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***Abstract*—This paper proposes a new generalization of the Voronoi diagram. Suppose that restaurants and bookstores are located in a city, and we want to visit both a restaurant and bookstore and return to our house. To each pair of a restaurant and a bookstore, we can assign a region such that a resident in this region can visit the restaurant and bookstore in a round tour that is shorter than that for a visit to any other pair. The city is partitioned into these regions according to which pair of a restaurant and bookstore permits the shortest round tour. We call this partitioning a "round-tour Voronoi Diagram" for the restaurants and bookstores. We study the basic properties of this Voronoi diagram and consider an efficient algorithm for its approximate construction.**

***Keywords*-generalized Voronoi diagram; round-tour; restaurants and bookstores; facility location analysis; shortest round tour;**

I. Introduction

The Voronoi diagram is one of the most fundamental concepts in computational geometry because of its useful generalizations and applications [12]. Generalization of the Voronoi diagram can be classified in three groups.

The first group comprises generalizations of the distance. The Euclidean distance, which is used for the ordinary Voronoi diagram, can be replaced by the L_p distance [5], the collision-avoidance distance [1], the power distance [4], [8], the weighted distance [3], or the boat-sail distance [9], to mention a few.

The second group comprises generalizations of the underlying space. The ordinary Voronoi diagram is in Euclidean space; however, the space can be replaced by a spherical surface [15], polygonal surface [2], or network [7].

The third group comprises generalizations of the generators. The ordinary Voronoi diagram is defined for points; however, they can be replaced by general figures such as circles, line segments, and polygons [14], or replaced by subsets of points in higher-order Voronoi diagrams [13].

In this paper, we propose a new generalization of the Voronoi diagram [6]. Our generalization might be understood easily in the context of a round tour involving a visit to both a restaurant and bookstore before returning home. Suppose that there are many restaurants and bookstores in a city. The city is partitioned into regions according to the pair of restaurant and bookstore one can visit in the shortest round tour. We call this partition the "round-tour Voronoi diagram" for the restaurants and bookstores. We study the basic properties of this diagram and consider an efficient algorithm for computing the approximation of this diagram in the form of a digital picture.

This work is closely related to Ohyama's work [10], [11] in which consumer behavior is studied using his new Voronoi diagrams. He considered a consumer who visits stores one by one until he selects a good. In that case, the "distance" from a point (where the consumer lives) to a set of stores is the expected length of the shortest path along which he travels while shopping.

In section 2 we introduce our new Voronoi diagram, the round-tour Voronoi diagram, and in section 3 we consider its basic properties. In sections 4 and 5, we construct an algorithm for computing a digital approximation of the Voronoi diagram, and in section 6 we give concluding remarks.

II. Round-Tour Voronoi Diagram for Two Sets of Generators

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of n points in the plane $\mathbb{R}^2$. For any two points $x, y \in \mathbb{R}^2$, we represent by $d(x, y)$ the Euclidean distance between x and y. We define $V(A; a_i)$ as

$$V(A; a_i) = \{z \in \mathbb{R}^2 \mid d(z, a_i) < d(z, a_j), j \neq i\}. \tag{1}$$

$V(A; a_i)$ represents the set of points that are closer to a_i than to any other point in A. The plane is partitioned into $V(A; a_1), V(A; a_2), \ldots, V(A; a_n)$ and their boundaries. We call this partition *the Voronoi diagram* for A, and $V(A; a_i)$ *the Voronoi region* of a_i. The elements of A are called *generators* (or *generating points*) of the Voronoi diagram. This is the definition of the ordinary Voronoi diagram.

We extend this diagram to two sets of different types of generating points in the following way.

Let A and B be two finite sets of points in the plane. A might be considered as a set of points at which restaurants are located, and B might be considered as a set of points at which bookstores are located. We assume that all restaurants are identical in the sense that people do not have any preference except for their distances, and that all the bookstores are identical in a similar sense.

Let z be a general point on the plane. For $a \in A$ and $b \in B$, we define

$$l_z(a, b) = d(z, a) + d(a, b) + d(b, z). \tag{2}$$

The value $l_z(a, b)$ is the length of the perimeter of the triangle formed by three vertices a, b, and z; that is, $l_z(a, b)$ is the length of the shortest round tour starting at z, visiting a and b, and returning to z.

For $a \in A$ and $b \in B$, let us define $V(A, B; a, b)$ by

$$V(A, B; a, b) = \{z \in \mathbb{R}^2 \mid l_z(a, b) = \min_{a' \in A,\ b' \in B} l_z(a', b')\}. \tag{3}$$

Intuitively, $V(A, B; a, b)$ represents the region in which any resident can visit a and b in a shorter round tour than he/she visits other pairs of a restaurant and bookstore.

The plane is decomposed into the regions $V(A, B; a, b)$ (where $a \in A$ and $b \in B$) without overlap except for the boundaries. We call this partition the *round-tour Voronoi diagram* for A and B. A and B are called the *generating sets* of the Voronoi diagram.

Figure 1 shows an example of the round-tour Voronoi diagram for three restaurants $A = \{R_0, R_1, R_2\}$ and three bookstores $B = \{S_0, S_1, S_2\}$. Each region is labeled by a pair of generating points; for example, (R_i, S_j) represents the Voronoi region $V(A, B; R_i, S_j)$.

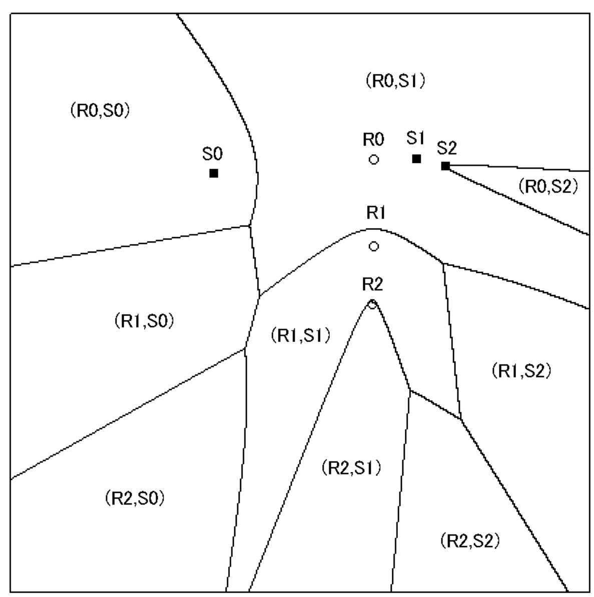

Fig. 1. Example of a round-tour Voronoi diagram.

We see that any pair (R_i, S_j) for $i, j = 0, 1, 2$, $V(A, B; R_i, S_j)$ has a nonempty region in this particular diagram. In general, however, some pairs of restaurants and bookstores may not have nonempty regions.

III. Basic Properties

In this section, we consider basic properties of the round-tour Voronoi diagram defined in the last section.

Property 1. *Let $a \in A$ and $b \in B$ and s be a positive real number satisfying $s > 2d(a, b)$. The trajectory of the point z satisfying $l_z(a, b) = s$ forms an ellipse.*

Proof: The condition $l_z(a, b) = s$ can be expressed as

$$d(a, p) + d(b, p) = s - d(a, b). \quad (4)$$

Because s and $d(a, b)$ are constants, Eq. (4) means that the sum of the distances of p from a and b is constant. Hence, p moves on an ellipse with foci a and b. ■

Property 2. *Let $A = \{a\}$ and $B = \{b, c\}$. Then the boundary between $V(A, B; a, b)$ and $V(A, B; a, c)$ is one branch of the hyperbola with foci b and c.*

Proof: The boundary is $\{z \in \mathbb{R}^2 \mid l_z(a, b) = l_z(a, c)\}$. We see

$$\begin{aligned} & \{z \mid l_z(a, b) = l_z(a, c)\} \\ = & \{z \mid d(z, a) + d(a, b) + d(b, z) \\ & \qquad = d(z, a) + d(a, c) + d(c, z)\} \\ = & \{z \mid d(a, b) + d(b, z) = d(a, c) + d(c, z)\} \\ = & \{z \mid d(b, z) - d(c, z) = d(a, c) - d(a, b)\}. \end{aligned}$$

Because $d(a, c) - d(a, b)$ is a fixed contant, the difference of the distances from z to b and c is constant, which means that the boundary point z moves on the hyperbola with foci b and c. ■

Property 3. *A Voronoi region of the round-tour Voronoi diagram is not necessarily connected.*

This property can be shown by an example. Consider the first generator set $A = \{a_1, a_2, a_3\}$ where

$$a_1 = (10, -20), \; a_2 = (20, 0), \; a_3 = (20, 22),$$

and the second generator set $B = \{b_1, b_2, b_3\}$ where

$$b_1 = (-10, -20), \; b_2 = (-20, 0), \; b_3 = (-20, 22).$$

The round-tour Voronoi diagram for A and B is shown in Fig. 2. In this figure, there are two small

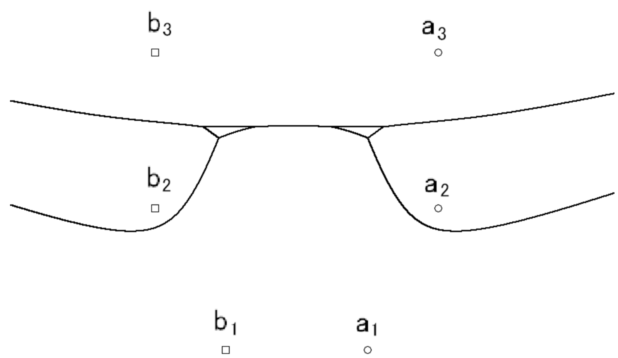

Fig. 2. Round-tour Voronoi diagram with a disconnected region.

triangle-like regions. These two regions together constitute the Voronoi region $V(A, B; a_2, b_2)$. Thus, the Voronoi region is not necessarily connected.

For point $z \in \mathbb{R}^2$ and positive real ϵ, let $U(z, \epsilon)$ be the set of all points that are within the distance ϵ from z. We call $U(z, \epsilon)$ the *ϵ-neighbor* of point z.

Property 4. *Let $a \in A, b \in B, z \in \mathbb{R}^2$, and ϵ be a positive real number. For any $z' \in U(z, \epsilon)$, the following inequality is satisfied.*

$$l_{z'}(a, b) \le l_z(a, b) + 2\epsilon \quad (5)$$

Proof: Suppose that we are at point z'. We can visit both points a and b by a round tour visiting z, a, b, z in this order and returning to z'. The length of this round tour is

$$d(z', z) + l_z(a, b) + d(z, z'). \quad (6)$$

The shortest round tour for z' is not longer than this tour, and hence we get

$$\begin{aligned} l_{z'}(a, b) &\le l_z(a, b) + 2d(z', z) \\ &\le l_z(a, b) + 2\epsilon. \end{aligned} \quad (7)$$

■

Property 5. *For any* $x, x' \in A$ *and* $y, y' \in B$, *if*

$$d(x,y) > d(x',y) + d(x,y'), \tag{8}$$

then V(A, B; x, y) is empty.

Proof: First, suppose that

$$d(z,x') \geq d(z,y'). \tag{9}$$

We then obtain

$$\begin{aligned} & l_z(x,y) \\ = \ & d(z,x) + d(x,y) + d(y,z) \\ > \ & d(z,x) + d(x',y) + d(x,y') + d(y,z) \\ & \text{(because of Ineq.(8))} \\ = \ & d(z,x) + d(x,y') + d(x',y) + d(y,z) \\ \geq \ & d(z,x) + d(x,y') + d(x',z) \\ & \text{(because of the triangular inequality)} \\ \geq \ & d(z,x) + d(x,y') + d(z,y') \\ & \text{(because of Ineq.(9))} \\ = \ & l_z(x,y'). \end{aligned} \tag{10}$$

Hence, we get

$$l_z(x,y) > l_z(x,y'). \tag{11}$$

Secondly, suppose that

$$d(z,x') \leq d(z,y'). \tag{12}$$

Then, by a symmetric argument, we obtain

$$l_z(x,y) \geq l_z(x',y). \tag{13}$$

From Ineqs. (11) and (13), we obtain Property 5.

■

Property 6. *If* $a \in V(B;b)$ *or* $b \in V(A;a)$, *then* $V(A,B;a,b)$ *is nonempty.*

Proof: Suppose that $a \in V(B;b)$. Then for any $b' \in B$, we get

$$d(a,b) \leq d(a,b'). \tag{14}$$

This is equivalent to

$$d(a,a)+d(a,b)+d(b,a) \leq d(a,a)+d(a,b')+d(b',a). \tag{15}$$

Therefore, we get

$$l_a(a,b) \leq l_a(a,b'). \tag{16}$$

On the other hand, we get

$$\begin{aligned} & l_a(a,b') \\ & = d(a,a) + d(a,b') + d(b',a) \\ & = d(a,b') + d(b',a) \\ & \leq d(a,a') + d(a',b') + d(b',a) \\ & \quad \text{(because of the triangular inequality)} \\ & = l_a(a',b'). \end{aligned} \tag{17}$$

Combining Ineqs. (16) and (17), we get

$$l_a(a,b) \leq l_a(a',b') \tag{18}$$

for any $a' \in A$ and $b' \in B$. This implies $a \in V(A,B;a,b)$, and hence $V(A,B;a,b) \neq \emptyset$.

Next, we suppose $b \in V(A;a)$. Then, by a symmetric argument, we get $b \in V(A,B;a,b)$ and have $V(A,B;a,b) \neq \emptyset$. This completes the proof.

■

For any finite set X, let $|X|$ denote the number of elements of X.

Property 7. *Let* $|A| = m$ *and* $|B| = n$. *Then, the number of nonempty Voronoi regions of the round-tour Voronoi diagram can be as small as* $\max(m,n)$.

Proof: We prove this property by giving an example of the Voronoi diagram. Without losing generality, we assume that $m \leq n$. We consider generator sets $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ such that

$$\begin{aligned} & a_i = b_i = (i,0) && \text{for } i = 1, \ldots, m, \\ & b_i = (i,0) && \text{for } i = m+1, \ldots, n \end{aligned}$$

as shown in Fig. 3. We show that this diagram has only n regions. First we prove that for any $a \in A$ and $b \in B$, if $V(A,B;a,b)$ is not empty, then b is in $V(A;a)$ in this setting.

Let z be an arbitrary point and a be the point in A that is nearest z. In this situation, going straight from z to a and returning to z gives the shortest round tour because we also visit a point in B at point a. We have $a \in V(A;a)$

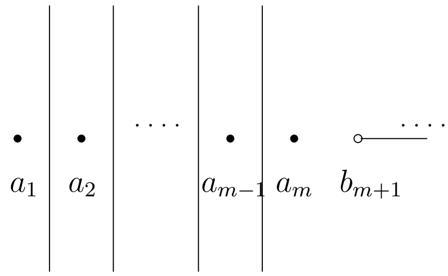

Fig. 3. Round-tour Voronoi diagram with n regions.

because a is the nearest point in A to z. Moreover, we have $a \in V(A;a)$ and $V(A;a)$ is convex. Hence, any point in the shortest round tour is in $V(A;a)$. Consequently, we get for any $a \in A$ and $b \in B$, if $V(A,B;a,b)$ is not empty, then b is in $V(A;a)$. Checking the distances, we find that for $i = 1,2,\ldots,m$, b_i is only in $V(A;a_i)$, and that for $i = m+1,\ldots,n$, b_i is only in $V(A;a_m)$. This completes the proof. ■

Note that some of the Voronoi regions can be without positive area although they are nonempty. An example of such a region is the Voronoi region $V(A,B;a_m,b_{m+1})$ in Fig. 3. The region $V(A,B;a_m,b_{m+1})$ forms the half line starting at b_{m+1} in the positive direction of the x axis. In fact, any point on this half line has the shortest round tour visiting a_m and b_{m+1}. However, any point near to, but not on, this half line can visit a_m and b_m in a round tour shorter than that visiting a_m and b_{m+1}. Thus, the region $V(A,B;a_m,b_{m+1})$ has no area. Let us call the Voronoi region with positive area a *proper region*.

Property 8. *Let $|A| = m$ and $|B| = n$. Then, the number of proper Voronoi regions of the round-tour Voronoi diagram can be as small as 1.*

Proof: We prove the property by giving an example. Suppose that $A = \{a_1,\ldots,a_m\}$ and $B = \{b_1,\ldots,b_n\}$ such that

$$a_i = (i,0) \qquad \text{for } i = 1,\ldots,m,$$
$$b_i = (-i,0) \qquad \text{for } i = 1,\ldots,n.$$

Let p be an arbitrary point outside the x axis. Then, as shown in Fig. 4, the triangle with the

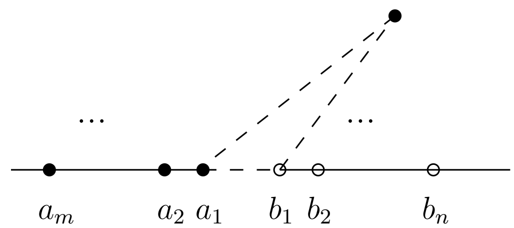

Fig. 4. Round-tour Voronoi diagram with only one positive-area region.

vertices p, a_1, and b_1, shown by broken lines has the smallest perimeter among all triangles with the vertices p, a_i, and b_j for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. Hence, any point outside the x axis belongs to the Voronoi region of the pair (a_1,b_1). Consequently, the pair (a_1,b_1) only has a positive area. ■

Property 9. *Let $|A| = m$ and $|B| = n$. Then the number of nonempty Voronoi regions of the round-tour Voronoi diagram can be as large as mn.*

Proof: We prove this property by giving an example of a Voronoi diagram. Let

$$A = \{(s_i\cos 60^\circ, s_i\sin 60^\circ) \mid s_i \in [1,1.5], \; i = 1,2,\ldots,m\},$$
$$B = \{(t_j,0) \mid t_j \in [1,1.5],\; j = 1,2,\ldots,n\}.$$

Suppose that $a \in A$ and $b \in B$ are any pair of generators. We will show that there exists a point $c \in \mathbb{R}^2$ such that $c \in V(A,B;a,b)$. Let L be the line passing through a and b. As shown in Fig. 5, let L_1 be the line that is a mirror image of L with respect to the line $y = 2x$, and let L_2 be the mirror image of L with respect to the line $y = 0$. Let c be the point of intersection of L_1 and L_2. Finally, let c_A and c_B be the mirror images of c with respect to $y = 2x$ and $y = 0$ respectively. Note that both c_A and c_B are on the line L, but neither c_A nor c_B is on the line segment connecting a and b. For any $a' \in A$ and $b' \in B$, we get

$$\begin{aligned} &d(c,a') + d(a',b') + d(b',c) \\ =&d(c_A,a') + d(a',b') + d(b',c_B) \\ \geq&d(c_A,c_B) \end{aligned}$$

because the points c_A, a, b and c_B are on the line L in this order. This implies that $c \in V(A,B;a,b)$.

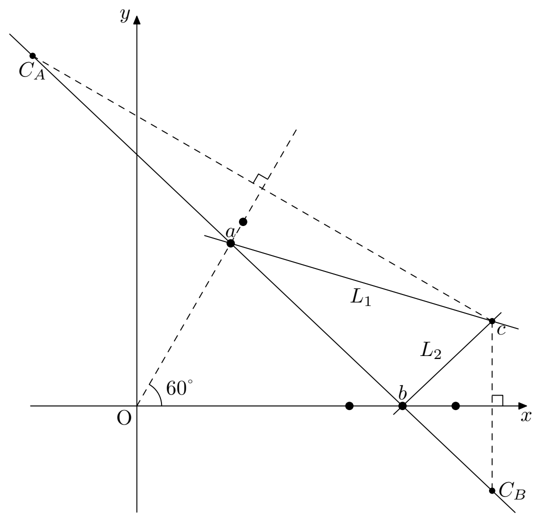

Fig. 5. Shortest-path proof with two mirrors.

Thus, for any pair $a \in A$ and $b \in B$, the Voronoi region $V(A, B; a, b)$ is nonempty. Consequently, the number of Voronoi regions can be as large as mn. ■

Figure 6 shows an example of a round-tour Voronoi diagram with $O(n^2)$ Voronoi regions for $|A| = |B| = n$.

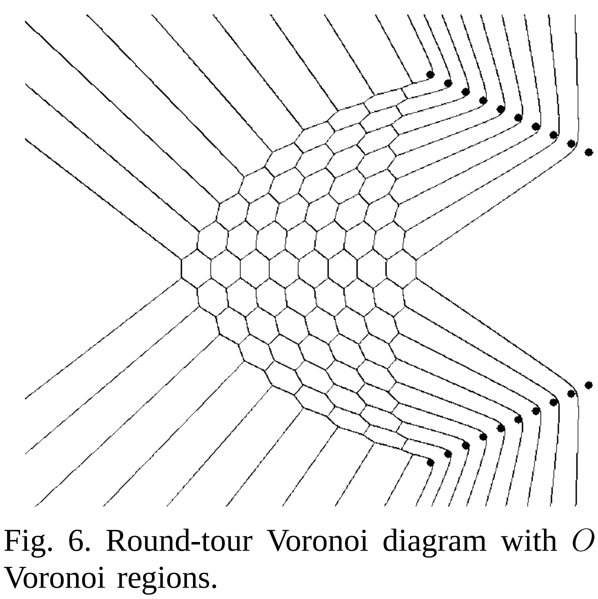

Fig. 6. Round-tour Voronoi diagram with $O(n^2)$ Voronoi regions.

Tab. I. Pruning of generator pairs without Voronoi regions

no. of all pairs	no. of pairs without regions	no. of pairs pruned by Condition 1	no. of pairs pruned by Condition 2
2500	2406	2406	2070
2500	2401	2399	2080
2500	2406	2405	2269

IV. Preliminary Study on the Number of Voronoi Regions

As we have seen by Property 8, the number of Voronoi regions can be as large as mn for the generator sets A and B with $|A| = m$ and $|B| = n$. However, this extreme case might be rare, and we are also interested in the average number of regions. To find the number of regions in ordinary cases, we experimentally investigate the case where the generating points in A and B are located at random.

We generated 50 points on integer grid intersections in $[0, 500) \times [0, 500)$ at random and considered them as elements of A. Similarly, we generated 50 more points in the same region and considered them as elements of B.

We generated three different pairs of A and B, and gathered statistical data from computational experiments. The results are summarized in Table I.

In the table, the three rows correspond to the three pairs of A and B, the leftmost column represents the number of all possible pairs (a, b), which is $50 \times 50 = 2500$ because $|A| = |B| = 50$. The other three columns show statistical data obtained in the following way.

We are interested in which pair (a, b) has a nonempty Voronoi region $V(A, B; a, b)$. To determine this property approximately, we checked all integer grid points in $[-100, 600) \times [-100, 600)$, and enumerated all pairs (a, b) whose regions $V(A, B; a, b)$ contain no grid points.

We expect that such a pair (a, b) does not have nonempty regions $V(A, B; a, b)$ with high probability. The results are shown in the second column of Table I.

We see that about 96% of possible pairs do not have nonempty regions.

From this observation we can say that the number of pairs of generators that admit nonempty Voronoi regions is very small. This implies that it is important to identify and prune the pairs that do not have nonempty regions as early as possible to construct the round-tour Voronoi diagram efficiently.

For this purpose, we consider two conditions that are sufficient for pair (a, b) not to admit a nonempty region.

Let us define $D = \{(i, j) \mid \text{i and j are integers such that } -100 \leq i, j < 600\}$.

Condition 1. *For any integer grid point* $z \in D$*,*

$$l_z(a, b) > l_z(a', b) \text{ for some } a' \in A$$
or
$$l_z(a, b) > l_z(a, b') \text{ for some } b' \in B.$$

Condition 2. *There exists* $a \in A$ *and* $b \in B$ *such that*

$$d(a, b) > d(a', b) + d(a, b').$$

Note that Condition 1 requires much more time to check than Condition 2 does because there are about 700×700 grid points in D while there are only 50×50 pairs (a, b) of generators.

Either Condition 1 or 2 is a sufficient condition for the pair (a, b) to have no nonempty Voronoi region. Hence, once we find that Condition 1 or Condition 2 is satisfied, we can conclude that $V(A, B; a, b)$ is empty.

The numbers of pairs (a, b) that satisfy Condition 1 are shown in the third column of Table I, while the numbers of pairs that satisfy Condition 2 are shown in the rightmost column.

We see that Condition 1 is almost perfect for checking the emptiness of the Voronoi region although it has high computational cost. Condition 2 detects more than 83% of the empty Voronoi regions.

Therefore, we expect Condition 2 to be a powerful tool for pruning nonempty regions.

V. Algorithm for a Digital-Picture Approximation of the Round-Tour Voronoi Diagram

The boundary curves of the round-tour Voronoi diagram are very complicated in general, and hence it is not easy to construct an exact diagram in a short time. Hence, we propose an algorithm for constructing a digital-picture approximation of the diagram.

We consider grid points $p_{ij} = (i, j)$ with integer coordinates for $i = 1, 2, \ldots, M$ and $j = 1, 2, \ldots, N$.

For each p_{ij}, there exists a pair (a, b), where $a \in A$ and $b \in B$, such that $p_{ij} \in V(A, B; a, b)$.

We define $D(i, j)$ as

$$D(i, j) = (a, b) \text{ if } p_{i,j} \in V(A, B; a, b).$$

If such a pair (a, b) is not unique, we assign the lexicographically smallest pair (a, b) to $D(i, j)$ for prespecified orders of generators in A and B. Thus, $D(i, j)$ where $i = 1, \ldots, M$ and $j = 1, \ldots, N$ are defined uniquely. We call the set of assignments

$$\{D(i, j) \mid i = 1, \ldots, M, j = 1, \ldots, N\}$$

the *digital-picture approximation of the round-tour Voronoi diagram*, or the *digital Voronoi diagram* for short. The digital Voronoi diagram can be constructed straightforwardly if we do not care about the efficiency; that is, for each grid point p_{ij}, we compute the lengths $l_{p_{ij}}(a, b)$ of the round tour for all pairs (a, b) and assign the one that realizes the minimum of $D(i, j)$. We call this naive algorithm *Algorithm 1*.

Algorithm 1 (naive method)**.** *For each grid point* p_{ij}*, we compute the lengths* $l_{p_{ij}}(a, b)$ *of the round tours for all* $a \in A$ *and* $b \in B$*. We assign the pair that attains the minimum of length* $D(i, j)$*.*

Suppose that we know $l_{p_{ij}}(a, b)$ for some $a \in A$ and $b \in B$. Then, the pair (a', b'), where $a \in A$ and $b \in B$, cannot attain the minimum round tour at p_{ij} if

$$d(p_{ij}, a') > l_{p_{ij}}(a, b). \tag{19}$$

This property can be used to prune the points that cannot attain the minimum round tour. Using this property, we consider the following algorithm.

Algorithm 2. *For each grid point p_{ij}, we first compute the length of the round tour under the assumption that p_{ij} belongs to the same Voronoi region as its neighbor does. We then prune the generators a with $d(p_{ij}, a)$ greater than the length of the round tour, and finally compute the true Voronoi region using only the remaining pairs of generators.*

We experimentally compared the processing times of Algorithms 1 and 2 for various pairs of generators. The results are summarized in Fig. 7, where the horizontal axis represents the number of generators of each kind on a linear scale, and the vertical axis represents the computation time on a logarithmic scale.

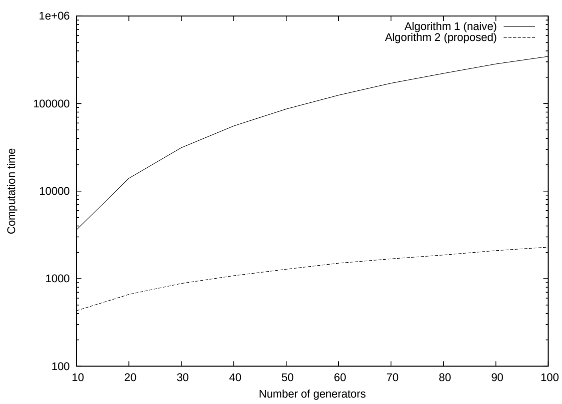

Fig. 7. Computation time for constructing digital round-tour Voronoi diagrams.

From this figure, we can see that the pruning strategy adopted in Algorithm 2 is very effective in constructing the digital Voronoi diagram.

VI. Concluding Remarks

We have proposed a new generalization of the Voronoi diagram called the round-tour Voronoi diagram. This Voronoi diagram is the partition of the plane according to what pair (a, b) of two different kinds of generators attains the minimum round tour. We have studied basic properties of this Voronoi diagram and constructed an algorithm for computing a digital-picture approximation of the diagram.

This generalization is motivated by facility location analysis considering the interaction between two different kinds of facilities such as restaurants and bookstores. Our next task is to apply the round-tour Voronoi diagram to such facility location analysis. In this paper, we considered two different sets of generators. This can be further generalized to three or more different kinds that a customer wants to visit in the shortest round tour.

This approach to generalization is more complicated because the order of the generators that the customer visits is also important. Full analysis of this approach will be a future work.

Another work for the future is to construct a method for computing the exact Voronoi diagram instead of the digital-picture approximation. For this purpose, we first need to understand what kinds of curves can be boundaries of the Voronoi regions. Since the boundary curves are represented by complicated equations, it is also important to make the algorithm robust against numerical errors.

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References

[1] B. Aronov: On the geodesic Voronoi diagram of point sites in a simple polygon. *Proceedings of the ACM Symposium on Computational Geometry*, Waterloo, Ontario, 1989, pp. 39–43.

[2] B. Aronov and J. O'Rourke: Nonoverlap of the star unfolding. *Discrete and Computational Geometry*, Vol. 8 (1992), pp. 219–250.

[3] P. F. Ash and E. D. Bolker: Generalized Dirichlet tessellations. *Geometriae Dedicata*, Vol. 20 (1986), pp. 209–243.

[4] F. Aurenhammer: Power diagrams — Properties, algorithms and applications. *SIAM Journal of Computing*, Vol. 16 (1987), pp. 78–96.

[5] G. M. Carter, J. M. Chaiken, and E. Ignall: Response areas for two emergency units. *Operations Research*, Vol. 20 (1972), pp. 571–594.

[6] H. Fujii: *Round-Tour Voronoi Diagrams and their Applications* (in Japanese). Thesis submitted for Bachelor's degree, Department of Mathmatical Engineering and Information Physics, University of Tokyo, 2008.

[7] S. L. Hakimi, M. Labbe, and E. Schmeichel: The Voronoi partition of a network and its implications in location theory. *ORSJ Journal on Computing*, Vol. 4 (1992), pp. 412–417.

[8] H. Imai, M. Iri, and K. Murota: Voronoi diagram in the Laguerre geometry and its applications. *SIAM Journal of Computing*, Vol. 14 (1985), pp. 93–105.

[9] T. Nishida and K. Sugihara: Boat-sail Voronoi diagram and its computation based on a cone approximation scheme. *Japan Journal of Industrial and Applied Mathematics*, Vol. 22 (2005), pp. 367–383.

[10] T. Ohyama: Some Voronoi diagrams that consider consumer behavior. *Proceedings of the International Symposium on Voronoi Diagrams in Science and Engineering*, Tokyo, September, 2004, pp. 191–202.

[11] T. Ohyama: Application of the additively weighted Voronoi diagram to flow analysis. *Proceedings of the 2nd International Symposium on Voronoi Diagrams in Science and Engineering*, Seoul, October 2005, pp. 22–32.

[12] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu: *Spatial Tessellations — Concepts and Applications of Voronoi Diagrams. 2nd Edition*, John Wiley, Chichester, 2000.

[13] M. I. Shamos and D. Hoey: Closest-point problems. *Proceedings of the 16th Annual IEEE Symposium on Foundations of Computer Science*, 1975, pp. 151–162.

[14] K. Sugihara: Approximation of generalized Voronoi diagrams by ordinary Voronoi diagrams. *Computer Vision, Graphics, and Image Processing: Graphical Models and Image Processing*, Vol. 55 (1993), pp. 522–531.

[15] K. Sugihara: Three-dimensional convex hull as a fruitful source of diagrams. *Theoretical Computer Science*, Vol. 235 (2000), pp. 325–337.