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Abstract

In this paper, two new approximate formulae for fractional derivatives are developed by means of Sinc methods. The difference of the two formulae is the variable transformations incorporated; the single exponential transformation and the double exponential transformation. We give error analysis of the formulae, and show that these formulae archive exponential convergence. Numerical examples that confirm the analysis are also given.

1 Introduction

In the last few decades, mathematical models with fractional derivatives have been used in the fields of physics [6], engineering [16], chemistry [14], biology [9], control theory [3], and many others [1, 2, 7]. We consider two types of derivatives of order p: Riemann–Liouville type ($\mathbf{D}_a^p f$) and Caputo's type ($\mathrm{D}_a^p f$), which are defined by

$$\mathbf{D}_a^p[f](t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\lfloor p \rfloor + 1} \left[\mathbf{I}_a^{\lfloor p \rfloor - p + 1} f\right](t), \quad t > a, \tag{1.1}$$

$$\mathrm{D}_a^p[f](t) = \mathbf{I}_a^{\lfloor p \rfloor - p + 1} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\lfloor p \rfloor + 1} f\right](t), \quad t > a, \tag{1.2}$$

respectively, where $\mathbf{I}_a^q f$ is the *Riemann–Liouville fractional integral* of order q,

$$\mathbf{I}_a^q[f](t) = \frac{1}{\Gamma(q)} \int_a^t \frac{f(s)\,\mathrm{d}s}{(t-s)^{1-q}}, \quad t > a. \tag{1.3}$$

In what follows, we assume $p,\, q \in (0,\, 1)$. In this case, approximating fractional derivatives with high accuracy is not an easy task, because there is a weakly singular kernel called the *Abel kernel* in (1.3). Typical numerical methods for fractional derivatives in the literature are reviewed in some books cited above and some papers [4, 5]. The convergence rates of those methods are all of polynomial: $\mathrm{O}(n^{-\gamma})$, where n denotes the number of evaluation of f, and γ is a positive constant.

Recently, an "exponentially" converging approximate formula based on Chebyshev polynomials has been proposed by Sugiura–Hasegawa [19]. In their beautiful work, they have extended the so-called Clenshaw–Curtis rule for the definite integral $\int_{-1}^{1} f(s)\,\mathrm{d}s$ to the fractional derivative of Caputo's type (1.2), and also pointed out that the formula is also applicable to the Riemann–Liouville type (1.1) through the relation $\mathbf{D}_a^p[f](t) = \mathrm{D}_a^p[f](t) + f(a)(t-a)^{-p}/\Gamma(1-p)$. They have shown that the formula converges uniformly on the given interval $[a,\, b]$ with the exponential rate, $\mathrm{O}(\mathrm{e}^{-\gamma n})$, under the assumption that f is analytic on an elliptic domain that contains the interval $[a,\, b]$. In general, however, f does not satisfy this assumption. In fact, the solution of fractional differential equations may have a singularity at the endpoint, $t = a$, due to the Abel kernel [8]. In such cases, their formula loses the fast convergence.

On the other hand, for such singular functions, it is known in the wide range of numerical analysis that Sinc methods are quite effective (see, for example, Stenger [17]). In fact, Riley [15] employed techniques in Sinc methods to approximate integrals of the form (1.3), and obtained exponential convergence, $\mathrm{O}(\mathrm{e}^{-\gamma\sqrt{n}})$, despite singularities in the kernel and the function f. This result has then been extended by Mori et al. [10] and the present authors [13], and it turned out that the convergence rate of the method can be improved to $\mathrm{O}(\mathrm{e}^{-\gamma n/\log n})$. The key in this improvement is the replacement of the variable transformation; the standard *Single Exponential* (*SE*) *transformation* employed in Riley's method was replaced with a stronger transformation, the so-called *Double Exponential* (*DE*) *transformation* [11, 18]. The latter methods, i.e. the Sinc methods incorporated with the DE transformation, are called *DE-Sinc methods*, while the former ones are referred to *SE-Sinc methods*, accordingly.

As a natural extension of these results, in the present paper we propose two new approximate formulae for Caputo's fractional derivative (1.2); either based on the SE-Sinc and DE-Sinc methods. It is then shown theoretically and numerically that the convergence rate is $\mathrm{O}(\mathrm{e}^{-\gamma\sqrt{n}})$ in the first formula, and $\mathrm{O}(\mathrm{e}^{-\gamma n/\log n})$ in the second formula. These formulae are also applicable to the Riemann–Liouville fractional derivative (1.1) in the same manner as in Sugiura–Hasegawa [19].

This paper is organized as follows. The main results are stated in Section 2. In Section 3, we show numerical examples of the new formulae, and compare them with the one by Sugiura–Hasegawa. The proofs of the main theorems are given in Section 4.

2 Approximate formulae and their error analysis

The main tool to derive approximate formulae is the *Sinc approximation*:

$$F(\tau) \approx \sum_{j=-N}^{N} F(jh)S(j,h)(\tau), \quad \tau \in \mathbb{R}, \tag{2.1}$$

where $S(j,h)(\tau)$ is the *Sinc function* defined by $S(j,h)(\tau) = \sin\{\pi(\tau/h - j)\}/\{\pi(\tau/h - j)\}$. The so-called *Sinc quadrature* rule is derived by integrating the both sides of (2.1):

$$\int_{-\infty}^{\infty} F(\tau)\,\mathrm{d}\tau \approx \sum_{j=-N}^{N} F(jh)\int_{-\infty}^{\infty} S(j,h)(\tau)\,\mathrm{d}\tau = h\sum_{j=-N}^{N} F(jh). \tag{2.2}$$

Note that the variable τ in these formulae moves on the whole real line. If the function to be approximated is defined on a finite domain, variable transformation should be employed in (2.1) or (2.2). There are two transformations, the SE transformation and the DE transformation, which are defined by

$$t = \psi^{\mathrm{SE}}_{a,b}(\tau) = \frac{b-a}{2}\tanh\left(\frac{\tau}{2}\right) + \frac{b+a}{2},$$
$$t = \psi^{\mathrm{DE}}_{a,b}(\tau) = \frac{b-a}{2}\tanh\left(\frac{\pi}{2}\sinh(\tau)\right) + \frac{b+a}{2}.$$

Both transformations map $\tau \in \mathbb{R}$ onto $t \in (a,\, b)$. Their inverse functions are:

$$\tau = \{\psi^{\mathrm{SE}}_{a,b}\}^{-1}(t) = \log\left(\frac{t-a}{b-t}\right),$$
$$\tau = \{\psi^{\mathrm{DE}}_{a,b}\}^{-1}(t) = \log\left[\frac{1}{\pi}\log\left(\frac{t-a}{b-t}\right) + \sqrt{1 + \left\{\frac{1}{\pi}\log\left(\frac{t-a}{b-t}\right)\right\}^2}\,\right].$$

2.1 Derivation of a formula by means of the SE-Sinc methods

Recall that Caputo's fractional derivative is defined by $\mathrm{D}^p_a[f](t) = \mathbf{I}^{1-p}_a[f'](t)$. Our basic idea is to approximate the integral part ($\mathbf{I}^{1-p}_a$) based on the idea in Riley [15], and the derivative part ($\frac{\mathrm{d}}{\mathrm{d}t}$) based on the idea in Stenger [17], respectively. Finally we combine them to approximate the target: $\mathrm{D}^p_a f$.

First we consider the approximation of $\mathbf{I}^{1-p}_a g$ for a given function g. Changing the original integral interval $(a,\, t)$ to $\mathbb{R}$ by the variable transformation $s = \psi^{\mathrm{SE}}_{a,t}(\sigma)$, we have

$$\mathbf{I}^{1-p}_a[g](t) = \int_{-\infty}^{\infty} \frac{g(\psi^{\mathrm{SE}}_{a,t}(\sigma))\{\psi^{\mathrm{SE}}_{a,t}\}'(\sigma)\,\mathrm{d}\sigma}{\Gamma(1-p)(t-\psi^{\mathrm{SE}}_{a,t}(\sigma))^p} = \frac{(t-a)^{1-p}}{\Gamma(1-p)}\int_{-\infty}^{\infty}\frac{g(\psi^{\mathrm{SE}}_{a,t}(\sigma))\,\mathrm{d}\sigma}{(1+\mathrm{e}^{-\sigma})(1+\mathrm{e}^{\sigma})^{1-p}}.$$

Note that the weakly singular integrand (the Abel kernel) is translated to a smooth function. Applying the quadrature rule (2.2) to the translated integral, we obtain the approximate formula for the integral part:

$$\mathbf{I}^{1-p}_a[g](t) \approx \mathcal{I}^{\mathrm{SE}}_N[g](t) = \frac{(t-a)^{1-p}}{\Gamma(1-p)} h \sum_{k=-N}^{N} \frac{g(\psi^{\mathrm{SE}}_{a,t}(kh))}{(1+\mathrm{e}^{-kh})(1+\mathrm{e}^{kh})^{1-p}}. \tag{2.3}$$

Here h is a mesh size suitably chosen depending on N, which will be described later.

Next we consider the approximation of f'. Let us define a function $Q_{a,b}$ as $Q_{a,b}(t) = (t-a)(b-t)$. Putting $F(\tau) = f(\psi^{\mathrm{SE}}_{a,b}(\tau))/Q_{a,b}(\psi^{\mathrm{SE}}_{a,b}(\tau))$ in (2.1), we have

$$\frac{f(\psi^{\mathrm{SE}}_{a,b}(\tau))}{Q_{a,b}(\psi^{\mathrm{SE}}_{a,b}(\tau))} \approx \sum_{j=-N}^{N} \frac{f(\psi^{\mathrm{SE}}_{a,b}(jh))}{Q_{a,b}(\psi^{\mathrm{SE}}_{a,b}(jh))} S(j,h)(\tau), \quad \tau \in \mathbb{R},$$

which is equivalent to:

$$f(t) \approx \mathcal{C}^{\mathrm{SE}}_N[f](t) = \sum_{j=-N}^{N} \frac{f(\psi^{\mathrm{SE}}_{a,b}(jh))}{Q_{a,b}(\psi^{\mathrm{SE}}_{a,b}(jh))} Q_{a,b}(t) S(j,h)(\{\psi^{\mathrm{SE}}_{a,b}\}^{-1}(t)), \quad t \in (a,\, b). \tag{2.4}$$

Differentiating the both sides gives an approximate formula for f', i.e. $f' \approx \{\mathcal{C}^{\mathrm{SE}}_N f\}'$.

Using this and (2.3) with $g = f'$, we finally obtain the desired formula as follows:

$$\mathrm{D}^p_a[f](t) = \mathbf{I}^{1-p}_a[f'](t) \approx \mathcal{I}^{\mathrm{SE}}_N[f'](t) \approx \mathcal{I}^{\mathrm{SE}}_N[\{\mathcal{C}^{\mathrm{SE}}_N f\}'](t). \tag{2.5}$$

2.2 Derivation of a formula by means of the DE-Sinc methods

We consider the use of the DE transformation instead of the SE transformation here. For the integral part $\mathbf{I}_a^{1-p} g$, we apply $s = \psi_{a,t}^{\mathrm{DE}}(\sigma)$, then $\mathbf{I}_a^{1-p} g$ is translated into

$$\mathbf{I}_a^{1-p}[g](t) = \frac{(t-a)^{1-p}}{\Gamma(1-p)} \int_{-\infty}^{\infty} \frac{\pi \cosh(\sigma) g(\psi_{a,t}^{\mathrm{DE}}(\sigma))\,\mathrm{d}\sigma}{(1+\mathrm{e}^{-\pi\sinh(\sigma)})(1+\mathrm{e}^{\pi\sinh(\sigma)})^{1-p}}.$$

Applying the quadrature rule (2.2) to this integral, we obtain the approximate formula:

$$\mathbf{I}_a^{1-p}[g](t) \approx \mathcal{I}_N^{\mathrm{DE}}[g](t) = \frac{(t-a)^{1-p}}{\Gamma(1-p)} h \sum_{k=-N}^{N} \frac{\pi \cosh(kh) g(\psi_{a,t}^{\mathrm{DE}}(kh))}{(1+\mathrm{e}^{-\pi\sinh(kh)})(1+\mathrm{e}^{\pi\sinh(kh)})^{1-p}}.$$

The derivative part can be handled in the same manner. Similar to (2.4), using

$$f(t) \approx \mathcal{C}_N^{\mathrm{DE}}[f](t) = \sum_{j=-N}^{N} \frac{f(\psi_{a,b}^{\mathrm{DE}}(jh))}{Q_{a,b}(\psi_{a,b}^{\mathrm{DE}}(jh))} Q_{a,b}(t) S(j,h)(\{\psi_{a,b}^{\mathrm{DE}}\}^{-1}(t)), \quad t \in (a,\, b),$$

and differentiating both sides, we have $f' \approx \{\mathcal{C}_N^{\mathrm{DE}} f\}'$. Then we obtain the formula:

$$\mathrm{D}_a^p[f](t) = \mathbf{I}_a^{1-p}[f'](t) \approx \mathcal{I}_N^{\mathrm{DE}}[f'](t) \approx \mathcal{I}_N^{\mathrm{DE}}[\{\mathcal{C}_N^{\mathrm{DE}} f\}'](t). \tag{2.6}$$

2.3 Results of error analysis

We here state the error analysis results of the presented approximate formulae, while their proofs are left to Section 4. Let us introduce the following function space.

Definition 2.1. Let $\mathscr{D}$ be a simply-connected domain which satisfies $(a,\, b) \subset \mathscr{D}$, and let α be a positive constant. Then $\mathbf{L}_\alpha(\mathscr{D})$ denotes the family of all functions f that are analytic on $\mathscr{D}$, and satisfy $|f(z)| \leq C|Q_{a,b}^{\alpha}(z)|$ for a positive constant C and all $z \in \mathscr{D}$.

In the statement of theorems below, $\mathscr{D}$ is either $\psi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d)$ or $\psi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d)$, where

$$\psi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d) = \left\{ z \in \mathbb{C} : \left| \arg\left(\frac{z-a}{b-z}\right) \right| < d \right\},$$
$$\psi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d) = \left\{ z \in \mathbb{C} : \left| \arg\left[\frac{1}{\pi}\log\left(\frac{z-a}{b-z}\right) + \sqrt{1 + \left\{\frac{1}{\pi}\log\left(\frac{z-a}{b-z}\right)\right\}^2} \right] \right| < d \right\}.$$

These are domains that are mapped by the SE or DE transformation from a strip domain

$$\mathscr{D}_d = \{\zeta \in \mathbb{C} : |\operatorname{Im}\zeta| < d\}, \tag{2.7}$$

for a positive constant d. With these notations, the approximate errors of the formula (2.5) and (2.6) are analyzed as follows.

Theorem 2.2. Let $(f/Q_{a,b}) \in \mathbf{L}_\alpha(\psi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$. Let $\mu = \min\{1-p,\, \alpha\}$, N be a positive integer, and h be selected by $h = \sqrt{\pi d/(\mu N)}$. Then there exists a constant C independent of N such that

$$\max_{t \in [a,\, b]} \left| \mathrm{D}_a^p[f](t) - \mathcal{I}_N^{\mathrm{SE}}[\{\mathcal{C}_N^{\mathrm{SE}} f\}'](t) \right| \leq C N \mathrm{e}^{-\sqrt{\pi d \mu N}}. \tag{2.8}$$

Theorem 2.3. Let $(f/Q_{a,b}) \in \mathbf{L}_{\alpha}(\psi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let $\mu = \min\{1-p,\, \alpha\}$, N be a positive integer with $N > \mu/(2d)$, and h be selected by $h = \log(2dN/\mu)/N$. Then there exists a constant C independent of N such that

$$\max_{t \in [a,\, b]} \left| \mathrm{D}_a^p[f](t) - \mathcal{I}_N^{\mathrm{DE}}[\{\mathcal{C}_N^{\mathrm{DE}} f\}'](t) \right| \leq C \frac{N}{\log(2dN/\mu)} \mathrm{e}^{-\pi d N/\log(2dN/\mu)}.$$

The number of evaluation of f in these approximation formulae is $n = 2N+1$, which means the convergence rate is $\mathrm{O}(\mathrm{e}^{-\gamma\sqrt{n}})$ in the SE-Sinc case, and $\mathrm{O}(\mathrm{e}^{-\gamma n/\log n})$ in the DE-Sinc case for some $\gamma > 0$; in both cases the errors decay exponentially.

Remark 1. The assumption $(f/Q_{a,b}) \in \mathbf{L}_{\alpha}(\mathscr{D})$ may seem to be not practical since the function f must be zero at the endpoints by the condition $|f(z)/Q_{a,b}(z)| \leq C|Q_{a,b}^{\alpha}(z)|$. But actually, functions in a certain wider, and reasonable space can be translated to those satisfying the assumption (see Stenger [17, § 4]).

3 Numerical examples

In this section we consider two test functions, $f_1(t) = t^{4/3}(1-t)^2/\Gamma(7/3)$ and $f_2(t) = t^2(1-t)^2\mathrm{e}^t$, and their 1/2-order derivatives in Caputo's sense on the interval (0, 1):

$$\mathrm{D}_0^{1/2}[f_1](t) = \frac{t^{5/6}}{\Gamma(11/6)} \frac{280t^2 - 476t + 187}{187},$$

$$\mathrm{D}_0^{1/2}[f_2](t) = \frac{1}{16}\left[\frac{t^{1/2}}{\Gamma(3/2)}(8t^3 - 4t^2 - 22t + 31) + \mathrm{e}^t \operatorname{erf}(\sqrt{t})\left\{8t(2t^3 - 7t + 8) - 31\right\}\right].$$

Let π_{m} denote an arbitrary positive number less than π. Then the function f_1 satisfies $(f_1/Q_{0,1}) \in \mathbf{L}_{1/3}(\psi_{0,1}^{\mathrm{SE}}(\mathscr{D}_{\pi_{\mathrm{m}}}))$ and $(f_1/Q_{0,1}) \in \mathbf{L}_{1/3}(\psi_{0,1}^{\mathrm{DE}}(\mathscr{D}_{\pi_{\mathrm{m}}/2}))$, and the function f_2 satisfies $(f_2/Q_{0,1}) \in \mathbf{L}_{1}(\psi_{0,1}^{\mathrm{SE}}(\mathscr{D}_{\pi_{\mathrm{m}}}))$ and $(f_2/Q_{0,1}) \in \mathbf{L}_{1}(\psi_{0,1}^{\mathrm{DE}}(\mathscr{D}_{\pi_{\mathrm{m}}/2}))$. In actual computations, we set $\pi_{\mathrm{m}} = 3.14$, and then h can be selected according to Theorem 2.2 or Theorem 2.3.

The numerical result of $\mathrm{D}_0^{1/2} f_1$ is shown in Figure 1, and the one of $\mathrm{D}_0^{1/2} f_2$ is shown in Figure 2. Both of the computation programs are written in C with double-precision floating-point arithmetic. The errors are checked on $t = 0.01, 0.02, \ldots, 0.99$, and the maximum error of them is plotted on the graphs. There are three plot lines in both graphs; the formula by Sugiura–Hasegawa [19] (dashed line with $\times$ points), by the SE-Sinc methods (solid line with $\triangle$ points), and by the DE-Sinc methods (solid line with $\circ$ points). The convergence profiles of the Chebyshev formula are different between Figure 1 and Figure 2. This should be caused by the singularity of the function f_1 at the endpoint, $t = 0$. In contrast, we can see that the results of the SE-Sinc formula and the DE-Sinc formula are consistent with Theorem 2.2 or Theorem 2.3 in both graphs.

4 Proofs of the theorems in Section 2

4.1 Proof of Theorem 2.2 (the SE-Sinc case)

The following two theorems are critical to prove Theorem 2.2.

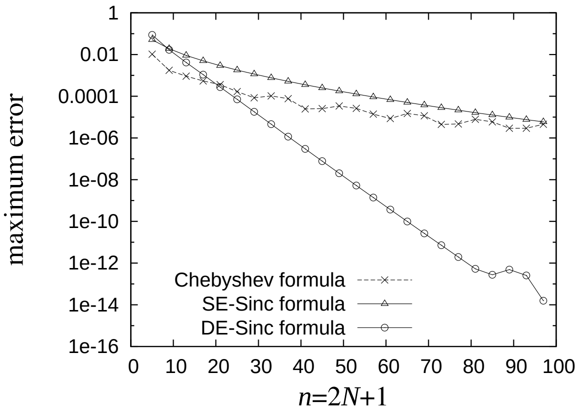

Figure 1. Approximation errors of $\mathrm{D}_0^{1/2} f_1$.

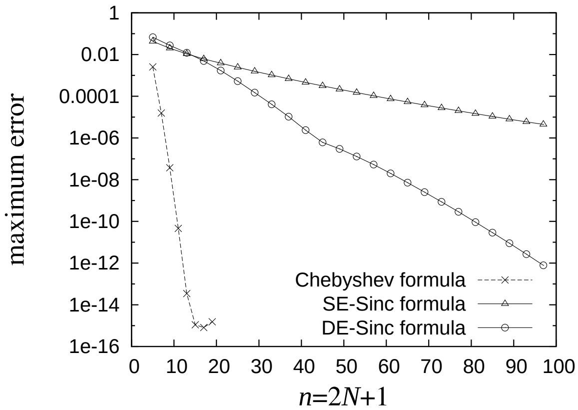

Figure 2. Approximation errors of $\mathrm{D}_0^{1/2} f_2$.

Theorem 4.1. Let the assumptions of Theorem 2.2 are fulfilled. Then there exists a constant C independent of N such that

$$\max_{t\in[a,\,b]} \left|\mathbf{I}_a^{1-p}[f'](t) - \mathcal{I}_N^{\mathrm{SE}}[f'](t)\right| \leq C\mathrm{e}^{-\sqrt{\pi d\mu N}}.$$

Theorem 4.2 (Stenger [17, Corollary of Theorem 4.4.2])**.** Let the assumptions of Theorem 2.2 are fulfilled. Then there exists a constant C independent of N such that

$$\sup_{t\in(a,\,b)} \left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{f(t) - \mathcal{C}_N^{\mathrm{SE}}[f](t)\right\}\right| \leq CN\mathrm{e}^{-\sqrt{\pi d\mu N}}.$$

Using these theorems and the trivial fact $\sup_N \|\mathcal{I}_N^{\mathrm{SE}}\|_{C([a,\,b])} < \infty$, we get (2.8). In what follows, we prove Theorem 4.1. The next theorem is the base of the error analysis.

Theorem 4.3 (Stenger [17, Theorem 4.2.6])**.** Let $(FQ_{a,b}) \in \mathbf{L}_\beta(\psi_{a,b}^{\mathrm{SE}}(\mathscr{D}_d))$ for d with $0 < d < \pi$, let N be a positive integer, and h be selected by $h = \sqrt{2\pi d/(\beta N)}$. Then there exists a constant C independent of N such that

$$\left|\int_a^b F(s)\,\mathrm{d}s - h\sum_{k=-N}^{N} F(\psi_{a,b}^{\mathrm{SE}}(kh))\{\psi_{a,b}^{\mathrm{SE}}\}'(kh)\right| \leq C\mathrm{e}^{-\sqrt{2\pi d\beta N}}.$$

Let us apply this theorem to the approximation (2.3). If we put $F(s) = g(s)/(t-s)^p$ in this theorem, and if g is analytic and bounded uniformly on $\psi_{a,t}^{\mathrm{SE}}(\mathscr{D}_d)$ for all $t \in [a,\,b]$, then $(FQ_{a,t}) \in \mathbf{L}_{1-p}(\psi_{a,t}^{\mathrm{SE}}(\mathscr{D}_d))$. Furthermore if we set $\mu = \min\{1-p,\,\alpha\}$, then $(FQ_{a,t}) \in \mathbf{L}_\mu(\psi_{a,t}^{\mathrm{SE}}(\mathscr{D}_d))$ since clearly $\mathbf{L}_\nu(\psi_{a,t}^{\mathrm{SE}}(\mathscr{D}_d)) \subseteq \mathbf{L}_\rho(\psi_{a,t}^{\mathrm{SE}}(\mathscr{D}_d))$ if $\nu \geq \rho$. Therefore we obtain the next result.

Lemma 4.4. Assume that there exists a constant d with $0 < d < \pi$ such that g is analytic and bounded uniformly on $\psi_{a,t}^{\mathrm{SE}}(\mathscr{D}_d)$ for all $t \in [a,\,b]$. Let $\mu = \min\{1-p,\,\alpha\}$, N be a positive integer, and h be selected by $h = \sqrt{2\pi d/(\mu N)}$. Then there exists a constant C independent of N such that

$$\max_{t\in[a,\,b]} \left|\mathbf{I}_a^{1-p}[g](t) - \mathcal{I}_N^{\mathrm{SE}}[g](t)\right| \leq C\mathrm{e}^{-\sqrt{2\pi d\mu N}}. \tag{4.1}$$

We can relax the condition on g using the following lemma.

Lemma 4.5. Let g be analytic and bounded on $\psi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d)$ for d with $0 < d < \pi$. Then g is analytic and bounded uniformly on $\psi^{\mathrm{SE}}_{a,t}(\mathscr{D}_d)$ for all $t \in [a, b]$.

Proof. We shall establish this lemma if we prove for all $t \in [a, b]$ that $\psi^{\mathrm{SE}}_{a,t}(\mathscr{D}_d) \subseteq \psi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d)$, which is equivalent to "$|\operatorname{Im}\{\psi^{\mathrm{SE}}_{a,t}\}^{-1}(z)| < d \Rightarrow |\operatorname{Im}\{\psi^{\mathrm{SE}}_{a,b}\}^{-1}(z)| < d$" (recall that $\mathscr{D}_d$ is defined by (2.7)). Set $\zeta(t) = \{\psi^{\mathrm{SE}}_{a,t}\}^{-1}(z)$ for simplicity. It is sufficient to show that $|\operatorname{Im}\zeta(t)|$ is a monotonically decreasing function, since from this we have $|\operatorname{Im}\zeta(b)| \leq |\operatorname{Im}\zeta(t)| < d$. Let $x, y \in \mathbb{R}$ and set $z = x + \mathrm{i}\, y$. Then $\operatorname{Im}\zeta(t)$ is expressed as

$$\operatorname{Im}\zeta(t) = \arg\left(\frac{z-a}{t-z}\right) = \arg\left(\frac{ax+tx-at-x^2-y^2}{(t-x)^2+y^2} + \mathrm{i}\,\frac{(t-a)y}{(t-x)^2+y^2}\right).$$

Considering $\cos(\operatorname{Im}\zeta(t))$ and its derivative, we have

$$\cos(\operatorname{Im}\zeta(t)) = \frac{ax+tx-at-x^2-y^2}{\sqrt{(ax+tx-at-x^2-y^2)^2+(t-a)^2y^2}},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\cos(\operatorname{Im}\zeta(t)) = \frac{(t-a)((a-x)^2+y^2)y^2}{\{((a-x)^2+y^2)((t-x)^2+y^2)\}^{3/2}} \geq 0.$$

Thus $\cos(\operatorname{Im}\zeta(t))$ is a monotonically increasing function. Since $-\pi < \operatorname{Im}\zeta(t) \leq \pi$ by definition and $\cos(-\operatorname{Im}\zeta(t)) = \cos(\operatorname{Im}\zeta(t))$, we can see that $|\operatorname{Im}\zeta(t)|$ is monotonically decreasing. ∎

Therefore Lemma 4.4 can be rewritten as follows.

Lemma 4.6. Let g be analytic and bounded on $\psi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d)$ for d with $0 < d < \pi$. Let $\mu = \min\{1-p, \alpha\}$, N be a positive integer, and h be selected by $h = \sqrt{2\pi d/(\mu N)}$. Then there exists a constant C independent of N such that (4.1) holds.

If $(f/Q_{a,b}) \in \mathbf{L}_\alpha(\psi^{\mathrm{SE}}_{a,b}(\mathscr{D}_d))$ (assumption in Theorem 2.2) holds, then f' is analytic and bounded on $\psi^{\mathrm{SE}}_{a,b}(\mathscr{D}_{d-\epsilon})$ for any ϵ with $0 < \epsilon < d$. Choosing $\epsilon = d/2$ and using Lemma 4.6, we obtain Theorem 4.1.

4.2 Proof of Theorem 2.3 (the DE-Sinc case)

Since $\sup_N \|\mathcal{I}^{\mathrm{DE}}_N\|_{C([a,\,b])} < \infty$, Theorem 2.3 can be proved in a similar way to the SE-Sinc case, by showing the following two theorems.

Theorem 4.7. Let the assumptions of Theorem 2.3 are fulfilled. Then there exists a constant C independent of N such that

$$\max_{t\in[a,\,b]} \left|\mathbf{I}^{1-p}_a[f'](t) - \mathcal{I}^{\mathrm{DE}}_N[f'](t)\right| \leq C\mathrm{e}^{-\pi dN/\log(2dN/\mu)}.$$

Theorem 4.8. Let the assumptions of Theorem 2.3 are fulfilled. Then there exists a constant C independent of N such that

$$\sup_{t\in(a,\,b)} \left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{f(t) - \mathcal{C}^{\mathrm{DE}}_N[f](t)\right\}\right| \leq C\frac{N}{\log(2dN/\mu)}\mathrm{e}^{-\pi dN/\log(2dN/\mu)}.$$

We first give the proof of Theorem 4.8, which is relatively short.

4.2.1 Proof of Theorem 4.8 (approximation error of derivatives)

We easily obtain that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{f(t)-\mathcal{C}_N^{\mathrm{DE}}[f](t)\right\}\right| \leq \left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{f(t)-\sum_{j=-\infty}^{\infty}\frac{f(\psi_{a,b}^{\mathrm{DE}}(jh))}{Q_{a,b}(\psi_{a,b}^{\mathrm{DE}}(jh))}Q_{a,b}(t)S(j,h)(\{\psi_{a,b}^{\mathrm{DE}}\}^{-1}(t))\right\}\right| + \sum_{|j|>N}\left|\frac{f(\psi_{a,b}^{\mathrm{DE}}(jh))}{Q_{a,b}(\psi_{a,b}^{\mathrm{DE}}(jh))}\right|\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{Q_{a,b}(t)S(j,h)(\{\psi_{a,b}^{\mathrm{DE}}\}^{-1}(t))\right\}\right|. \tag{4.2}$$

Let us examine the first term. We need the following definition for it.

Definition 4.9. Let $\mathscr{D}_d(\epsilon)$ be defined for $0<\epsilon<1$ by $\mathscr{D}_d(\epsilon)=\{\zeta\in\mathbb{C}:|\operatorname{Re}\zeta|<1/\epsilon,\ |\operatorname{Im}\zeta|<d(1-\epsilon)\}$. Then $\mathbf{H}^1(\mathscr{D}_d)$ denotes the family of all functions F that are analytic on $\mathscr{D}_d$, and such that $\mathcal{N}_1(F,d)=\lim_{\epsilon\to 0}\oint_{\partial\mathscr{D}_d(\epsilon)}|F(\zeta)||\mathrm{d}\zeta|<\infty$.

Then the next assertion holds for any conformal map ψ that satisfies $\psi(\mathbb{R})=(a,\,b)$.

Theorem 4.10 (Stenger [17, part of Theorem 4.4.2])**.** Assume the next two conditions:

(A1) $f(\psi(\cdot))/Q_{a,b}(\psi(\cdot))\in\mathbf{H}^1(\mathscr{D}_d)$,

(A2) $\displaystyle\sup_{t\in(a,\,b),-\pi/h\leq s\leq\pi/h}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{Q_{a,b}(t)\mathrm{e}^{\mathrm{i}s\psi^{-1}(t)}\right\}\right|\leq C/h$ with C depending only on ψ and $Q_{a,b}$.

Then there exists a constant $\tilde{C}$, depending only on ψ, Q, d and f, such that

$$\sup_{t\in(a,\,b)}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{f(t)-\sum_{j=-\infty}^{\infty}\frac{f(\psi(jh))}{Q_{a,b}(\psi(jh))}Q_{a,b}(t)S(j,h)(\psi^{-1}(t))\right\}\right|\leq\tilde{C}\frac{\mathrm{e}^{-\pi d/h}}{h}.$$

We show that (A1) and (A2) are fulfilled with $\psi(t)=\psi_{a,b}^{\mathrm{DE}}(t)$ under the assumption that $(f/Q_{a,b})\in\mathbf{L}_\alpha(\psi_{a,b}^{\mathrm{DE}}(\mathscr{D}_d))$. For (A1), it is sufficient to prove $\mathcal{N}_1(Q_{a,b}^{\alpha}(\psi_{a,b}^{\mathrm{DE}}(\cdot)),d)$ is finite, since $|f(z)/Q_{a,b}(z)|\leq C|Q_{a,b}^{\alpha}(z)|$ holds by the assumption (recall Definition 2.1). The next lemma shows the desired claim.

Lemma 4.11 (Okayama et al. [12, Lemma 4.6])**.** Let α and d be positive constants. Then $\mathcal{N}_1(Q_{a,b}^{\alpha}(\psi_{a,b}^{\mathrm{DE}}(\cdot)),d)$ is finite for any $d\in(0,\,\pi/2)$.

Using the Leibniz rule and the following inequality:

$$\frac{Q_{a,b}(t)}{\{\psi_{a,b}^{\mathrm{DE}}\}'(\{\psi_{a,b}^{\mathrm{DE}}\}^{-1}(t))}=\frac{(t-a)(b-t)}{\dfrac{\pi(t-a)(b-t)}{b-a}\sqrt{1+\left\{\dfrac{1}{\pi}\log\left(\dfrac{t-a}{b-t}\right)\right\}^2}}\leq\frac{b-a}{\pi},$$

we can easily show the condition (A2).

Lemma 4.12. The condition (A2) in Theorem 4.10 holds with $\psi(t)=\psi_{a,b}^{\mathrm{DE}}(t)$.

Therefore, by using Theorem 4.10 the first term in (4.2) is evaluated as follows.

Lemma 4.13. Let $(f/Q_{a,b}) \in \mathbf{L}_\alpha(\psi^{\mathrm{DE}}_{a,b}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Then there exists a constant C independent of h such that

$$\sup_{t\in(a,\,b)}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{f(t)-\sum_{j=-\infty}^{\infty}\frac{f(\psi^{\mathrm{DE}}_{a,b}(jh))}{Q_{a,b}(\psi^{\mathrm{DE}}_{a,b}(jh))}Q_{a,b}(t)S(j,h)(\{\psi^{\mathrm{DE}}_{a,b}\}^{-1}(t))\right\}\right| \leq C\frac{\mathrm{e}^{-\pi d/h}}{h}.$$

There remains to evaluate the second term in (4.2); this is done by the next lemma.

Lemma 4.14. Let $(f/Q_{a,b}) \in \mathbf{L}_\alpha(\psi^{\mathrm{DE}}_{a,b}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Then there exists a constant C independent of h and N such that

$$\sup_{t\in(a,\,b)}\sum_{|j|>N}\left|\frac{f(\psi^{\mathrm{DE}}_{a,b}(jh))}{Q_{a,b}(\psi^{\mathrm{DE}}_{a,b}(jh))}\right|\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{Q_{a,b}(t)S(j,h)(\{\psi^{\mathrm{DE}}_{a,b}\}^{-1}(t))\right\}\right| \leq C\frac{1}{h^2\mathrm{e}^{Nh}}\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(Nh)}. \quad (4.3)$$

Proof. First, by the identity

$$Q_{a,b}(t)S(j,h)(\{\psi^{\mathrm{DE}}_{a,b}\}^{-1}(t)) = \frac{hQ_{a,b}(t)}{2\pi}\int_{-\pi/h}^{\pi/h}\mathrm{e}^{\mathrm{i}s[\{\psi^{\mathrm{DE}}_{a,b}\}^{-1}(t)-jh]}\,\mathrm{d}s$$

and Lemma 4.12, it follows that for a constant C_1

$$\sup_{t\in(a,\,b)}\left|\frac{\mathrm{d}}{\mathrm{d}t}\left\{Q_{a,b}(t)S(j,h)(\{\psi^{\mathrm{DE}}_{a,b}\}^{-1}(t))\right\}\right| \leq C_1/h. \quad (4.4)$$

Second, by the assumption $(f/Q_{a,b}) \in \mathbf{L}_\alpha(\psi^{\mathrm{DE}}_{a,b}(\mathscr{D}_d))$, there exists a constant $\tilde{C}$ such that

$$\left|\frac{f(\psi^{\mathrm{DE}}_{a,b}(jh))}{Q_{a,b}(\psi^{\mathrm{DE}}_{a,b}(jh))}\right| \leq \tilde{C}|Q^\alpha_{a,b}(\psi^{\mathrm{DE}}_{a,b}(jh))| = \frac{\tilde{C}\{(b-a)/2\}^{2\alpha}}{\cosh^{2\alpha}(\pi\sinh(jh)/2)} \leq \tilde{C}(b-a)^{2\alpha}\mathrm{e}^{-\pi\alpha\sinh(|jh|)}.$$

Furthermore using $\sinh(|jh|) \geq (\mathrm{e}^{|jh|}-1)/2$, and putting $C_2 = \tilde{C}(b-a)^{2\alpha}\mathrm{e}^{\frac{\pi}{2}\alpha}$, we have

$$\begin{aligned}\sum_{|j|>N}\left|\frac{f(\psi^{\mathrm{DE}}_{a,b}(jh))}{Q_{a,b}(\psi^{\mathrm{DE}}_{a,b}(jh))}\right| &\leq C_2\sum_{|j|>N}\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(|jh|)}\\ &= 2C_2\sum_{j>N}\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(jh)}\\ &\leq 2C_2\int_N^\infty \mathrm{e}^{-\frac{\pi}{2}\alpha\exp(sh)}\,\mathrm{d}s\\ &\leq 2C_2\left\{\frac{2}{\pi\alpha h\mathrm{e}^{Nh}}\right\}\int_N^\infty\left\{\frac{\pi\alpha h\mathrm{e}^{sh}}{2}\right\}\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(sh)}\,\mathrm{d}s\\ &= \frac{4C_2}{\pi\alpha h\mathrm{e}^{Nh}}\mathrm{e}^{-\frac{\pi}{2}\alpha\exp(Nh)}. \end{aligned} \quad (4.5)$$

Combining (4.4) with (4.5), we get (4.3). ∎

Theorem 4.8 is then established by taking h as $h = \log(2dN/\mu)/N$ in Lemma 4.13 and Lemma 4.14.

4.2.2 Proof of Theorem 4.7 (approximation error of integrals)

Theorem 4.7 can be shown in almost the same manner as the SE-Sinc case (Theorem 4.1). Let us start with the next theorem.

Theorem 4.15 (Tanaka et al. [20, Theorem 3.1])**.** Let $(f/Q_{a,b}) \in \mathbf{L}_{\beta}(\psi^{\text{DE}}_{a,b}(\mathscr{D}_d))$ for d with $0 < d < \pi/2$. Let N be a positive integer with $N > \beta/(4d)$, and let h be selected by $h = \log(4dN/\beta)/N$. Then there exists a constant C independent of N such that

$$\left| \int_a^b F(s)\,\mathrm{d}s - h \sum_{k=-N}^{N} F(\psi^{\text{DE}}_{a,b}(kh))\{\psi^{\text{DE}}_{a,b}\}'(kh) \right| \leq C\mathrm{e}^{-2\pi dN/\log(4dN/\beta)}.$$

Applying this theorem to the approximation $\mathbf{I}_a^{1-p}\, g \approx \mathcal{I}^{\text{DE}}_N g$, we have the next lemma.

Lemma 4.16. Assume that there exists a constant d with $0 < d < \pi/2$ such that g is analytic and bounded uniformly on $\psi^{\text{DE}}_{a,t}(\mathscr{D}_d)$ for all $t \in [a,\, b]$. Let $\mu = \min\{1-p,\, \alpha\}$, N be a positive integer with $N > \mu/(4d)$, and h be selected by $h = \log(4dN/\mu)/N$. Then there exists a constant C independent of N such that

$$\max_{t\in[a,\, b]} \left| \mathbf{I}_a^{1-p}[g](t) - \mathcal{I}^{\text{SE}}_N[g](t) \right| \leq C\mathrm{e}^{-2\pi dN/\log(4dN/\mu)}. \tag{4.6}$$

We can relax the condition on g using the following lemma.

Lemma 4.17. Let g be analytic and bounded on $\psi^{\text{DE}}_{a,b}(\mathscr{D}_d)$ for d with $0 < d < \pi/2$. Then g is analytic and bounded uniformly on $\psi^{\text{DE}}_{a,t}(\mathscr{D}_d)$ for all $t \in [a,\, b]$.

Since its proof is far more complicated than the SE-Sinc case (Lemma 4.5), we leave it to the end of this section. If we accept this lemma, Lemma 4.16 can be rewritten as follows.

Lemma 4.18. Let g be analytic and bounded on $\psi^{\text{DE}}_{a,b}(\mathscr{D}_d)$ for d with $0 < d < \pi/2$. Let $\mu = \min\{1-p,\, \alpha\}$, N be a positive integer with $N > \mu/(4d)$, and h be selected by $h = \log(4dN/\mu)/N$. Then there exists a constant C independent of N such that (4.6) holds.

If $(f/Q_{a,b}) \in \mathbf{L}_{\alpha}(\psi^{\text{DE}}_{a,b}(\mathscr{D}_d))$ holds, then f' is analytic and bounded on $\psi^{\text{DE}}_{a,b}(\mathscr{D}_{d-\epsilon})$ for any ϵ with $0 < \epsilon < d$. Choosing $\epsilon = d/2$ and using Lemma 4.18, we obtain Theorem 4.7.

It remains to prove Lemma 4.17. To this end, it is essential to examine the imaginary part of G_1 and G_2, which are defined by

$$G_1(\eta) = \eta + \sqrt{1+\eta^2}, \quad G_2(\eta) = \frac{\sqrt{1+\eta^2}}{1+\mathrm{e}^{-\pi\eta}}.$$

The following lemma is useful to determine whether $\operatorname{Im} G_1$ is positive or negative.

Lemma 4.19. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$ with $|y| \leq 1$. Then $\operatorname{Im}[G_1(x + \mathrm{i}\, y)]$ is represented as follows:

$$\operatorname{Im}\left[G_1(x + \mathrm{i}\, y)\right] = yG_1^{+}(x, y),$$

where $G_1^{+}(x, y)$ is a positive function.

Proof. Let X and Y be defined by

$$X = \frac{1}{\sqrt{2}}\sqrt{\sqrt{(1+x^2-y^2)^2+(2xy)^2}+(1+x^2-y^2)}, \quad Y = \frac{xy}{X}. \tag{4.7}$$

Considering the principal branch of square root, we have $\sqrt{1+(x+\mathrm{i}\,y)^2} = X + \mathrm{i}\,Y$. Thus it follows that

$$\mathrm{Im}[G_1(x+\mathrm{i}\,y)] = y + Y = y\left\{1+\frac{x}{X}\right\}.$$

We show $x/X > -1$ below, from which this lemma follows. In the case where $x \geq 0$, clearly $x/X \geq 0 > -1$. Furthermore, in the case where $x < 0$ and $y = 0$, we have

$$X = \frac{1}{\sqrt{2}}\sqrt{\sqrt{(1+x^2-0^2)^2+(2x\cdot 0)^2}+(1+x^2-0^2)} = \sqrt{1+x^2} > \sqrt{x^2} = -x,$$

from which $x/X > -1$ follows. Similarly, in the case where $x < 0$ and $y \neq 0$, we have

$$X > \frac{1}{\sqrt{2}}\sqrt{\sqrt{(1+x^2-y^2)^2+(2\cdot 0\cdot 0)^2}+(1+x^2-y^2)} = \sqrt{1+x^2-y^2} \geq \sqrt{1+x^2-1^2} = -x,$$

which completes the proof. ∎

For the function G_2, we have the next lemma.

Lemma 4.20. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$ with $|y| \leq 1$. Then $\mathrm{Im}[G_2(x+\mathrm{i}\,y)]$ is represented as follows:

$$\mathrm{Im}\left[G_2(x+\mathrm{i}\,y)\right] = yG_2^{+}(x,y), \tag{4.8}$$

where $G_2^{+}(x,y)$ is a positive function.

Proof. Using X and Y defined by (4.7), we can write the function $G_2(x+\mathrm{i}\,y)$ as

$$\begin{aligned} G_2(x+\mathrm{i}\,y) &= \frac{X+\mathrm{i}\,Y}{\{1+\mathrm{e}^{-\pi x}\cos(\pi y)\} - \mathrm{i}\,\{\mathrm{e}^{-\pi x}\sin(\pi y)\}} \\ &= \frac{X\{1+\mathrm{e}^{-\pi x}\cos(\pi y)\} - Y\mathrm{e}^{-\pi x}\sin(\pi y)}{\{1+\mathrm{e}^{-\pi x}\cos(\pi y)\}^2+\{\mathrm{e}^{-\pi x}\sin(\pi y)\}^2} + \mathrm{i}\,\frac{X\mathrm{e}^{-\pi x}\sin(\pi y) + Y\{1+\mathrm{e}^{-\pi x}\cos(\pi y)\}}{\{1+\mathrm{e}^{-\pi x}\cos(\pi y)\}^2+\{\mathrm{e}^{-\pi x}\sin(\pi y)\}^2}. \end{aligned}$$

Hence we have (4.8), where the function G_2^{+} is defined by

$$G_2^{+}(x,y) = \frac{\{X\sin(\pi y)/y\} + (x/X)\{\mathrm{e}^{\pi x}+\cos(\pi y)\}}{\mathrm{e}^{-\pi x}[\{\mathrm{e}^{\pi x}+\cos(\pi y)\}^2+\{\sin(\pi y)\}^2]}.$$

We show $G_2^{+}(x,y) > 0$ below, from which this lemma follows. In the case where $x > 0$, clearly $G_2(x,y) > 0$. Thus let $x \leq 0$ below. First, X is evaluated as

$$X \geq \frac{1}{\sqrt{2}}\sqrt{\sqrt{(1+0^2-y^2)^2+(2\cdot 0\cdot y)^2}+(1+0^2-y^2)} = \sqrt{1-y^2} \geq 1-y^2.$$

The last inequality holds since $|y| \leq 1$. Second, $-1 < x/X \leq 0$ holds from $x \leq 0$ and the proof of Lemma 4.19. Then the numerator of G_2^{+} is evaluated as

$$\{X\sin(\pi y)/y\} + (x/X)\{\mathrm{e}^{\pi x}+\cos(\pi y)\} \geq \{(1-y^2)\sin(\pi y)/y\} - \{1+\cos(\pi y)\}. \tag{4.9}$$

For the denominator of G_2^+, we have

$$\mathrm{e}^{-\pi x}[\{\mathrm{e}^{\pi x}+\cos(\pi y)\}^2+\{\sin(\pi y)\}^2]\leq \mathrm{e}^{-\pi x}[\{1+\cos(\pi y)\}^2+\{\sin(\pi y)\}^2]. \tag{4.10}$$

We here set a function f as

$$f(y)=\frac{\{(1-y^2)\sin(\pi y)/y\}-\{1+\cos(\pi y)\}}{\{1+\cos(\pi y)\}^2+\{\sin(\pi y)\}^2},$$

and in what follows we prove $f(y)>0$ for all $y\in[-1,\,1]$. If it is shown, then by using (4.9) and (4.10) we obtain the desired conclusion:

$$0<\frac{f(y)}{\mathrm{e}^{-\pi x}}=\frac{\{(1-y^2)\sin(\pi y)/y\}-\{1+\cos(\pi y)\}}{\mathrm{e}^{-\pi x}[\{1+\cos(\pi y)\}^2+\{\sin(\pi y)\}^2]}\leq G_2^+(x,y).$$

Since f is an even function, it is sufficient to show $f(y)>0$ for $y\in[0,\,1]$. The derivative of f is written as

$$f'(y)=-\frac{(1+y^2)\sin(\pi y)-\pi y(1-y^2)}{4y^2\cos^2(\pi y/2)},$$

and we can show $(1+y^2)\sin(\pi y)-\pi y(1-y^2)\geq 0$ for $y\in[0,\,1]$. Therefore the function f is monotonically decreasing on the interval $[0,\,1]$. Thus it follows that

$$f(y)\geq f(1)=\frac{4-\pi}{2\pi}>0,$$

which establishes the lemma. ■

Using the two lemmas above, we prove Lemma 4.17.

Proof. In the same argument as in Lemma 4.5, we consider the function $\cos(\operatorname{Im}\zeta(t))$, where $\zeta(t)=\{\psi_{a,t}^{\mathrm{DE}}\}^{-1}(z)$, and show it is a monotonically increasing function. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\cos(\operatorname{Im}(\zeta(t)))=-\sin(\operatorname{Im}\zeta(t))\frac{\mathrm{d}}{\mathrm{d}t}\{\operatorname{Im}\zeta(t)\},$$

we examine $\sin(\operatorname{Im}\zeta(t))$ and $\{\operatorname{Im}\zeta\}'$ below. Let us define a function $\eta(t)$ as $\eta(t)=\frac{1}{\pi}\log\left(\frac{z-a}{t-z}\right)$. Then $\zeta(t)=\log\{\eta(t)+\sqrt{1+\eta^2(t)}\}=\log\{G_1(\eta(t))\}$, and

$$\sin(\operatorname{Im}\zeta(t))=\sin(\arg\{G_1(\eta(t))\})=\frac{\operatorname{Im}[G_1(\eta(t))]}{|G_1(\eta(t))|},$$

because $\sin(\arg(\xi))=\operatorname{Im}\xi/|\xi|$ for all $\xi\in\mathbb{C}$. We set $x_t=\operatorname{Re}\eta(t)$ and $y_t=\operatorname{Im}\eta(t)$ here. Note $|y_t|\leq 1$ by the definition of η. According to Lemma 4.19, using a positive function G_1^+, we have

$$\sin(\operatorname{Im}\zeta(t))=\frac{\operatorname{Im}[G_1(\eta(t))]}{|G_1(\eta(t))|}=\frac{y_t G_1^+(x_t,y_t)}{|G_1(\eta(t))|}. \tag{4.11}$$

Next we examine $\{\operatorname{Im}\zeta\}'$. The function $\operatorname{Im}\zeta(t)$ can be written as

$$\operatorname{Im}\zeta(t)=\frac{1}{2\mathrm{i}}\left[\log\left\{\eta(t)+\sqrt{1+\eta^2(t)}\right\}-\log\left\{\eta^*(t)+\sqrt{1+\{\eta^*(t)\}^2}\right\}\right],$$

where η^* denotes a conjugate complex number of η. By differentiating and rewriting this equation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\zeta(t) = \frac{\operatorname{Im}\left[(t-z)\sqrt{1+\eta^2(t)}\right]}{\left|(t-z)\sqrt{1+\eta^2(t)}\right|^2}.$$

From the definition of $\eta(t)$, we have $(t-z) = (t-a)/(1+\mathrm{e}^{\pi\eta(t)})$, and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\zeta(t) = \left|\frac{1+\mathrm{e}^{\pi\eta(t)}}{(t-a)\sqrt{1+\eta^2(t)}}\right|^2 \operatorname{Im}\left[\frac{(t-a)\sqrt{1+\eta^2(t)}}{1+\mathrm{e}^{\pi\eta(t)}}\right] = \frac{\operatorname{Im}\left[G_2(-\eta(t))\right]}{(t-a)|G_2(-\eta(t))|^2}.$$

Thus by using the representation (4.8) in Lemma 4.20, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\zeta(t) = \frac{(-y_t)G_2^+(-x_t,-y_t)}{(t-a)|G_2(-\eta(t))|^2}. \tag{4.12}$$

Finally, combining the expression (4.11) with (4.12), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\cos(\operatorname{Im}(\zeta(t))) = \frac{y_t^2 G_1^+(x_t,y_t)G_2^+(-x_t,-y_t)}{|G_1(\eta(t))|(t-a)|G_2(-\eta(t))|^2} \geq 0,$$

which is the desired conclusion. ■

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