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Enumeration of All Wedged Equilibrium Configurations in Contact Problem with Coulomb Friction

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Abstract

For a linear structure subjected to the unilateral contact condition with a fixed obstacle, we refer to a nontrivial equilibrium state as a wedged configuration. Finding a wedged configuration is called a wedged problem. This paper discusses theoretical properties of solution set of the finite-dimensional wedged problem, as well as numerical methods for computing all the wedged configurations. We propose algorithms for enumerating all the finitely many representative solutions, with which we can completely describe the solution set of the wedged problem. There exists a positive critical friction coefficient defined as the minimum value of friction coefficient with which at least one wedged configuration exists. We also propose an algorithm for computing the critical friction coefficient, which is based on the bisection method and the second-order cone program.

Keywords

Coulomb friction; Complementarity problem; Enumeration; Double description method; Second-order cone program.

1 Introduction

Wedged configuration is a nontrivial equilibrium state of a linear elastic structure subjected to the unilateral contact condition against a fixed rigid obstacle with the existence of Coulomb friction [3, 7, 8]. Problem for finding a wedged configuration is referred to as a wedged problem. The existence of a wedged configuration is regarded as a particular type of non-uniqueness of an equilibrium state in the frictional contact problem.

Hassani *et al.* [7, 8] studied a continuum formulation of the wedged problem, and show the presence of a critical value of the friction coefficient for which wedged configurations can exist. They applied a genetic algorithm to the problem of finding the critical friction coefficient. Barber and Hild [3] show that the critical friction coefficient can be obtained as the solution of a nonlinear eigenvalue problem, and presented a heuristic algorithm which does not necessarily converge.

A wedged configuration can be related to a non-uniqueness of the equilibrium state of a linear elastic structure with the Coulomb friction. The non-uniqueness of quasi-static solution and rate

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solution of the contact problem with the Coulomb friction have been investigated extensively [9, 10, 14, 15]. A uniqueness condition [10] and an enumeration algorithm [14] have been presented for the quasi-static and/or rate problems by formulating those problems as the complementarity problems.

In this paper we propose numerical algorithms for solving the wedged problem of a finitely discretized structure. It is mentioned that the wedged problem has infinitely many solutions in general. We show that all the solutions can be completely expressed as a union of finitely many convex cones. We propose numerical algorithms for enumerating all the maximal convex cones, as well as a global optimization algorithm for finding the critical value of friction coefficient.

For the in-plane wedged problem we show that the set of all the wedged configurations can be described as a family of finitely many polyhedral cones, each of which can be represented as the nonnegative combination of some extremal rays. It is also shown that the enumeration of those polyhedral cones is reduced to the enumeration of the maximal cliques of the given undirected graph. Thus the solution set of the in-plane wedged problem is described completely by finitely many representatives. We present a numerical algorithm for the enumeration based on the conventional double description method [6, 13] and algorithm for enumerating all the maximal cliques [12].

In contrast to the in-plane problem, the solution set of the wedged problem in the three-dimensional space cannot be represented by using finitely many representatives. The solution set, however, can be represented as the union of finitely many convex cones, each of which is identified by the sign-pattern of the variables subjected to the complementarity condition. We propose two algorithms for enumerating all the sign-patterns: one of which is based on the branch-and-bound method, and the other the polyhedral approximation of the friction cone.

The critical friction coefficient, μ^{c}, is defined as the minimum value of the friction coefficient with which a wedged configuration exists [7]. It is not difficult to show that at least one wedged configuration exists for any friction coefficient μ satisfying $\mu > \mu^{\mathrm{c}}$. Hassani *et al.* [7] proposed a genetic algorithm to find an upper bound of μ^{c}. Barber and Hild [3] formulated the problem finding μ^{c} as a nonlinear eigenvalue problem, and proposed a heuristic numerical algorithm, which is not guaranteed to converge to the solution. In contrast to those heuristic algorithms, we propose the global optimization algorithm for finding μ^{c} based on the combination of the bisection method and the enumeration method of all the sign-patterns of the wedged problem.

This paper is organized as follows. In section 2, we present a rigorous statement of the wedged problem for a finite-dimensional structure. Properties of solution set of the wedged problem, which are useful to construct enumeration algorithms of solutions, are investigated in section 3. We propose algorithms for finding all the solutions for two-dimensional and three-dimensional wedged problems, respectively, in sections 4 and 5. An algorithm for finding the critical coefficient friction is presented in section 6. Numerical experiments are presented in section 7. Some conclusions are drawn in section 8.

A few words regarding our notation: all vectors are assumed to be column vectors. The $(m+n)$-dimensional column vector $(u^{\top}, v^{\top})^{\top}$ consisting of $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ is often written simply as (u, v). For vectors $p = (p_i) \in \mathbb{R}^n$ and $q = (q_i) \in \mathbb{R}^n$, we write $p \geq q$ if $p_i \geq q_i$ $(i = 1, \dots, n)$. Hence, $p \geq 0$ means $p_i \geq 0$ $(i = 1, \dots, n)$. For a set $C \in \mathbb{R}^n$, let $\mathrm{int}(C)$ and $\mathrm{bd}(C)$ denote the interior and boundary of C, respectively. We denote by $\mathcal{S}^n$ the set of $n \times n$ real symmetric matrices.

2 Wedged problem

The formal statement of a wedged problem for a continuum can be found in [8]. We present here a formulation for a finitely discretized structure in order to make the paper self-contained.

Consider a finite-dimensional linear elastic structure and a fixed rigid obstacle in the d-dimensional space, where $d \in \{2, 3\}$. We denote by m_0 the number of nodes of the structure. The set of all nodes, denoted by Γ, has a partition consisting of $\Gamma_{\rm D}$, $\Gamma_{\rm N}$, and $\Gamma_{\rm C}$. Here, $\Gamma_{\rm C}$ is the set of contact candidate nodes, which are assumed to be on the surface of the rigid obstacle, i.e. no gap exists between each contact candidate node and the obstacle surface at the reference configuration. Suppose that the nodal displacement is vanishing at the node contained in $\Gamma_{\rm D}$, while no external load is applied to the node contained in $\Gamma_{\rm N}$.

Consider the node i which is a contact candidate, i.e. $i \in \Gamma_{\rm C}$. For defining the nodal displacement vector, we consider the orthogonal reference frame which consists of an outer normal vector and tangential vectors of the surface of rigid obstacle, and the origin of which coincides with the location of the node i at the reference configuration. Then we denote by $u_{{\rm n}i} \in \mathbb{R}$ and $u_{{\rm t}i} \in \mathbb{R}^{d-1}$ the normal and tangential displacements, respectively. Let $u_i = (u_{{\rm n}i}, u_{{\rm t}i}) \in \mathbb{R}^d$ denote the nodal displacement vector. Similarly, we denote by $r_i = (r_{{\rm n}i}, r_{{\rm t}i}) \in \mathbb{R} \times \mathbb{R}^{d-1}$ the reaction vector applied to the node i.

For each contact candidate node we consider the unilateral contact condition as well as the Coulomb friction law. Since we assume at the reference configuration that the node i is located on the obstacle surface, the unilateral contact condition is written as

$$u_{{\rm n}i} \geq 0, \quad r_{{\rm n}i} \geq 0, \quad u_{{\rm n}i} r_{{\rm n}i} = 0.$$

We denote by $\mu > 0$ the friction coefficient. The friction cone of the Coulomb law is given by

$$\|r_{{\rm t}i}\| \leq \mu r_{{\rm n}i}.$$

Let $u = (u^1, \dots, u^{m_0}) \in \mathbb{R}^{dm_0}$ and $r = (r^1, \dots, r^{m_0}) \in \mathbb{R}^{dm_0}$ denote the displacement vector and reaction vector, respectively. We denote by $K^0 \in \mathcal{S}^{dm_0}$ the stiffness matrix, which is assumed to be positive definite. Under the assumptions stated above, a problem of finding nontrivial equilibrium states is referred to as the *wedged problem*, which is formulated as

$$\left.\begin{array}{lll} \text{given} & K^0 \in \mathcal{S}^{m_0 d}, \ \mu > 0, \ \Gamma_{\rm C}, \ \Gamma_{\rm D}, \ \Gamma_{\rm N} & \\ \text{find} & (u, r) \in \mathbb{R}^{m_0 d} \times \mathbb{R}^{m_0 d} & \\ \text{s.t.} & K^0 u = r, & \\ & u_{{\rm n}i} \geq 0, \ r_{{\rm n}i} \geq 0, \ u_{{\rm n}i} r_{{\rm n}i} = 0, \ \|r_{{\rm t}i}\| \leq \mu r_{{\rm n}i}, & \forall i \in \Gamma_{\rm C}, \\ & u_j = 0, & \forall j \in \Gamma_{\rm D}, \\ & r_k = 0, & \forall k \in \Gamma_{\rm N}. \end{array}\right\} \qquad (1)$$

A nontrivial solution, i.e. $(u, r) \neq 0$, of (1) is referred to as the *wedged configuration*.

Note that the constraint conditions of the wedged problem (1) consist of some linear inequalities, friction cones, and complementarity conditions. Particularly, in the in-plane case, i.e. for $d = 2$, the wedged problem is regarded as a kind of generalized linear complementarity problems [5]; see section 3.1 for details.

Let $u_{\rm C} = (u^i \mid i \in \Gamma_{\rm C})$ and $r_{\rm C} = (r^i \mid i \in \Gamma_{\rm C})$. Define the vectors $u_{\rm N}$ and $r_{\rm D}$ similarly for the nodes contained in $\Gamma_{\rm N}$ and $\Gamma_{\rm D}$, respectively. Without loss of generality, the equilibrium equation considering the boundary conditions in (1) can be rewritten as

$$\begin{bmatrix} K^0_{\mathrm{CC}} & K^0_{\mathrm{CN}} & K^0_{\mathrm{CD}} \\ K^0_{\mathrm{NC}} & K^0_{\mathrm{NN}} & K^0_{\mathrm{ND}} \\ K^0_{\mathrm{DC}} & K^0_{\mathrm{DN}} & K^0_{\mathrm{DD}} \end{bmatrix} \begin{bmatrix} u_{\mathrm{C}} \\ u_{\mathrm{N}} \\ 0 \end{bmatrix} = \begin{bmatrix} r_{\mathrm{C}} \\ 0 \\ r_{\mathrm{D}} \end{bmatrix}. \tag{2}$$

Since K^0 is assumed to be positive definite, we can reduce (2) to

$$[K^0_{\mathrm{CC}} - K^0_{\mathrm{CN}}(K^0_{\mathrm{NN}})^{-1}K^0_{\mathrm{NC}}]u_{\mathrm{C}} = r_{\mathrm{C}}, \tag{3}$$

$$u_{\mathrm{N}} = -(K^0_{\mathrm{NN}})^{-1}K^0_{\mathrm{NC}}u_{\mathrm{C}}, \tag{4}$$

$$r_{\mathrm{D}} = K^0_{\mathrm{DC}}u_{\mathrm{C}} + K^0_{\mathrm{DN}}u_{\mathrm{N}}. \tag{5}$$

It is easy to see that if u_{C} and r_{C} are given, then u_{N} and r_{D} are obtained by using (4) and (5). Consequently, the wedged problem (1) can be reformulated only in terms of u_{C} and r_{C} as below.

Define $K \in \mathcal{S}^{dm}$ by $K = K^0_{\mathrm{CC}} - K^0_{\mathrm{CN}}(K^0_{\mathrm{NN}})^{-1}K^0_{\mathrm{NC}}$, where $m = |\Gamma_{\mathrm{C}}|$. Note that K is positive definite, because K is the Schur complement of K^0_{NN} in $\begin{bmatrix} K^0_{\mathrm{CC}} & K^0_{\mathrm{CN}} \\ K^0_{\mathrm{NC}} & K^0_{\mathrm{NN}} \end{bmatrix}$ and K^0 is assumed to be positive definite. For simplicity, rewrite u_{C} and r_{C} as $u \in \mathbb{R}^{dm}$ and $r \in \mathbb{R}^{dm}$, respectively. Let $u_{\mathrm{n}} = (u_{n1}, \dots, u_{nm}) \in \mathbb{R}^m$ and $r_{\mathrm{n}} = (r_{n1}, \dots, r_{nm}) \in \mathbb{R}^m$. Then the problem (1) is reduced to

$$\left.\begin{array}{lll} (\mathrm{WP}): & \mathrm{find} & (u, r) \in \mathbb{R}^{dm} \times \mathbb{R}^{dm} \\ & \mathrm{s.t.} & Ku = r, \\ & & u_{\mathrm{n}} \geq 0,\ r_{\mathrm{n}} \geq 0,\ u_{\mathrm{n}}^\top r_{\mathrm{n}} = 0, \\ & & \|r_{\mathrm{t}i}\| \leq \mu r_{\mathrm{n}i}, \quad i = 1, \dots, m. \end{array}\right\} \tag{6}$$

Throughout the paper we deal with the problem (6), which is referred to as (WP).

The following result summarizes fundamental properties of solutions of (WP), which was first presented in [8] for continua.

Proposition 2.1.

(i) (WP) *has a trivial solution at the origin, i.e.* $(u, r) = (0, 0)$ *solves* (WP)*;*

(ii) *Let* $\mu > 0$ *be fixed. If* (u, r) *solves* (WP)*, then* $(\lambda u, \lambda r)$ *for any* $\lambda > 0$ *solves* (WP)*;*

(iii) *For the given* $\mu > 0$*, suppose that* (u, r) *is a solution of* (WP)*. Then* (u, r) *is also a solution of* (WP) *for any* μ' *satisfying* $\mu' > \mu$*.*

3 Solution analysis of wedged problem

This section provides an in-depth study of properties of the solution set of (WP) in (6). Particularly, as an important observation we show that the solution set can be represented as a union of finitely many convex cones. Such a property plays a key role to design algorithms for enumerating all the wedged configurations presented in sections 4 and 5 as well as the one for finding the critical value of friction coefficient discussed in section 6.

3.1 Solution set of two-dimensional problem

In this section we consider the solution set of the in-plane wedged problem, i.e. $d = 2$.

A key observation is that the friction cone in the two-dimensional space can be represented as linear inequalities. Hence the solution set of (WP) with $d = 2$ is characterized by polyhedral cones, which is defined as follows [20].

Definition 3.1. A set $\mathcal{P}(A) \subseteq \mathbb{R}^\beta$ is a *polyhedral cone* if

$$\mathcal{P}(A) = \{y \mid Ay \geq 0\}$$

for some matrix $A \in \mathbb{R}^{\alpha\times\beta}$. ■

In other words, a polyhedral cone is the intersection of finitely many linear half-spaces. We shall investigate a polyhedral cone having an important property, which we define below.

Definition 3.2. A cone $\mathcal{L}$ is *pointed* if $\mathcal{L} \cap (-\mathcal{L}) = \{0\}$. ■

Define $\mathcal{C} \subseteq \mathbb{R}^{dm} \times \mathbb{R}^{dm}$ by

$$\mathcal{C} = \{x = (u, r) \mid Ku = r,\ u_\mathrm{n} \geq 0,\ r_\mathrm{n} \geq 0,\ -\mu r_\mathrm{n} \leq r_t \leq \mu r_\mathrm{n}\}, \tag{7}$$

where $d = 2$. Note that $\mathcal{C}$ is the set of (u, r) satisfying the constraint conditions of (6) except for the complementarity conditions. We first state a property of $\mathcal{C}$ in the following proposition.

Proposition 3.3. *The set $\mathcal{C}$ defined by* (7) *is a pointed polyhedral cone.*

Proof. Define $\hat{A}^{7m\times 4m}$ by

$$\hat{A} = \left[\begin{array}{c|c} K & -I_{2n} \\ \hline O & \begin{matrix} \mu & 1 & & & \\ \mu & -1 & & & \\ & & \ddots & & \\ & & & \mu & 1 \\ & & & \mu & -1 \end{matrix} \\ \hline -K & I_{2n} \\ \hline \begin{matrix} 1 & 0 & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{matrix} & O \end{array}\right].$$

Note that $K \in \mathcal{S}^{2m}$, because we now consider the in-plane problem. It is easy to see that $\mathcal{C}$ satisfies $\mathcal{C} = \mathcal{P}(\hat{A}) := \{x = (u, r) \mid \hat{A}x \geq 0\}$, which implies that $\mathcal{C}$ is a polyhedral cone. Since K is positive definite, we obtain $\operatorname{rank} \hat{A}_{\{1,\dots,4m\},\{1,\dots,4m\}} = 4m$. Hence, $\operatorname{rank} \hat{A} = 4m$, i.e. $\hat{A}x = 0$ if and only if $x = 0$. Thus we obtain $\mathcal{P}(\hat{A}) \cap (-\mathcal{P}(\hat{A})) = \{x \mid \hat{A}x = 0\} = \{0\}$, which implies that $\mathcal{P}(\hat{A})$ is pointed. □

A pointed polyhedral cone can be expressed completely by using a finitely many representative elements, which we define below.

Definition 3.4. A vector $v \in \mathbb{R}^\beta \setminus \{0\}$ is an *extremal ray* of $\mathcal{P}(A)$ if there exists a hyperplane $\mathcal{H} = \{x \in \mathbb{R}^\beta \mid h^\top x = 0\}$ such that $\mathcal{P}(A) \cap \mathcal{H} = \{x \mid x = \lambda v, \lambda \geq 0\}$. ■

In other words, an extremal ray v of $\mathcal{P}(A)$ is a nonzero vector $v \in \mathcal{P}(A)$ which cannot be represented as a nonnegative combination of elements of $\mathcal{P}(A)$ other than v. A convex cone is polyhedral if and only if the number of its extremal rays is finite [17, Corollary 7.1a]. The following proposition states a useful property of a pointed polyhedral cone.

Proposition 3.5 ([17, Chap. 8])**.** *A pointed polyhedral cone can be represented by all possible nonnegative combinations of its extremal rays.*

For simplicity, we say that a vector $x \in \mathbb{R}^{2dm}$, where $x = (u, r) = (u_\mathrm{n}, u_\mathrm{t}, r_\mathrm{n}, r_\mathrm{t})$, satisfies the complementarity condition if it satisfies $u_\mathrm{n}^\top r_\mathrm{n} = 0$. Let $\{v^1, \dots, v^p, w^1, \dots, w^q\}$ be the set of extremal rays of $\mathcal{C}$, where v^j $(1 \leq j \leq p)$ satisfies the complementarity condition. Here we say that $v^j = (u^j, r^j) = (u_\mathrm{n}^j, u_\mathrm{t}^j, r_\mathrm{n}^j, r_\mathrm{t}^j)$ satisfies the complementarity condition if $(u_\mathrm{n}^j)^\top r_\mathrm{n}^j = 0$ is satisfied. It should be clear that w^k $(1 \leq k \leq q)$ does not satisfy the complementarity condition.

Let $\mathrm{cone}(\{x^1, \dots, x^\ell\})$ denote the set of nonnegative combinations of the vectors $x^1, \dots, x^\ell$, i.e.

$$\mathrm{cone}(\{x^1, \dots, x^\ell\}) = \left\{ \sum_{1 \leq k \leq \ell} \lambda_k x^k \;\middle|\; \lambda_k \geq 0 \ (1 \leq k \leq \ell) \right\}.$$

It follows from Proposition 3.3 and Proposition 3.5 that $\mathcal{C}$ defined by (7) is written as

$$\mathcal{C} = \mathrm{cone}(\{v^1, \dots, v^p, w^1, \dots, w^q\}).$$

Let $\mathcal{S} \subseteq \mathbb{R}^{2dm}$ denote the set of solutions to (WP) in (6). It is clear that $\mathcal{S}$ is the set of vectors in $\mathcal{C}$ satisfying the complementarity condition, i.e.

$$\mathcal{S} = \{(u, r) \in \mathcal{C} \mid u_\mathrm{n}^\top r_\mathrm{n} = 0\}.$$

The following proposition shows that $x = (u, r) \in \mathcal{S}$ only if x belongs to a cone generated by $Y \subseteq \{v^1, \dots, v^p\}$.

Proposition 3.6. *Let $Y \subseteq \{v^1, \dots, v^p, w^1, \dots, w^q\}$. If there exists a $w^k \in Y$, then any interior point of* $\mathrm{cone}(Y)$ *does not satisfy the complementarity condition, i.e.* $u_\mathrm{n}^\top r_\mathrm{n} \neq 0$ *holds for any* $x = (u, r) \in \mathrm{int}(\mathrm{cone}(Y))$.

Proof. Observe that any $(u, r) \in \mathcal{C}$ satisfies $u_\mathrm{n} \geq 0$ and $r_\mathrm{n} \geq 0$. Hence, $(u, r) \in \mathcal{C}$ does not satisfy the complementarity condition if and only if there exists i $(1 \leq i \leq n)$ such that $u_{\mathrm{n}i} > 0$ and $r_{\mathrm{n}i} > 0$. This implies that, for $w^k = (u^k, r^k) \in Y$, there exists i' such that $u_{\mathrm{n}i'}^k > 0$ and $r_{\mathrm{n}i'}^k > 0$. Since $\mathrm{cone}(Y)$ is the set of nonnegative combinations of Y, any $(\check{u}, \check{r}) \in \mathrm{int}(\mathrm{cone}(Y))$ satisfies $\check{u}_{\mathrm{n}i'} > 0$, $\check{r}_{\mathrm{n}i'} > 0$, $\check{u}_\mathrm{n} \geq 0$, and $\check{r}_\mathrm{n} \geq 0$. Hence, $(\check{u}, \check{r})$ does not satisfy the complementarity condition. □

Furthermore, the solution set $\mathcal{S}$ is the union of finitely many polyhedral cones belonging to $\mathrm{cone}(\{v^1, \dots, v^p\})$ as shown below. To see this, the notion of cross-complementarity plays an important role, which was first introduced in [4].

Definition 3.7. Let $B \subseteq \{1,\ldots,p\}$. The set of vectors $\{v^j = (u^j, r^j) \mid j \in B\}$ is *cross-complementary* if $\sum_{j\in B} v^j$ satisfies the complementarity condition, i.e. if $\left(\sum_{j\in B} u_{\mathrm{n}}^j\right)^\top \left(\sum_{j\in B} r_{\mathrm{n}}^j\right) = 0$. We also say that $\{v^j\}^{j\in B}$ satisfies the *cross-complementarity condition* if $\{v^j\}^{j\in B}$ is cross-complementary. ■

Note again that $\{v^1,\ldots,v^p\}$ denotes the set of all extreme rays of $\mathcal{C}$ satisfying the complementarity condition. The following proposition provides a necessary and sufficient condition for any $x \in \mathrm{cone}(Y)$ ($Y \subseteq \{v^1,\ldots,v^p\}$) to satisfy the complementarity condition.

Proposition 3.8. *Let $Y \subseteq \{v^1,\ldots,v^p\}$. Any $x = (u,r) \in \mathrm{cone}(Y)$ satisfies $u_{\mathrm{n}}^\top r_{\mathrm{n}} = 0$ if and only if Y is cross-complementary.*

Proof. It suffices to show the sufficiency, i.e. the 'if' part. Let $Y := \{(u^{j_1}, r^{j_1}),\ldots,(u^{j_\ell}, r^{j_\ell})\} \subseteq \{v^1,\ldots,v^p\}$, and suppose that Y satisfies the cross-complementarity condition. It follows from Definition 3.7 that

$$\sum_{1\le k\le \ell} u_{\mathrm{n}i}^{j_k} = 0 \quad \text{or} \quad \sum_{1\le k\le \ell} r_{\mathrm{n}i}^{j_k} = 0 \tag{8}$$

holds for any $i \in \{1,\ldots,m\}$. Since $(u^{j_k}, r^{j_k}) \in \mathcal{C}$, it satisfies $u_{\mathrm{n}i}^{j_k} \ge 0$ and $r_{\mathrm{n}i}^{j_k} \ge 0$, from which and (8) it follows that

$$\forall k : u_{\mathrm{n}i}^{j_k} = 0 \quad \text{or} \quad \forall k : r_{\mathrm{n}i}^{j_k} = 0$$

holds for any $i \in \{1,\ldots,m\}$. Then we can easily see that any $x \in \mathrm{cone}(Y)$, where

$$\mathrm{cone}(Y) = \left\{ \sum_{1\le k\le \ell} \lambda_k v^{j_k} \mid \lambda_k \ge 0 \ (1 \le k \le \ell) \right\},$$

satisfies the complementarity condition. □

We finally show that the solution set of the in-plane wedged problem can be completely described by finitely many representative elements as follows.

Proposition 3.9. *Let $\{v^j \mid j = 1,\ldots,p\}$ be the set of extremal rays of $\mathcal{C}$, which satisfy the complementarity condition. Then*

$$\mathcal{S} = \bigcup_{B\subseteq\{1,\ldots,p\}} \left\{ \sum_{j\in B} \lambda_j v^j \,\middle|\, \{v^j\}^{j\in B} \text{ is cross-complementary, } \lambda_j \ge 0 \ (j \in B) \right\}. \tag{9}$$

Proof. By Proposition 3.6 we see that $\mathcal{S} \subseteq \mathrm{cone}(\{v^1,\ldots,v^p\})$, from which and Proposition 3.8 the assertion of Proposition 3.9 is immediately obtained. □

An important consequence of Proposition 3.9 is that $\mathcal{S}$ can be expressed only by finitely many vectors $v^1,\ldots,v^q$ and their cross-complementarity relationship. Let $B_t \subseteq \{1,\ldots,q\}$ be an index set of v^j's, where $\{v^j, v^{j'}\}$ satisfies the cross-complementary condition for any j, $j' \in B_t$. Then $\mathcal{S}$ can be written as

$$\mathcal{S} = \bigcup_t \mathrm{cone}(\{v^j \mid j \in B_t\}). \tag{10}$$

Such a description of $\mathcal{S}$ is not unique. Among families of index sets, B_t's, satisfying (10), we denote by $\hat{B}_1, \dots, \hat{B}_{\hat{t}}$ the one consisting of the minimum number of ray index sets. In other words, the description

$$\mathcal{S} = \bigcup_{t=1}^{\hat{t}} \mathrm{cone}(\{v^j \mid j \in \hat{B}_t\}) \tag{11}$$

corresponds to the most concise description among (10). We call $(\hat{B}_t \mid t = 1, \dots, \hat{t})$ the family of maximal ray index sets satisfying the cross-complementarity condition. It should be emphasized that (11) implies that $\mathcal{S}$ can be completely represented by finitely many vectors $v^1, \dots, v^q$ and index sets $\hat{B}_1, \dots, \hat{B}_{\hat{t}}$. The enumeration algorithm of $\mathcal{S}$ presented in section 4 is based on (11), i.e. the algorithm consists of finding all v^j's and finding all $\hat{B}_t$'s.

3.2 Solution set of three-dimensional problem

In contrast to the in-plane problem studied in section 3.1, the solution set of the 3-D wedged problem no longer has the polyhedrarlity. In this section we show that the solution set of the 3-D problem is represented as a union of convex cones.

Let $d = 3$ throughout this section. Define $\mathcal{C}^3 \subseteq \mathbb{R}^{2dm}$ by

$$\mathcal{C}^3 = \left\{(u, r) \in \mathbb{R}^{dm} \times \mathbb{R}^{dm} \mid Ku = r,\ u_{\mathrm{n}} \geq 0,\ r_{\mathrm{n}} \geq 0,\ \|r_{\mathrm{t}i}\| \leq \mu r_{\mathrm{n}i}\ (1 \leq i \leq m)\right\},$$

which is the set of (u, r) satisfying the constraint conditions of (WP) except for the complementarity conditions. Note that $\mathcal{C}^3$ is a convex cone, although it is not a polyhedral cone in contrast to $\mathcal{C}$ in (7) for the in-plane problem. The solution set of (WP) in the three-dimensional space is given by

$$\mathcal{S}^3 = \{(u, r) \in \mathcal{C}^3 \mid u_{\mathrm{n}}^{\top} r_{\mathrm{n}} = 0\}.$$

For $x = (u, r) \in \mathbb{R}^{dm} \times \mathbb{R}^{dm}$, define $Z^u(x)$, $Z^u(x) \subseteq \{1, \dots, m\}$ by

$$Z^u(x) = \{i \in \{1, \dots, m\} \mid u_{\mathrm{n}i} = 0\}, \tag{12}$$

$$Z^r(x) = \{i \in \{1, \dots, m\} \mid r_{\mathrm{n}i} = 0\}. \tag{13}$$

Since $u_{\mathrm{n}i} = 0 \Leftrightarrow i \in Z^u(x)$ and $r_{\mathrm{n}i} = 0 \Leftrightarrow i \in Z^r(x)$, we call $Z(x) := (Z^u(x), Z^r(x))$ the *sign-pattern* of x. It is easy to see that $x = (u, r)$ satisfies the complementarity condition if and only if $Z^u(x) \cup Z^r(x) = \{1, \dots, m\}$. We call $Z(x)$ a sign-pattern satisfying the complementarity condition if x satisfies the complementarity condition. We say that x satisfies the strict complementarity condition if x satisfies $Z^u(x) \cap Z^r(x) = \emptyset$ as well as the complementarity condition. We call $Z(x)$ a sign-pattern satisfying the strict complementarity condition if x satisfies the strict complementarity condition. It is clear that a sign-pattern satisfying the strict complementarity condition satisfies the complementarity condition. We write $Z(x) \subseteq Z(x')$ if $Z^u(x) \subseteq Z^u(x')$ and $Z^u(x) \subseteq Z^r(x')$.

The following result, which will be used in Proposition 5.7, states the relation between two solutions of (WP) which does not satisfy the strict complementarity condition.

Proposition 3.10. *Let $x = (u, r)$ and $x' = (u', r')$ be solutions of* (WP). *If that there exists a sign-pattern Y satisfying the strong complementarity condition, $Z(x) \supseteq Y$, and $Z(x') \supseteq Y$, then any nonnegative combination of x and x' solves* (WP).

Proof. From $x \in \mathcal{S}^3$ and $x' \in \mathcal{S}^3$, we see that $x \in \mathcal{C}^3$ and $x' \in \mathcal{C}^3$. Since $\mathcal{C}^3$ is a convex cone, any nonnegative combination of x and x', denoted by x'', satisfies $x'' \in \mathcal{C}^3$. Note that any $(u, r) \in \mathcal{S}^3$ satisfies $u_{\mathrm{n}} \geq 0$ and $r_{\mathrm{n}} \geq 0$. Hence, $Z(x) \supseteq Y$ and $Z(x') \supseteq Y$ imply that $Z(x'') \supseteq Y$. Since the sign-pattern Y satisfies the complementarity condition, so does $Z(x'')$, from which and $x'' \in \mathcal{C}^3$ it follows that $x'' \in \mathcal{S}^3$. □

Let $Y = (Y^u, Y^r)$ be a sign-pattern satisfying the strict complementarity condition. Define $D(Y) \subseteq \mathbb{R}^{2dm}$ by

$$D(Y) = \{x \in \mathcal{S}^3 \mid Z(x) \supseteq Y\}. \tag{14}$$

We first state a fundamental property of $D(Y)$ below.

Proposition 3.11. *A nonempty $D(Y)$ is a convex cone.*

Proof. If $x \in D(Y)$, then it is easy to see that $\alpha x \in D(Y)$ for any $\alpha \geq 0$. Hence, $D(Y)$ is a cone. The convexity of $D(Y)$ is shown as follows. Let x, $x' \in D(Y)$. Since $\mathcal{C}^3$ is convex, we see that $\lambda x + (1 - \lambda)x' \in \mathcal{C}^3$ for any $\lambda \in [0, 1]$. From the definition of $D(Y)$, we have $Z(x) \supseteq Y$ and $Z(x') \supseteq Y$. Hence, $Z(\lambda x + (1 - \lambda)x') \supseteq Y$, i.e. $Z(\lambda x + (1 - \lambda)x')$ is a sign-pattern satisfying the complementarity condition. Consequently, $\lambda x + (1 - \lambda)x' \in \mathcal{C}^3$ satisfies the complementarity condition for any $\lambda \in [0, 1]$. □

From the definition (14) of $D(Y)$, we see that $D(Y) \subseteq \mathcal{S}^3$ and that any $x \in \mathcal{S}^3$ satisfies $x \in D(Y)$ for some Y. Hence, we obtain

$$\mathcal{S}^3 = \bigcup_{Y} D(Y). \tag{15}$$

In (15) the union is taken for any Y satisfying the strict complementarity condition. However, in (15) we can exclude some Y's, e.g. it is obvious that Y satisfying $D(Y) = \{0\}$ can be excluded. Among these variants of description we denote by

$$\mathcal{S}^3 = \bigcup_{t=1}^{\hat{t}} D(\hat{Y}_t) \tag{16}$$

the one consisting of the minimum number of convex cones. For simplicity, we often write $\hat{D}_t = D(\hat{Y}_t)$. The following result states a property of this concise description of $\mathcal{S}^3$.

Proposition 3.12. *In* (16), $\mathrm{int}(\hat{D}_t) \cap \mathrm{int}(\hat{D}_{t'}) = \emptyset$ *for any* $t \neq t'$.

Proof. From the definition of $\hat{D}_t$ it follows that there exists a $\hat{Y}_t$ satisfying the strict complementarity condition and $\hat{D}_t = D(\hat{Y}_t)$.

Let Y_1 and Y_2 ($Y_1 \neq Y_2$) be sign-patterns satisfying the strict complementarity condition. We begin with seeing that $D(Y_1)$ and $D(Y_2)$ satisfies one of the following conditions:

(i) $D(Y_1) \cap D(Y_2) = \{0\}$;

(ii) $\mathrm{bd}(D(Y_1)) \cap \mathrm{bd}(D(Y_2)) \supset \{0\}$ and $\mathrm{int}(D(Y_1)) \cap \mathrm{int}(D(Y_2)) = \emptyset$;

(iii) either $D(Y_1) \subseteq D(Y_2)$ or $D(Y_1) \supseteq D(Y_2)$ holds.

In other words, (i) $D(Y_1)$ and $D(Y_2)$ share only the origin, (ii) they share some portions of boundaries, or (iii) they have an inclusion relation.

Suppose that there exists an $x = (u, r) \in \mathcal{S}^3$ $(x \neq 0)$ satisfying $Z(x) \supseteq Y_1$ and $Z(x) \supseteq Y_2$. Since Y_1 and Y_2 are two different sign-patterns satisfying the strict complementarity condition, there exists an $i \in \{1, \dots, m\}$ satisfying $u_{\mathrm{n}i} = r_{\mathrm{n}i} = 0$. From Proposition 3.11 we see that $D(Y_1)$ is a convex cone, which implies that $x \in D(Y_1)$ can be represented as a nonnegative combination of elements of $\mathrm{bd}(D(Y_1))$. Note that any $x'' \in \mathrm{bd}(D(Y_1))$ satisfies $u''_{\mathrm{n}i} \geq 0$ and $r''_{\mathrm{n}i} \geq 0$. Hence, from $u_{\mathrm{n}i} = r_{\mathrm{n}i} = 0$ we obtain $u''_{\mathrm{n}i} = r''_{\mathrm{n}i} \geq 0$, which implies that $x \notin \mathrm{int}(D(Y_1))$. Similarly, $x \notin \mathrm{int}(D(Y_2))$. Consequently, $D(Y_1)$ and $D(Y_2)$ satisfy either (ii) or (iii). Conversely, suppose that there exists no $x = (u, r) \in \mathcal{S}^3$ $(x \neq 0)$ satisfying $Z(x) \supseteq Y_1$ and $Z(x) \supseteq Y_2$. Then only the origin corresponds to the common point of $D(Y_1)$ and $D(Y_2)$, i.e. (i) is satisfied.

In (16), if $D(Y_1)$ and $D(Y_2)$ satisfy the condition (i), then one of Y_1 and Y_2 can be excluded. For example, if $D(Y_1) \subset D(Y_2)$, then we can exclude Y_1 from (16). This leads the contradiction because (16) includes the minimum number of $D(Y)$'s, which concludes the proof. $\square$

In section 5 we present algorithms for enumerating $\hat{Y}_1, \dots, \hat{Y}_{\hat{t}}$ in (16) for (WP) with $d = 3$.

4 Enumeration of two-dimensional wedged configurations

In this section we propose an algorithm for enumerating all the solutions of the in-plane wedged problem.

4.1 Prototype of algorithm

As shown in section 3.1, the solution set of WP (6) with $d = 2$ is described completely as (11) by using finitely many extremal rays and maximal ray index sets representing the cross-complementarity relationship. Our algorithm for enumerating those representatives in (11) consists of two parts as follows.

Algorithm 4.1 (prototype of enumeration algorithm)**.**

Step 1: For the polyhedral cone $\mathcal{C}$ defined by (7), find all the extremal rays, $v^1, \dots, v^p$, satisfying the complementarity condition.

Step 2: Find all the maximal ray index sets, $\hat{B}_1, \dots, \hat{B}_{\hat{t}} \subseteq \{1, \dots, p\}$, such that $\{v^j\}^{j \in \hat{B}_t}$ satisfies the cross-complementarity condition.

We present an algorithm for Step 1 in section 4.2, while the one for Step 2 in section 4.3.

4.2 Enumeration of extremal rays finding complementarity condition

There exist some numerical algorithms for enumerating all the extremal rays of a given polyhedral cone, e.g. the double description method [6, 13], the reverse search [2], etc. In this section we propose an algorithm which enumerate only the extremal rays of $\mathcal{C}$ satisfying the complementarity conditions.

More precisely, we consider the polyhedral cone

$$\mathcal{C} = \{(u, r) \in \mathbb{R}^{2m} \times \mathbb{R}^{2m} \mid Ku = r,\ u_{\rm n} \geq 0,\ r_{\rm n} \geq 0, -\mu r_{\rm n} \leq r_{\rm t} \leq \mu r_{\rm n}\}$$

and enumerate all its extremal rays satisfying $u_{\rm n}^{\top} r_{\rm n} = 0$. In Algorithm 4.2 presented below, however, we eliminate the variables u by using the equilibrium equations and consider the polyhedral cone

$$\mathcal{C}_r = \{r \in \mathbb{R}^{2m} \mid -\mu r_{\rm n} \leq r_{\rm t} \leq \mu r_{\rm n},\ (K^{-1}r)_{\rm n} \geq 0\}$$

in terms of r. The following algorithm finds all the extremal rays of $\mathcal{C}_r$ satisfying the complementarity condition $(K^{-1}r)_{\rm n}^{\top} r_{\rm n} = 0$.

Algorithm 4.2 (enumeration of extremal rays satisfying the complementarity condition)**.**

Step 1: Enumerate all the extremal rays of a polyhedral cone defined by $P := \{r \mid -\mu r_{\rm n} \leq r_{\rm t} \leq \mu r_{\rm n}\}$. Let $\mathcal{V} = \{v^j\}^{j \in J}$ be the set of the obtained extremal rays. Set $i := 1$.

Step 2: Introduce a new inequality $u_{{\rm n}i} = (K^{-1}r)_{{\rm n}i} \geq 0$ to P as follows. Define J^+, J^0, and J^- by

$$\begin{aligned} J^+ &= \{j \in J \mid (K^{-1}v^j)_{{\rm n}i} > 0\}, \\ J^0 &= \{j \in J \mid (K^{-1}v^j)_{{\rm n}i} = 0\}, \\ J^- &= \{j \in J \mid (K^{-1}v^j)_{{\rm n}i} < 0\}. \end{aligned}$$

Let $\mathcal{V}_i := \{v^j\}^{j \in J^0}$. For each $(j_1, j_2) \in J^+ \times J^-$, perform the following procedure.

(a) If v^{j_1} and v^{j_2} are adjacent on P, then perform (b).

(b) If v^{j_1} and v^{j_2} satisfy $[(K^{-1}v^{j_1})_{{\rm n}l} + (K^{-1}v^{j_2})_{{\rm n}l}](v^{j_1}_{{\rm n}l} + v^{j_2}_{{\rm n}l}) = 0$ $(l = 1, \dots, i-1)$, then perform (c).

(c) Compute an internally dividing point v^{j_*} of v^{j_1} and v^{j_2} such that $(K^{-1}v^{j_*})_{{\rm n}i} = 0$ is satisfied. Let $\mathcal{V}_i := \mathcal{V}_i \cup \{v^{j_*}\}$.

Step 3: For each $v^j \in \mathcal{V}_i$, if $(K^{-1}v^j)_{{\rm n}i} v^j_{{\rm n}i} \neq 0$, then let $\mathcal{V}_i := \mathcal{V}_i \setminus \{v^j\}$.

Step 4: Let $\mathcal{V} := \mathcal{V} \cup \mathcal{V}_i$ and $P := P \cap \{r \in \mathbb{R}^{2m} \mid (K^{-1}r)_{{\rm n}i} \geq 0\}$.

Step 5: If $i = m$, then declare $\mathcal{V}$ as the output. Otherwise, set $i \leftarrow i + 1$, and go to Step 2.

Remark 4.3. It should be clear that $v^j \in \mathbb{R}^{2m}$ in Algorithm 4.2 corresponds to the reaction vector, which is an extremal ray of $\mathcal{C}_r$. In contrast, we have considered extremal rays of $\mathcal{C}$ in section 3.1, and hence we have written there as $v^j = (u^j, r^j) \in \mathbb{R}^{4m}$. ■

Remark 4.4. At Step 1 of Algorithm 4.2 all the extremal rays of P can be obtained analytically. Since each linear inequality defining P contains only two variables, $r_{{\rm n}i}$ and $r_{{\rm t}i}$, each extremal ray of P is orthogonal to the corresponding row vector of the matrix representing the linear inequalities of P. Hence, $\mathcal{V}$ at Step 1 is obtained immediately, i.e. if we define $v^j = ((r^j_{{\rm n}i}, r^j_{{\rm t}i}) \mid i = 1, \dots, m)$ by

$$\begin{aligned} &r^{2i-1}_{{\rm n}i} = 1, \quad r^{2i-1}_{{\rm t}i} = \mu, \quad r^{2i-1}_{{\rm n}l} = r^{2j-1}_{{\rm t}l} = 0\ (\forall l \neq i), \\ &r^{2i}_{{\rm n}i} = 1, \qquad r^{2i}_{{\rm t}i} = -\mu, \quad r^{2i}_{{\rm n}l} = r^{2j}_{{\rm t}l} = 0\ (\forall l \neq i), \end{aligned}$$

then $\mathcal{V} = \{v^1, \dots, v^{2m}\}$. This is the reason why we prefer to eliminate u, rather than to eliminate r, by using the equilibrium equations. ■

Remark 4.5. Step 2(b) of Algorithm 4.2, we check whether (u^{j_1}, r^{j_1}) and (u^{j_2}, r^{j_2}) satisfy the cross-complementarity condition with respect to the nodes $l = 1, \dots, i-1$, where $r^{j_k} := v^{j_k}$ and $u^{j_k} := K^{-1}v^{j_k}$ $(k = 1, 2)$. At Step 3 we check whether (u^{j_*}, r^{j_*}) satisfies the complementarity condition with respect to the node i, where $r^{j_*} := v^{j_*}$ and $u^{j_*} := K^{-1}v^{j_*}$. ■

Remark 4.6. At Step 2(a) we say that v^{j_1} and v^{j_2} are adjacent on the polyhedral cone P if the minimal face of P containing v^{j_1} and v^{j_2} contains no other extremal rays [6]. Let A denote a representation matrix of P, i.e. $P = \{r \mid Ar \geq 0\}$. We denote by $A(v^{j_1}, v^{j_2})$ the submatrix consisting of A_k satisfying $A_k v^{j_1} = A_k v^{j_2} = 0$, where A_k is the kth row vector of A. It is known that v^{j_1} and v^{j_2} are adjacent on P if and only if $\mathrm{rank}(A(v^{j_1}, v^{j_2})) = 2m - 2$. Hence, we proceed to Step 2(b) if $\mathrm{rank}(A(v^{j_1}, v^{j_2})) = 2m - 2$. ■

The validity of the conventional double description method [6, 13] is stated formally as follows.

Proposition 4.7. *All the extremal rays of $\mathcal{C}_r$ are enumerated by the double description method.*

The validity of Algorithm 4.2 can be shown as follows.

Proposition 4.8. *Algorithm 4.2 finds all the extremal rays of $\mathcal{C}$ satisfying $u_n^\top r_n = 0$.*

Proof. Let $\mathcal{V}$ denote the output of Algorithm 4.2. We denote by $\bar{\mathcal{V}}$ the set of extremal rays of $\mathcal{C}_r$ obtained by applying the double description method. Note that Algorithm 4.2 contains two additional procedures compared with the double description method; (i) we consider the cross-complementarity condition at Step 2(b), and (ii) we eliminate some extremal rays at Step 3. By Proposition 4.7 it suffices to show that by performing the procedures (i) and (ii) we do not lose any extremal rays contained in $\bar{\mathcal{V}}$ satisfying the complementarity condition. This is equivalent to the assertion that any extremal ray contained in $\bar{\mathcal{V}} \setminus \mathcal{V}$, i.e. any extremal ray eliminated from $\bar{\mathcal{V}}$ as a result of performing (i) and (ii), does not satisfy the complementarity condition. This assertion is shown as follows.

For seeing the validity of Step 2(b), suppose that there exists a pair of extremal rays (r^{j_1}, r^{j_2}), where $(j_1, j_2) \in J^+ \times J^-$, r^{j_1} and r^{j_2} are adjacent, and $(u^{j_1}_{\mathrm{n}l} + u^{j_2}_{\mathrm{n}l})(r^{j_1}_{\mathrm{n}l} + r^{j_2}_{\mathrm{n}l}) \neq 0$ for some l $(l < i)$. Let r^* be an internally dividing point of r^{j_1} and r^{j_2}, and let $u^* = K^{-1}r^*$. Since r^{j_1} and r^{j_2} satisfy the constraints of the current P, we see that $u^{j_1}_{\mathrm{n}k}$, $u^{j_2}_{\mathrm{n}k}$, $r^{j_1}_{\mathrm{n}k}$, and $r^{j_2}_{\mathrm{n}k}$ are nonnegative for any $k < i$. Hence, $(u^{j_1}_{\mathrm{n}l} + u^{j_2}_{\mathrm{n}l})(r^{j_1}_{\mathrm{n}l} + r^{j_2}_{\mathrm{n}l}) \neq 0$ implies that $u^*_{\mathrm{n}l} > 0$ and $r^*_{\mathrm{n}l} > 0$. As a consequence, any extremal rays computed at Step 2(c) of the ith iteration does not satisfy the complementarity condition. If Step 2(c) is not performed, at the i'th iteration $(i' > i)$ we may compute an internally dividing point of r^* and a newly obtained vector $r^{j'}$ by introducing $(Kr)_{\mathrm{n}i'} \geq 0$. Note that such an internally dividing point is contained in $\bar{\mathcal{V}} \setminus \mathcal{V}$. Since the constraint conditions $r_{\mathrm{n}} \geq 0$ has ben considered at Step 1, we see that $r^{j'}$ satisfies $r^{j'}_{\mathrm{n}l} \geq 0$. Moreover, $r^{j'}$ satisfies $u^{j'}_{\mathrm{n}l} \geq 0$ $(\forall l \leq i')$, because $r^{j'}$ is contained in P at the i'th iteration. Hence, $u^*_{\mathrm{n}l} > 0$ and $r^*_{\mathrm{n}l} > 0$ imply that any internal dividing point of r^* and $r^{j'}$ does not satisfy the complementarity condition. Consequently, any extremal ray which we lose by performing Step 2(b) does not satisfy the complementarity condition.

For seeing the validity of Step 3, similarly we can show that any internal dividing point of r^j satisfying $u^j_{\mathrm{n}i} r^j_{\mathrm{n}i} \neq 0$ and a vector $r^{j'}$ obtained at Step 2 does not satisfy the complementarity condition. Hence, any the extremal ray does not found by performing Step 2(b) and Step 3 does not satisfy the complementarity condition, which concludes the proof. □

4.3 Enumeration of index sets of cross-complementary rays

We here show that Step 2 of Algorithm 4.1 can be performed by using a conventional algorithm for enumerating all the maximal cliques of an undirected graph, which is a extensively studied problem of the graph theory. A key observation is given in the following proposition.

Proposition 4.9. *Let $\{v^1, \dots, v^p\}$ be the set of extremal rays satisfying the complementarity condition, and let $B \subseteq \{1, \dots, p\}$. Then $\{v^j\}^{j \in B}$ satisfies the cross-complementarity condition if and only if $\{v^{j_1}, v^{j_2}\}$ satisfies the cross-complementarity condition for any j_1, $j_2 \in B$.*

Proof. It suffices to show the necessity, i.e. the 'only if' part. Let $v^j = (u^j, r^j)$ $(j \in B)$. For each $i = 1, \dots, m$, we have one of the following three cases:

(a) $u^j_{\mathrm{n}i} = r^j_{\mathrm{n}i} = 0$ for any $j \in B$;

(b) There exists $l \in B$ such that $u^l_{\mathrm{n}i} > 0$;

(c) There exists $l \in B$ such that $r^l_{\mathrm{n}i} > 0$.

Nothing should be proved for the case (a). In the case (b), the complementarity condition for v^l implies $r^l_{\mathrm{n}i} = 0$. By putting $j_1 = l$ and $j_2 \in B \setminus \{l\}$, we obtain $r^{j_2}_{\mathrm{n}i} = 0$ $(\forall \in B \setminus \{l\})$. Consequently, we see that $r^j_{\mathrm{n}i} = 0$ $(\forall j \in B)$, and hence $\{v^j\}^{j \in B}$ satisfies the cross-complementarity condition. Similarly, in the case (c) we obtain $u^j_{\mathrm{n}i} = 0$ $(\forall j \in B)$, which concludes the proof. □

Define a binary relation $\sim$ on $\{1, \dots, p\}$ by

$$j_1 \sim j_2 \quad \Leftrightarrow \quad \{v^{j_1}, v^{j_2}\} \text{ satisfies the cross-complementarity condition,}$$

where j_1, $j_2 \in \{1, \dots, p\}$. We consider a graph $G = (V, E)$ with the vertex set $V = \{1, \dots, p\}$ and the edge set $E = \{\{j_1, j_2\} \mid j_1 \sim j_2\}$. Proposition 4.9 implies that $B \subseteq E$ is a clique of G if and only if $\{v^j\}^{j \in B}$ satisfies the cross-complementarity condition. Recall that in (11) we focus on the family of the maximal ray index sets, say, $\hat{B}_1, \dots, \hat{B}_{\hat{t}}$. The enumeration of $\hat{B}_t$ corresponds to the enumeration of the maximal cliques of the graph G, which can be performed by an existing algorithm; see, e.g. [12] and the references therein. In summary, at Step 2 of Algorithm 4.1 we enumerate the maximal cliques of the graph G by using a conventional method.

5 Enumeration of three-dimensional wedged configurations

In this section we consider the enumeration of solutions to the wedged problem (6) in the three-dimensional space, i.e. for $d = 3$. As discussed in section 3.2 the solution set of (WP) (6) with $d = 3$ cannot be described by finitely many representative solutions. However, the solution set consists of finitely many convex cones, each of which is characterized by a sign-pattern of the complementary variables, u_{n} and r_{n}. We present here two alternative algorithms for finding all the sign-patterns. In section 5.1 we propose an algorithm based on the branch-and-bond method, while in section 5.2 the one based on a polyhedral outer approximation of the friction cone.

5.1 Enumeration based on branch-and-bound method

We present here an algorithm for the enumeration of sign-patterns. Algorithm 5.1 below is based on the brand-and-bound method [11], at each node of the branch-and-bound tree we solve an SOCP problem [1].

Note that the complementarity condition implies that either $u_{\mathrm{n}i} = 0$ or $r_{\mathrm{n}i} = 0$ should be satisfied for each $i = 1, \dots, m$. Based on this observation we construct a binary search tree as follows. Each node N of the search tree is characterized by $U(N)$ and $R(N)$, where $U(N)$, $R(N) \subseteq \{1, \dots, m\}$ are sets of indices satisfying $U(N) \cap R(N) = \emptyset$. We set $u_{\mathrm{n}i} = 0$ for each $i \in U(N)$ and $r_{\mathrm{n}j} = 0$ for each $j \in R(N)$. It is clear that the complementarity condition, $u_{\mathrm{n}}^{\top} r_{\mathrm{n}} = 0$, is satisfied if $U(N) \cup R(N) = \{1, \dots, m\}$. The root node N^0 is defined by $U(N^0) = R(N^0) = \emptyset$. If $U(N) \cup R(N) \neq \{1, \dots, m\}$ at the node N, we select an $i \notin U(N) \cup R(N)$, and define a child node of N by adding i to $U(N)$ or $R(N)$. Thus, the node N has two child nodes. Consequently, all the sign-patterns satisfying the complementarity conditions are represented by 2^m leaf nodes of the search tree.

Note that we consider the constraint conditions

$$\forall i \in U(N) : u_{\mathrm{n}i} = 0, \quad \forall j \in R(N) : r_{\mathrm{n}j} = 0 \tag{17}$$

at the node N. By solving an appropriate subproblem at the node N, we determine whether there exists a nontrivial solution (WP) in (6) satisfying (17). If there exists such a solution, then we proceed to the branching procedure, in which we add two child nodes of N to the search tree. Otherwise, the node N is pruned by infeasibility.

For the determination of the feasibility, we may solve the following optimization problem at the node N:

$$\left.\begin{array}{lll}
\displaystyle\max_{u,r,t} & t & \\
\text{s.t.} & Ku = r, & \\
& u_{\mathrm{n}i} = 0,\ r_{\mathrm{n}i} \geq t,\ \|r_{\mathrm{t}i}\| \leq \mu r_{\mathrm{n}i}, & \forall i \in U(N), \\
& u_{\mathrm{n}j} \geq t,\ r_{\mathrm{n}j} = 0,\ r_{\mathrm{t}j} = 0, & \forall j \in R(N), \\
& u_{\mathrm{n}k} \geq t,\ r_{\mathrm{n}k} \geq t,\ \|r_{\mathrm{t}k}\| \leq \mu r_{\mathrm{n}k}, & \forall k \notin U(N) \cup R(N), \\
& \displaystyle\sum_{1 \leq i \leq m} r_{\mathrm{n}i} = 1, &
\end{array}\right\} \tag{18}$$

where the variables are u, r, and t. Let t^* be the optimal value of the problem (18). If $t^* < 0$, then only the trivial solution, i.e. $u = r = 0$, satisfies (6) and (17), and hence the node N is pruned. On the other hand, suppose $t^* \geq 0$, which implies that there exists a nontrivial solution to (6) and (17). In this case, if $U(N) \cup R(N) = \{1, \dots, m\}$, then such a solution satisfies the complementarity condition, and hence N corresponds to one of sign-patterns of (WP). If $U(N) \cup R(N) = \{1, \dots, m\}$, we apply an appropriate branching rule to add two child nodes of N to the search tree.

Consequently, the following algorithm finds all the sign-patterns of the solution of (WP):

Algorithm 5.1 (branch-and-bound method for enumeration of sign-patterns)**.**

Step 1: Define the root node, N^0, of the search tree by $U(N^0) = R(N^0) = \emptyset$. Set $\mathcal{Y} = \emptyset$.

Step 2: Select a node N, which has not been chosen, in the search tree. If none exists, then go to Step 5.

Step 3: Solve (18) at the node N to find the optimal value t^*. If $t^* \geq 0$, then go to Step 4. Otherwise, go to Step 2.

Step 4: If $U(N) \cup R(N) = \{1, \dots, m\}$, then set $\mathcal{Y} \leftarrow \mathcal{Y} \cup \{N\}$, and go to Step 2. Otherwise, select $i \in \{1, \dots, m\} \setminus (U(N) \cup R(N))$. Add N_1, N_2 defined by

$$\begin{aligned} U(N_1) &:= U(N) \cup \{i\}, & R(N_1) &:= R(N), \\ U(N_2) &:= U(N), & R(N_2) &:= R(N) \cup \{i\}, \end{aligned}$$

to the search tree as child nodes of N, and go to Step 2.

Step 5: Declare $\mathcal{Y}$ as the set of sign-patterns, and stop.

Remark 5.2. At Step 2 of Algorithm 5.1 we choose the next candidate node by using a conventional node selection method, e.g. the depth-first search, the breadth-first search, etc. ∎

Remark 5.3. Algorithm 5.1 is well-defined in the sense that the subproblem (18) to be solved at Step 3 always has a feasible solution. Moreover, if there exists a nontrivial solution satisfying (6) and (17), then the optimal value, t^*, of (18) satisfies $0 \leq t^* < +\infty$. Note that the optimal value becomes unbounded above if we remove the constraint condition $\sum_{1 \leq i \leq m} r_{\mathrm{n}i} = 1$ from (18). Thus the optimal value of (18) always satisfies $-\infty < t^* < +\infty$, which is desired from the view point of numerical computation. ∎

Remark 5.4. The subproblem (18) to be solved at Step 3 is a minimization problem of a linear function over linear constraint conditions and second-order cone inequalities. Hence, (18) is an SOCP (second-order cone programming) problem, which can be solved effectively by using the primal-dual interior-point method [1]. Several well-developed software packages for solving SOCPs are available, e.g. [16, 18]. ∎

Remark 5.5. At Step 4 we apply an appropriate rule to select $i \in \{1, \dots, m\} \setminus (U(N) \cup R(N))$. For example, in our numerical experiments presented in section 7.3, we put $i := \arg\max\{u^*_{\mathrm{n}i} r^*_{\mathrm{n}i} \mid i = 1, \dots, m\}$, where (u^*, r^*, t^*) is the optimal solution of the problem (18) at the node N. ∎

5.2 Enumeration based on polyhedral approximation

The difficulty of (WP) with $d = 3$ arises from the nonlinearity of the friction cone, while the friction cone with $d = 2$ can be represented by linear inequalities. In this section we propose an algorithm for $d = 3$ based on a polyhedral outer approximation of the friction cone. Since the solution set of the approximated problem has the same property as that of (WP) with $d = 2$, we can enumerate the solutions of the approximated problem by using the algorithm presented in section 4. By using these approximate solutions we can find all the sign-patterns of the original (WP) with $d = 3$ efficiently.

For each $i = 1, \dots, m$, let $\overline{K}_i \subseteq \mathbb{R}^3$ be a polyhedral cone satisfying

$$\overline{\mathcal{K}}_i \supset \{(r_{\mathrm{n}i}, r_{\mathrm{t}i}) \in \mathbb{R} \times \mathbb{R}^2 \mid \|r_{\mathrm{t}i}\| \leq \mu r_{\mathrm{n}i}\}, \tag{19}$$

i.e. $\overline{\mathcal{K}}_i$ contains the friction cone associated with the node i. By replacing the friction cone $\|r_{\mathrm{t}i}\| \leq$

$\mu r_{\mathrm{n}i}$ in (6) with $\overline{\mathcal{K}}_i$, we obtain an approximation problem of $(\overline{\mathrm{WP}})$ as follows.

$$\left.\begin{array}{lll} (\overline{\mathrm{WP}}): & \text{find} & (u,r) \in \mathbb{R}^{dm} \times \mathbb{R}^{dm} \\ & \text{s.t.} & Ku = r, \\ & & u_{\mathrm{n}} \geq 0,\ r_{\mathrm{n}} \geq 0,\ u_{\mathrm{n}}^{\top} r_{\mathrm{n}} = 0, \\ & & (r_{\mathrm{n}i}, r_{\mathrm{t}i}) \in \overline{\mathcal{K}}_i, \quad i = 1, \dots, m. \end{array}\right\} \tag{20}$$

Since $\overline{\mathcal{K}}_i$ is a polyhedral cone, the constraint conditions of $(\overline{\mathrm{WP}})$ consist of finitely many homogeneous linear inequalities and m complementarity conditions.

We denote by $\overline{\mathcal{S}} \subseteq \mathbb{R}^{2dm}$ the solution set of $(\overline{\mathrm{WP}})$. Since $\overline{\mathcal{K}}_i$ satisfies (19), it is easy to see that $\mathcal{S}^3 \subseteq \overline{\mathcal{S}}$.

Recall that in section 3.2 we have investigated the solution set $\mathcal{S}^3$ of (WP) by using $D(Y)$ defined by (14). Similarly, define $E(Y) \subseteq \mathbb{R}^{2dm}$ by

$$E(Y) = \{x \in \overline{\mathcal{S}} \mid Z(x) \supseteq Y\} \tag{21}$$

for a sign-pattern $Y = (Y^u, Y^r)$ satisfying the strict complementarity condition. Since $\overline{\mathcal{S}} = \bigcup_Y E(Y)$, we obtain an alternative description of $\overline{\mathcal{S}}$ as $\overline{\mathcal{S}} = \bigcup_s E(Y_s)$, where Y_s is a sign-pattern satisfying the strict complementarity condition. Note that such a description is not unique. Among those descriptions we denote by

$$\overline{\mathcal{S}} = \bigcup_{s=1}^{\hat{s}} E_s(\hat{Y}_s) \tag{22}$$

the one consisting of the minimum number of polyhedral cones. For simplicity we write $\hat{E}_s = E(\hat{Y}_s)$. The following statement can be obtained in a manner similar to Proposition 3.12.

Proposition 5.6. *In* (22), $\mathrm{int}(\hat{E}_s) \cap \mathrm{int}(\hat{E}_{s'}) = \emptyset$ *for any* $s \neq s'$.

The following result, which plays a key role in showing the veridity of Algorithm 5.8 below, summarizes a relation between $\mathcal{S}$ and $\overline{\mathcal{S}}$.

Proposition 5.7. *$\mathcal{S}^3$ and $\overline{\mathcal{S}}$ can be described by using finitely many convex cones D_t and polyhedral cones E_s as $\mathcal{S}^3 = \bigcup_t D_t$ and $\overline{\mathcal{S}} = \bigcup_s E_s$, where D_t and E_s satisfy the following conditions:*

(i) $\mathrm{int}(D_t) \cap \mathrm{int}(D_{t'}) = \emptyset$ *if* $t \neq t'$;

(ii) $\mathrm{int}(E_s) \cap \mathrm{int}(E_{s'}) = \emptyset$ *if* $s \neq s'$;

(iii) *For each D_t, there exists E_s satisfying $D_t \subseteq E_s$ uniquely;*

(iv) *For each E_s, there exists at most one D_t satisfying $D_t \subseteq E_s$.*

Proof. From Proposition 3.12 and Proposition 5.6, it is immediate to see that there exist D_t and E_s satisfying the conditions (i) and (ii).

Suppose that Y' is a sign-pattern satisfying $D(Y') \neq \{0\}$. From the definition (14) of $D(Y')$ it follows that any $x \in D(Y')$ satisfies $x \in \mathcal{S}^3$ and $Z(x) \supseteq Y'$. Moreover, since $\mathcal{S}^3 \subseteq \overline{\mathcal{S}}$, we see that

$x \in \overline{\mathcal{S}}$. Hence, from (21), we obtain $x \in E(Y')$. As a consequence, for any Y satisfying the strict complementarity condition, we obtain

$$D(Y) \subseteq E(Y).$$

Notice here that this inclusion holds even if $D(Y) = \{0\}$. Thus the assertion (iii) is obtained.

To see (iv), observe that there exists Y which satisfies $Z(x) \supseteq Y$ for any $x \in E_s$ as well as the strict complementarity condition. Suppose that there exist D_t, $D_{t'} \neq \{0\}$ ($D_t \neq D_{t'}$) satisfying $D_t \subseteq E_s$ and $D_{t'} \subseteq E_s$. Then, Y satisfies $Z(x) \supseteq Y$ for any $x \in D_t$ and $Z(x') \supseteq Y$ for any $x' \in D_{t'}$. It follows from Proposition 3.10 that any nonnegative combination of $x \in D_t$ and $x' \in D_{t'}$ is a solution to (WP). This implies that there exists a convex cone D' which contains D_t and $D_{t'}$, and is contained by E_s. Hence, D_t and $D_{t'}$ can be replaced with D' so that E_s contains only one convex cone D'. □

It should be emphasized that, similarly to (WP) with $d = 2$, the constraint conditions of ($\overline{\text{WP}}$) consist of homogeneous linear inequalities and complementarity conditions. Hence, we can enumerate solutions, as well as the maximal sign-patterns of those solutions, of ($\overline{\text{WP}}$) by using the algorithm presented in section 4. Thus, $\hat{E}_s$ in Proposition 5.7 can be found. Moreover, Proposition 5.7 implies that each $\hat{E}_s$ either contains one $\hat{D}_t$ or does not contain any nontrivial solution of (WP). Hence, for the enumeration of $\hat{D}_t$, it suffices to determine whether each $\hat{E}_s = E(\hat{Y}_s)$ contains a nontrivial solution to (WP) or not. This determination can be performed by solving the following optimization problem:

$$\left.\begin{array}{lll} \displaystyle\max_{t,u,r,x,y} & t & \\ \text{s.t.} & Ku = r, & \\ & u_{\text{n}i} \geq x_i,\ r_{\text{n}i} = 0,\ r_{\text{t}i} = 0, & \forall i \in Y^r, \\ & u_{\text{n}j} = 0,\ r_{\text{n}j} \geq 0,\ \|r_{\text{t}j}\| + y_j \leq \mu r_{\text{n}j}\ , & \forall j \in Y^u, \\ & x_i \geq t,\ y_i \geq t, & i = 1, \dots, m, \\ & \displaystyle\sum_{1 \leq i \leq m} r_{\text{n}} = 1. & \end{array}\right\} \tag{23}$$

We denote by t^* the optimal value of the problem (23). If $t^* < 0$, then we see that only the trivial solution, i.e. $(u, r) = (0, 0)$, satisfies $(u, r) \in E(Y)$ as well as the constraint conditions of (WP). Otherwise, $E(Y)$ contains a nontrivial solution of (WP), and hence Y corresponds to one of the sign-patterns of solutions of (WP).

Consequently, all the sign-patterns of the solution of (WP) can be found by using the following algorithm.

Algorithm 5.8 (enumeration of sign-patterns by polyhedral approximation of friction cone)**.**

Step 1: Formulate ($\overline{\text{WP}}$) by approximating the friction cone of (WP) by a polyhedral cone. Enumerate the sign patterns, $Y_1, \dots, Y_k$, of the solutions of ($\overline{\text{WP}}$) by using the algorithm presented in section 4.

Step 2: For each Y_t, we solve the SOCP problem (23) to find the optimal value t^*. If $t^* \geq 0$, then declare Y_t as the sign-pattern of (WP). Otherwise, discard Y_t.

Remark 5.9. Algorithm 5.8 is well-defined in the sense that the problem (23) to be solved at Step 2 has a feasible solution, and its optimal value is bounded. In addition, (23) is an SOCP problem, and hence it can be solved effectively by using the primal-dual interior-point method. ∎

6 Critical value of friction coefficient

In this section we consider the critical friction coefficient, μ^{c}, defined as the minimum value of friction coefficient with which a nontrivial wedged configuration exists. Since the problem for computing μ^{c} is regarded as a nonconvex optimization problem, its global optimal solution cannot be found easily in general. Barber and Hild [3] formulated this problem as a nonlinear eigenvalue problem, and solved it numerically. However, their algorithm is not guaranteed to converge. Hassani *et al.* [7] proposed a genetic algorithm for finding an upper bound of μ^{c}. We here describe an algorithm which is guaranteed to find μ^{c}.

It follows from Proposition 2.1 that there exists a minimum value of the friction coefficient, μ^{c}, with which there exists at least one nontrivial solution to (WP). Moreover, we easily see that (WP) has a nontrivial solution for any μ satisfying $\mu \geq \mu^{\mathrm{c}}$. Hence, if there exists an algorithm which determines whether (WP) has a nontrivial solution for a given μ, then we can compute μ^{c} by using the bisection method. This is a basic idea of our algorithm.

Recall that by using the algorithms in section 5, we can enumerate the sign-patterns Y satisfying the strict complementarity condition. For an in-plane problem, Algorithm 4.2 can be also used to enumerate sign-patterns. The first step of our algorithm, Algorithm 6.1 below, consists of the enumeration of sign-patterns of (WP) for a sufficiently large friction coefficient. For the specified sign-pattern, it is easy to check the existence of nontrivial solution, as shown in section 5.2. For a given sign-pattern $Y = (Y^u, Y^r)$ satisfying the strict complementarity condition and a given $\mu > 0$, define $\mathcal{Q}(Y;\mu) \subseteq \mathbb{R}^{2dm}$ by

$$\mathcal{Q}(Y;\mu) = \left\{ (u,r) \neq 0 \;\middle|\; \begin{array}{c} Ku = r, \\ u_{\mathrm{n}i} \geq 0,\ r_{\mathrm{n}i} = 0,\ r_{\mathrm{t}i} = 0, \quad \forall i \in Y^r, \\ u_{\mathrm{n}j} = 0,\ r_{\mathrm{n}j} \geq 0,\ \|r_{\mathrm{t}j}\| \leq \mu r_{\mathrm{n}j}, \quad \forall j \in Y^u \end{array} \right\}.$$

Out algorithm for finding the critical friction coefficient is described as follows.

Algorithm 6.1 (bisection method for computing the critical friction coefficient)**.**

Step 1: Choose a sufficiently large μ^0. For (WP) defined with $\mu := \mu^0$, enumerate a family of sign-patterns $\hat{Y}_1, \dots, \hat{Y}_{\hat{t}}$ of solutions which satisfy the strict complementarity condition.

Step 2: For each t, find $\mu_t^{\mathrm{c}} := \inf_{\mu}\{\mu \mid \mathcal{Q}(\hat{Y}_t;\mu) \supset \{0\}\}$ by a bisection method as follows.

(a) Set $\underline{\mu} := 0$ and $\overline{\mu} := \mu^0$.

(b) If $\overline{\mu} - \underline{\mu}$ is small enough, then declare $\overline{\mu}$ as μ_t^{c}, and stop. Otherwise, choose $\mu \in]\underline{\mu}, \overline{\mu}[$.

(c) If $\mathcal{Q}(\hat{Y}_t;\mu) = \{0\}$, then set $\underline{\mu} := \mu$. Otherwise, set $\overline{\mu} := \mu$. Go to (b).

Step 3: Declare $\min_{1 \leq t \leq \hat{t}} \{\mu_t^{\mathrm{c}}\}$ as μ^{c}, and stop.

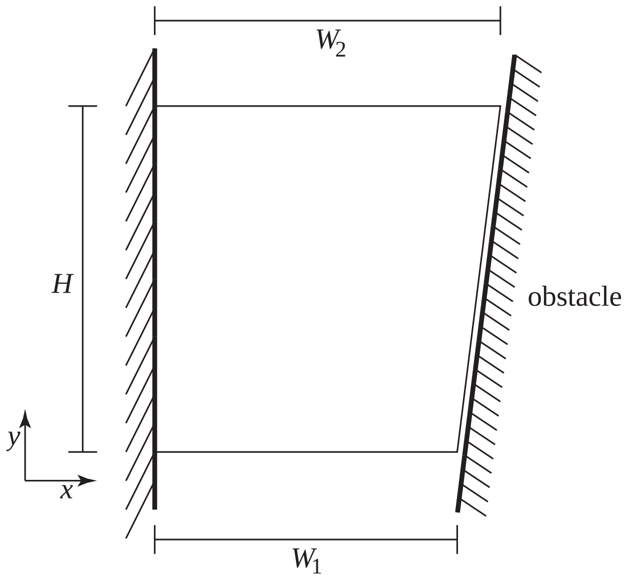

Figure 1: A square elastic block on the rigid obstacle.

At Step 2(c) of Algorithm 6.1, we solve the SOCP problem (23) to determine whether $\mathcal{Q}(\hat{Y}_t;\mu) = \{0\}$ or not. Note that (23) is reduced to a linear programming problem for $d = 2$.

7 Numerical experiments

Numerical results are presented by using the presented algorithms for various structures; the wedged configurations are enumerated for an isotropic elastic body and trusses in the two-dimensional space in sections 7.1 and 7.2, respectively, while for a 3D truss we compute the sign-patterns of wedged configurations in section 7.3. Computation has been carried out on Core2 Duo P8400 (2.26 GHz with 4.0 GB memory) with Matlab R2009a [19].

7.1 Elastic body in plane stress

Consider an isotropic linear elastic body in the plane stress as shown in Figure 1, where $W_1 = 1.8\,\mathrm{m}$ and $W_2 = H = 2.0\,\mathrm{m}$. The Poisson ratio is taken to be 0.25. The solid is discretized into 11×11 four-node quadrilateral (Q4) elements. All the nodes on the right boundary are supposed to be contact candidates, and be in contact with the rigid obstacle without reactions. The coefficient of friction is $\mu = 1.5$. The nodes of the left boundary are fixed. Hence, $d = 2$ and $m = 12$ in (6).

We enumerate the wedged configurations by the algorithms presented in section 4. All the extremal rays obtained by using Algorithm 4.2 are shown in Figure 2, which illustrate the wedged equilibrium configurations as well as the corresponding reactions. The CPU time required by Algorithm 4.2 is 18.72 sec. The cross-complementarity relationship among those extremal rays is represented by the undirected graph shown in Figure 3 as discussed in section 4.3. It is observed from Figure 3 that the solution set of (WP) consists of the extremal rays (a)–(e) and all convex combinations of (a)–(d). Note that the extremal rays (a)–(d) share the same sign-pattern.

We next compute the critical value of friction coefficient, μ^{c}, by using Algorithm 6.1. We solve

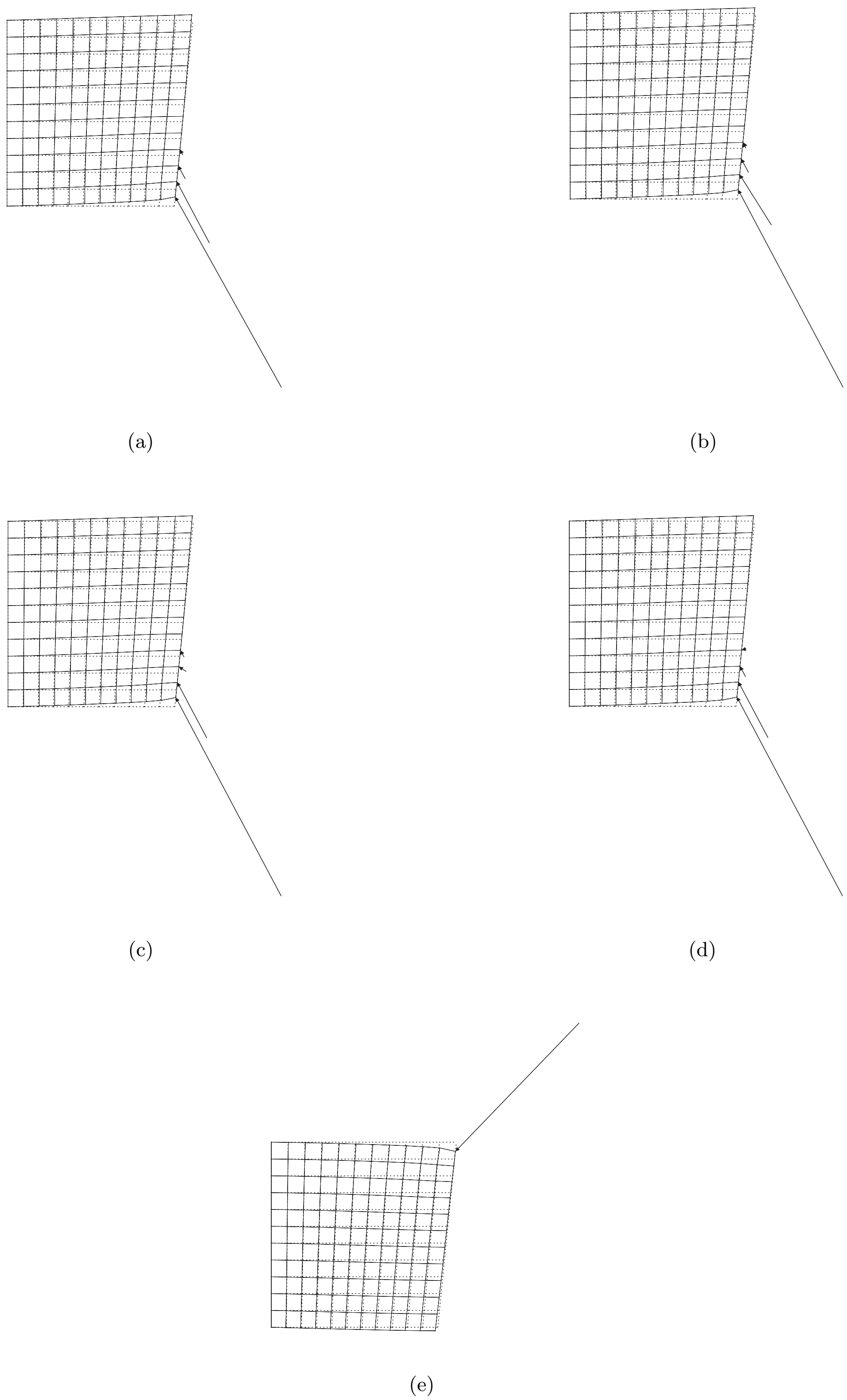

Figure 2: Extremal rays of wedged configurations of the linear elastic body.

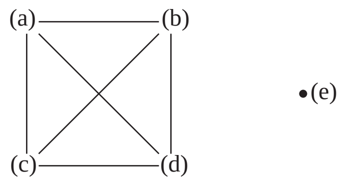

Figure 3: Cross-complementarity relationship among extremal rays illustrated in Figure 2.

Table 1: Critical values of friction coefficient for sign-patterns of the wedged configurations illustrated in Figure 2.

sign-pattern	μ_t^{c}
(a)–(d)	1.4545
(e)	1.2689

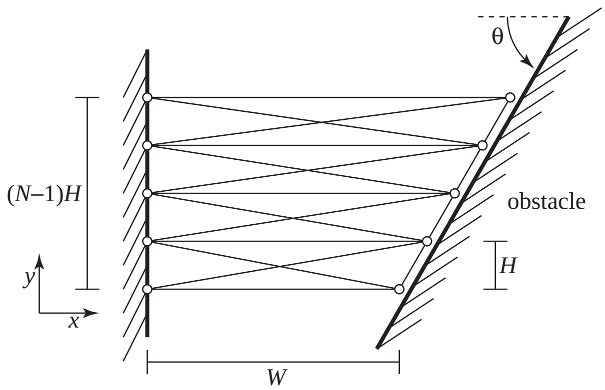

Figure 4: A plane truss contacting with the rigid obstacle.

the LP (23) by using a MATLAB built-in function `linprog` with the default settings. From Figure 3 we see that $\hat{t} = 2$ in Algorithm 6.1. The critical friction coefficient for each sign-pattern is listed in Table 1, which implies $\mu^{\mathrm{c}} = 1.2689$.

7.2 Plane trusses with various sizes

We next consider a plane truss as shown in Figure 4 for various N, where $H = 1.0\,\mathrm{m}$, $W = 5.0\,\mathrm{m}$, and $\theta = \pi/3$. Note that Figure 4 depicts the case of $N = 5$. All the members of the truss have the same cross-sectional area. The nodes on the right boundary are supposed to be contact candidates, while those of the left boundary are pin-supported. Hence, $d = 2$ and $m = N$ in (6).

All the wedged configurations are found by using Algorithm 4.2 for various N. The numbers of extremal rays and sign-patterns, as well as the CPU time required by Algorithm 4.2, are listed in Table 2 for $\mu = 0.65$. The 25 extremal wedged configurations obtained for $N = 5$ and $\mu = 1.0$ are collected in Appendix A; see Figures 8 and 9. The cross-complementarity relationship among those extremal rays is described by the undirected graph shown in Figure 5. Note that extremal rays in

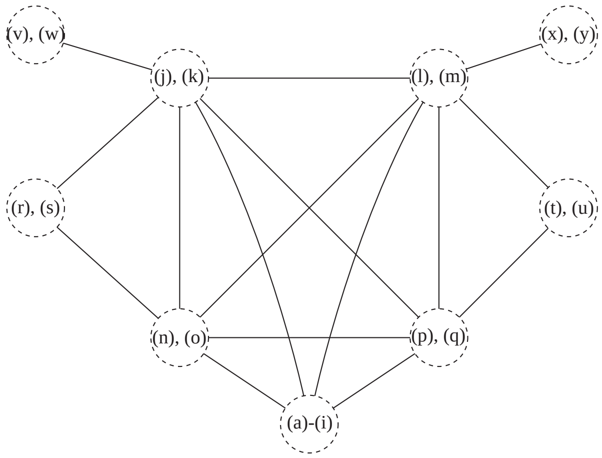

Figure 5: Cross-complementarity relationship among extremal rays of the wedged configuration of the truss example of Figure 4 with $N = m = 5$ and $\mu = 1.0$. All extremal rays are shown in Figure 8 and Figure 9. Any two extremal rays in the same dashed circle are connected.

Table 2: Variations of the number of extremal rays and the number of sign-patterns of the plane-truss example with respect to the number of contact candidates ($\mu = 0.65$).

m $(= N)$	# of rays	# of cliques	CPU (sec)
5	8	1	0.11
7	24	3	0.31
9	47	5	0.87
11	100	9	2.78
13	273	15	10.89
15	744	30	74.05
17	2048	57	572.57
19	5413	108	3245.94

the same dashed circle have the common sign-pattern. Hence, for example, any convex combination of (a)–(i) is a solution of the wedged problem.

The critical value of friction coefficient μ^{c} is computed by using Algorithm 6.1 in the case of $N = m = 5$. We choose $\mu^0 = 1.0$ at the Step 1 of Algorithm 6.1, hence $\hat{t} = 5$ as shown in Figure 5. The critical friction coefficient for each sign-pattern is listed in Table 3, which implies $\mu^{\mathrm{c}} = 0.6072$. The wedged configuration corresponding to each critical friction coefficient is shown Figure 6.

7.3 3D truss example

Consider a linear elastic truss illustrated in Figure 7, which may be regarded as the three-dimensional version of the example investigated in section 7.2. The dimensions of the truss are $W_1 = 10.0\,\mathrm{m}$, $W_2 = W_1 + (\tan \pi/6)H\,\mathrm{m}$, $L = 10.0\,\mathrm{m}$, and $H = 4.0\,\mathrm{m}$. All the members of the truss have the same cross-sectional area. The nodes on the right boundary are supposed to be contact candidates, while

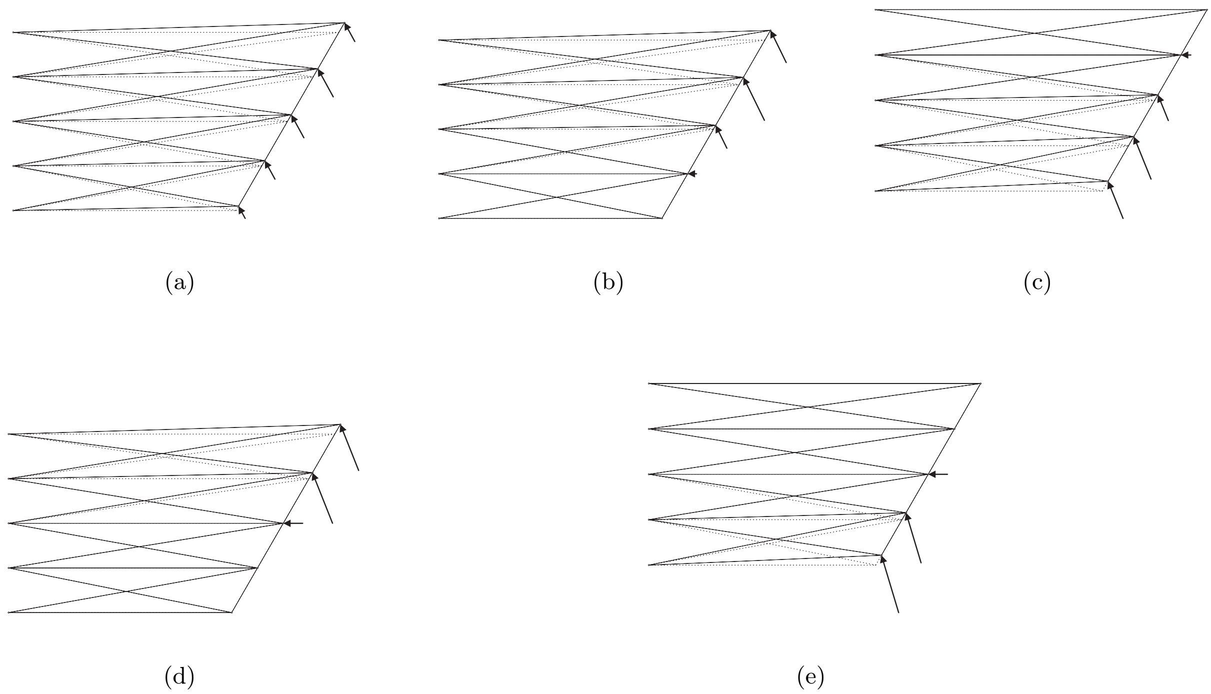

Figure 6: Extremal rays of wedged configurations at the critical coefficients of friction for the plane-truss with $N = m = 5$.

Table 3: Critical values of friction coefficient for sign-patterns of the wedged configurations illustrated in Figure 6.

sign-pattern	$\mu_t^{\rm c}$
(a)	0.6072
(b)	0.6718
(c)	0.7804
(d)	0.7909
(e)	0.9422

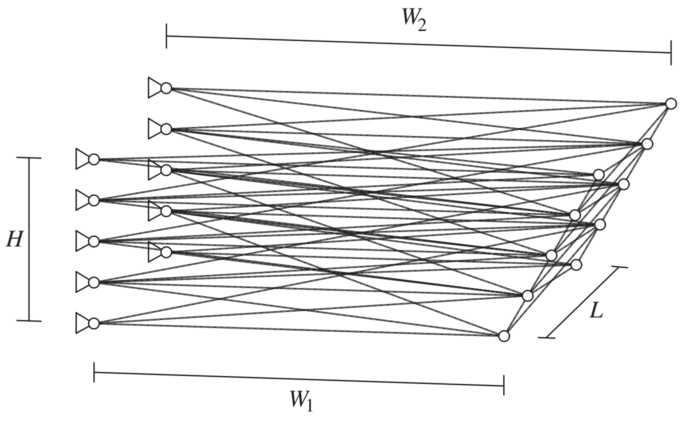

Figure 7: A truss example in the three-dimensional space contacting with the rigid obstacle.

Table 4: Relation between the friction coefficient and the number of sign-patterns of the wedged configurations of the truss in the three-dimensional space illustrated in Figure 7.

μ	sign-patterns
0.55	0
0.6	1
0.65	26
0.7	61
1.0	236

those of the left boundary are pin-supported. Hence, $d = 3$ and $m = 10$ in (6).

All the sign-patterns of wedged configurations are found by using Algorithm 5.1. At the Step 3 we solve the SOCP problem (18) by using SeDuMi Ver. 1.1 [16, 18]. The number of sign-patterns obtained by Algorithm 5.1 is listed in Table 4 for various friction coefficients.

8 Conclusions

We have investigated the solution set properties of the wedged problem, which is finding a nontrivial equilibrium state of a finitely-discretized linear elastic structure subjected to the unilateral contact conditions with the Coulomb friction. It has been shown that the solution set of the in-plane wedged problem can be represented by finitely many representative solutions and there convex combinations. A numerical enumeration algorithm has been proposed consisting of the enumeration of the extremal rays of a polyhedral cone which satisfy the complementarity conditions and the enumeration of the maximal cliques of an directed graph. For the wedged problem in the three-dimensional space we have proposed two algorithms for enumerating the sign-patterns of the complementarity variables of all the wedged configurations. We finally presented the bisection algorithm for finding the globally minimal value of the friction coefficient, at which at least one wedged configuration can exist. At each step of the bisection algorithm we solve an SOCP (second-order cone programming) problem by using the primal-dual interior-point method.

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A Extremal solutions of plane-truss example in section 7.2

In section 7.2 we compute all the extremal rays of the wedged configurations for the plane-truss illustrated in Figure 4. By using Algorithm 4.2, we obtain 25 extremal rays for $N = 5$ and $\mu = 1.0$, which are shown in Figures 8 and 9.

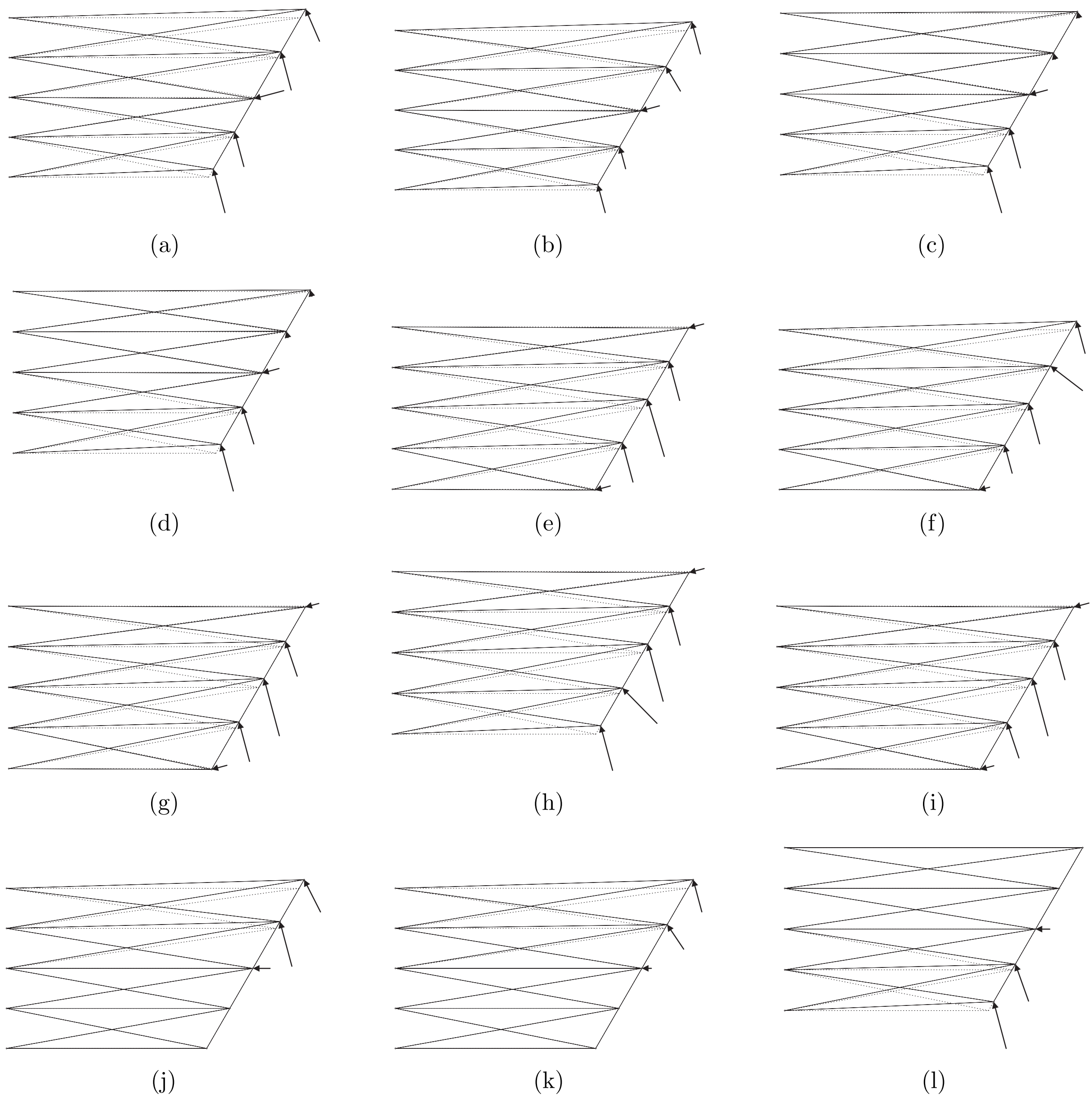

Figure 8: Extremal rays of wedged configurations of the truss example in section 7.2 with $N = m = 5$ and $\mu = 1.0$.

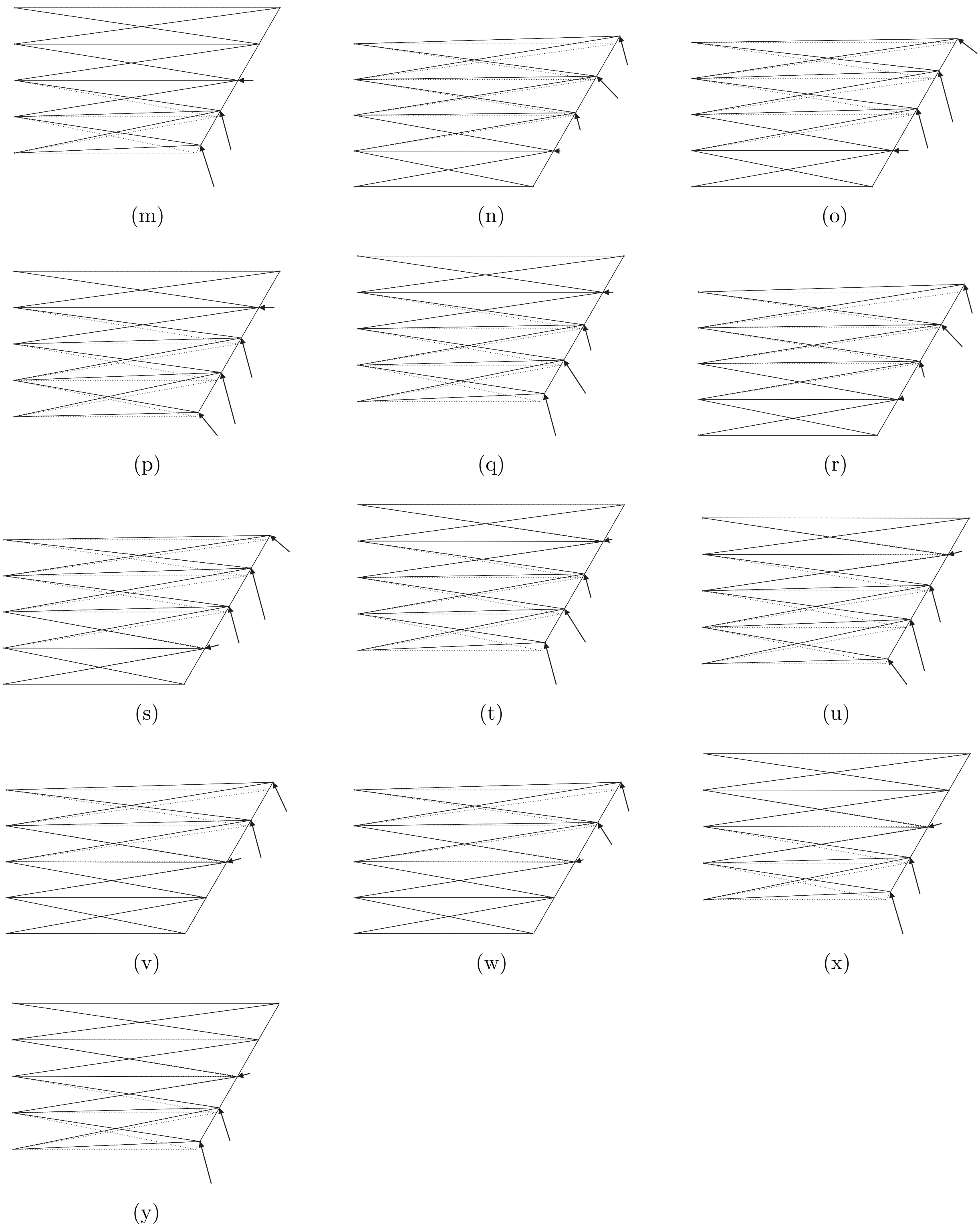

Figure 9: Extremal rays of wedged configurations of the truss example in section 7.2 with $N = m = 5$ and $\mu = 1.0$ (continued).