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Abstract

In this paper, we propose a hierarchical modeling of the systems described by conservation laws. A conservation law on a hierarchically parted domain can be considered as a subsystem which interacts with neighbor elements through fluxes. Therefore, we regard the system as a networked system of each subsystem. The important idea is to weaken the interconnection in the sense of a rank. For detailed analysis of our method, we apply the proposed method to a diffusion equation. The hierarchical model is described by a non-circulant block Toeplitz matrix. Because the resulting system is symmetrically-networked system, we can show that the eigenvalues of the system consist of those of related uniform models. We also show that our method relaxes the stability condition of the fully discretized model. Finally we examine the performance of the hierarchical model by numerical simulations.

1 Introduction

Control system design for distributed parameter systems is one of the important issue for the system control theory. A lot of researches have been presented in this area [1], however resulting compensator usually becomes an infinite dimensional system in proposed method. This causes a problem in implementation. A standard approach to treat such systems is space discretization. Although a concept of the approach is very simple, it causes numerical complexity. Therefore, we need another efficient discretization method.

A discretized model usually becomes the special system, a state variable of which interacts with neighbor state variables only. This structure appears as the sparse and banded system matrix. Similar structure is seen in control of multi-agent system or consensus of distributed sensor network [2]. In this area, a novel framework is proposed by Smith *et al* [3]. They introduce hierarchical cyclic pursuit scheme which focuses on a fractal structure. Recently, from another point of view, the result has been extended to a more general case by Shimizu *et al* [4]. In their approaches, dynamics of system are induced by internal and external interactions. The latter study focuses on cyclic structures and defined an essential role of the external interaction with an interconnection matrix and regarded strength of interconnection as rank of the interconnection matrix.

Generally, physical distributed parameter systems can be represented by conservation law and this formulation have a fractal structure. Hence, we can expect that such a system may be represented by a hierarchical system based on the fractal structure.

The purpose of this paper is to propose a hierarchical modeling for systems described by conservation laws. The key idea is to regard the system as the networked system with low rank interconnection based on the previous study [4]. We propose the general framework of the method, however our theoretical analysis limits the class because of theoretical difficulty. That is, we apply the method to a diffusion equation for detailed analysis and represent it as a symmetric network of subsystems with rank 1 interconnection. Because the previous result can not treat non-circulant subsystems and non-circulant interconnection, we extend to treat such classes.

Notation. Bold letters represent finite dimensional vectors. $\boldsymbol{e}_i^d$ is unit coordinate vector of i-direction and I_d is the identity matrix in $\mathbb{R}^d$. We introduce special matrices $\hat{I}_d, P_d$ and Q_d defined by

$$\hat{I}_d := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}, P_d := \begin{pmatrix} I_{d-1} \\ 0 \end{pmatrix}, Q_d := \begin{pmatrix} 0 \\ I_{d-1} \end{pmatrix}.$$

Note that $\hat{I}_d \in \mathbb{R}^{d\times d}$, $\hat{I}_d^{-1} = \hat{I}_d$ and $P_d,\ Q_d \in \mathbb{R}^{d\times(d-1)}$. $\sigma_p(A)$ is the set of all eigenvalues of A. Finally, "$\otimes$" is Kronecker product.

2 Hierarchical modeling

We propose a fundamental framework of our hierarchical modeling of distributed parameter systems in this section. Although, only 1-dimensional systems are considered in this paper, we can easily extend the idea to the higher dimensional systems.

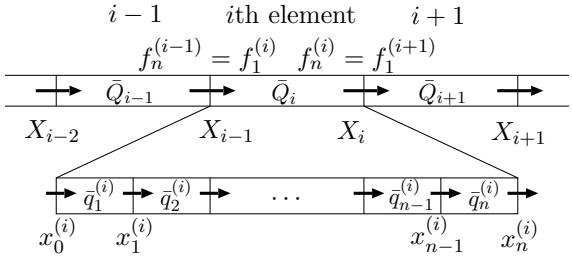

Figure 1: Hierarchical partition of the physical domain Ω.

2.1 Conservation law and its fractal structure

Consider a distributed parameter system governed by conservation law.

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0 \quad \text{in } \Omega = (0,1). \tag{1}$$

Here, $q : \Omega \times [0,\infty) \to \mathbb{R}$ is a conserved quantity and f is a flux. We assume that $q(\cdot,t) \in L^2(\Omega)$ and the range of f is also included in $L^2(\Omega)$. Equation (1) contains a lot of practical systems. For example, if we take $f(q) = -a\partial q/\partial x$, Eq. (1) becomes diffusion (heat) equation.

We employ a formulation of the finite volume methods [5] to show a fractal structure. We first divide physical domain Ω into $N\,(\geq 2)$ subdomains:

$$\Omega = \Omega_1 \cup \bar{\Omega}_2 \cup \cdots \cup \bar{\Omega}_{N-1} \cup \Omega_N,$$

where $\Omega_i := (X_{i-1}, X_i)$ and $\bar{\Omega}_i$ denotes the closure of Ω_i. Furthermore, we also divide Ω_i into $n\,(\geq 2)$ part:

$$\Omega_i = \Omega_1^{(i)} \cup \bar{\Omega}_2^{(i)} \cup \cdots \cup \bar{\Omega}_{n-1}^{(i)} \cup \Omega_n^{(i)},$$

where $\Omega_j^{(i)} := (x_{j-1}^{(i)}, x_j^{(i)})$. Note that $X_0 = 0$, $X_N = 1$, $x_n^{(i-1)} = x_1^{(i)} = X_{i-1}$ and $x_n^{(i)} = x_1^{(i+1)} = X_i$. Figure 1 illustrates the situation.

Integrating Eq. (1) on the interval $\Omega_j^{(i)}$ yields

$$\begin{aligned}\frac{d}{dt}\int_{x_{j-1}^{(i)}}^{x_j^{(i)}} q(x,t)\,dx &= -\int_{x_{j-1}^{(i)}}^{x_j^{(i)}} \frac{\partial f(q)}{\partial x}\,dx \\ &= -\Big(f(q)|_{x_j^{(i)}} - f(q)|_{x_{j-1}^{(i)}}\Big).\end{aligned} \tag{2}$$

This equation shows that the time rate of change of the integral of q over $\Omega_j^{(i)}$ depends only on the fluxes flowing into the inside through the boundary. Subsequently, we use the following notation for the fluxes:

$$\begin{aligned} f_j^{(i)} &:= f(q)|_{x_j^{(i)}}, \quad j = 1, \cdots, n-1, \\ F_i &:= f(q)|_{x_n^{(i)}} = f(q)|_{x_0^{(i+1)}}, \quad i = 1, \ldots, N-1. \end{aligned}$$

The fluxes at the boundary, $f(q)|_{x_0^{(1)}}$ and $f(q)|_{x_n^{(N)}}$, are denoted by F_0 and F_N respectively[1]. We introduce the average value of q over $\Omega_j^{(i)}$ defined as

$$\bar{q}_j^{(i)}(t) := \frac{1}{\Delta x_j^{(i)}} \int_{x_{j-1}^{(i)}}^{x_j^{(i)}} q(x,t)\, dx,$$

where $\Delta x_j^{(i)} := x_j^{(i)} - x_{j-1}^{(i)}$. Eq. (2) becomes

$$\Delta x_j^{(i)} \frac{d\bar{q}_j^{(i)}(t)}{dt} = -\left(f_j^{(i)}(t) - f_{j-1}^{(i)}(t) \right). \tag{3}$$

We rewrite the above equations with matrices as

$$E \begin{pmatrix} \dot{\bar{q}}_1^{(i)} \\ \dot{\bar{q}}_2^{(i)} \\ \vdots \\ \dot{\bar{q}}_n^{(i)} \end{pmatrix} = -(P_n - Q_n) \begin{pmatrix} f_1^{(i)} \\ f_2^{(i)} \\ \vdots \\ f_{n-1}^{(i)} \end{pmatrix} + \begin{pmatrix} F_{i-1} \\ 0 \\ \vdots \\ 0 \\ -F_i \end{pmatrix}, \tag{4}$$

where, $E := \mathrm{diag}\left(\Delta x_1^{(i)}, \ldots, \Delta x_n^{(i)} \right)$. Left-multiply Eq. (4) by the vector

$$\boldsymbol{c}^\top := \frac{1}{n} \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix},$$

then Eq. (4) becomes

$$(X_i - X_{i-1}) \frac{d\bar{Q}_i(t)}{dt} = -\Big(F_i(t) - F_{i-1}(t) \Big), \tag{5}$$

where

$$\bar{Q}_i(t) := \frac{1}{X_i - X_{i-1}} \int_{X_{i-1}}^{X_i} q(x,t) dx.$$

It is easily seen that Eq. (5) is in the same form as Eq. (3) up to the scale parameter. This fact implies that the systems described by the conservation law (1) have the fractal structure.

Before moving to next subsection, we should emphasize that there are no approximations in this subsection. In other words, both Eq. (3) and Eq. (5) are exact relations.

[1] F_0 and F_N are related with boundary conditions.

2.2 Hierarchical discretization

We swap the flux f for a numerical flux $\tilde{f}(\bar{q})$, which is evaluated by average values only, to discretize the equation. For example, let $f(q) = \bar{c}q$, where $\bar{c}$ is a positive constant. This corresponds to an advection equation. If we use an upwind method, then

$$f_j \approx \tilde{f}_j(\bar{q}_j) = \bar{c}\bar{q}_j.$$

In a conventional method, the fluxes are uniformly discretized. The most important feature of our method is that F and f are discretized in a different way.

Let the numerical fluxes in ith element be evaluated by the average values in the same element only, that is,

$$f_j^{(i)} \approx \tilde{f}_j^{(i)}(\bar{q}_1^{(i)}, \bar{q}_2^{(i)}, \ldots, \bar{q}_{n-1}^{(i)}), \quad j = 1, \ldots, n-1.$$

Then, we can regard Eq. (4) as a state space equation, where $\bar{\boldsymbol{q}}^{(i)} := \left(\bar{q}_1^{(i)}, \bar{q}_2^{(i)}, \ldots, \bar{q}_n^{(i)}\right)^\top$ is the state and F_{i-1} and F_i are the inputs. Furthermore, we employ $\bar{Q}_i = \boldsymbol{c}^\top \bar{\boldsymbol{q}}^{(i)}$ as the output equation. We call this system ith subsystem. Each subsystem interacts with each other through the flux at the boundary of each subdomain. Hence, we can consider that F determines the interconnection structure. Usually, F is evaluated by the average values in the neighbor elements. The key idea of our method is to weaken the interconnection of each subsystems in a certain sense. Therefore, we evaluate F by the output values of each subsystem:

$$F_i \approx \tilde{F}_i(\bar{Q}_1, \bar{Q}_2, \ldots, \bar{Q}_N), \quad i = 1, \ldots, N-1.$$

The output values are degenerated information determined by the vector $\boldsymbol{c}$. Thus, we can conclude this is rank 1 interconnection [4]. The interconnection is weakened in the sense of the rank. Figure 2 is a block diagram of the interconnection structure. As the figure shows, the system is modeled as the interconnected system of N subsystems and each subsystem is n-dimensional system.

3 Main results

In this section, we apply our method introduced in the previous section to the diffusion equation. Subsequently, we consider uniform grid, i.e. $\Delta x_j^{(i)} = \Delta x$ for all $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, n\}$. In this case, E, which appears in Eq. (4), becomes $\Delta x I_n$.

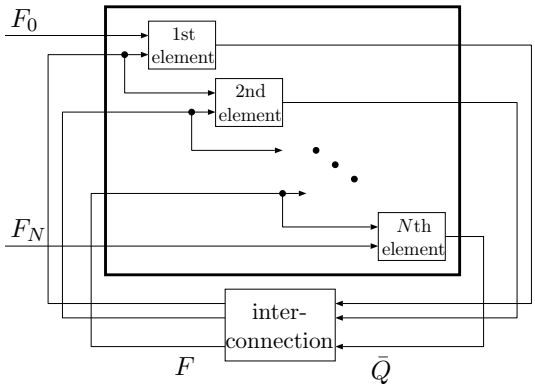

Figure 2: Interconnection structure of each systems.

3.1 Application to diffusion equation

Consider a 1-dimensional diffusion equation with constant coefficient $a > 0$:

$$\frac{\partial q(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(a\frac{\partial q(x,t)}{\partial x}\right)$$

with the Neumann boundary condition:

$$a\frac{\partial q}{\partial x}\bigg|_{(0,t)} = u_0(t), \quad a\frac{\partial q}{\partial x}\bigg|_{(1,t)} = u_1(t).$$

See standard textbooks (e.g. [6]) for details of the equation. Let the value of q at the midpoint of the jth interval in ith element approximate the average value $\bar{q}_j^{(i)}$.

$$\bar{q}_j^{(i)} \approx q\left(x_{j-1/2}^{(i)}, t\right), \quad j = 1, \ldots, n.$$

We denote it briefly by $q_j^{(i)}$. The numerical flux is evaluated by a central difference

$$f_j^{(i)} \approx \tilde{f}(q_{j+1}^{(i)}, q_j^{(i)}) := -a\frac{q_{j+1}^{(i)} - q_j^{(i)}}{\Delta x}, \quad j = 1, \ldots, n.$$

Then, Eq. (4) becomes

$$\dot{\boldsymbol{q}}^{(i)} = A\boldsymbol{q}^{(i)} + \begin{pmatrix} \boldsymbol{b}_1 & -\boldsymbol{b}_2 \end{pmatrix}\begin{pmatrix} \tilde{F}_{i-1} \\ \tilde{F}_i \end{pmatrix}, \tag{6}$$

where $\alpha := a/(\Delta x)^2$, $\boldsymbol{q}^{(i)} := \left(q_1^{(i)}, q_2^{(i)}, \cdots, q_n^{(i)}\right)^\top$ and

$$A := -\alpha(P_n - Q_n)(P_n - Q_n)^\top = \alpha \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{pmatrix}.$$

For consistent formulation, we should define $\boldsymbol{b}_1$ and $\boldsymbol{b}_2$ as $\boldsymbol{b}_1 := (1/\Delta x)\boldsymbol{e}_1^n$ and $\boldsymbol{b}_2 := (1/\Delta x)\boldsymbol{e}_n^n$. We will show, in the later section, that the important property is the fact

$$\boldsymbol{b}_1 = \hat{I}_n \boldsymbol{b}_2. \tag{7}$$

Therefore, we assume that $\boldsymbol{b}_1$ is an arbitrary vector in $\mathbb{R}^n$ and that $\boldsymbol{b}_2$ satisfies the symmetric condition Eq. (7).

Define a representative value of ith element as

$$Q_i = \boldsymbol{c}^\top \boldsymbol{q}^{(i)}. \tag{8}$$

Of course, Q_i approximates $\bar{Q}_i$. As stated in the previous section, we regard ith element as the system with the state space equation (6) and the output equation (8). The interconnection with the other systems is determined by the output Q_i. We define the interconnection structure as

$$\tilde{F}_i = \frac{1}{n}\tilde{f}(Q_{i+1}, Q_i) = -a\frac{Q_{i+1} - Q_i}{n\Delta x}. \tag{9}$$

For simplicity of notation, we write β for $a/(n\Delta x)$. By comparison with the definition $\tilde{f}$, it is clear that the interconnection is determined so that fractal structure is preserved. Note that, unlike previous study [4], we do not employ cyclic structure. We define two matrices B_1 and B_2 as

$$B_1 := \beta \boldsymbol{b}_1 \boldsymbol{c}^\top, \quad B_2 := \beta \boldsymbol{b}_2 \boldsymbol{c}^\top.$$

B_1 and B_2 correspond to interconnection matrix and they are rank 1, $n \times n$ matrices. Substitute (9) into (6), an augmented system becomes

$$\dot{\boldsymbol{q}} = \mathcal{A}\boldsymbol{q} + \mathcal{B}\boldsymbol{u}, \tag{10}$$

where $\boldsymbol{q} := \left(\boldsymbol{q}^{(1)\top}, \boldsymbol{q}^{(2)\top}, \ldots, \boldsymbol{q}^{(N)\top}\right)^\top$,

$$\begin{aligned} \mathcal{A} :=& I_N \otimes A - \beta(P_N \otimes \boldsymbol{b}_2 - Q_N \otimes \boldsymbol{b}_1)(P_N^\top \otimes \boldsymbol{c}^\top - Q_N^\top \otimes \boldsymbol{c}^\top) \\ =& \begin{pmatrix} A - B_2 & B_2 & & & \\ B_1 & A - (B_1 + B_2) & \ddots & & \\ & \ddots & \ddots & & \\ & & & A - (B_1 + B_2) & B_2 \\ & & & B_1 & A - B_1 \end{pmatrix}, \end{aligned} \tag{11}$$

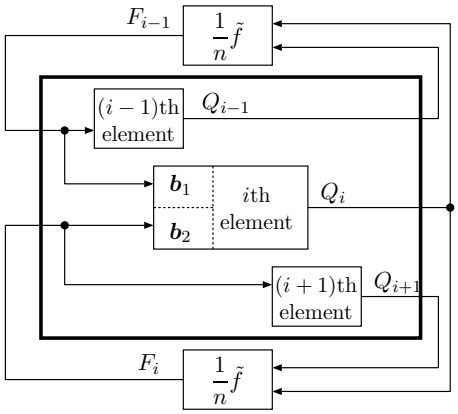

Figure 3: Symmetric network structure around ith element.

$$\mathcal{B} := \begin{pmatrix} -\boldsymbol{e}_1 \otimes \boldsymbol{b}_1 & \boldsymbol{e}_N \otimes \boldsymbol{b}_2 \end{pmatrix}$$

and $\boldsymbol{u} := (u_0,\ u_1)^\top$ which is determined by boundary conditions. Equation (10) is the hierarchically-discretized model of the diffusion equation.

There are two major differences from the previous study of hierarchical study [4]. The first is the interconnection matrix. We have two matrices B_1 and B_2 as the interconnection matrices. However, they are not independent of each other and they satisfy symmetric condition $\hat{I}_n B_1 = B_2$. Of course, the rank of vertically-concatenated matrix of the interconnection matrices is one, i.e.,

$$\operatorname{rank} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 1.$$

Another difference is that $\mathcal{A}$ is not block circulant matrix but a block tridiagonal matrix. Note that $B_1 \neq B_2$ and that $\mathcal{A}$ is not symmetric matrix despite of the fact that A is. These facts cause difficulty in the analysis of the eigenstructure of $\mathcal{A}$. However, as seen in Fig. 3, our formulation constructs a symmetric interconnection structure due to Eq. (7). Therefore, we can expect that $\mathcal{A}$ have a certain kind of symmetry and this property helps us to analyze $\mathcal{A}$ theoretically.

3.2 Eigenvalue analysis

A, B_1 and B_2 have characteristic properties such that

$$\hat{I}_n A \hat{I}_n = A, \quad \hat{I}_n B_1 = B_2, \quad B_1 \hat{I}_n = B_1. \tag{12}$$

Although the matrix (11) is not simple block (banded) Toeplitz matrix, these properties allow us to obtain exact eigenvalues of $\mathcal{A}$.

Theorem 1. *The eigenvalues of $\mathcal{A}$ defined by Eq. (11) is given as follows:*

$$-2\alpha'\left(1-\cos\left(\frac{\pi k}{N}\right)\right),\ -2\alpha\left(1-\cos\left(\frac{\pi l}{n}\right)\right),$$

where $\alpha' := \beta \boldsymbol{c}^\top \boldsymbol{b}_1$, $k = 0, 1, \ldots, N-1$ and $lk = 1, 2, \ldots, n-1$. Furthermore, each eigenvalue in the second group has multiplicity N.

The results in Theorem 1 are independent on the choice of $\boldsymbol{b}_1$, whereas $\boldsymbol{c}$ is fixed. By considering $\mathcal{A}^\top$, the same results in Theorem 1 can be proved in the case where each subsystem is a system with single input and 2 outputs.

Before proving Theorem 1, we show the more general result.

Theorem 2. *Let M_0, M_1 and M_2 be $n_1 \times n_1$ matrices. Assume that $M_0 + \hat{I}_{n_1} M_0 \neq 0$ or $M_2 + M_1 \neq 0$. Define a $(n_1 n_2) \times (n_1 n_2)$ matrix $\mathcal{M}$ as*

$$\mathcal{M} := \begin{pmatrix} M_0+M_1 & M_2 & & & \\ M_1 & M_0 & M_2 & & \\ & \ddots & \ddots & \ddots & \\ & & M_1 & M_0 & M_2 \\ & & & M_1 & M_0+M_2 \end{pmatrix}.$$

If $\hat{I}_{n_1} M_0 \hat{I}_{n_1} = M_0$, $\hat{I}_{n_1} M_1 = M_2$ and $M_1 \hat{I}_{n_1} = M_1$, then

$$\begin{aligned}\sigma_p(\mathcal{M}) &= \bigcup_{k=0}^{n_2-1} \sigma_p\left(M_0 - M_1 + M_2 + 2\left(\cos\left(\frac{\pi k}{n_2}\right)\right) M_1\right) \\ &= \bigcup_{k=0}^{n_2-1} \sigma_p\left(M_0 - M_2 + M_1 + 2\left(\cos\left(\frac{\pi k}{n_2}\right)\right) M_2\right).\end{aligned}$$

Proof. See appendix. □

From this theorem, we can notice the set of all eigenvalues of $\mathcal{A}$ is given by the union:

$$\bigcup_{k=0}^{N-1} \sigma_p\left(A + 2\left(\cos\left(\frac{\pi k}{N}\right) - 1\right) B_1\right). \tag{13}$$

Hence, to prove the Theorem 1 is to obtain the eigenvalues of these matrices. Let us move to the proof of Theorem 1.

Proof. (Theorem 1) Since (12) holds, Theorem 2 indicates the set of the eigenvalues of $\mathcal{A}$ is given by Eq. (13). We introduce the following notation,

$$\gamma_k := \cos\left(\frac{\pi k}{N}\right) - 1, \quad S := \begin{pmatrix} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix},$$

and $\hat{A}_k := A + 2\gamma_k B_1$, where $k = 0, \ldots, N-1$ and

$$S^{-1} = \begin{pmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}.$$

Then, we have

$$S\hat{A}_k S^{-1} = \left(\begin{array}{c|ccc} 2\gamma_k\beta\, \boldsymbol{c}^\top \boldsymbol{b}_1 & 0 & & \\ \hline \alpha & -2\alpha & \alpha & \\ & \ddots & \ddots & \ddots \\ & & \alpha & -2\alpha & \alpha \\ & & & \alpha & -2\alpha \end{array}\right).$$

This shows that the eigenvalues of $\hat{A}_k$ consist of

$$2\gamma_k\beta\, \boldsymbol{c}^\top \boldsymbol{b}_1 = -2\alpha' \left(1 - \cos\left(\frac{k\pi}{N}\right)\right)$$

and those of a symmetric tridiagonal Toeplitz matrix of order $n-1$. The exact value of the latter is well-known as

$$-2\alpha \left(1 - \cos\left(\frac{\pi l}{n}\right)\right), \quad l = 1, \cdots, n-1.$$

This fact is found in many textbook of linear algebra such as [7]. This relation holds for all $k = 0, \ldots, N-1$. Thus, this proves the theorem. □

The meaning of Theorem 1 is explained as follows. A has a zero eigenvalue originally, because A is graph Laplacian. All eigenvalues of $\mathcal{A}$ in the second group are eigenvalues of A except for 0. Therefore, rank 1 interconnection affects only zero eigenvalue of A. Finally, note that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same symmetric condition which each subsystem satisfies. Hence, these results can be naturally extended to multi-layer case.

3.3 Comparison with uniform discretization

To understand the property of the hierarchical model, we compare the eigenvalue distribution of $\mathcal{A}$ with those of uniformly discretized model. Subsequently, we fix $\boldsymbol{b}_1 = 1/(\Delta x)\boldsymbol{e}_1^n$. Thus, $\alpha' = a/(n\Delta x)^2$. In the case of uniform discretization on a uniform grid with nN points, the eigenvalues are

$$-2\alpha \left(1 - \cos\left(\frac{\pi l}{nN}\right)\right), \quad l = 0, \ldots, nN-1.$$

It is clear that the set contains the second group of the eigenvalues of $\mathcal{A}$, which appears when k is the multiple of N.

Next, we consider the case of uniform discretization on a uniform grid with N points. In this case, the eigenvalues of a discretized system are given by

$$-2\alpha' \left(1 - \cos\left(\frac{\pi k}{N}\right)\right), \quad k = 0, \cdots, N-1.$$

This is the same as the first group of the eigenvalues of $\mathcal{A}$. These facts show the eigenvalues of $\mathcal{A}$ consist of those of uniformly discretized model on the grid with nN and N points.

3.4 Stability of fully-discretized model

In this subsection, we consider the stability of the fully discretized equation. We discretize the time variable using the Euler explicit method, which is the simplest method:

$$\boldsymbol{q}[k+1] = (I_{nN} + \Delta t\mathcal{A})\boldsymbol{q}[k] + \Delta t\mathcal{B}\boldsymbol{u}[k]. \tag{14}$$

Since exact values of the eigenvalues of $\mathcal{A}$ have been already obtained, we can readily derive the following stability condition.

Proposition 3. *The fully discretized system (14) is stable, if and only if*

$$\Delta t < \frac{(\Delta x)^2}{a}\left(\frac{1}{1 - \cos\frac{(n-1)\pi}{n}}\right).$$

Proof. It is easy to verify that the eigenvalues of $I_{nN} + \Delta t\mathcal{A}$ are given by

$$1 - 2\Delta t\alpha\left(1 - \cos\left(\frac{\pi l}{n}\right)\right), \quad l = 1, \ldots, n-1,$$
$$1 - \frac{2\Delta t\alpha}{n^2}\left(1 - \cos\left(\frac{\pi k}{N}\right)\right), \quad k = 0, \ldots, N-1.$$

For discrete time system (14), All the eigenvalues of $I_{nN} + \Delta t\mathcal{A}$ must be on the open unit disk in complex plain. There is a single eigenvalue at 1 and this determines the steady state. Let us consider other eigenvalues. We can observe that all eigenvalues except 1 are real and less than 1. This implies that the necessary and sufficient condition for stability is

$$-1 < 1 - 2\Delta t\alpha \min\left\{1 - \cos\left(\frac{(n-1)\pi}{n}\right), n^2\left(1 - \cos\left(\frac{(N-1)\pi}{N}\right)\right)\right\}.$$

For $n,\ N \geq 2$, the minimum in the right hand side is achieved by the former. Hence, we have

$$-1 < 1 - 2\Delta t\alpha\left(1 - \cos\left(\frac{(n-1)\pi}{n}\right)\right)$$

Thus, the proposition holds. □

The stability condition for the uniform discretized model on the grid with nN points is given by

$$\Delta t < \frac{(\Delta x)^2}{a}\left(\frac{1}{1-\cos\frac{(nN-1)\pi}{nN}}\right).$$

The value of the right hand side of the above inequality is smaller than the right hand side of the inequality in Proposition 3. Therefore, hierarchical structure relaxes the stability condition. However, we can easily imagine that the accurate may be sacrificed. The analytic results of the accuracy have not been obtained yet.

4 Numerical simulation

In this section, we examine the performance of the hierarchical modeling by numerical simulations. Parameters are given as follows: $a = 1$, $n = 10$, $N = 10$ and $\Delta t = 1.0 \times 10^{-5}$. The boundary condition is $u_1 = 0$ and $u_2 = 0$. The width of each subelment turns to be $\Delta x = 1/(nN) = 1.0 \times 10^{-2}$.

First, we consider the case where the initial condition is $q(x, 0) = \cos(10\pi x)$. In this case, we can obtain the exact solution, which is $q(x, t) = e^{-(10\pi)^2 t}\cos(10\pi x)$. We denote the discretized exact solution q by $\boldsymbol{q}^{(e)}$. Figure 4 shows the numerical solution of the hierarchical model and Fig. 5 plots the Euclidean norm of the errors:

$$\boldsymbol{e}^h(t) := \boldsymbol{q}^e(t) - \boldsymbol{q}(t), \quad \boldsymbol{e}^u(t) := \boldsymbol{q}^e(t) - \boldsymbol{q}^u(t),$$

where the solution of the uniformly-discretized model with the same grid. The black solid line and the gray dashed line in Fig. 5, represent $\|\boldsymbol{e}^h(t)\|_{\mathbb{R}^{nN}}$ and $\|\boldsymbol{e}^u(t)\|_{\mathbb{R}^{nN}}$, respectively. These results show the hierarchical model achieves almost the same result and their solutions provide close agreement with the exact solution. This is the best case and similar results are obtained when initial condition is $q(x, 0) = \cos(kN\pi x),\ k = 1, \ldots, n-1$.

Next, we show the result where the hierarchical model does not provide good agreement. We employ $q(0, t) = \cos(5\pi x))$ as the initial condition. The exact solution is $q(x, t) = e^{-(10\pi)^2 t}\cos(10\pi x)$. Figures 6 and 7 illustrate the numerical solutions of a hierarchical model and the errors. In these figures, we can observe the shape of the numerical solution is distorted, and thus the norm of the error $\boldsymbol{e}^h$ becomes large.

The performance of a hierarchical model strongly depends on the initial condition. We have to analyze the eigenvectors of $\mathcal{A}$ to obtain detailed information about this issue. The last remark is on stability. In the first case, the solution of the uniform model diverged, when we set $\Delta t = 5.2 \times 10^{-5}$. Whereas, the solution of the hierarchical model is still stable. This fact numerically confirms Proposition 3.

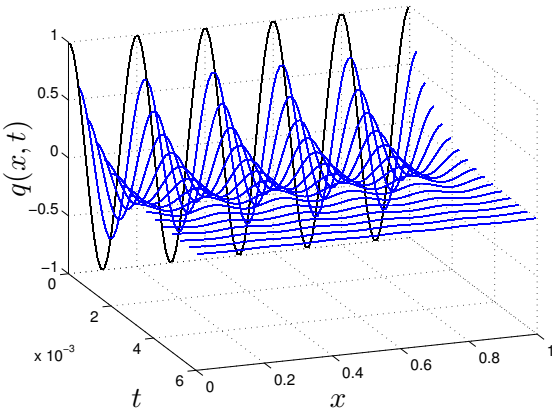

Figure 4: Numerical solution of the hierarchical model with initial condition $q(x, 0) = \cos(10\pi x)$.

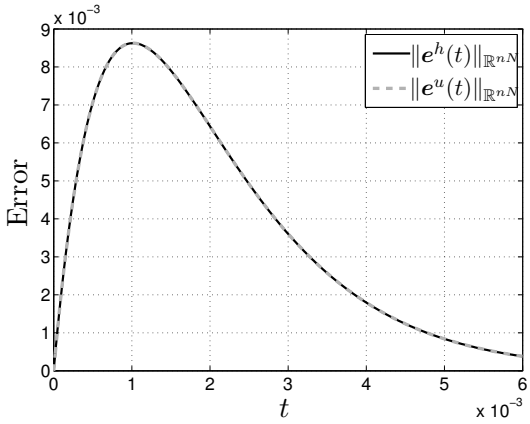

Figure 5: Errors of the solution of the hierarchical model and the uniform model.

5 Conclusion

In this paper, we proposed hierarchical modeling for the system described by conservation law and applied it to the diffusion equation. Proposed method regards the system as a weakly interconnected system of subsystems. The sense of weakness is a rank of the interconnection matrix [4].

Although the augmented system was described by non-circulant block Toeplitz matrix, we showed that the eigenvalues were equal to those of lower order matrices. In the special case, we can obtain exact eigenvalues. They consist of those of related uniformly discretized models. These results depend on the symmetry of network of the subsystems. Furthermore, augmented system also has the same symmetric property. This means that we

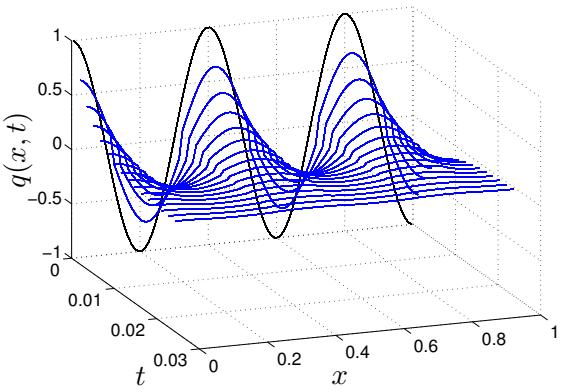

Figure 6: Numerical solution of the hierarchical model with initial condition $q(x,0) = \cos(5\pi x)$.

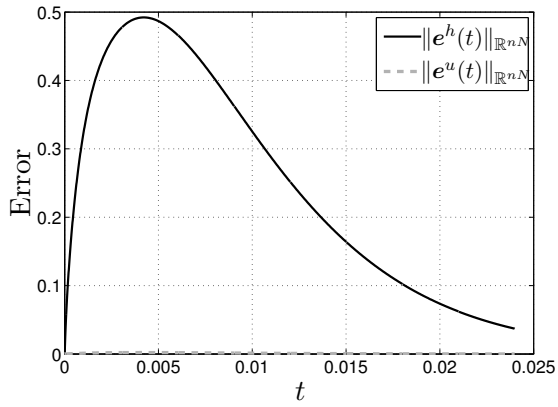

Figure 7: Errors of the solution of the hierarchical model and the uniform model.

can easily extend those results to multi-layer case. Because we obtained the exact eigenvalues, the stability bound for Δt of fully discretized model was shown. The hierarchically discretization relaxes the restriction imposed on Δt. The performance of the hierarchical model was illustrated by numerical simulations. The accuracy of the solutions of the hierarchical model is influenced by initial conditions.

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A Proof of Theorem 2

We first show the relation

$$\sigma_p(\mathcal{M}) = \bigcup_{k=0}^{n_2-1} \sigma_p\left(M_0 - M_1 + M_2 + 2\cos\left(\frac{\pi k}{n_2}\right) M_1\right).$$

We introduce a block upper triangular matrix $\mathcal{S}$ with order $n_1 n_2$ defined by

$$\mathcal{S} := \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1n_2} \\ & S_{22} & \cdots & S_{2n_2} \\ & & \ddots & \vdots \\ & & & S_{n_2 n_2} \end{pmatrix},$$

where S_{ij} is a matrix in $\mathbb{R}^{n_1 \times n_1}$. Let S_{ij} be as follows:

$$S_{ij} = (\hat{I}_{n_1})^{i+j}, \quad i,\, j = 1, \ldots, n_2.$$

When $n_2 = 3$ and $n_2 = 4$, $\mathcal{S}$ becomes

$$\mathcal{S} = \begin{pmatrix} I_{n_1} & \hat{I}_{n_1} & I_{n_1} \\ & I_{n_1} & \hat{I}_{n_1} \\ & & I_{n_1} \end{pmatrix}, \ \mathcal{S} = \begin{pmatrix} I_{n_1} & \hat{I}_{n_1} & I_{n_1} & \hat{I}_{n_1} \\ & I_{n_1} & \hat{I}_{n_1} & I_{n_1} \\ & & I_{n_1} & \hat{I}_{n_1} \\ & & & I_{n_1} \end{pmatrix},$$

respectively. $\mathcal{S}$ is a nonsingular matrix which has an inverse:

$$\mathcal{S}^{-1} = \begin{pmatrix} I_{n_1} & -\hat{I}_{n_1} & & \\ & \ddots & \ddots & \\ & & I_{n_1} & -\hat{I}_{n_1} \\ & & & I_{n_1} \end{pmatrix}.$$

Consider a similarity transformation of $\mathcal{M}$ by $\mathcal{S}$, we obtain

$$\mathcal{S}\mathcal{M}\mathcal{S}^{-1} = \begin{pmatrix} M_0 + M_1 + M_2 & 0 \\ * & \mathcal{M}_d \end{pmatrix},$$

where "$*$" is a non-zero matrix of suitable order and $\mathcal{M}_d$ is a matrix of order $(n_2 - 1)$ defined by

$$\mathcal{M}_d = \begin{pmatrix} M_0 - M_1 + M_2 & M_1 & & \\ M_1 & \ddots & \ddots & \\ & \ddots & M_0 - M_1 + M_2 & M_1 \\ & & M_1 & M_0 - M_1 + M_2 \end{pmatrix}.$$

This fact shows that the set of all eigenvalues of $\mathcal{M}$ satisfies

$$\sigma_p(\mathcal{M}) = \sigma_p(M_0 + M_1 + M_2) \cup \sigma_p(\mathcal{M}_d). \tag{15}$$

Note that $\mathcal{M}_d$ is a block Toeplitz matrix which has identical off-diagonal terms. There is a known result for such matrices [8] and we obtain

$$\sigma_p(\mathcal{M}_d) = \bigcup_{k=1}^{n_2-1} \sigma_p\left(M_0 - M_1 + M_2 + 2\cos\left(\frac{\pi k}{n_2}\right) M_1\right).$$

The first set in the right hand side of Eq. (15) can be rewritten as

$$\sigma_p(M_0 + M_1 + M_2) = \sigma_p\left(M_0 - M_1 + M_2 + 2\cos\left(\frac{\pi \cdot 0}{n_2}\right) M_1\right).$$

Thus, the first relation in the theorem holds.

Next, we show the second relation

$$\sigma_p(\mathcal{M}) = \bigcup_{k=0}^{n_2-1} \sigma_p\left(M_0 - M_2 + M_1 + 2\cos\left(\frac{\pi k}{n_2}\right) M_2\right).$$

This result is easily proved from the first relation. Consider the following similarity transformation:

$$\begin{aligned}&\hat{I}_{n_1}\left(M_0 - M_1 + M_2 + 2\cos\left(\frac{\pi k}{n_2}\right)M_1\right)\hat{I}_{n_1}\\&= M_0 - M_2 + M_1 + 2\cos\left(\frac{\pi k}{n_2}\right)M_2.\end{aligned}$$

The second relation hold. Therefore, these facts complete the theorem. □