MATHEMATICAL ENGINEERING TECHNICAL REPORTS

Superharmonic Priors for Autoregressive Models

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(Communicated by Fumiyasu KOMAKI)

METR 2009–18 May 2009

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Superharmonic Priors for Autoregressive Models

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May, 2009

Abstract

Tanaka and Komaki showed that when there exists a superharmonic prior on a stationary ARMA model, the Bayesian spectral density estimator based on the superharmonic prior asymptotically dominates that based on the Jeffreys prior. This result is an extension of Komaki's result for Bayesian predictive densities in the i.i.d. cases. In the second order autoregressive process, a superharmonic prior was obtained by Tanaka and Komaki. Numerical simulation indicates the effectiveness of the superharmonic prior even in small sample. Since the Laplacian is in a complicated form, no superharmonic prior for the higher order autoregressive model has been discovered. In the present paper, we give a superharmonic prior for the autoregressive process in an explicit form. Some systematic methods of dealing with complex polynomial are also developed.

1 Introduction

Let us consider a parametric model of stationary Gaussian process with mean zero. It is known that a stationary Gaussian process corresponds to its spectral density one-to-one (for proof, see, e.g., Brockwell and Davis [4]). In the present paper, we focus on the estimation of the true spectral density $S(\omega|\theta_0)$ in a parametric family of spectral densities

$$\mathcal{M} := \{S(\omega|\theta) : \theta \in \Theta \subseteq \mathbf{R}^k\}.$$

The performance of a spectral density estimator $\hat{S}(\omega|x)$, where x denotes an observation, is evaluated by the Kullback-Leibler divergence.

$$D(S(\omega|\theta_0)||\hat{S}(\omega|x)) := \int_{-\pi}^{\pi} \frac{\mathrm{d}\omega}{4\pi} \left\{ \frac{S(\omega|\theta_0)}{\hat{S}(\omega|x)} - 1 - \log\left(\frac{S(\omega|\theta_0)}{\hat{S}(\omega|x)}\right) \right\}.$$

The above setting is proposed by Komaki [6].

First, let us consider minimizing the average risk assuming that a proper prior density $\pi(\theta)$ is known in advance. The spectral density estimator minimizing the average risk,

$$\begin{aligned} &\mathrm{E}^{\pi}\mathrm{E}^{p_n}[D(S(\omega|\theta)||\hat{S}(\omega|x))] \\ &:= \int \mathrm{d}\theta\pi(\theta) \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \; p_n(x_1, \ldots, x_n|\theta) D(S(\omega|\theta)||\hat{S}(\omega|x)), \end{aligned}$$

is given by the Bayesian spectral density (with respect to $\pi(\theta)$), which is defined by

$$S_{\pi}(\omega|x) := \int S(\omega|\theta)\pi(\theta|x)\mathrm{d}\theta. \tag{1}$$

We call $S_{\pi}(\omega|x)$ in (1) a *Bayesian spectral density* even when an improper prior distribution is considered.

Generally, speaking, if one has no information on the unknown parameter θ, it is natural to adopt a noninformative prior in the Bayesian framework. The Jeffreys prior is a well-known candidate for a noninformative prior from several reasons, but often improper and then there is much room to argue the choice of a noninformative prior.

Komaki showed that the Bayesian predictive density based on a superharmonic prior asymptotically dominates that based on the Jeffreys prior if there exists a superharmonic prior in the parametric model [7]. While his result is in the i.i.d. setting, Tanaka and Komaki [10] extended to the estimation of spectral densities in the ARMA model. When there exists a superharmonic prior on a stationary ARMA model, the Bayesian spectral density estimator based on a superharmonic prior asymptotically dominates that based on the Jeffreys prior, where the Jeffreys prior is calculated by the Fisher metric (Fisher information matrix) as usual. The Fisher metric of a parametric model of spectral densities $\mathcal{M}$ is defined by

$$g_{ij} := g\left(\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\theta^j}\right) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\omega}{4\pi} \frac{\partial_i S(\omega|\theta)}{S(\omega|\theta)} \frac{\partial_j S(\omega|\theta)}{S(\omega|\theta)} \tag{2}$$

(Amari [1]). Indeed, Tanaka and Komaki [9] find a superharmonic prior in the AR(2) process and validated their result by numerical simulation.

Here we emphasize that a parameteric model of spectral densities does not always admit a superharmonic prior. Definition of a superharmonic prior is given later, but the existence of a superharmonic prior is reduced to the existence of a nonconstant positive solution of a second order differential inequality on a model manifold. Until now, it is an open problem to determine if there exists a superharmonic prior for the AR(p) process ($p \geq 3$). In the present paper, we present a superharmonic prior for the AR(p) process ($p \geq 3$) in an explicit form, which is a positive result.

In the next section, we briefly review our notation in AR model manifolds. For statistical model manifolds and differential geometrical concepts in statistics, see, e.g., Amari and Nagaoka [2]. In Section 3, we mention our main result, the explicit form of a superharmonic prior in the AR model. Concluding remarks follow in Section 4. Proof is given in Appendix. It is straightforward but still needs a systematic way of dealing with lots of irreducible fractional polynomial.

2 Basic Definition and Notation

2.1 Fisher metric on the AR model manifold

Autoregressive (AR) models are widely-known in the field of time series analysis and defined as follows. A *p-th order AR model* with AR parameter $a_1, \ldots, a_p$ is defined by

$$X_t = -\sum_{i=1}^{p} a_i X_{t-i} + W_t,$$

where $\{W_t\}$ is a Gaussian white noise with mean 0 and variance σ^2. Now, we define the shift operator Z by $ZX_t = X_{t+1}$. Then, $Z^{-i}X_t = X_{t-i}$ and

$$X_t = H_a(Z)^{-1} W_t, \quad H_a(Z) := \sum_{i=0}^{p} a_i Z^{-i} \text{ with } a_0 = 1 \ .$$

In the present paper only stationary AR models are considered.

According to Komaki [6], we calculate the Fisher metric on the AR model manifolds. The explicit form of the spectral density of the AR model is given by

$$S(\omega | a_1, \ldots, a_p, \sigma^2) = \frac{\sigma^2}{2\pi} \frac{1}{|H_a(z)|^2}, \quad z = e^{i\omega}.$$

Here, we adopt another coordinate system, which brings us a more convenient form to consider. Equation $z^p H_a(z) = z^p + a_1 z^{p-1} + \cdots + a_{p-1} z + a_p$ is a polynomial of degree p and has p complex roots, $z_1, z_2, \ldots, z_p$ (Note that $|z_i| < 1$ from the stationarity condition). Since $a_1, a_2, \ldots, a_p$ are all real, it consequently has the conjugate roots. Thus, we can put them in the order like, $z_1, \ldots, z_q, z_{q+1}, \ldots, z_{2q} \in \mathbf{C}, z_{2q+1}, \ldots, z_{2q+r} \in \mathbf{R}$ and $z_{q+j} = \bar{z}_j (1 \leq j \leq q)$ (for simplicity, we assume that there are no multiple roots). The roots $z_1, z_2, \ldots, z_p$ correspond to the original parameter $a_1, a_2, \ldots, a_p$ in a one-to-one manner. Now we introduce a coordinate system $(\theta^1, \theta^2, \ldots, \theta^p)$ using these roots

$$\theta^0 := \sigma^2, \quad \theta^1 := z_1, \quad \theta^2 := z_2, \ \ldots\ , \theta^p := z_p.$$

In the remainder of the paper indices $I, J, K, \dots$ run $0, 1, \dots, p$ (from zero) and indices $i, j, k, \dots$ run $1, 2, \dots, p$ (from one). The formal complex derivatives are defined by

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad ,$$

where x and y are both real part and imaginary part of z. See, for example, Gunning and Rossi [5]. Since the conjugate complex coordinates z_i and $\bar{z}_i$ correspond to x_i and y_i in a one-to-one manner, each quantity is evaluated in the original real coordinate if necessary. Index i and the imaginary unit $i := \sqrt{-1}$ often appear simultaneously but they are clearly distinguished from context.

In the coordinate system given above, the Fisher metric g_{IJ} is

$$g_{IJ} = \left(\begin{array}{c|ccc} g_{00} & \cdots & g_{0i} & \cdots \\ \hline \vdots & \cdots & \cdots & \cdots \\ g_{i0} & \vdots & g_{ij} & \vdots \\ \vdots & \vdots & \cdots & \vdots \end{array}\right) \quad \text{and} \quad \left\{ \begin{array}{ccc} g_{00} & = & \frac{1}{2(\theta^0)^2} = \frac{1}{2\sigma^4} \\ g_{0i} & = & g_{i0} = 0 \\ g_{ij} & = & \frac{1}{1 - z_i z_j} \end{array} \right. \quad , \qquad (3)$$

see [6].

2.2 Superharmonic prior

We describe the general definition of a superharmonic prior. Let $\mathcal{M}$ denote a Riemannian manifold with a coordinate θ. A scalar function $\phi(\theta)$ on $\mathcal{M}$ is called *a superharmonic function* if it satisfies,

$$\Delta\phi(\theta) \leq 0 \quad \forall\theta,$$

where Δ is the Laplace-Beltrami operator. Let g_{IJ} be a Riemannian metric (Fisher metric), g^{IJ}, the inverse of g_{IJ}, and $g := \det(g_{IJ})$. The Laplace-Beltrami operator is defined by

$$\Delta\phi := \frac{1}{\sqrt{g}} \frac{\partial}{\partial \theta^I} \left(\sqrt{g} g^{IJ} \frac{\partial}{\partial \theta^J} \phi \right).$$

If a superharmonic function is positive, i.e., $\phi(\theta) > 0$, $\forall\theta$, then it is called *a positive superharmonic function.* When a model manifold endowed with the Fisher metric has a non-constant positive superharmonic function $\phi(\theta)$, we call $\pi_H(\theta) := \pi_J(\theta)\phi(\theta)$ *a superharmonic prior.* Note that not all model manifolds with the Fisher metric admit a superharmonic prior while all of them admit the Jeffreys prior because the Jeffreys prior is given as a volume element on the model manifold (i.e., $\pi_J(\theta) \propto \sqrt{g(\theta)}$).

In the AR(p) model manifold, Δ can be decomposed into two parts. One part is relevant with $\theta^0 = \sigma^2$ and the others with $\theta^1, \cdots, \theta^p$. Thus, without loss of generality, we can set $\sigma^2 = 1$. (See Tanaka and Komaki [9], for details) Since we assume the stationarity condition, the parameter region on the AR(p) model is given by

$$\Omega := \{\theta = (\theta^1, \dots, \theta^p) = (z_1, \dots, z_p) : \ |z_1| < 1, |z_2| < 1, \dots, |z_p| < 1\}.$$

3 Superharmonic Prior for the AR(p) Process

In this section, we obtain a superharmonic prior for the AR(p) process. We begin with a positive superharmonic function on the AR model manifold. General formula is given by

$$\phi(\theta) = \prod_{i<j}(1 - z_i z_j). \tag{4}$$

For example, when $p = 3$,

$$\phi = (1 - z_1 z_2)(1 - z_1 z_3)(1 - z_2 z_3).$$

We see that the above superharmonic function (4) is not only a positive superharmonic function, but also the eigenfunction of the Laplace-Beltrami operator Δ.

Theorem 3.1.
When $p \geq 2$, for the above ϕ (4),

$$\Delta\phi = -\frac{p(p-1)}{2}\phi \tag{5}$$

holds. Thus, ϕ is a positive nonconstant superharmonic function for the AR(p) model manifold.

Proof.
First we check positivity of ϕ because we introduce formal complex variables. Recall that we assume the stationarity condition, which says

$$|z_i| < 1.$$

For all real roots ($z_i \in \mathbf{R}$), clearly ϕ is positive. If there are complex conjugate pair of roots $z_i, z_{i+r} = \bar{z}_i$, then such terms are rewritten as

$$\begin{aligned}\prod_k (1 - z_i z_k)(1 - z_{i+r} z_k) &= \prod_{k:\ z_k \in \mathbf{R}} (1 - z_i z_k)(1 - z_{i+r} z_k) \\ &\quad \times \prod_{k:\ z_k \in \mathbf{C}} (1 - z_i z_k)(1 - z_{i+r} z_k)\end{aligned}$$

If $z_k \in \mathbf{R}$, we obtain

$$(1-z_iz_k)(1-z_{i+r}z_k) = (1-z_iz_k)(1-\overline{z_iz_k}) = |1-z_iz_k|^2 \geq 0.$$

If $z_k \in \mathbf{C}$, gathering the terms including complex conjugate pair $z_{k+r} = \bar{z}_k$, we obtain

$$\begin{aligned} & (1-z_iz_k)(1-z_{i+r}z_k)(1-z_iz_{k+r})(1-z_{i+r}z_{k+r}) \\ = & (1-z_iz_k)(1-\overline{z_iz_k}) \times (1-\bar{z}_iz_k)(1-\overline{\bar{z}_iz_k}) \\ = & |1-z_iz_k|^2|1-\bar{z}_iz_k|^2 \geq 0. \end{aligned}$$

Thus,

$$\prod_k (1-z_iz_k)(1-z_{i+r}z_k) \geq 0.$$

and $\phi \geq 0$.

Next, we show Eq.(5). We set $g := \det g_{ij}$ (Here, recall that indices $i, j, \ldots,$ run $1, 2, \ldots, p$). Then, $\frac{\Delta\phi}{\phi}$ is rewritten in the following form.

$$\begin{aligned} \frac{\Delta\phi}{\phi} &= \frac{1}{\sqrt{g}} \frac{\partial_i\left(\sqrt{g}\partial^i\phi\right)}{\phi} \\ &= \frac{1}{2}(\partial_i \log g)\partial^i \log\phi + \frac{\partial_i(\phi\partial^i \log\phi)}{\phi} \\ &= \frac{1}{2}(\partial_i \log g)\partial^i \log\phi + (\partial_i \log\phi)(\partial^i \log\phi) + \partial_i\partial^i \log\phi \\ &= f_i\partial^i \log\phi + \partial_i\partial^i \log\phi, \end{aligned}$$

where we set $f_i := \frac{1}{2}\partial_i \log g + \partial_i \log\phi$. Now we calculate terms f_i, $\partial_i \log\phi$, and $\partial^i \log\phi$.

First we calculate $\log(\sqrt{g}\phi)$.

$$\begin{aligned} \log\left(\sqrt{g}\phi\right) &= \log\left[\left|\frac{\prod_{i<j}(z_i-z_j)^2}{\prod_{i=1}^p\prod_{j=1}^p(1-z_iz_j)^2}\right|^{\frac{1}{2}} \times \prod_{j>i}(1-z_iz_j)\right] \\ &= \frac{1}{2}\log\prod_{i<j}|z_i-z_j|^2 - \frac{1}{2}\log\left\{\prod_{i=1}^p(1-z_i^2)\right\} \\ &= \log|\mathit{\Delta}| - \frac{1}{2}\log\left\{\prod_{i=1}^p(1-z_i^2)\right\}, \end{aligned}$$

where $\mathit{\Delta}$ is Vandermond determinant.(See Appendix.). Thus,

$$\begin{aligned} f_i &= \frac{1}{2}(\partial_i \log g) + \partial_i \log\phi \\ &= \partial_i \log(\sqrt{g}\phi) \\ &= \partial_i \log|\mathit{\Delta}| + \frac{z_i}{1-z_i^2}. \end{aligned}$$

From now on, since summation rule is irregular, we indicate summation of terms by $\sum$. We evaluate $\partial_i \log\phi$,

$$
\begin{aligned}
\partial_i \log\phi &= \frac{\partial}{\partial z_i}\left(\sum_{j>k}\log(1-z_k z_j)\right)\\
&= \sum_{k\neq i}\frac{-z_k}{1-z_k z_i}\\
&= \sum_{k=1}^{p}\frac{-z_k}{1-z_k z_i}+\frac{z_i}{1-{z_i}^2}.
\end{aligned}
$$

Finally, we rewrite $\partial^j \log\phi$.

$$
\begin{aligned}
g^{ji}\partial_i \log\phi &= \sum_{i=1}^{p}\sum_{k=1}^{p} g^{ji}\left(\frac{-z_k}{1-z_k z_i}\right)+\sum_{i=1}^{p} g^{ji}\left(\frac{z_i}{1-{z_i}^2}\right)\\
&= -z_j+\sum_{i=1}^{p} g^{ji}\left(\frac{z_i}{1-{z_i}^2}\right).
\end{aligned}
$$

Thus, putting these terms together, we obtain

$$
\begin{aligned}
\frac{\Delta\phi}{\phi} &= \sum_{i=1}^{p}\left[\left(\partial_i \log|\mathit{\Delta}|+\frac{z_i}{1-z_i^2}\right)\times\left\{-z_i+\sum_{j=1}^{p} g^{ij}\left(\frac{z_j}{1-z_j^2}\right)\right\}\right]\\
&\quad+\sum_{i=1}^{p}\partial_i\left\{-z_i+\sum_{j=1}^{p} g^{ij}\left(\frac{z_j}{1-z_j^2}\right)\right\}\\
&= -\sum_{i=1}^{p} z_i\left(\frac{\partial_i\mathit{\Delta}}{\mathit{\Delta}}\right)-\sum_{i=1}^{p}\partial_i z_i\\
&\quad+\left[\sum_{i=1}^{p}\sum_{j=1}^{p}\left(\frac{\partial_i\mathit{\Delta}}{\mathit{\Delta}}\right) g^{ij}\left(\frac{z_j}{1-z_j^2}\right)+\sum_{i=1}^{p}\sum_{j=1}^{p}\partial_i\left\{g^{ij}\left(\frac{z_j}{1-z_j^2}\right)\right\}\right]\\
&\quad+\left\{-\sum_{i=1}^{p} z_i\left(\frac{z_i}{1-z_i^2}\right)+\sum_{i=1}^{p}\sum_{j=1}^{p} g^{ij}\left(\frac{z_i}{1-z_i^2}\right)\left(\frac{z_j}{1-z_j^2}\right)\right\}
\end{aligned}
$$

The first term is shown to be equal to $-\frac{p(p-1)}{2}$. The second term is clearly equal to $-p$. The other terms are calculated in Appendix. Final result is as follows.

Lemma 3.1.

$$
\begin{aligned}
(A) &:= -\sum_{i=1}^{p} z_i \left(\frac{z_i}{1-{z_i}^2}\right) + \sum_{i=1}^{p}\sum_{j=1}^{p} g^{ij}\left(\frac{z_i}{1-{z_i}^2}\right)\left(\frac{z_j}{1-{z_j}^2}\right) \\
&= \begin{cases} \frac{1}{2}p & \text{even } p \\ \frac{1}{2}(p-1) & \text{odd } p \end{cases}
\end{aligned}
$$

Lemma 3.2.

$$
\begin{aligned}
(B) &:= \sum_{i=1}^{p}\sum_{j=1}^{p}\left[\left(\frac{\partial_i \varDelta}{\varDelta}\right) g^{ij}\left(\frac{z_j}{1-z_j^2}\right) + \frac{\partial}{\partial z_i}\left\{g^{ij}\left(\frac{z_j}{1-z_j^2}\right)\right\}\right] \\
&= \begin{cases} \frac{1}{2}p & \text{even } p \\ \frac{1}{2}(p+1) & \text{odd } p \end{cases}
\end{aligned}
$$

Thus, when p is even, $\frac{\Delta\phi}{\phi} = -\frac{p(p-1)}{2} + (-p) + \frac{1}{2}p + \frac{1}{2}p = -\frac{p(p-1)}{2}$. When p is odd, we also obtain the same result. *Q.E.D.*

Now we obtain the final result.

Theorem 3.2.
When $p \geq 2$, a superharmonic prior for the AR(p) process is given by

$$
\pi_H = \phi(z_1, \ldots, z_p)\pi_J \propto \left|\frac{\prod_{i<j}(z_i - z_j)^2}{\prod_{i=1}^{p}(1-z_i^2)}\right|^{\frac{1}{2}},
$$

where the parameter $\theta^i = z_i, i = 1, \ldots, p$ are roots of characteristic equation defined by $\sum_{l=0}^{p} a_l z^{p-l} = 0$. (See Section 2).

Proof.
Recall that by definition,

$$
\pi_J \propto \sqrt{g} = \left|\frac{\prod_{i<j}(z_i - z_j)^2}{\prod_{i=1}^{p}\prod_{j=1}^{p}(1 - z_i z_j)}\right|^{\frac{1}{2}}.
$$

Note that the absolute value $|\cdot|$ is required if z is complex. Since

$$\begin{aligned}\pi_H &= \phi(z_1,\ldots,z_p)\pi_J \\ &\propto \prod_{i<j}(1-z_iz_j)\times\left|\frac{\prod_{i<j}(z_i-z_j)^2}{\prod_{i=1}^p\prod_{j=1}^p(1-z_iz_j)}\right|^{\frac{1}{2}} \\ &= \left|\frac{\prod_{i<j}(z_i-z_j)^2}{\prod_{i=1}^p(1-z_i^2)}\right|^{\frac{1}{2}},\end{aligned}$$

Due to Theorem 3.1, π_H/π_J is a positive nonconstant superharmonic function.
Q.E.D.

4 Concluding Remarks

In Tanaka and Komaki [9], numerical simulation of the spectral density estimation for the AR(2) process is also presented. For higher order autoregressive process, numerical simulation itself is not so trivial because the expression of a superharmonic prior includes complex conjugate pair of roots and the stationarity region is divided into some regions corresponding to r real roots and q complex conjugate pairs. The expression of both priors in the AR parameter also seems complex for the AR(p) model when $p \geq 3$. Numerical simulation in another parametrization is left to the future work.

From differential geometrical viewpoint, our result is deeply related to the theorem connecting a global property of a Riemannian manifold and local one by Aomoto [3], which claims, a sufficient condition for the existence of a positive nonconstant superharmonic function, is that the sectional curvature is negative for *any plane* and at *any point*. Indeed, in Theorem 3.1, we see that the higer order (i.e., $p \geq 3$) AR model manifolds admit a positive nonconstant superharmonic function although the sectional curvature of the AR model manifold ($p \geq 3$) is strictly positive for *some plane* and at *some point*. (See, Tanaka and Komaki [8]). As far as the author knows, this is the first nontrivial counterexample. Thus, it is expected that Aomoto's sufficient condition is modified to some extent in a more appropriate (weaker) form.

Acknowledgment

This research was supported by JST PRESTO.

A Vandermonde Determinant and Related Summation Formulas

Let n be fixed and Z be the field of rational functions of $z_1, z_2, \ldots, z_n$, i.e., $Z := \mathbf{R}(z_1, z_2, \ldots, z_n)$ and $f(X)$ be polynomials whose coefficients are rational expression of n-variables $z_1, z_2, \ldots, z_n$. Then, we introduce a Z-linear map V of $f(X)$

$$V:\ f(X) \longmapsto V(f(X)) := \sum_{m=1}^{n} (-1)^{m+1} f(z_m) \mathit{\Delta}_m, \tag{6}$$

where $\mathit{\Delta}_m$ is obtained from the Vandermonde determinant by subtracting z_m term, i.e.,

$$\mathit{\Delta} := \prod_{i<j} (z_j - z_i) \quad \text{and} \quad \mathit{\Delta}_m := \frac{\mathit{\Delta}}{\prod_{l \neq m} (z_m - z_l)} (-1)^{n-m}.$$

As a useful notation, we define $\tilde{\mathit{\Delta}} := V(X^n)$.

Example
For $n = 3$, we obtain

$$\mathit{\Delta} = (z_3 - z_2)(z_3 - z_1)(z_2 - z_1),$$
$$\mathit{\Delta}_1 = z_3 - z_2,\ \mathit{\Delta}_2 = z_3 - z_1,\ \mathit{\Delta}_3 = z_2 - z_1.$$

When $f(X) := X^2 \in Z(X)$,

$$\begin{aligned} V(f) = V(X^2) &= \sum_{m=1}^{3} (-1)^{m+1} f(z_m) \mathit{\Delta}_m \\ &= f(z_1)(z_3 - z_2) - f(z_2)(z_3 - z_1) + f(z_3)(z_2 - z_1) \\ &= 0. \end{aligned}$$

It is due to the asymmetry of the summation form (6). This property is easily generalized when f is at most $n-1$-th degree polynomial. We briefly review some useful formula shown in Tanaka and Komaki [8].

Lemma A.1.

$$V(X^p) = 0 \quad (p = 0, 1, \ldots, n-2) \quad \text{and} \quad V(X^{n-1}) = (-1)^{n-1} \mathit{\Delta}.$$
$$F(a) := V\left(\frac{1}{1 - aX}\right) = (-a)^{n-1} \mathit{\Delta} \prod_{l=1}^{n} \frac{1}{1 - a z_l}, \quad a \in \mathbf{R}.$$

A bit tedious form is also evaluated if we differentiate $F(a)$ with respect to a.

Lemma A.2.

$$V\left(\frac{X^p}{1-aX}\right) = \begin{cases} \frac{1}{a^p}F(a) & (0 \le p \le n-1) \\ \frac{1}{a^n}F(a) - \frac{1}{a}(-1)^{n-1}\varDelta & (p=n) \\ \frac{1}{a^{n+1}}F(a) - \frac{1}{a}\tilde{\varDelta} - \frac{1}{a^2}(-1)^{n-1}\varDelta & (p=n+1) \end{cases},$$

$$V\left(\frac{X^p}{(1-aX)^2}\right) = \begin{cases} -\frac{p-1}{a^p}F(a) + \frac{1}{a^{p-1}}\frac{\partial F(a)}{\partial a} & (0 \le p \le n) \\ -\frac{n}{a^{n+1}}F(a) + \frac{1}{a^n}\frac{\partial F(a)}{\partial a} + \frac{1}{a^2}(-1)^{n-1}\varDelta & (p=n+1) \end{cases},$$

$$\begin{aligned} V\left(\frac{X^p}{(1-aX)^3}\right) &= \frac{\frac{1}{2}(p-1)(p-2)}{a^p}F(a) - \frac{p-2}{a^{p-1}}\frac{\partial F(a)}{\partial a} \\ &\quad + \frac{1}{2}\frac{1}{a^{p-2}}\frac{\partial^2 F(a)}{\partial a^2} \qquad (0 \le p \le n+1). \end{aligned}$$

The following formula reminds us of Cauchy's formula in complex analysis. Using Lemma A.1 and Lemma A.2, we easily obtain all of them.

Lemma A.3.
Let $G(X)$ be a polynomial of X, of at most n-th degree, i.e., $G(X) := \sum_{p=0}^{n} A_p X^p, \quad A_p \in Z$. *Then, the following holds.*

$$V\left(\frac{G(X)}{1-aX}\right) = G\left(\frac{1}{a}\right)F(a) - (-1)^{n-1}\frac{1}{a}A_n\varDelta, \tag{7}$$

$$V\left(\frac{G(X)}{(1-aX)(1-bX)}\right) = \frac{1}{a-b}\left\{aG\left(\frac{1}{a}\right)F(a) - bG\left(\frac{1}{b}\right)F(b)\right\}, \text{ if } a \neq b, \tag{8}$$

$$V\left(\frac{G(X)}{(1-aX)^2}\right) = \frac{\partial}{\partial a}\left\{aG\left(\frac{1}{a}\right)F(a)\right\}. \tag{9}$$

Proposition A.1.(Special case in Cauchy's double alternants.) [8]
Let $n \ge 1$, and a matrix g_{mh} be defined by

$$g_{mh} = \frac{1}{1-z_m z_h}, |z_j| < 1, \quad 1 \le m, h \le n.$$

Then, the inverse of g_{mh} is given by

$$g^{mh} = \frac{(1-z_m z_h)\prod_{l\neq h}(1-z_l z_m)\prod_{l\neq m}(1-z_l z_h)}{\prod_{l\neq h}(z_h - z_l)\prod_{l\neq m}(z_m - z_l)}. \tag{10}$$

For later convenience, we rewrite g^{mh} using the Vandermonde determinant

$$g^{mh} = (-1)^{m+h}\frac{G(z_m)G(z_h)}{1-z_m z_h}\frac{\Delta_m \Delta_h}{\Delta^2},$$

where $G(X) := \prod_{l=1}^{n}(1-z_l X) = \sum_{p=0}^{n} A_p X^p$.

B Preparation for Proof of Lemmas 3.1 and 3.2

First, we derive some formulas using the above Lemmas. Here, we set $n = p$ in the above notation and fix $G(X) := \prod_{l=1}^{p}(1-z_l X)$. It is convenient to define

$$L(a) := \prod_{l=1}^{p}\left(\frac{a-z_l}{1-az_l}\right).$$

Clearly, $L(1) = 1,\ L(-1) = (-1)^p,\ L(z_i) = 0, i = 1, \ldots, p$. We also obtain

$$\begin{aligned} aG\left(\frac{1}{a}\right)F(a) &= a\prod_{l=1}^{p}\left(1-\frac{z_l}{a}\right)\frac{(-a)^{p-1}\Delta}{\prod_{l=1}^{p}(1-az_l)} \\ &= (-1)^{p-1}\Delta\prod_{l=1}^{p}\left(\frac{a-z_l}{1-az_l}\right) \\ &= (-1)^{p-1}\Delta L(a). \end{aligned}$$

Since

$$\begin{aligned} \frac{\partial}{\partial a}\log L(a) &= \frac{\partial}{\partial a}\left\{\sum_{l=1}^{p}\log(a-z_l) - \sum_{l=1}^{p}\log(1-az_l)\right\} \\ &= \sum_{l=1}^{p}\frac{1}{a-z_l} + \sum_{l=1}^{p}\frac{z_l}{1-az_l}, \end{aligned}$$

we obtain

$$\begin{aligned} L'(1) &= \sum_{l=1}^{p}\frac{1+z_l}{1-z_l}, \\ L'(-1) &= (-1)^p\sum_{l=1}^{p}\frac{-1+z_l}{1+z_l}. \end{aligned}$$

By Lemma A.3, we obtain useful formulas below. First,

$$\begin{aligned} V\left(\frac{G(X)}{(1-aX)(1-bX)}\right) &= \frac{1}{a-b}\left\{aG\left(\frac{1}{a}\right)F(a) - bG\left(\frac{1}{b}\right)F(b)\right\} \\ &= \frac{1}{a-b}\left\{(-1)^{p-1}\Delta L(a) - (-1)^{p-1}\Delta L(b)\right\} \\ &= (-1)^{p-1}\Delta\frac{L(a)-L(b)}{a-b} \end{aligned}$$

holds when $a \neq b$. In particular,

$$V\left(\frac{G(X)}{(1-X)(1-z_iX)}\right) = \frac{(-1)^{p-1}}{1-z_i}\Delta \tag{11}$$

$$V\left(\frac{G(X)}{(1+X)(1-z_iX)}\right) = \frac{1}{1+z_i}\Delta \tag{12}$$

$$V\left(\frac{G(X)}{(1-X^2)}\right) = \frac{(-1)^{p-1}+1}{2}\Delta \tag{13}$$

hold. If $a = b$, we use the following formula,

$$\begin{aligned} V\left(\frac{G(X)}{(1-aX)^2}\right) &= \frac{\partial}{\partial a}\left\{aG\left(\frac{1}{a}\right)F(a)\right\} \\ &= (-1)^{p-1}\Delta\frac{\partial}{\partial a}L(a). \end{aligned}$$

Substituting a to ± 1, we obtain

$$V\left(\frac{G(X)}{(1-X)^2}\right) = (-1)^{p-1}\Delta\sum_{l=1}^{p}\frac{1+z_l}{1-z_l}, \tag{14}$$

$$V\left(\frac{G(X)}{(1+X)^2}\right) = \Delta\sum_{l=1}^{p}\frac{1-z_l}{1+z_l}. \tag{15}$$

In what follows, we use the above formulas, (11)-(15) in order to show Lemma 3.1, Lemma 3.2.

C Proof of Lemma 3.1

We again present the statement to be proved in this section.

Lemma 3.1.

$$\begin{aligned} (A) &:= -\sum_{i=1}^{p} z_i\left(\frac{z_i}{1-{z_i}^2}\right) + \sum_{i=1}^{p}\sum_{j=1}^{p} g^{ij}\left(\frac{z_i}{1-{z_i}^2}\right)\left(\frac{z_j}{1-{z_j}^2}\right) \\ &= \begin{cases} \frac{1}{2}p & \text{even } p \\ \frac{1}{2}(p-1) & \text{odd } p \end{cases}. \end{aligned}$$

Proof.
From Proposition A.1,

$$g^{ij} = (-1)^{i+j}\frac{G(z_i)G(z_j)}{1 - z_i z_j}\frac{\Delta_i}{\Delta}\frac{\Delta_j}{\Delta}.$$

Then,

$$\begin{aligned}
\sum_{j=1}^{p} g^{ij}\frac{z_j}{1 - {z_j}^2} &= \sum_{j=1}^{p}(-1)^{i+j}\frac{G(z_i)G(z_j)}{1 - z_i z_j}\frac{\Delta_i}{\Delta}\frac{\Delta_j}{\Delta} \times \frac{z_j}{1 - {z_j}^2} \\
&= (-1)^{i+1}G(z_i)\frac{\Delta_i}{\Delta^2}\left\{\sum_{j=1}^{p}\frac{z_j}{1 - {z_j}^2}\frac{G(z_j)}{1 - z_i z_j}(-1)^{j+1}\Delta_j\right\} \\
&= (-1)^{i+1}G(z_i)\frac{\Delta_i}{\Delta^2}V\biggl(\frac{XG(X)}{(1 - X^2)(1 - z_i X)}\biggr).
\end{aligned}$$

Using Eq.(11),(12), this term is rewritten as

$$\begin{aligned}
& V\left(\frac{XG(X)}{(1 - X^2)(1 - z_i X)}\right) \\
= \; & V\left[\left\{\frac{1}{2}\frac{1}{1 - X} - \frac{1}{2}\frac{1}{1 + X}\right\}\frac{G(X)}{1 - z_i X}\right] \\
= \; & \frac{1}{2}\left\{V\left(\frac{G(X)}{(1 - X)(1 - z_i X)}\right) - V\left(\frac{G(X)}{(1 + X)(1 - z_i X)}\right)\right\} \\
= \; & \frac{1}{2}\left\{\frac{1}{1 - z_i}(-1)^{p-1}\Delta - \frac{1}{1 + z_i}\Delta\right\} \\
= \; & -\frac{1}{2}\left(\frac{(-1)^p}{1 - z_i} + \frac{1}{1 + z_i}\right)\Delta \\
= \; & -h_p(z_i)\Delta,
\end{aligned}$$

where $h_p(x)$ is defined by

$$h_p(x) := \begin{cases} \frac{1}{1-x^2} & p \text{ even} \\ \frac{-x}{1-x^2} & p \text{ odd} \end{cases}.$$

Thus, we obtain

$$\sum_{j=1}^{p} g^{ij}\frac{z_j}{1 - {z_j}^2} = (-1)^i G(z_i)h_p(z_i)\frac{\Delta_i}{\Delta}. \tag{16}$$

Next, we calculate (A).

$$\begin{aligned}
(A) &= \sum_{i=1}^{p} \frac{-z_i^2}{1-z_i^2} + \sum_{i=1}^{p}\sum_{j=1}^{p} g^{ij} \frac{z_i}{1-{z_i}^2}\frac{z_j}{1-{z_j}^2} \\
&= \sum_{i=1}^{p} \frac{-z_i^2}{1-z_i^2} + \sum_{i=1}^{p} (-1)^i G(z_i) h_p(z_i) \frac{z_i}{1-{z_i}^2}\frac{\Delta_i}{\Delta} \\
&= \sum_{i=1}^{p} \frac{-z_i^2}{1-z_i^2} - \frac{1}{\Delta} V\bigg(G(X) h_p(X) \frac{X}{1-X^2} \bigg).
\end{aligned}$$

Now we deal with the last term separately for even p and odd p.

(i) When p is even

When p is even, observing that

$$\frac{X}{(1-X^2)^2} = \frac{1}{4}\left\{ \frac{1}{(1-X)^2} - \frac{1}{(1+X)^2} \right\}$$

and using Eq.(14),(15), we obtain

$$\begin{aligned}
V\bigg(G(X) h_p(X) \frac{X}{1-X^2} \bigg) &= V\bigg(\frac{XG(X)}{(1-X^2)^2} \bigg) \\
&= \frac{1}{4}\left\{ V\bigg(\frac{G(X)}{(1-X)^2} \bigg) - V\bigg(\frac{G(X)}{(1+X)^2} \bigg) \right\} \\
&= \frac{1}{4}\left\{ (-1)^{p-1} \Delta \sum_{l=1}^{p} \frac{1+z_l}{1-z_l} - \Delta \sum_{l=1}^{p} \frac{1-z_l}{1+z_l} \right\} \\
&= -\frac{\Delta}{2} \sum_{l=1}^{p} \left(\frac{1+z_l^2}{1-z_l^2} \right). \qquad (17)
\end{aligned}$$

Thus,

$$\begin{aligned}
(A)_{\text{even}} &= \sum_{i=1}^{p} \frac{-z_i^2}{1-z_i^2} + \left(-\frac{1}{\Delta} \right) \left\{ -\frac{\Delta}{2} \sum_{l=1}^{p} \left(\frac{1+z_l^2}{1-z_l^2} \right) \right\} \\
&= \frac{p}{2}.
\end{aligned}$$

(ii) When p is odd

When p is odd, observing that

$$\frac{X^2}{(1-X^2)^2} = \frac{1}{4}\left\{ \frac{1}{(1-X)^2} + \frac{1}{(1+X)^2} \right\} - \frac{1}{2}\frac{1}{1-X^2}$$

and using Eq.(13),(14) and (15),

$$
\begin{aligned}
& V\left(G(X)h_p(X)\frac{X}{1-X^2}\right) && (18)\\
= \;& -V\left(\frac{X^2G(X)}{(1-X^2)^2}\right)\\
= \;& -\frac{1}{4}\left\{V\!\left(\frac{G(X)}{(1-X)^2}\right)+V\!\left(\frac{G(X)}{(1+X)^2}\right)\right\}+\frac{1}{2}V\!\left(\frac{G(X)}{1-X^2}\right)\\
= \;& -\frac{1}{4}\left\{(-1)^{p-1}\mathit{\Delta}\sum_{l=1}^{p}\frac{1+z_l}{1-z_l}+\mathit{\Delta}\sum_{l=1}^{p}\frac{1-z_l}{1+z_l}\right\}+\frac{1}{2}\left\{\frac{(-1)^{p-1}+1}{2}\mathit{\Delta}\right\}\\
= \;& \frac{1}{2}\mathit{\Delta}-\frac{\mathit{\Delta}}{2}\sum_{l=1}^{p}\left(\frac{1+z_l^2}{1-z_l^2}\right). && (19)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(A)_{\mathrm{odd}} &= \sum_{i=1}^{p}\frac{-z_i^2}{1-z_i^2}+\left(-\frac{1}{\mathit{\Delta}}\right)\left\{\frac{1}{2}\mathit{\Delta}-\frac{\mathit{\Delta}}{2}\sum_{l=1}^{p}\left(\frac{1+z_l^2}{1-z_l^2}\right)\right\}\\
&= \frac{p-1}{2}.
\end{aligned}
$$

Q.E.D.

D Proof of Lemma 3.2

Finally, we show the following lemma in Section 3.

Lemma 3.2.

$$
\begin{aligned}
(B) \;&:=\; \sum_{i=1}^{p}\sum_{j=1}^{p}\left[\left(\frac{\partial_i\mathit{\Delta}}{\mathit{\Delta}}\right)g^{ij}\left(\frac{z_j}{1-z_j^2}\right)+\frac{\partial}{\partial z_i}\left\{g^{ij}\left(\frac{z_j}{1-z_j^2}\right)\right\}\right]\\
&=\; \begin{cases}\frac{1}{2}p & \text{even } p\\ \frac{1}{2}(p+1) & \text{odd } p\end{cases}.
\end{aligned}
$$

Proof.

Using Eq.(16),

$$
\begin{aligned}
&(B)\\
= \ & \sum_{i=1}^{p}\left(\frac{\partial_i \Delta}{\Delta}\right)\sum_{j=1}^{p} g^{ij}\left(\frac{z_j}{1-{z_j}^2}\right)+\sum_{i=1}^{p}\frac{\partial}{\partial z_i}\left\{\sum_{j=1}^{p} g^{ij}\left(\frac{z_j}{1-{z_j}^2}\right)\right\}\\
= \ & \sum_{i=1}^{p}\left(\frac{\partial_i \Delta}{\Delta}\right)(-1)^i G(z_i)h_p(z_i)\frac{\Delta_i}{\Delta}+\sum_{i=1}^{p}\frac{\partial}{\partial z_i}\left\{(-1)^i G(z_i)h_p(z_i)\frac{\Delta_i}{\Delta}\right\}\\
= \ & \sum_{i=1}^{p}(-1)^i G(z_i)h_p(z_i)\left(\frac{\Delta_i}{\Delta^2}\frac{\partial \Delta}{\partial z_i}\right)+\sum_{i=1}^{p}(-1)^i\left[\frac{\partial}{\partial z_i}\left\{G(z_i)h_p(z_i)\right\}\right]\frac{\Delta_i}{\Delta}\\
&+\sum_{i=1}^{p}(-1)^i\left\{G(z_i)h_p(z_i)\right\}\frac{\partial}{\partial z_i}\left(\frac{\Delta_i}{\Delta}\right)\\
= \ & \sum_{i=1}^{p}(-1)^i\left[\frac{\partial}{\partial z_i}\left\{G(z_i)h_p(z_i)\right\}\right]\frac{\Delta_i}{\Delta}.
\end{aligned}
$$

In the last equality, we use

$$\frac{\partial}{\partial z_i}\left(\frac{\Delta_i}{\Delta}\right)=-\frac{\Delta_i}{\Delta^2}\frac{\partial \Delta}{\partial z_i}$$

because Δ_i does not include z_i. It is useful to rewrite the derivative of $G(z_i)$ with respect to z_i.

$$\frac{\partial}{\partial z_i}G(z_i)=\left(\sum_{l=1}^{p}\frac{-z_l}{1-z_l z_i}\right)G(z_i)+\left(\frac{-z_i}{1-z_i^2}\right)G(z_i).$$

Again, we calculate (B) for even p and odd p separately.

(i) When p is even

When p is even, the derivative of h_p is given by

$$\frac{\partial}{\partial z_i}h_p(z_i)=\frac{2z_i}{(1-z_i^2)^2}.$$

Thus, (B) is rewritten as

$$\begin{aligned}
(B)_{\text{even}}\Delta &= \sum_{i=1}^{p}\left\{-\frac{\partial h_p(z_i)}{\partial z_i}G(z_i)+(-h_p(z_i))\frac{\partial G(z_i)}{\partial z_i}\right\}(-1)^{i+1}\Delta_i \\
&= \sum_{i=1}^{p}\left[\frac{-2z_i}{(1-z_i^2)^2}+\frac{-1}{1-z_i^2}\left\{\left(\sum_{l=1}^{p}\frac{-z_l}{1-z_lz_i}\right)+\left(\frac{-z_i}{1-z_i^2}\right)\right\}\right] \\
&\quad \times G(z_i)(-1)^{i+1}\Delta_i \\
&= \sum_{i=1}^{p}\left\{-\frac{z_i}{(1-z_i^2)^2}+\sum_{l=1}^{p}\frac{z_l}{(1-z_i^2)(1-z_lz_i)}\right\}G(z_i)(-1)^{i+1}\Delta_i \\
&= -V\!\left(\frac{XG(X)}{(1-X^2)^2}\right)+\sum_{l=1}^{p}\left\{z_l\ V\!\left(\frac{G(X)}{(1-X^2)(1-z_lX)}\right)\right\}.
\end{aligned}$$

Here, using Eq.(11), (12), the summand in the second term is calculated as

$$\begin{aligned}
&V\!\left(\frac{G(X)}{(1-X^2)(1-z_lX)}\right) \\
&= \frac{1}{2}\left\{V\!\left(\frac{G(X)}{(1-X)(1-z_lX)}\right)+V\!\left(\frac{G(X)}{(1+X)(1-z_lX)}\right)\right\} \\
&= \frac{1}{2}\left\{-\frac{\Delta}{1-z_l}+\frac{\Delta}{1+z_l}\right\} \\
&= \Delta\left(\frac{-z_i}{1-z_i^2}\right).
\end{aligned}$$

Thus, using Eq.(17), we obtain the final result,

$$\begin{aligned}
(B)_{\text{even}}\Delta &= -V\!\left(\frac{XG(X)}{(1-X^2)^2}\right)+\sum_{l=1}^{p}z_l\ V\!\left(\frac{G(X)}{(1-X^2)(1-z_lX)}\right) \\
&= \frac{\Delta}{2}\sum_{l=1}^{p}\left(\frac{1+z_l^2}{1-z_l^2}\right)+\sum_{l=1}^{p}z_l\Delta\left(\frac{-z_l}{1-z_l^2}\right) \\
&= \frac{p}{2}\Delta.
\end{aligned}$$

(ii) When p is odd

When p is odd, the derivative of h_p is given by

$$\frac{\partial}{\partial z_i}h_p(z_i) = \frac{-1}{1-z_i^2}+\frac{-2z_i^2}{(1-z_i^2)^2}.$$

Thus, (B) is rewritten as

$$
\begin{aligned}
&(B)_{\mathrm{odd}}\mathit{\Delta} \\
= & \sum_{i=1}^{p}\left\{-\frac{\partial h_p(z_i)}{\partial z_i}G(z_i)+(-h_p(z_i))\frac{\partial G(z_i)}{\partial z_i}\right\}(-1)^{i+1}\mathit{\Delta}_i \\
= & \sum_{i=1}^{p}\left[\left\{\frac{1}{1-z_i^2}+\frac{2z_i^2}{(1-z_i^2)^2}\right\}\right. \\
& \left. +\frac{z_i}{1-z_i^2}\left\{\left(\sum_{l=1}^{p}\frac{-z_l}{1-z_l z_i}\right)+\left(\frac{-z_i}{1-z_i^2}\right)\right\}\right]G(z_i)(-1)^{i+1}\mathit{\Delta}_i \\
= & \sum_{i=1}^{p}\left\{\frac{1}{(1-z_i^2)^2}+\sum_{l=1}^{p}\frac{-z_l z_i}{(1-z_i^2)(1-z_l z_i)}\right\}G(z_i)(-1)^{i+1}\mathit{\Delta}_i \\
= & V\!\left(\frac{G(X)}{(1-X^2)^2}\right)+\sum_{l=1}^{p}\left\{-z_l\, V\!\left(\frac{XG(X)}{(1-X^2)(1-z_l X)}\right)\right\}.
\end{aligned}
$$

The summand in the second term is rewritten as

$$
\begin{aligned}
V\left(\frac{XG(X)}{(1-X^2)(1-z_l X)}\right) &= -h_p(z_l)\mathit{\Delta} \\
&= \frac{z_l}{1-z_l^2}\mathit{\Delta}.
\end{aligned}
$$

In order to evaluate the first term, we use Eq.(19) and Eq.(13). Observing that

$$
\frac{1}{(1-X^2)^2}=\frac{1}{1-X^2}+\frac{X^2}{(1-X^2)^2},
$$

we obtain

$$
\begin{aligned}
V\!\left(\frac{G(X)}{(1-X^2)^2}\right) &= V\!\left(\frac{G(X)}{1-X^2}\right)+V\!\left(\frac{X^2G(X)}{(1-X^2)^2}\right) \\
&= \mathit{\Delta}+\left\{-\frac{1}{2}\mathit{\Delta}+\frac{\mathit{\Delta}}{2}\sum_{l=1}^{p}\frac{1+z_l^2}{1-z_l^2}\right\} \\
&= \frac{1}{2}\mathit{\Delta}+\frac{\mathit{\Delta}}{2}\sum_{l=1}^{p}\frac{1+z_l^2}{1-z_l^2}.
\end{aligned}
$$

Collecting them yields the final result,

$$
\begin{aligned}
(B)_{\mathrm{odd}}\mathit{\Delta} &= \left(\frac{1}{2}\mathit{\Delta}+\frac{\mathit{\Delta}}{2}\sum_{l=1}^{p}\frac{1+z_l^2}{1-z_l^2}\right)+\sum_{l=1}^{p}\left\{-z_l\left(\frac{z_l}{1-z_l^2}\mathit{\Delta}\right)\right\} \\
&= \frac{p+1}{2}\mathit{\Delta}.
\end{aligned}
$$

Q.E.D.

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