$$

and $\mathcal{K} = \mathcal{A} + \mathcal{B}$, where $k \in \mathbf{HC}(\mathscr{D} \times \mathscr{D})$. With these notions, the equation (1.1) can be symbolically expressed as $(\lambda I - \mathcal{K})u = g$. Then the next theorem states that the parameters d and α can be determined by investigating the known functions k and g for $\mathscr{D} = \phi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d)$ or $\mathscr{D} = \phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d)$.

Theorem 4.1. Let $k(z,\cdot) \in \mathbf{HC}(\mathscr{D})$ for all $z \in \overline{\mathscr{D}}$, $k(\cdot,w) \in \mathbf{M}_p(\mathscr{D})$ for all $w \in \overline{\mathscr{D}}$, and let also $g \in \mathbf{M}_p(\mathscr{D})$. Furthermore, assume that the homogeneous equation $(\lambda I - \mathcal{K})f = 0$ has only the trivial solution $f \equiv 0$. Then the equation (1.1) has a unique solution $u \in \mathbf{M}_p(\mathscr{D})$.

Below we prove Theorem 4.1. Note that, in view of Definition 2.5, $u \in \mathbf{M}_p(\mathscr{D})$ if and only if $u \in \mathbf{HC}(\mathscr{D})$ and u is p-Hölder continuous at the endpoints. The property $u \in \mathbf{HC}(\mathscr{D})$ can be checked by the next theorem.

Theorem 4.2. Let $k \in \mathbf{HC}(\mathscr{D} \times \mathscr{D})$ and suppose that the homogeneous equation $(\lambda I - \mathcal{K})f = 0$ has only the trivial solution $f \equiv 0$. Then the operator $(\lambda I - \mathcal{K}) : \mathbf{HC}(\mathscr{D}) \to \mathbf{HC}(\mathscr{D})$ has a bounded inverse, $(\lambda I - \mathcal{K})^{-1} : \mathbf{HC}(\mathscr{D}) \to \mathbf{HC}(\mathscr{D})$. Furthermore, if $g \in \mathbf{HC}(\mathscr{D})$, then the equation (1.1) has a unique solution $u \in \mathbf{HC}(\mathscr{D})$.

This theorem can be proved using the Fredholm alternative theorem, where the compactness of the integral operators is assured by the next lemma.

Lemma 4.3. If $k \in \mathbf{HC}(\mathscr{D} \times \mathscr{D})$, the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{K} : \mathbf{HC}(\mathscr{D}) \to \mathbf{HC}(\mathscr{D})$ are compact.

Proof. It is easily seen that the operators $\mathcal{A}$ and $\mathcal{B}$ map the set $\{f : \|f\|_{\mathbf{HC}(\mathscr{D})} \leq 1\}$ onto an uniformly bounded and equicontinuous set. Therefore $\mathcal{A}$ and $\mathcal{B}$ are compact operators by the Arzelà–Ascoli theorem for complex functions (cf. Rudin [14, Theorem 11.28]). Then $\mathcal{K}$ is also a compact operator since $\mathcal{K} = \mathcal{A} + \mathcal{B}$. ∎

The p-Hölder continuity of u immediately follows from the following lemma, since $u = (g + \mathcal{K}u)/\lambda$ where $g \in \mathbf{M}_p(\mathscr{D})$, and $u \in \mathbf{HC}(\mathscr{D})$. We leave the proof to Appendix A.

Lemma 4.4. Let $k(z,\cdot) \in \mathbf{HC}(\mathscr{D})$ for all $z \in \overline{\mathscr{D}}$, $k(\cdot,w) \in \mathbf{M}_p(\mathscr{D})$ for all $w \in \overline{\mathscr{D}}$, and let also $f \in \mathbf{HC}(\mathscr{D})$. Then $\mathcal{K}f \in \mathbf{M}_p(\mathscr{D})$.

Combining Lemma 4.4 and Theorem 4.2, we establish Theorem 4.1.

5 Error analysis

In this section we give an error analysis of the SE- and DE-Sinc schemes derived in Section 3. We first consider the SE case. Based on Corollary 2.8, we can deduce the following lemma which is used in the subsequent error analysis.

Lemma 5.1. Assume that there exists a constant d with $0 < d < \pi$ such that $k(t, \cdot) \in \mathbf{HC}(\phi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d))$ uniformly for all t in $[a,\, b]$, and $f \in \mathbf{HC}(\phi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d))$. Furthermore, assume that there exists a constant K for all t in $[a,\, b]$ such that $\|k(t,\cdot)\|_{\mathbf{HC}(\phi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d))} \leq K$. Then there exists a constant C which is independent of a, b, t and N, such that

$$
\begin{aligned}
|\mathcal{A}[f](t) - \mathcal{A}^{\mathrm{SE}}_N[f](t)| &\leq C(t-a)^p \exp\left(-\sqrt{\pi d p N}\right),\\
|\mathcal{B}[f](t) - \mathcal{B}^{\mathrm{SE}}_N[f](t)| &\leq C(b-t)^p \exp\left(-\sqrt{\pi d p N}\right).
\end{aligned}
$$

The next lemma estimates the error in the solution vector $\boldsymbol{c}_n$.

Lemma 5.2. Suppose that the assumptions in Theorem 4.1 are fulfilled with $\mathscr{D} = \phi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d)$ for $0 < d < \pi$. Let $\boldsymbol{c}_n$ be the solution of the linear equations (3.4), and $\boldsymbol{v}^{\mathrm{SE}}_n$ be the coefficient vector of the $\mathcal{P}^{\mathrm{SE}}_N u$ in (3.1), i.e.,

$$
\boldsymbol{v}^{\mathrm{SE}}_n = [u(a),\, \mathcal{T}[u](t^{\mathrm{SE}}_{-N}),\, \ldots,\, \mathcal{T}[u](t^{\mathrm{SE}}_{N}),\, u(b)]^{\mathrm{T}}. \tag{5.1}
$$

Furthermore, let us define $\mu^{\mathrm{SE}}_N = \|(E^{\mathrm{SE}}_n - K^{\mathrm{SE}}_n)^{-1}\|_\infty$. Then there exists a constant C independent of N such that

$$
\|\boldsymbol{v}^{\mathrm{SE}}_n - \boldsymbol{c}_n\|_\infty \leq C\mu^{\mathrm{SE}}_N\sqrt{N}\exp\left(-\sqrt{\pi d p N}\right).
$$

Proof. Considering $\mathcal{P}^{\mathrm{SE}}_N u - u^{\mathrm{SE}}_N$ on the sampling points $t = t^{\mathrm{SE}}_i$, we have

$$
\begin{aligned}
\mathcal{P}^{\mathrm{SE}}_N[u](t^{\mathrm{SE}}_i) - u^{\mathrm{SE}}_N(t^{\mathrm{SE}}_i) &= u(t^{\mathrm{SE}}_i) - u^{\mathrm{SE}}_N(t^{\mathrm{SE}}_i)\\
&= \{g(t^{\mathrm{SE}}_i) + \mathcal{K}[u](t^{\mathrm{SE}}_i)\} - \{g(t^{\mathrm{SE}}_i) + \mathcal{K}^{\mathrm{SE}}_N[u^{\mathrm{SE}}_N](t^{\mathrm{SE}}_i)\}\\
&= \mathcal{K}[u](t^{\mathrm{SE}}_i) - \mathcal{K}^{\mathrm{SE}}_N[\mathcal{P}^{\mathrm{SE}}_N u](t^{\mathrm{SE}}_i) + \mathcal{K}^{\mathrm{SE}}_N[\mathcal{P}^{\mathrm{SE}}_N u - u^{\mathrm{SE}}_N](t^{\mathrm{SE}}_i)
\end{aligned}
$$

for $i = -N-1,\, -N,\, \ldots,\, N,\, N+1$. In matrix-vector form,

$$
(E^{\mathrm{SE}}_n - K^{\mathrm{SE}}_n)(\boldsymbol{v}^{\mathrm{SE}}_n - \boldsymbol{c}_n) = (\boldsymbol{q}_n - \boldsymbol{q}^{\mathrm{SE}}_n),
$$

where

$$
\begin{aligned}
\boldsymbol{q}_n &= [\mathcal{K}[u](a),\, \mathcal{K}[u](t^{\mathrm{SE}}_{-N}),\, \ldots,\, \mathcal{K}[u](t^{\mathrm{SE}}_N),\, \mathcal{K}[u](b)]^{\mathrm{T}},\\
\boldsymbol{q}^{\mathrm{SE}}_n &= [\mathcal{K}^{\mathrm{SE}}_N[\mathcal{P}^{\mathrm{SE}}_N u](a),\, \mathcal{K}^{\mathrm{SE}}_N[\mathcal{P}^{\mathrm{SE}}_N u](t^{\mathrm{SE}}_{-N}),\, \ldots,\, \mathcal{K}^{\mathrm{SE}}_N[\mathcal{P}^{\mathrm{SE}}_N u](t^{\mathrm{SE}}_N),\, \mathcal{K}^{\mathrm{SE}}_N[\mathcal{P}^{\mathrm{SE}}_N u](b)]^{\mathrm{T}}.
\end{aligned}
$$

Then we have the bound of $\|\boldsymbol{v}^{\mathrm{SE}}_n - \boldsymbol{c}_n\|_\infty$ as

$$
\begin{aligned}
\|\boldsymbol{v}^{\mathrm{SE}}_n - \boldsymbol{c}_n\|_\infty &\leq \mu^{\mathrm{SE}}_N\|\boldsymbol{q}_n - \boldsymbol{q}^{\mathrm{SE}}_n\|_\infty\\
&= \mu^{\mathrm{SE}}_N \max_{i=-N-1,\ldots,N+1} |\mathcal{K}[u](t^{\mathrm{SE}}_i) - \mathcal{K}^{\mathrm{SE}}_N[\mathcal{P}^{\mathrm{SE}}_N u](t^{\mathrm{SE}}_i)|\\
&\leq \mu^{\mathrm{SE}}_N \max_{t\in[a,\,b]} |\mathcal{K}[u](t) - \mathcal{K}^{\mathrm{SE}}_N[\mathcal{P}^{\mathrm{SE}}_N u](t)|\\
&\leq \mu^{\mathrm{SE}}_N \left\{\max_{t\in[a,\,b]} |\mathcal{K}[u](t) - \mathcal{K}^{\mathrm{SE}}_N[u](t)| + \max_{t\in[a,\,b]} |\mathcal{K}^{\mathrm{SE}}_N[u - \mathcal{P}^{\mathrm{SE}}_N u](t)|\right\}.
\end{aligned}
$$

For the second term, we have

$$
\max_{t\in[a,\,b]} |\mathcal{K}^{\mathrm{SE}}_N[u - \mathcal{P}^{\mathrm{SE}}_N u](t)| \leq \max_{t\in[a,\,b]} |u(t) - \mathcal{P}^{\mathrm{SE}}_N u(t)| \cdot \max_{t\in[a,\,b]} |\mathcal{K}^{\mathrm{SE}}_N[1](t)|.
$$

According to Theorem 4.1, the solution u belongs to $\mathbf{M}_p(\mathscr{D})$, therefore we can apply Theorem 2.2 to obtain

$$\max_{t\in[a,\,b]} |u(t) - \mathcal{P}_N^{\mathrm{SE}} u(t)| \leq C_1 \sqrt{N} \exp\left(-\sqrt{\pi d p N}\right) \tag{5.2}$$

for a constant C_1. Moreover, since $\mathcal{K}_N^{\mathrm{SE}}[1](t) \to \mathcal{K}[1](t)$ as $N \to \infty$, there exists a constant C_2, such that

$$\sup_{N\in\mathbb{N}} \left\{ \max_{t\in[a,\,b]} |\mathcal{K}_N^{\mathrm{SE}}[1](t)| \right\} \leq C_2.$$

Next we consider the first term. Since $u \in \mathbf{HC}(\mathscr{D})$ and $k(t,\cdot) \in \mathbf{HC}(\mathscr{D})$ for all $t \in [a,\,b]$, we can apply Lemma 5.1 to obtain

$$\begin{aligned} |\mathcal{K}[u](t) - \mathcal{K}_N^{\mathrm{SE}}[u](t)| &\leq |\mathcal{A}[u](t) - \mathcal{A}_N^{\mathrm{SE}}[u](t)| + |\mathcal{B}[u](t) - \mathcal{B}_N^{\mathrm{SE}}[u](t)| \\ &\leq C_3 \left\{(t-a)^p + (b-t)^p\right\} \exp\left(-\sqrt{\pi d p N}\right) \\ &\leq 2C_3 (b-a)^p \exp\left(-\sqrt{\pi d p N}\right) \end{aligned}$$

for a constant C_3. Thus this lemma follows. ■

Next we bound the error of the approximate solution u_N^{SE}. For the purpose, the following lemma is required.

Lemma 5.3 (Stenger [17, p. 142])**.** Let $h > 0$, $j \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

$$\sup_{x\in\mathbb{R}} \sum_{j=-N}^{N} |S(j,h)(x)| \leq \frac{2}{\pi}(3 + \log N).$$

The next theorem states the error in u_N^{SE}.

Theorem 5.4. Suppose that assumptions of Theorem 4.1 are fulfilled with $\mathscr{D} = \phi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d)$ for $0 < d < \pi$. Furthermore, let us define $\mu_N^{\mathrm{SE}} = \|(E_n^{\mathrm{SE}} - K_n^{\mathrm{SE}})^{-1}\|_\infty$. Then there exists a constant C independent of N such that

$$\max_{t\in[a,\,b]} |u(t) - u_N^{\mathrm{SE}}(t)| \leq C \mu_N^{\mathrm{SE}} \log(N+1) \sqrt{N} \exp\left(-\sqrt{\pi d p N}\right).$$

Proof. By the triangle inequality,

$$\max_{t\in[a,\,b]} |u(t) - u_N^{\mathrm{SE}}(t)| \leq \max_{t\in[a,\,b]} |u(t) - \mathcal{P}_N^{\mathrm{SE}}[u](t)| + \max_{t\in[a,\,b]} |\mathcal{P}_N^{\mathrm{SE}}[u](t) - u_N^{\mathrm{SE}}(t)|,$$

and the first term is bounded by (5.2). Next we bound the second term. Using $\boldsymbol{v}_n^{\mathrm{SE}}$ defined by (5.1), we have

$$\begin{aligned} &|\mathcal{P}_N^{\mathrm{SE}}[u](t) - u_N^{\mathrm{SE}}(t)| \\ &\leq |(u(a) - c_{-N-1}) w_a(t)| + \sum_{j=-N}^{N} \left|(\mathcal{T}[u](t_j^{\mathrm{SE}}) - c_j) S(j,h)(\{\phi_{a,b}^{\mathrm{SE}}\}^{-1}(t))\right| + |(u(b) - c_{N+1}) w_b(t)| \\ &\leq \|\boldsymbol{v}_n^{\mathrm{SE}} - \boldsymbol{c}_n\|_\infty \left\{ |w_a(t)| + \sum_{j=-N}^{N} |S(j,h)(\{\phi_{a,b}^{\mathrm{SE}}\}^{-1}(t))| + |w_b(t)| \right\}. \end{aligned}$$

We have already estimated $\|\boldsymbol{v}_n^{\mathrm{SE}} - \boldsymbol{c}_n\|_\infty$ in Lemma 5.2. Since $|w_a(t)| \leq 1$, $|w_b(t)| \leq 1$ and in light of Lemma 5.3, we get

$$\left\{ |w_a(t)| + \sum_{j=-N}^{N} |S(j,h)(\{\phi_{a,b}^{\mathrm{SE}}\}^{-1}(t))| + |w_b(t)| \right\} \leq C_3 \log(N+1).$$

for a constant C_3. This completes the proof. ■

Next we consider the DE case. Since the proof goes almost in the same way as in the SE case, we only show the result here.

Lemma 5.5. Assume that there exists a constant d with $0 < d < \pi/2$ such that $k(t,\cdot) \in \mathbf{HC}(\phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d))$ uniformly for all t in $[a,\, b]$, and $f \in \mathbf{HC}(\phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d))$. Furthermore, assume that there exists a constant K for all t in $[a,\, b]$ such that $\|k(t,\cdot)\|_{\mathbf{HC}(\phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d))} \leq K$. Then there exists a constant C which is independent of a, b, t and N, such that

$$|\mathcal{A}[f](t) - \mathcal{A}_N^{\mathrm{DE}}[f](t)| \leq C(t-a)^p \exp\left\{ \frac{-2\pi d N}{\log(2dN/p)} \right\},$$
$$|\mathcal{B}[f](t) - \mathcal{B}_N^{\mathrm{DE}}[f](t)| \leq C(b-t)^p \exp\left\{ \frac{-2\pi d N}{\log(2dN/p)} \right\}.$$

Lemma 5.6. Suppose that the assumptions of Theorem 4.1 are fulfilled with $\mathscr{D} = \phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d)$ for $0 < d < \pi/2$. Let $\boldsymbol{c}_n$ be the solution of the linear equations (3.6) and $\boldsymbol{v}_n^{\mathrm{DE}}$ be the coefficient vector of the $\mathcal{P}_N^{\mathrm{DE}} u$ in (3.5), i.e.,

$$\boldsymbol{v}_n^{\mathrm{DE}} = [u(a),\, \mathcal{T}[u](t_{-N}^{\mathrm{DE}}),\, \ldots,\, \mathcal{T}[u](t_N^{\mathrm{DE}}),\, u(b)]^{\mathrm{T}}.$$

Furthermore, let us define $\mu_N^{\mathrm{DE}} = \|(E_n^{\mathrm{DE}} - K_n^{\mathrm{DE}})^{-1}\|_\infty$. Then there exists a constant C independent of N such that

$$\|\boldsymbol{v}_n^{\mathrm{DE}} - \boldsymbol{c}_n\|_\infty \leq C \mu_N^{\mathrm{DE}} \exp\left\{ \frac{-\pi d N}{\log(2dN/p)} \right\}.$$

Theorem 5.7. Suppose that the assumptions of Theorem 4.1 are fulfilled with $\mathscr{D} = \phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d)$ for $0 < d < \pi/2$. Furthermore, let us define $\mu_N^{\mathrm{DE}} = \|(E_n^{\mathrm{DE}} - K_n^{\mathrm{DE}})^{-1}\|_\infty$. Then there exists a constant C independent of N such that

$$\max_{t \in [a,\, b]} |u(t) - u_N^{\mathrm{DE}}(t)| \leq C \mu_N^{\mathrm{DE}} \log(N+1) \exp\left\{ \frac{-\pi d N}{\log(2dN/p)} \right\}.$$

Remark 2. It is not difficult to show theoretically that the infinity norms of the matrices $(E_n^{\mathrm{SE}} - K_n^{\mathrm{SE}})$ and $(E_n^{\mathrm{DE}} - K_n^{\mathrm{DE}})$ grow relatively slowly like $\mathrm{O}(\log N)$, if we note Lemma 5.3. On the other hand, the norms of their inverse matrices, μ_N^{SE} and μ_N^{DE}, are not easy to estimate; in fact, this point has not been fully investigated in Riley [13], either. In the present paper, we investigate them numerically in the next section.

6 Numerical examples

In this section we show numerical results of the SE- and DE-Sinc schemes derived in Section 3. The computation is done using C++ with double-precision floating-point arithmetic. The interval (a, b) is set to $(0, 1)$ throughout this section. In the graphs, $E_{\max}$ is the maximum absolute error at 1001 equally-spaced points, defined by

$$E_{\max} = \max_{t=0,\,0.001,\,\ldots,\,0.999,1} |u(t) - u_N(t)|$$

where u_N is the approximate solution u_N^{SE} or u_N^{DE}. In all of the following examples, the functions k and g satisfy the conditions assumed in Theorem 4.1 with $\mathscr{D} = \phi_{a,b}^{\mathrm{SE}}(\mathscr{D}_{\pi_{\mathrm{m}}})$ or $\mathscr{D} = \phi_{a,b}^{\mathrm{DE}}(\mathscr{D}_{\pi_{\mathrm{m}}/2})$ (π_{m} is an arbitrary positive number less than π). Thus we set $d = 3.14$ in the SE-Sinc scheme, and $d = 1.57$ in the DE-Sinc scheme.

We consider the following four problems.

Example 1. Consider the problem [5, 16]

$$\frac{3\sqrt{2}}{4}u(t) - \int_0^1 \frac{u(s)}{\sqrt{|t-s|}}\,\mathrm{d}s = 3\{t(1-t)\}^{3/4} - \frac{3}{8}\pi\{1 + 4t(1-t)\}, \quad 0 \leq t \leq 1.$$

The solution is $u(t) = 2\sqrt{2}\{t(1-t)\}^{3/4}$, and in this case $p = 1/2$.

Example 2. Consider the problem [11]

$$\begin{aligned} &u(t) - \frac{1}{10}\int_0^1 \frac{u(s)}{\sqrt[3]{|t-s|}}\,\mathrm{d}s \\ &= t^2(1-t)^2 - \frac{27}{30800}\left[t^{8/3}(54t^2 - 126t + 77) + (1-t)^{8/3}(54t^2 + 18t + 5)\right], \quad 0 \leq t \leq 1. \end{aligned}$$

The solution is $u(t) = t^2(1-t)^2$, and in this case $p = 2/3$.

Example 3. Consider the problem [4, 5]

$$u(t) - \int_0^1 \frac{u(s)}{\sqrt{|t-s|}}\,\mathrm{d}s = t - 2\sqrt{1-t} - \frac{4}{3}t^{3/2} + \frac{4}{3}(1-t)^{3/2}, \quad 0 \leq t \leq 1.$$

The solution is $u(t) = t$, and in this case $p = 1/2$.

Example 4. Consider the problem

$$u(t) - \frac{1}{4}\int_0^1 \frac{\sqrt{ts}}{\sqrt{|t-s|}}u(s)\,\mathrm{d}s = \frac{1}{5}\sqrt{t}(1-t)\{15 - \sqrt{1-t}(1+4t)\} + \frac{1}{5}(4t-5)t^2, \quad 0 \leq t \leq 1.$$

The solution is $u(t) = 3\sqrt{t}(1-t)$, and in this case $p = 1/2$.

Figure 1–4 show the numerical results corresponding to Example 1–4. In each set of figures, the upper figure shows the decay of errors, and the bottom shows the computed values of the norm of the inverse matrices μ_N^{SE} and μ_N^{DE}.

We can observe the same convergence profiles (in top figures) in Example 1, 2 and 4; in all of these examples, the convergence rate is $\mathrm{O}(\exp(-c_1\sqrt{N}))$ in the SE-Sinc scheme (dashed-line

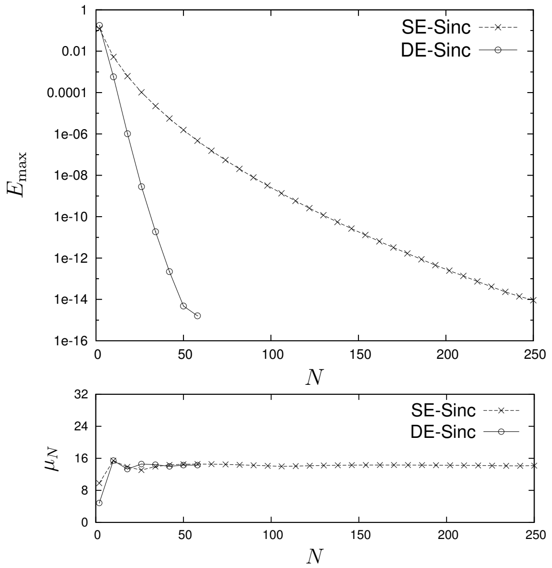

Figure 1. Results in Example 1: (top) convergence of errors; (bottom) profile of μ_N^{SE} and μ_N^{DE}.

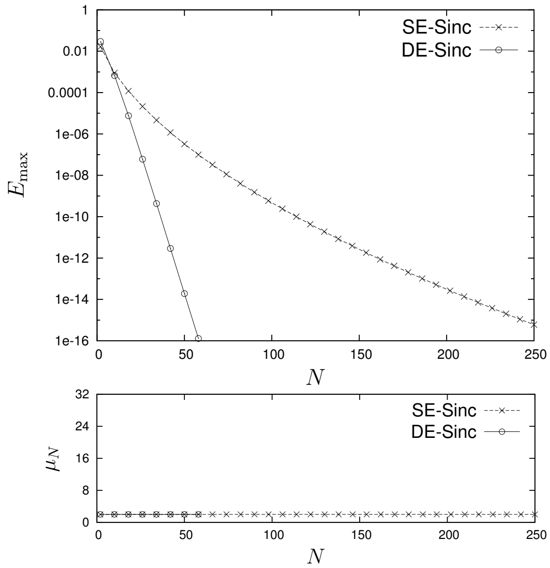

Figure 2. Results of Example 2: (top) convergence of errors; (bottom) profile of μ_N^{SE} and μ_N^{DE}.

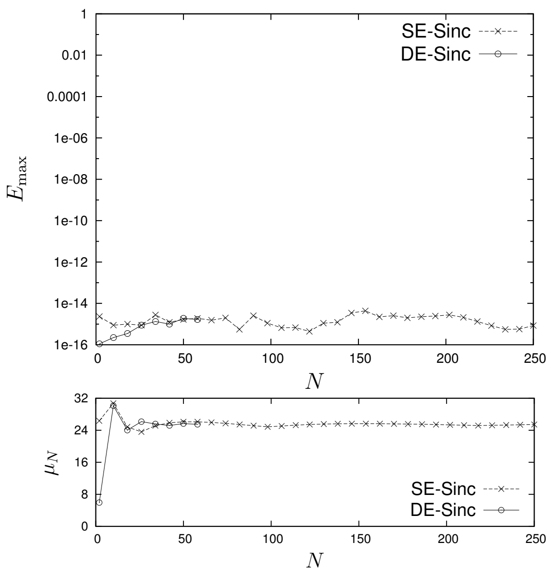

Figure 3. Results of Example 3: (top) convergence of errors; (bottom) profile of μ_N^{SE} and μ_N^{DE}.

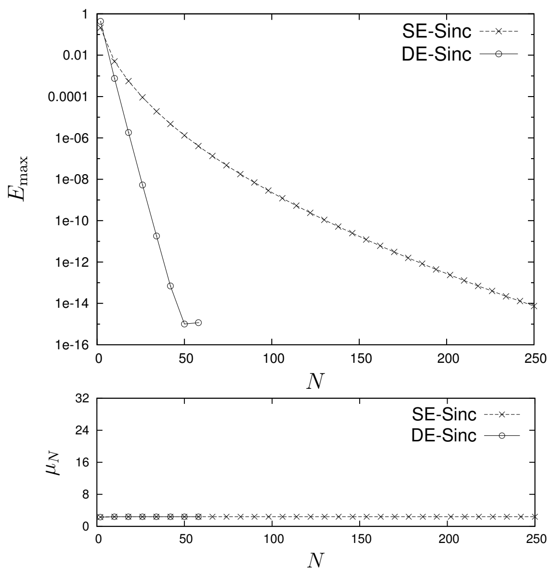

Figure 4. Results of Example 4: (top) convergence of errors; (bottom) profile of μ_N^{SE} and μ_N^{DE}.

with $\times$ points), and $\mathrm{O}(\exp(-c_2 N/\log N))$ in the DE-Sinc scheme (solid-line with $\circ$ points). In Example 3 the profile is different; there the error $E_{\max}$ is at the machine accuracy level for all N in both the SE-Sinc and DE-Sinc schemes. This should be attributed to the fact that the solution in this case is a linear function, and the formula (2.1) is then used to approximate the trivial function $\mathcal{T}u = 0$.

From the bottom figures which show the dependence of μ_N^{SE} and μ_N^{DE} on N, we can conclude, at least numerically, they are bounded and thus the system (3.4) and (3.6) are not ill-conditioned. In particular, in Example 2 and 4, they remain quite low.

7 Concluding remarks

In this paper two new numerical methods have been developed by extending Riley's method for weakly singular Volterra integral equations; more specifically, the methods are constructed based on the Sinc methods and either of the SE or DE transformation. A theoretical analysis has been also given regarding the integral equation for tuning the parameters d and α which strongly affect the actual convergence profile. By the numerical experiments it has been shown that both schemes are extremely accurate, and achieve exponential convergence with respect to N, which is roughly speaking relative to the number of the basis functions (or equivalently, of the sampling points). It has been also shown numerically that the systems of the linear equations generated in these schemes are highly well-conditioned.

Future works include the followings: first, a similar parameter tuning for d and α can be done also in the Volterra case (we like to stress again that this point has not been considered in Riley [13]); second, theory on the sizes of μ_N^{SE} and μ_N^{DE} (the norms of the inverse matrices) should be given. We are now working on these issues, and the results will be reported somewhere else soon.

A Proof of the Hölder continuity

Lemma 4.4 is proved. The SE and DE transformation cases are considered separately.

First, the SE case is considered. We commence by preparing the following lemma.

Lemma A.1. Let d be a constant with $0 < d < \pi$ and let us define a function ϕ_1 by

$$\phi_1(x) = \frac{1}{2}\tanh\left(\frac{x}{2}\right) + \frac{1}{2}.$$

Then there exists a constant c_d depending only on d, such that for all $x \in \mathbb{R}$ and $y \in [-d,\, d]$

$$|\{\phi_{a,b}^{\mathrm{SE}}\}'(x + \mathrm{i}\, y)| \leq (b-a) c_d \phi_1'(x), \tag{A.1}$$

$$|\phi_{0,1}^{\mathrm{SE}}(x + \mathrm{i}\, y)| \geq \phi_1(x) \tag{A.2}$$

hold. Furthermore, if $t \leq x$,

$$|\phi_{a,b}^{\mathrm{SE}}(x + \mathrm{i}\, y) - \phi_{a,b}^{\mathrm{SE}}(t + \mathrm{i}\, y)| \geq (b-a)\{\phi_1(x) - \phi_1(t)\}. \tag{A.3}$$

Proof. The proof of the first two inequalities are straightforward: using

$$|\{\phi_{a,b}^{\mathrm{SE}}\}'(x + \mathrm{i}\, y)| = \frac{(b-a)/4}{\cosh^2(x/2) - \sin^2(y/2)} \leq \frac{(b-a)/4}{\cosh^2(x/2)\{1 - \sin^2(y/2)\}} = \frac{b-a}{\cos^2(y/2)}\phi_1'(x),$$

we obtain the inequality (A.1) with $c_d = 1/\cos^2(d/2)$. And for (A.2),

$$|\phi_{0,1}^{\mathrm{SE}}(x + \mathrm{i}\, y)| = \frac{1}{\sqrt{1 + 2\,\mathrm{e}^{-x}\cos y + \mathrm{e}^{-2x}}} \geq \frac{1}{\sqrt{1 + 2\,\mathrm{e}^{-x} + \mathrm{e}^{-2x}}} = \phi_1(x).$$

Next we prove the inequality (A.3). Since the inequality

$$|\cosh(r + \mathrm{i}\, y)| = \sqrt{\cosh^2(r) - \sin^2(y)} \leq \cosh(r)$$

holds for all $r \in \mathbb{R}$, we have

$$\begin{aligned} |\phi_{a,b}^{\mathrm{SE}}(x + \mathrm{i}\, y) - \phi_{a,b}^{\mathrm{SE}}(t + \mathrm{i}\, y)| &= (b-a)\left|\frac{\cosh(x)}{\cosh(x + \mathrm{i}\, y)}\frac{\cosh(t)}{\cosh(t + \mathrm{i}\, y)}\right|\{\phi_1(x) - \phi_1(t)\} \\ &\geq (b-a)\{\phi_1(x) - \phi_1(t)\}. \end{aligned}$$

■

Then we can prove Lemma 4.4 in the case of the SE transformation.

Lemma A.2. Let d be a constant with $0 < d < \pi$ and let $\mathscr{D} = \phi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d)$. Suppose that $k(z, \cdot) \in \mathbf{HC}(\mathscr{D})$ for all $z \in \overline{\mathscr{D}}$, $k(\cdot, w) \in \mathbf{M}_p(\mathscr{D})$ for all $w \in \overline{\mathscr{D}}$, and $f \in \mathbf{HC}(\mathscr{D})$. Then $\mathcal{K}f \in \mathbf{M}_p(\mathscr{D})$.

Proof. It is sufficient to show the p-Hölder continuity of $\mathcal{K}f$, since $\mathcal{K}f \in \mathbf{HC}(\mathscr{D})$. We give the proof only for the operator $\mathcal{A}$ because the proof for $\mathcal{B}$ goes in a similar manner, and then the result for $\mathcal{K} = \mathcal{A} + \mathcal{B}$ is straightforward. First we prove the p-Hölder continuity at the point a. Set $x = \mathrm{Re}[\{\phi_{a,b}^{\mathrm{SE}}\}^{-1}(z)]$ and $y = \mathrm{Im}[\{\phi_{a,b}^{\mathrm{SE}}\}^{-1}(z)]$. By a variable transformation $w = \phi_{a,b}^{\mathrm{SE}}(t + \mathrm{i}\, y)$,

$$\begin{aligned} \mathcal{A}[f](z) - \mathcal{A}[f](a) &= \int_a^z \frac{k(z,w)f(w)}{(z-w)^{1-p}}\,\mathrm{d}w - 0 \\ &= \int_{-\infty}^x \frac{k(\phi_{a,b}^{\mathrm{SE}}(x + \mathrm{i}\, y), \phi_{a,b}^{\mathrm{SE}}(t + \mathrm{i}\, y)) f(\phi_{a,b}^{\mathrm{SE}}(t + \mathrm{i}\, y))\{\phi_{a,b}^{\mathrm{SE}}\}'(t + \mathrm{i}\, y)}{(\phi_{a,b}^{\mathrm{SE}}(x + \mathrm{i}\, y) - \phi_{a,b}^{\mathrm{SE}}(t + \mathrm{i}\, y))^{1-p}}\,\mathrm{d}t. \end{aligned}$$

Using the inequalities (A.1) and (A.3) in Lemma A.1, we obtain

$$\begin{aligned} |\mathcal{A}[f](z) - \mathcal{A}[f](a)| &\leq \int_{-\infty}^x \frac{M_k \|f\|_{\mathbf{HC}(\mathscr{D})}(b-a)c_d\phi_1'(t)}{[(b-a)\{\phi_1(x) - \phi_1(t)\}]^{1-p}}\,\mathrm{d}t \\ &= \frac{M_k\|f\|_{\mathbf{HC}(\mathscr{D})}c_d}{p}\{(b-a)\phi_1(x)\}^p. \end{aligned}$$

Here $M_k = \max_{z,w\in\overline{\mathscr{D}}}|k(z,w)|$. Furthermore, using the inequality (A.2), we have

$$(b-a)\phi_1(x) \leq |(b-a)\phi_{0,1}^{\mathrm{SE}}(x + \mathrm{i}\, y)| = |\phi_{a,b}^{\mathrm{SE}}(x + \mathrm{i}\, y) - a| = |z - a|.$$

Thus it follows that

$$|\mathcal{A}[f](z) - \mathcal{A}[f](a)| \leq \frac{M_k\|f\|_{\mathbf{HC}(\mathscr{D})}c_d}{p}|z - a|^p. \tag{A.4}$$

Now we consider the p-Hölder continuity at the point b. We split the path of the integral as

$$\mathcal{A}[f](z) = \int_a^b (z-s)^{p-1}k(z,s)f(s)\,\mathrm{d}s + \int_b^z (z-w)^{p-1}k(z,w)f(w)\,\mathrm{d}w.$$

Subtracting $\mathcal{A}[f](b)$ from $\mathcal{A}[f](z)$ gives

$$\begin{aligned}\mathcal{A}[f](b)-\mathcal{A}[f](z) &= \int_a^b (b-s)^{p-1}\{k(b,s)-k(z,s)\}f(s)\,\mathrm{d}s \\ &\quad + \int_a^b \{(b-s)^{p-1}-(z-s)^{p-1}\}k(z,s)f(s)\,\mathrm{d}s \\ &\quad - \int_b^z (z-w)^{p-1}k(z,w)f(w)\,\mathrm{d}w.\end{aligned}$$

Since $k(\cdot,s)\in\mathbf{M}_p(\mathscr{D})$, there exists a constant H_k and the first term can be bounded as

$$\begin{aligned}\left|\int_a^b (b-s)^{p-1}\{k(b,s)-k(z,s)\}f(s)\,\mathrm{d}s\right| &\leq H_k|b-z|^p\|f\|_{\mathbf{HC}(\mathscr{D})}\int_a^b (b-s)^{p-1}\,\mathrm{d}s \\ &= H_k|b-z|^p\|f\|_{\mathbf{HC}(\mathscr{D})}\frac{(b-a)^p}{p}.\end{aligned}$$

The third term is bounded as

$$\left|\int_b^z (z-w)^{p-1}k(z,w)f(w)\,\mathrm{d}w\right| \leq \frac{M_k\|f\|_{\mathbf{HC}(\mathscr{D})}c_d}{p}|b-z|^p,$$

similarly to the case of (A.4). Then there remains the second term. Using integration by parts, we have

$$\begin{aligned}&\int_a^b \{(b-s)^{p-1}-(z-s)^{p-1}\}k(z,s)f(s)\,\mathrm{d}s \\ &= \int_a^b \frac{\partial}{\partial s}\left\{-\frac{(b-s)^p}{p}+\frac{(z-s)^p}{p}\right\}k(z,s)f(s)\,\mathrm{d}s \\ &= \frac{1}{p}(z-b)^p k(z,b)f(b) + \frac{1}{p}\left\{(b-a)^p-(z-a)^p\right\}k(z,a)f(a) \\ &\quad + \frac{1}{p}\int_a^b \{(b-s)^p-(z-s)^p\}\frac{\partial}{\partial s}\{k(z,s)f(s)\}\,\mathrm{d}s.\end{aligned}$$

Since the function $F(z)=z^p$ is p-Hölder continuous, there exists a constant H_F such that

$$\left|\frac{1}{p}(z-b)^p k(z,b)f(b)\right| + \left|\frac{1}{p}\left\{(b-a)^p-(z-a)^p\right\}k(z,a)f(a)\right| \leq \frac{1+H_F}{p}|b-z|^p M_k\|f\|_{\mathbf{HC}(\mathscr{D})},$$

and since $k(z,\cdot)f(\cdot)\in\mathbf{HC}(\mathscr{D})$, there exists a constant C such that

$$\left|\frac{1}{p}\int_a^b \{(b-s)^p-(z-s)^p\}\frac{\partial}{\partial s}\{k(z,s)f(s)\}\,\mathrm{d}s\right| \leq \frac{1}{p}|b-z|^p\int_a^b\left|\frac{\partial}{\partial s}k(z,s)f(s)\right|\,\mathrm{d}s \leq \frac{1}{p}|b-z|^p C.$$

Thus it finally follows that

$$|\mathcal{A}[f](b)-\mathcal{A}[f](z)| \leq \frac{H_k\|f\|_{\mathbf{HC}(\mathscr{D})}(b-a)^p + (c_d+1+H_F)M_k\|f\|_{\mathbf{HC}(\mathscr{D})}+C}{p}|b-z|^p.$$

■

Next we consider the case of the DE transformation. The following lemma is necessary to prove the target lemma.

Lemma A.3 (Tanaka et al. [20, Lemma 5.4])**.** Let d be a constant with $0 < d < \pi/2$. Then there exists a constant $\tilde{c}_d$ depending only on d such that, for all $x \in \mathbb{R}$ and $y \in [-d,\, d]$,

$$\left|\frac{\cosh(x + \mathrm{i}\, y)}{\cosh^2\{(\pi/2)\sinh(x + \mathrm{i}\, y)\}}\right| \leq \tilde{c}_d \exp(|x|)\exp\{-(\pi/2)\cos(y)\exp(|x|)\}.$$

Lemma A.4. Let d be a constant with $0 < d < \pi/2$ and let us define a function ϕ_2 as

$$\phi_2(x) = \frac{1}{2}\tanh\left(\frac{\pi\cos(y)}{2}\sinh(x)\right) + \frac{1}{2}.$$

Then there exists such a constant c_d depending only on d that, for all $x \in \mathbb{R}$ and $y \in [-d,\, d]$,

$$|\{\phi_{a,b}^{\mathrm{DE}}\}'(x + \mathrm{i}\, y)| \leq (b-a)c_d\phi_2'(x), \tag{A.5}$$

$$|\phi_{0,1}^{\mathrm{DE}}(x + \mathrm{i}\, y)| \geq \phi_2(x). \tag{A.6}$$

Furthermore, if $t \leq x$,

$$|\phi_{a,b}^{\mathrm{DE}}(x + \mathrm{i}\, y) - \phi_{a,b}^{\mathrm{DE}}(t + \mathrm{i}\, y)| \geq (b-a)\{\phi_2(x) - \phi_2(t)\}. \tag{A.7}$$

Proof. We first prove the inequality (A.5). Here we set $c_y = (\pi/2)\cos y$ and $s_y = (\pi/2)\sin y$. According to Lemma A.3, there exists a constant $\tilde{c}_d$ depending only on d, such that

$$|\{\phi_{a,b}^{\mathrm{DE}}\}'(x + \mathrm{i}\, y)| \leq \frac{\pi(b-a)}{4}\tilde{c}_d\exp(|x|)\exp\{-c_y\exp(|x|)\}$$

holds for all $x \in \mathbb{R}$ and $y \in [-d,\, d]$. Using the inequalities

$$\exp(|x|) \leq 2\cosh(x),$$
$$\exp(c_y\exp(|x|)) \geq \cosh^2(c_y\sinh(x)),$$

we have

$$\begin{aligned}
\frac{\pi(b-a)}{4}\tilde{c}_d\exp(|x|)\exp\{-c_y\exp(|x|)\} &\leq \frac{\pi(b-a)}{4}\tilde{c}_d\frac{2\cosh(x)}{\cosh^2\{c_y\sinh(x)\}}\\
&= \frac{(b-a)\tilde{c}_d}{\cos y}\phi_2'(x)\\
&\leq \frac{(b-a)\tilde{c}_d}{\cos d}\phi_2'(x).
\end{aligned}$$

And for the inequality (A.6),

$$\begin{aligned}
|\phi_{0,1}^{\mathrm{DE}}(x + \mathrm{i}\, y)| &= \frac{1}{\sqrt{1 + 2\exp(-2c_y\sinh x)\cos(2s_y\cosh x) + \exp(-4c_y\sinh x)}}\\
&\geq \frac{1}{\sqrt{1 + 2\exp(-2c_y\sinh x) + \exp(-4c_y\sinh x)}} = \phi_2(x).
\end{aligned}$$

Next we prove the inequality (A.7). Since the following inequalities:

$$\begin{aligned}
&|\sinh[(\pi/2)\{\sinh(x + \mathrm{i}\, y) - \sinh(t + \mathrm{i}\, y)\}]|\\
&= \sqrt{\sinh^2[c_y\{\sinh(x) - \sinh(t)\}] + \sin^2[s_y\{\cosh(x) - \cosh(t)\}]}\\
&\geq \sinh[c_y\{\sinh(x) - \sinh(t)\}],
\end{aligned}$$

and

$$|\cosh\{(\pi/2)\sinh(x+\mathrm{i}\,y)\}| = \sqrt{\cosh^2\{c_y\sinh(x)\} - \sin^2\{s_y\cosh(x)\}}$$
$$\leq \cosh\{c_y\sinh(x)\}$$

hold, we readily have

$$\begin{aligned}|\phi^{\mathrm{DE}}_{a,b}(x+\mathrm{i}\,y) - \phi^{\mathrm{DE}}_{a,b}(t+\mathrm{i}\,y)| &= \frac{b-a}{2}\left|\frac{\sinh[(\pi/2)\{\sinh(x+\mathrm{i}\,y)-\sinh(t+\mathrm{i}\,y)\}]}{\cosh\{(\pi/2)\sinh(x+\mathrm{i}\,y)\}\cosh\{(\pi/2)\sinh(t+\mathrm{i}\,y)\}}\right|\\ &\geq \frac{b-a}{2}\frac{\sinh\{c_y(\sinh(x)-\sinh(t))\}}{\cosh\{c_y\sinh(x)\}\cosh\{c_y\sinh(t)\}}\\ &= (b-a)\{\phi_2(x)-\phi_2(t)\}.\end{aligned}$$

■

Then we can prove Lemma 4.4 in the case of the DE transformation. It can be proved by just replacing the SE transformation with the DE transformation, and the function ϕ_1 with ϕ_2 in the proof of Lemma A.2.

Lemma A.5. Let d be a constant with $0 < d < \pi/2$ and let $\mathscr{D} = \phi^{\mathrm{DE}}_{a,b}(\mathscr{D}_d)$. Suppose that $k(z,\cdot) \in \mathbf{HC}(\mathscr{D})$ for all $z \in \overline{\mathscr{D}}$, $k(\cdot,w) \in \mathbf{M}_p(\mathscr{D})$ for all $w \in \overline{\mathscr{D}}$, and $f \in \mathbf{HC}(\mathscr{D})$. Then $\mathcal{K}f \in \mathbf{M}_p(\mathscr{D})$.

Remark 3. In the statements of these lemmas, the parameter d is limited to $0 < d < \pi$ or $0 < d < \pi/2$. This is required to ensure that $\phi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d)$ and $\phi^{\mathrm{DE}}_{a,b}(\mathscr{D}_d)$ are bounded domains. One may notice, however, that such conditions have not appeared in Section 4. This is because the boundedness of the domains is independently assumed by Definition 2.4.

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