MATHEMATICAL ENGINEERING
TECHNICAL REPORTS

Cone Superadditivity of Discrete Convex Functions

Yusuke KOBAYASHI, Kazuo MUROTA,
Robert WEISMANTEL

METR 2009–30 August 2009

DEPARTMENT OF MATHEMATICAL INFORMATICS
GRADUATE SCHOOL OF INFORMATION SCIENCE AND TECHNOLOGY
THE UNIVERSITY OF TOKYO
BUNKYO-KU, TOKYO 113-8656, JAPAN

WWW page: http://www.i.u-tokyo.ac.jp/mi/mi-e.htm

Cone Superadditivity of Discrete Convex Functions

Yusuke Kobayashi[*], Kazuo Murota[†], Robert Weismantel[‡]

August 2009

Abstract

A function f is said to be cone superadditive if there exists a partition of $\mathbf{R}^n$ into a family of polyhedral convex cones such that $f(z+x) + f(z+y) \leq f(z) + f(z+x+y)$ holds whenever x and y belong to the same cone in the family. This concept is useful in nonlinear integer programming in that, if the objective function is cone superadditive, the global minimality can be characterized by local optimality criterion involving Hilbert bases. This paper shows cone superadditivity of L-convex and M-convex functions with respect to conic partitions that are independent of particular functions. L-convex and M-convex functions in discrete variables (integer vectors) as well as in continuous variables (real vectors) are considered.

Keywords: discrete convex function, superadditivity, integer programming, optimality criterion, Hilbert bases

1 Introduction

Discrete convex functions have been attracting research interest in operations research and related disciplines. Discrete convex analysis [3, 12, 13, 14], in particular, provides a theoretical framework for solvable discrete optimization problems through a combination of convex analysis and matroid/submodular function theory. Two convexity concepts, called L-convexity and M-convexity, play major roles in discrete convex analysis. L-convex functions generalize submodular set functions, and M-convex functions base polyhedra (see Sections 3.1 and 4.1 for details). This extends the

[*]Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan. Yusuke_Kobayashi@mist.i.u-tokyo.ac.jp

[†]Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Tokyo 113-8656, Japan. murota@mist.i.u-tokyo.ac.jp

[‡]Department of Mathematics/IMO, Otto-von-Guericke-University of Magdeburg, Universitätsplatz 2, D-39106 Magdeburg, Germany. weismant@mail.math.uni-magdeburg.de

direction set forth by J. Edmonds, A. Frank, S. Fujishige, and L. Lovász in the 1980's.

In this paper we are interested in an inequality of the form

$$f(z+x)+f(z+y) \leq f(z)+f(z+x+y) \qquad (1.1)$$

for a function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ in a real vector or $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ in an integer vector[1], where $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$. The above inequality says in effect that if $f(z)$ and $f(z+x+y)$ are finite, then both $f(z+x)$ and $f(z+y)$ are finite and the inequality (1.1) holds. In terms of the increment function $g_z(x) = f(z+x) - f(z)$, defined for z with $f(z)$ finite, the above inequality (1.1) can be reformulated as *superadditivity*:

$$g_z(x) + g_z(y) \leq g_z(x+y). \qquad (1.2)$$

When f is a convex function, inequality (1.1) certainly holds if $y = \alpha x$ for some nonnegative $\alpha \in \mathbf{R}$, and fails if $y = -\alpha x$ (unless f is linear in this direction). The same is true when f is a function on $\mathbf{Z}^n$ that is extensible to a convex function on $\mathbf{R}^n$. Hence it is natural to consider (1.1) for x and y belonging to a certain pointed convex cone C.

Let $\{C_k\}_k$ be a family of polyhedral convex cones $C_k \subseteq \mathbf{R}^n$ such that $\bigcup_k C_k = \mathbf{R}^n$, which we refer to as a conic partition of $\mathbf{R}^n$. We say that f is *cone superadditive* with respect to $\{C_k\}_k$ if the inequality (1.1) holds for every z with $f(z)$ finite and for all directions x and y belonging to the same cone C_k among the given cones. We are naturally interested in finite conic partitions, consisting of a finite number of cones, although we allow for an infinite family in technical arguments. Not every convex function is cone superadditive; for example, $f(x_1, x_2) = \exp({x_1}^2 + {x_2}^2)$ is not cone superadditive with respect to any finite conic partition of $\mathbf{R}^2$ (see Example 6.1 in Section 6 for the details). We are particularly interested in a class of functions that are cone superadditive with respect to a fixed finite conic partition independent of individual functions.

Cone superadditivity has a significant implication in nonlinear integer optimization, as is discussed by Murota–Saito–Weismantel [15] without using this terminology and by Lee–Onn–Weismantel [10] under the name of oriented superadditivity. Consider, to be specific, the problem of minimizing $f(x)$ subject to $Ax = b$, $x \in \mathbf{Z}^n$, $x \geq \mathbf{0}$, and assume that the objective function f is cone superadditive with respect to some finite cone partition. Then the global optimality of an integer vector x can be characterized by a local optimality condition involving Hilbert bases; Section 2 explains more about this. It is observed in [15] that a function f of the form

$$f(x) = \sum_{i=1}^{s} \phi_i({c_i}^\top x) \qquad (1.3)$$

[1]We denote the set of all real numbers by $\mathbf{R}$ and that of all integers by $\mathbf{Z}$. The set of nonnegative reals and that of integers are denoted respectively as $\mathbf{R}_+$ and $\mathbf{Z}_+$.

with some vectors $c_i \in \mathbf{R}^n$ $(i = 1, \ldots, s)$ and univariate convex functions $\phi_i : \mathbf{R} \to \mathbf{R}$ $(i = 1, \ldots, s)$ for some integer s is cone superadditive with respect to a finite conic partition chosen suitably for f. This implies, in particular, that a convex quadratic function is cone superadditive.

The objective of this paper is to reveal cone superadditivity of L-convex and M-convex functions with respect to certain finite conic partitions that are independent of individual functions. This demonstrates another instance of general properties of discrete convex functions, sometimes referred to as "discreteness in direction" in contrast to "discreteness in value." The main results of this paper are Theorems 3.2 and 3.3 for L-convex functions, and Theorems 4.2 and 4.3 for M-convex functions, where functions in integer vectors are dealt with in Theorems 3.2 and 4.2, and polyhedral functions in real vectors in Theorems 3.3 and 4.3. The proofs consist of two ingredients. The first is a general result for locally polyhedral convex functions, showing that every locally polyhedral convex function is cone superadditive with respect to some conic partition suitably chosen for the function (Theorem 5.1). The second is a reformulation of fundamental facts in discrete convex analysis, showing that there exists a finite conic partition that is valid universally for all polyhedral L-convex functions, and another finite conic partition valid for all polyhedral M-convex functions.

In addition, cone superadditivity of more general (nonpolyhedral or smooth) convex functions is investigated. For twice-differentiable functions cone superadditivity is characterized by a kind of positivity of the Hessian matrix (Theorem 6.1), which is a natural generalization of the well-known characterization of convexity by positive-semidefiniteness. This general result is applied to twice-differentiable L-convex functions, to establish their cone superadditivity (Theorems 6.4). Cone superadditivity of general closed proper L-convex functions is also established (Theorem 7.1). Corresponding results for M-convex functions are easy to conceive but rigorous proofs are still awaited.

This paper is organized as follows. Significance of the cone superadditivity in nonlinear integer optimization is explained in Section 2. L-convex functions are treated in Section 3, and M-convex functions in Section 4. Section 5 shows the general result for polyhedral convex functions, and affords the proofs for the main theorems. Section 6 deals with twice-differentiable convex functions and Section 7 discusses some extensions and problems left unsettled.

2 Application to Nonlinear Integer Programming

As a motivation of our interest in cone superadditivity we discuss here its role in the design of algorithms for nonlinear integer programming.

Given $A \in \mathbf{Z}^{m \times n}$ and $b \in \mathbf{Z}^m$, we consider an integer program of the form:

$$\text{minimize } f(x) \text{ subject to } x \in S, \tag{2.1}$$

where the feasible region S is described as

$$S = \{x \in \mathbf{Z}^n \mid Ax = b,\ x \geq \mathbf{0}\}$$

and the objective function f is generally a nonlinear function defined on $\mathbf{Z}^n$.

For a pointed rational polyhedral cone C, let $H(C)$ denote the unique Hilbert basis of C, i.e., the inclusionwise minimal subset of the integer points in C such that every integer point in C is representable as a nonnegative integer combination of the elements in the set. The existence of a Hilbert basis follows from the classical lemma of Gordan [5], and the Hilbert basis of a pointed cone C is uniquely determined, as pointed out by van der Corput [21].

With the notion of a Hilbert basis we can give optimality conditions for a linear integer program:

$$\text{minimize } c^\top x \text{ subject to } x \in S. \tag{2.2}$$

Let $O_1, \ldots, O_{2^n}$ denote the partition of $\mathbf{R}^n$ into all its orthants. Then

$$C_l = \{x \in \mathbf{R}^n \mid Ax = \mathbf{0}\} \cap \mathcal{O}_l \tag{2.3}$$

is a pointed polyhedral cone in $\mathbf{R}^n$ for each $l \in \{1, \ldots, 2^n\}$. Let H_l be the unique minimal Hilbert basis of C_l. Then the following optimality criterion, due to Graver [6], holds:

Theorem 2.1 ([6]). *A feasible point $x \in S$ for the linear integer program* (2.2) *is optimal if and only if $c^\top h \geq 0$ for every $h \in \bigcup_{l=1}^{2^n} H_l$ such that $x + h \in S$.*

This optimality criterion forms the basis of the integral basis method proposed by Haus, Köppe, and Weismantel [7]. This method solves a linear integer program by iteratively computing Hilbert bases of discrete relaxations of the underlying integer program and reformulating the problem in a higher dimensional space. The algorithm uses many advanced techniques that are not related to the above optimality criterion, but in abstract mathematical terms, it is an integer simplex algorithm based on Hilbert bases and inspired by Graver's optimality criterion.

A generalization of the optimality criterion above is considered by Murota, Saito and Weismantel [15] for a certain class of nonlinear objective functions, which are cone superadditive in our present terminology.

Suppose that f is cone superadditive with respect to a finite conic partition, say $\{C_k(f)\}_k$, of $\{x \in \mathbf{R}^n \mid Ax = \mathbf{0}\}$, where each cone $C_k(f)$ is a rational polyhedral cone contained in some C_l in (2.3). More precisely, assume that the inequality (1.1) holds for every $z \in S$ and every $x, y \in \mathbf{Z}^n$ with

$\{x, y\} \subseteq C_k(f)$ for some k and $z+x+y \in S$. We denote by $\mathcal{F} = \mathcal{F}(A, b)$ the family of such functions f. A typical member of $\mathcal{F}$ is $f(x) = \sum_{i=1}^{s} \phi_i(c_i{}^\top x)$ in (1.3) defined with rational vectors $c_1, \ldots, c_s$.

If the objective function f belongs to $\mathcal{F}$, the global optimality in (2.1) is guaranteed by a local optimality, as follows. Suppose that a feasible point $x \in S$ is locally optimal in the sense that $f(x + h) \geq f(x)$ for all $h \in \bigcup_k H(C_k(f))$ such that $x + h \in S$, where $H(C_k(f))$ denotes the unique minimal Hilbert basis of $C_k(f)$. For all $y \in S$, there exists k such that $y - x \in C_k(f)$, and hence $y = x + \sum_{j=1}^{t} \alpha_j h_j$ for some $h_j \in H(C_k(f))$ and positive integers α_j $(j = 1, \ldots, t)$. Then

$$f(y) - f(x) = f(x + \sum_{j=1}^{t} \alpha_j h_j) - f(x) \geq \sum_{j=1}^{t} \alpha_j [f(x + h_j) - f(x)] \geq 0,$$

where the first inequality is by the cone superadditivity and the second is by the assumed local optimality. It is noted that $x + \sum_{j=1}^{t} \alpha_j h_j \in S$ implies $x + h_j \in S$ for each j, since $\{h_1, \ldots, h_t\} \subseteq C_k(f) \subseteq C_l$ for some C_l. Thus we obtain the following theorem.

Theorem 2.2 ([15]). *Suppose that the objective function f of* (2.1) *belongs to the class $\mathcal{F}$ and is cone superadditive with respect to a finite conic partition $\{C_k(f)\}_k$ of $\mathbf{R}^n$ with rational cones. Then a feasible point $x \in S$ is optimal if and only if $f(x + h) \geq f(x)$ for all $h \in \bigcup_k H(C_k(f))$ such that $x + h \in S$.*

The local optimality criterion above naturally suggests a minimization algorithm, which will be described in the following.

We assume that $f : \mathbf{Z}^n \to \mathbf{Z}$ is an integer-valued convex-extensible function that is cone superadditive with respect to a finite conic partition $\{C_k(f)\}_k$. Moreover, we assume that an optimal solution of Problem (2.1) exists. The optimal value is denoted by f^* and an optimal solution by x^*. Starting off from a feasible point $x^0 \in S$ we iteratively apply a greedy augmentation scheme below to solve Problem (2.1).

Greedy minimization algorithm

Input: f, A, b, $\bigcup_k H(C_k(f))$ and $x^0 \in S$.
Output: an optimal solution to (2.1).

1. (Initialization) Set $i := 0$ and $f^0 := f(x^0)$;

2. (Greedy augmentation) Determine a vector $h \in \bigcup_k H(C_k(f))$ and a step length $\alpha \in \mathbf{Z}_+$ such that $x^i + \alpha h$ attains

$$\min \left\{ f(x^i + \beta z) \mid x^i + \beta z \in S, \; z \in \bigcup_k H(C_k(f)), \; \beta \in \mathbf{Z}_+ \right\}.$$

   If this value is equal to $f(x^i)$, then STOP.

3. (Update) Set $x^{i+1} := x^i + \alpha h$, $f^{i+1} := f(x^{i+1})$ and $i := i + 1$. Return to Step 2.

It is clear from Theorem 2.2 that the above algorithm gives an optimal solution of (2.1). In addition there is a theoretical guarantee for the convergence speed that the number of iterations of Step 2 is bounded by $O((2n - 2)\log(f^0 - f^*))$, from which a complexity bound can be obtained, as stated in Theorem 2.3 below, where $\langle\cdot\rangle$ means the length of the binary encoding of a vector, a matrix etc. This analysis is a straightforward adaptation of the argument presented in [8].

Theorem 2.3. *The greedy minimization algorithm solves Problem* (2.1) *in time that is polynomial in* $\log(f^0 - f^*)$, *in* $\sum_k \langle H(C_k(f))\rangle$ *and in* $\langle x^0\rangle, \langle A\rangle, \langle b\rangle$.

Proof. If $f(x^i + h) \geq f(x^i)$ for all $h \in \bigcup_k H(C_k(f))$, we stop with x^i. If $f(x^i + h) < f(x^i)$ for some h, then consider a function $g(\alpha) = f(x^i + \alpha h)$, which is a univariate convex function in nonnegative integer α. The minimum of $g(\alpha)$ over feasible α can be computed by a simple binary search. Hence, every application of Step 2 can be implemented to run in time that is polynomial in $\log(f^i - f^*)$, in $\sum_k \langle H(C_k(f))\rangle$ and in $\langle A\rangle$, $\langle b\rangle$, $\langle x^i\rangle$.

For an analysis of the number of augmentation steps one proceeds as follows. By definition, there exists an index k such that $x^* - x^i \in C_k(f) \cap \mathbf{Z}^n$. An integer version of the Caratheodory theorem (cf., [20]) shows that there exists $\{h_1, \ldots, h_{2n-2}\} \subseteq H(C_k(f))$ and nonnegative integers $\alpha_1, \ldots, \alpha_{2n-2} \in \mathbf{Z}_+$ such that

$$x^* = x^i + \sum_{j=1}^{2n-2} \alpha_j h_j.$$

Then the superadditivity with respect to the cone $C_k(f)$ yields

$$\begin{aligned} f^i - f^* &= f(x^i) - f(x^i + \sum_{j=1}^{2n-2} \alpha_j h_j) \\ &\leq \sum_{j=1}^{2n-2} \left(f(x^i) - f(x^i + \alpha_j h_j)\right) \\ &\leq (2n-2)\left(f(x^i) - f(x^i + \alpha h)\right), \end{aligned}$$

where αh denotes the greedy augmentation determined in Step 2 of the algorithm. This shows

$$f(x^i) - f(x^i + \alpha h) \geq \frac{1}{2n-2}\left(f^i - f^*\right).$$

It then follows (cf., Theorem 3.1 in [1]) that the number of iterations is bounded by $O((2n - 2)\log(f^0 - f^*))$. □

3 L-convex Functions

3.1 Preliminaries

Let $V = \{1, 2, \ldots, n\}$ be a finite set. For a subset X of V, we denote by χ_X the characteristic vector of X; the ith component of χ_X equals one or zero according to whether i belongs to X or not. For $i \in V$, we write χ_i for $\chi_{\{i\}}$, which is the ith unit vector. For vectors $x, y \in \mathbf{R}^n$ we denote by $x \vee y$ and $x \wedge y$ the vectors of componentwise maximum and minimum of x and y, respectively, i.e.,

$$(x \vee y)_i = \max(x_i, y_i), \quad (x \wedge y)_i = \min(x_i, y_i) \qquad (i \in V).$$

We also define

$$\mathrm{supp}^+(x) = \{i \in V \mid x_i > 0\}, \qquad \mathrm{supp}^-(x) = \{i \in V \mid x_i < 0\},$$

$$\arg\max(x) = \{i \in V \mid x_i = \max_{j \in V} x_j\}, \qquad x(X) = \sum_{i \in X} x_i \quad (X \subseteq V).$$

We consider functions $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ or $f : \mathbf{R}^n \to \overline{\mathbf{R}}$, where $\overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$. The effective domain of f is denoted by $\mathrm{dom}\, f = \{x \mid f(x) < +\infty\}$. In this paper we always assume $\mathrm{dom}\, f \neq \emptyset$. For a vector $p \in \mathbf{R}^n$ the set of the minimizers of $(f - p)(x) = f(x) - \sum_{i=1}^n p_i x_i$ is denoted as $\arg\min(f - p)$, which is defined to be an empty set if $\inf_x (f - p)(x) = -\infty$ or $\inf_x (f - p)(x)$ is not attained.

A function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is said to be *submodular* if $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$ for all $x, y \in \mathbf{Z}^n$. A function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is said to be *L-convex* if (i) it is submodular and (ii) there exists $r \in \mathbf{R}$ such that $f(x + \mathbf{1}) = f(x) + r$ for all $x \in \mathbf{Z}^n$, where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{Z}^n$. A function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is called *L$^\natural$-convex* if the function $\tilde{f} : \mathbf{Z}^{n+1} \to \overline{\mathbf{R}}$ defined by

$$\tilde{f}(x, x_{n+1}) = f(x - x_{n+1}\mathbf{1}) \qquad (x \in \mathbf{Z}^n, x_{n+1} \in \mathbf{Z}) \tag{3.1}$$

is submodular in $n + 1$ variables ("L$^\natural$" should be read "L-natural"). L$^\natural$-convexity can be characterized by *discrete midpoint convexity*:

$$f(x) + f(y) \geq f\left(\left\lceil \frac{x + y}{2} \right\rceil\right) + f\left(\left\lfloor \frac{x + y}{2} \right\rfloor\right) \qquad (x, y \in \mathbf{Z}^n), \tag{3.2}$$

where, for $t \in \mathbf{R}$ in general, $\lceil t \rceil$ denotes the smallest integer not smaller than t (rounding-up to the nearest integer) and $\lfloor t \rfloor$ the largest integer not larger than t (rounding-down to the nearest integer), and this operation is extended to a vector by componentwise application.

An L-convex function is an L$^\natural$-convex function, and conversely, an L$^\natural$-convex function f is an L-convex function if it has the property (ii) above. Thus L-convex functions in n variables form a proper subclass of L$^\natural$-convex functions in n variables. At the same time, L-convex functions are conceptually equivalent to L$^\natural$-convex functions through the relation (3.1). For an L$^\natural$-convex function f in n variables, $\tilde{f}$ in (3.1) is an L-convex function in $n+1$ variables, and conversely, for an L-convex function f in n variables, $f'(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, 0)$ is an L$^\natural$-convex function in $n-1$ variables.

L-convexity can be defined for polyhedral convex functions as well as for polyhedra. A set $S \subseteq \mathbf{R}^n$ is called an L-convex polyhedron if it can be represented as

$$S = \{x \in \mathbf{R}^n \mid x_j - x_i \leq \gamma_{ij} \ (\forall i, j \in V, \, i \neq j)\} \tag{3.3}$$

for some $\gamma_{ij} \in \overline{\mathbf{R}}$ $(i, j \in V, \, i \neq j)$. A polyhedron $S \subseteq \mathbf{R}^n$ is called L$^\natural$-convex if it can be represented as the intersection of an L-convex polyhedron $\tilde{S} \subseteq \mathbf{R}^{n+1}$ with the coordinate plane $x_{n+1} = 0$.

An L-convex polyhedron is an L$^\natural$-convex polyhedron, and conversely, an L$^\natural$-convex polyhedron S is an L-convex polyhedron if it has the additional property that $x \in S$ implies $x + \alpha \mathbf{1} \in S$ for all $\alpha \in \mathbf{R}$. Thus L-convex polyhedra in $\mathbf{R}^n$ form a proper subclass of L$^\natural$-convex polyhedra in $\mathbf{R}^n$. At the same time, L-convex polyhedra are conceptually equivalent to L$^\natural$-convex polyhedra.

A convex function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ is said to be locally polyhedral if, for every bounded interval $[a, b]$ with $[a, b] \cap \mathrm{dom}\, f \neq \emptyset$, the restriction of f to $[a, b]$ is a polyhedral convex function. A locally polyhedral convex function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ is called L-convex (resp. L$^\natural$-convex) if, for each $p \in \mathbf{R}^n$, $\arg\min(f - p)$ is an L-convex (resp. L$^\natural$-convex) polyhedron. The effective domain $\mathrm{dom}\, f$ of a locally polyhedral L-convex (resp. L$^\natural$-convex) function f is an L-convex (resp. L$^\natural$-convex) polyhedron. Locally polyhedral L-convex functions and L$^\natural$-convex functions are conceptually equivalent to each other. See [13, Section 7.8] for the original definitions of polyhedral L-convex and L$^\natural$-convex functions.

The following fact shows that the discrete-variable case can be embedded in the polyhedral case. It should be clear that the convex closure of a convex-extensible function f means the pointwise largest convex extension of f.

Theorem 3.1. *An L$^\natural$-convex function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ in integer vectors is convex-extensible, and its convex closure is a locally polyhedral L$^\natural$-convex function. Conversely, for a locally polyhedral L$^\natural$-convex function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ such that* $\arg\min(f - p)$ *is an integral polyhedron for each $p \in \mathbf{R}^n$, the restriction of f to $\mathbf{Z}^n$ is an L$^\natural$-convex function in integer vectors.*

L-convex functions are introduced by Murota [12], and L$^\natural$-convex functions are by Fujishige and Murota [4] as a variant thereof. L$^\natural$-convex functions turn out to be the same as submodular integrally convex functions introduced earlier by Favati and Tardella [2]. Polyhedral L-convex functions are due to Murota–Shioura [17], and the extension to locally polyhedral functions are to Fujishige [3]. L-convexity can be considered for more general (nonpolyhedral or smooth) functions, as we see in Section 6. More details about L-convex and L$^\natural$-convex functions can be found in [3, 13, 14].

3.2 Cone superadditivity

For a partition $\mathcal{P} = (P_1, \ldots, P_m)$ of $\{0, 1, \ldots, n\}$ into mutually disjoint nonempty subsets, we define a cone $C_{\mathcal{P}}$ by

$$C_{\mathcal{P}} = \{x \in \mathbf{R}^n \mid \tilde{x}_i \geq \tilde{x}_j \text{ if } i \in P_k, j \in P_l, k \leq l\}, \tag{3.4}$$

where $\tilde{x} = (x_0, x_1, \ldots, x_n) = (0, x)$ with $x_0 = 0$ and $(x_1, \ldots, x_n) = x$. It is noted that the subsets P_k are ordered by the index, and $\tilde{x}_i = \tilde{x}_j$ if $\{i, j\} \subseteq P_k$ for some k. The family of such cones,

$$\mathcal{C}_{\mathrm{L}} = \{C_{\mathcal{P}} \mid \mathcal{P} : \text{partition}\}, \tag{3.5}$$

gives a finite conic partition of $\mathbf{R}^n$, i.e., $\mathbf{R}^n = \bigcup_{\mathcal{P}} C_{\mathcal{P}}$. We mention that each $C_{\mathcal{P}}$ is an L$^\natural$-convex polyhedral cone.

We say that two vectors x and y are *L-compatible* if $\{x, y\} \subseteq C_{\mathcal{P}}$ for some $\mathcal{P}$. Cone superadditivity with respect to $\mathcal{C}_{\mathrm{L}}$ will be referred to as *L-cone superadditivity.*

L$^\natural$-convex functions are L-cone superadditive, as is stated in the following theorems.

Theorem 3.2. *Let $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ be an L$^\natural$-convex function. For any $z \in \mathrm{dom}\, f$ and any L-compatible integer vectors x and y, we have*

$$f(z + x) + f(z + y) \leq f(z) + f(z + x + y). \tag{3.6}$$

Theorem 3.3. *Let $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ be a locally polyhedral L$^\natural$-convex function. For any $z \in \mathrm{dom}\, f$ and any L-compatible real vectors x and y, the inequality* (3.6) *is satisfied.*

We prove Theorem 3.3 in Section 5.2. Then Theorem 3.2 follows from this with the aid of Theorem 3.1. An alternative proof of Theorem 3.2, purely combinatorial as opposed to geometrical, is given in Appendix.

Remark 3.1. L-compatibility of x and y is necessary for the inequality (3.6) to hold for any L$^\natural$-convex function. If (3.6) is true for the L$^\natural$-convex function $f(z) = {z_i}^2$, we must have $x_i y_i \geq 0$. If (3.6) is true for the L$^\natural$-convex function $f(z) = (z_i - z_j)^2$, we must have $(x_i - x_j)(y_i - y_j) \geq 0$. ■

Remark 3.2. Convex extension of an L$^\natural$-convex function f, mentioned in Theorem 3.1, is closely related to the conic partition (3.5). Consider partitions $\mathcal{P}$ of $\{0, 1, \ldots, n\}$ into singletons, $\mathcal{P} = (P_1, \ldots, P_{n+1})$, with $0 \in P_{n+1}$.

With the use of cones $C_{\mathcal{P}}$ for such $\mathcal{P}$ the n-dimensional unit cube $[\mathbf{0},\mathbf{1}]$ can be divided into $n!$ polyhedral regions $[\mathbf{0},\mathbf{1}] \cap C_{\mathcal{P}}$. The piecewise linear extension of f in each region $z + ([\mathbf{0},\mathbf{1}] \cap C_{\mathcal{P}})$ with $z \in \mathbf{Z}^n$ and partition $\mathcal{P}$ gives the convex closure of f. See [13, Section 7.7]. ■

4 M-convex Functions

4.1 Preliminaries

Recall that we consider $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ with $\mathrm{dom}\, f \neq \emptyset$, and χ_i denotes the ith unit vector for $i \in V$, where $V = \{1, 2, \ldots, n\}$.

A function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is said to be *M-convex* if it satisfies the exchange axiom:

> **(M-EXC)** For any $x, y \in \mathrm{dom}\, f$ and $i \in \mathrm{supp}^+(x - y)$, there exists $j \in \mathrm{supp}^-(x - y)$ such that
> $$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j). \tag{4.1}$$

Inequality (4.1) implicitly imposes the condition that $x - \chi_i + \chi_j \in \mathrm{dom}\, f$ and $y + \chi_i - \chi_j \in \mathrm{dom}\, f$. The effective domain of an M-convex function f lies on a hyperplane of a constant component sum, i.e.,

$$\mathrm{dom}\, f \subseteq \{x \in \mathbf{Z}^n \mid x(V) = r\} \tag{4.2}$$

for some $r \in \mathbf{Z}$.

By the projection of f along the nth coordinate axis we mean a function $f' : \mathbf{Z}^{n-1} \to \overline{\mathbf{R}}$ defined by

$$f'(x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, r - \sum_{i=1}^{n-1} x_i). \tag{4.3}$$

A function derived from an M-convex function by projection is called an *M$^\natural$-convex function* ("M$^\natural$" should be read "M-natural"). Or equivalently, a function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ is said to be M$^\natural$-convex if the function $\tilde{f} : \mathbf{Z}^{n+1} \to \overline{\mathbf{R}}$ defined by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & \text{if } x_0 = -x(V) \\ +\infty & \text{otherwise} \end{cases} \qquad (x_0 \in \mathbf{Z}, x \in \mathbf{Z}^n) \tag{4.4}$$

is M-convex. M$^\natural$-convex functions can be characterized by a variant of the exchange axiom, labeled (M$^\natural$-EXC[$\mathbf{Z}$]) in [13].

An M-convex function is an M$^\natural$-convex function, and conversely an M$^\natural$-convex function f is an M-convex function if it has the property (4.2) for some $r \in \mathbf{Z}$. Thus M-convex functions in n variables form a proper subclass of M$^\natural$-convex functions in n variables. At the same time, M-convex functions are conceptually equivalent to M$^\natural$-convex functions through the relations (4.3) and (4.4). For an M$^\natural$-convex function f in n variables, $\tilde{f}$ in (4.4) is an M-convex function in $n+1$ variables, and conversely, for an M-convex function f in n variables, its projection f' in (4.3) is an M$^\natural$-convex function in $n-1$ variables.

M-convexity can be defined for polyhedral convex functions as well as for polyhedra. A set $S \subseteq \mathbf{R}^n$ is called a base polyhedron (or an M-convex polyhedron) if it can be represented as

$$S = \{x \in \mathbf{R}^n \mid x(X) \leq \rho(X) \ (\forall X \subset V), \ x(V) = \rho(V)\} \tag{4.5}$$

for some submodular set function $\rho : 2^V \to \overline{\mathbf{R}}$ with $\rho(\emptyset) = 0$ and $\rho(V)$ finite. A polyhedron $S \subseteq \mathbf{R}^n$ is called a g-polymatroid (or an M$^\natural$-convex polyhedron) if it can be represented as the projection of an M-convex polyhedron $\tilde{S} \subseteq \mathbf{R}^{n+1}$ onto the coordinate plane $x_{n+1} = 0$. It is convenient to use the terminology of M-convex and M$^\natural$-convex polyhedra, rather than base polyhedron and g-polymatroid, to see the parallelism with L-convexity.

An M-convex polyhedron is an M$^\natural$-convex polyhedron, and conversely, an M$^\natural$-convex polyhedron S is an M-convex polyhedron if it has the additional property that $x(V)$ is a constant for all $x \in S$. Thus M-convex polyhedra in $\mathbf{R}^n$ form a proper subclass of M$^\natural$-convex polyhedra in $\mathbf{R}^n$. At the same time, M-convex polyhedra are conceptually equivalent to M$^\natural$-convex polyhedra.

A locally polyhedral convex function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ is called M-convex (resp. M$^\natural$-convex) if, for each $p \in \mathbf{R}^n$, $\arg\min(f - p)$ is an M-convex (resp. M$^\natural$-convex) polyhedron. The effective domain $\mathrm{dom}\, f$ of a locally polyhedral M-convex (resp. M$^\natural$-convex) function f is an M-convex (resp. M$^\natural$-convex) polyhedron. Locally polyhedral M-convex functions and M$^\natural$-convex functions are conceptually equivalent to each other. See [13, Section 6.11] for the original definitions of polyhedral M-convex and M$^\natural$-convex functions.

The following fact shows that the discrete-variable case can be embedded in the polyhedral case.

Theorem 4.1. *An M$^\natural$-convex function $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ in integer vectors is convex-extensible, and its convex closure is a locally polyhedral M$^\natural$-convex function. Conversely, for a locally polyhedral M$^\natural$-convex function $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ such that* $\arg\min(f - p)$ *is an integral polyhedron for each $p \in \mathbf{R}^n$, the restriction of f to $\mathbf{Z}^n$ is an M$^\natural$-convex function in integer vectors.*

M-convex functions are introduced by Murota [11], and M$^\natural$-convex functions are by Murota and Shioura [16] as a variant thereof. Polyhedral M-convex functions are due to Murota–Shioura [17], and the extension to locally polyhedral functions to Fujishige [3]. M-convexity can be considered for more general (nonpolyhedral or smooth) functions, as we see in Section 6. More details about M-convex and M$^\natural$-convex functions can be found in [3, 13, 14].

4.2 Cone superadditivity

For a set family $\mathcal{F} \subseteq 2^V$ we define a cone $C_{\mathcal{F}}$ by

$$C_{\mathcal{F}} = \{x \in \mathbf{R}^n \mid x(X) \leq 0 \ (X \in \mathcal{F}), \quad x(Y) \geq 0 \ (Y \in 2^V \setminus \mathcal{F})\}. \tag{4.6}$$

The family of such cones,

$$\mathcal{C}_{\mathrm{M}} = \{C_{\mathcal{F}} \mid \mathcal{F} \subseteq 2^V\}, \tag{4.7}$$

gives a finite conic partition of $\mathbf{R}^n$, i.e., $\mathbf{R}^n = \bigcup_{\mathcal{F}} C_{\mathcal{F}}$. It is noted that each $C_{\mathcal{F}}$ is not necessarily an M$^\natural$-convex polyhedral cone, but it can be represented as an intersection of M$^\natural$-convex polyhedral cones.

We say that two vectors x and y are *M-compatible* if $\{x, y\} \subseteq C_{\mathcal{F}}$ for some $\mathcal{F}$. Cone superadditivity with respect to $\mathcal{C}_{\mathrm{M}}$ will be referred to as *M-cone superadditivity.*

M$^\natural$-convex functions are M-cone superadditive, as is stated in the following theorems.

Theorem 4.2. *Let $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ be an M$^\natural$-convex function. For any $z \in$* $\mathrm{dom}\, f$ *and any M-compatible integer vectors x and y, we have*

$$f(z + x) + f(z + y) \leq f(z) + f(z + x + y). \tag{4.8}$$

Theorem 4.3. *Let $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ be a locally polyhedral M$^\natural$-convex function. For any $z \in$* $\mathrm{dom}\, f$ *and any M-compatible real vectors x and y, the inequality* (4.8) *is satisfied.*

We prove Theorem 4.3 in Section 5.3. Then Theorem 4.2 follows from this with the aid of Theorem 4.1.

Remark 4.1. M-compatibiliy of x and y is necessary for the inequality (4.8) to hold for any M$^\natural$-convex function. If (4.8) is true for the M$^\natural$-convex function $f(z) = (z(X))^2$, we have $f(z) + f(z+x+y) - f(z+x) - f(z+y) = 2x(X)y(X) \geq 0$. ■

Remark 4.2. An M$^\natural$-convex function f, either on $\mathbf{Z}^n$ or on $\mathbf{R}^n$, is known to be supermodular, i.e.,

$$f(x) + f(y) \leq f(x \vee y) + f(x \wedge y) \qquad (\forall\, x, y). \tag{4.9}$$

Theorems 4.2 and 4.3 contain this as a special case. Given x and y, put $z = x \wedge y$, $x' = x - (x \wedge y)$, $y' = y - (x \wedge y)$. Then x' and y' are nonnegative vectors, which are M-compatible, and $x \vee y = z + x' + y'$. The cone superadditivity (4.8) for (z, x', y') yields the supermodularity (4.9). ■

Example 4.1. M-convex functions arise naturally from a network flow problem. Let $G = (V, A)$ be a digraph, and $f_a : \mathbf{Z} \to \overline{\mathbf{R}}$ be a convex cost function for each $a \in A$. Let $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty, -\infty\}$ denote the minimum-cost of an integral flow $\xi : A \to \mathbf{Z}$ that meets the demand requirement specified by an integer vector x on the vertex set $V = \{1, \ldots, n\}$. To be more precise, for $x \in \mathbf{Z}^n$, define $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty, -\infty\}$ by

$$f(x) = \inf \left\{ \sum_{a \in A} f_a(\xi(a)) \mid \xi : A \to \mathbf{Z},\ \partial\xi = x \right\}, \tag{4.10}$$

where $\partial\xi$ denotes the boundary of ξ, a vector on V, defined by

$$(\partial\xi)_i = \sum\{\xi(a) \mid a \text{ leaves } i\} - \sum\{\xi(a) \mid a \text{ enters } i\} \qquad (i \in V).$$

It is known that this function f is M-convex if $f > -\infty$. The M-cone superadditivity of such an M-convex function can be shown by a natural combinatorial argument, which is included here for a better understanding of the combinatorial nature of this phenomenon, as well as for an extension to the continuous-variable case (cf. Section 7).

First note that the infimum in (4.10) is attained when $f > -\infty$. Let $\xi_0 : A \to \mathbf{Z}$ be an integral flow with $\partial\xi_0 = z$ that attains the infimum for $f(z)$. By considering $\tilde{f}_a(\xi(a)) = f_a(\xi(a) + \xi_0(a))$ instead of $f_a(\xi(a))$ we may assume that $z = \mathbf{0}$ and $f(\mathbf{0}) = \sum_{a \in A} f_a(0)$, and hence $0 \in \mathrm{dom}\, f_a$ for each $a \in A$.

For two M-compatible integer vectors x and y in $\mathbf{Z}^n$, let $\xi : A \to \mathbf{Z}$ be an integral flow that minimizes $\sum_{a \in A} f_a(\xi(a))$ subject to $\partial\xi = x + y$. By changing the orientation of arcs if necessary, we may assume further that $\xi(a) \geq 0$ for every $a \in A$.

We claim that there exists $\xi_x : A \to \mathbf{Z}$ such that $\partial\xi_x = x$ and $0 \leq \xi_x(a) \leq \xi(a)$ for each $a \in A$. By a well-known feasibility criterion (see, e.g., [13, Theorem 9.3]), this is equivalent to the condition

$$x(X) \leq \kappa(X) \qquad (\forall X \subset V),$$

where $\kappa(X) = \sum\{\xi(a) \mid a \text{ is an arc from } X \text{ to } V \setminus X\}$, and $x(V) = 0$. This inequality holds indeed, since $x(X) \leq \max\{(x + y)(X), 0\}$ by the M-compatibility of x and y, and $(x + y)(X) \leq \kappa(X)$ by the feasibility of $x + y$ (i.e., the existence of ξ with $\partial\xi = x + y$).

Take a ξ_x as above and define $\xi_y = \xi - \xi_x$, where $\partial\xi_x = x$ and $\partial\xi_y = y$. For each $a \in A$ we have

$$f_a(\xi_x(a)) + f_a(\xi_y(a)) \leq f_a(0) + f_a(\xi(a))$$

by the convexity of f_a, whereas

$$f(x) \leq \sum_{a \in A} f_a(\xi_x(a)), \qquad f(y) \leq \sum_{a \in A} f_a(\xi_y(a)),$$

$$f(\mathbf{0}) = \sum_{a \in A} f_a(0), \qquad f(x + y) = \sum_{a \in A} f_a(\xi(a)).$$

Hence it follows that $f(x) + f(y) \le f(\mathbf{0}) + f(x+y)$. ■

5 Polyhedral Convex Functions

5.1 General case

Let $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ be a locally polyhedral convex function, which means that, for every bounded interval $[a,b]$ with $[a,b] \cap \mathrm{dom}\, f \neq \emptyset$, the restriction of f to $[a,b]$ is a polyhedral convex function. The effective domain $\mathrm{dom}\, f$ is the union of a family of countably many polyhedra, say, $\mathcal{S}_f = \{S_k \mid k = 1,2,\ldots\}$ such that f is linear on S_k and S_k is a maximal subset of $\mathrm{dom}\, f$ having this property.

We define a conic partition $\mathcal{C}_f$ as follows. Let

$$S_k = \{x \in \mathbf{R}^n \mid {c_{ki}}^\top x \le d_{ki} \ (\forall i)\} \tag{5.1}$$

be a description of S_k by finitely many linear inequalities indexed by i, where $c_{ki} \in \mathbf{R}^n$ and $d_{ki} \in \mathbf{R}$. An assignment of sign ($+$ or $-$) to each ${c_{ki}}^\top x$ induces a cone as follows:

$$C_{\mathcal{J}} = \{x \in \mathbf{R}^n \mid {c_{ki}}^\top x \le 0 \ (\forall (k,i) \in \mathcal{J}),\ {c_{ki}}^\top x \ge 0 \ (\forall (k,i) \notin \mathcal{J})\}. \tag{5.2}$$

Define $\mathcal{C}_f$ to be the family of such cones, i.e., $\mathcal{C}_f = \{C_{\mathcal{J}} \mid \mathcal{J}\}$. Note here that $\mathcal{C}_f$ may possibly consist of an infinite number of cones, and that $\mathcal{C}_f$ is a finite family if f is a polyhedral convex function. In any case we have $\mathbf{R}^n = \bigcup_{\mathcal{J}} C_{\mathcal{J}}$.

The main result of this section is the following theorem, showing that any locally polyhedral convex function f is cone superadditive with respect to $\mathcal{C}_f$. We say that two vectors x and y are *f-compatible* if $\{x,y\} \subseteq C_{\mathcal{J}}$ for some $\mathcal{J}$.

Theorem 5.1. *Let $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ be a locally polyhedral convex function. For any $z \in \mathrm{dom}\, f$ and any f-compatible real vectors x and y, we have*

$$f(z+x) + f(z+y) \le f(z) + f(z+x+y). \tag{5.3}$$

In particular, a polyhedral convex function f is cone superadditive with respect to a finite family $\mathcal{C}_f$.

To prove Theorem 5.1 fix z, x and y in (5.3). By restricting f to a sufficiently large bounded interval, we may assume that $\mathrm{dom}\, f$ is bounded. This implies, in particular, that $\mathcal{S}_f$ is a finite family.

We first observe the following fact, which is the set version of cone superadditivity. For $w \in \mathbf{R}^n$ and $\alpha, \beta \in \mathbf{R}_+$ we denote by $P(w,\alpha,\beta)$ the parallelogram with vertices at w, $w + \alpha x$, $w + \beta y$, and $w + \alpha x + \beta y$.

Lemma 5.2. *Let f, x and y be as in Theorem 5.1, and let $w \in \mathbf{R}^n$ and $\alpha, \beta \in \mathbf{R}_+$.*

(1) *If $w \in S_k$ and $w + \alpha x + \beta y \in S_k$ for some $S_k \in \mathcal{S}_f$, then $w + \alpha x \in S_k$ and $w + \beta y \in S_k$, and therefore $P(w, \alpha, \beta) \subseteq S_k$.*

(2) *If $w \in \mathrm{dom}\, f$ and $w + \alpha x + \beta y \in \mathrm{dom}\, f$, then $w + \alpha x \in \mathrm{dom}\, f$ and $w + \beta y \in \mathrm{dom}\, f$, and therefore $P(w, \alpha, \beta) \subseteq \mathrm{dom}\, f$.*

Proof. (1) Recall the description (5.1) of S_k. Since $w \in S_k$, we have ${c_{ki}}^\top w \leq d_{ki}$ for all i. For each i we have either (i) ${c_{ki}}^\top x \geq 0$, ${c_{ki}}^\top y \geq 0$, or (ii) ${c_{ki}}^\top x \leq 0$, ${c_{ki}}^\top y \leq 0$. In case (i), we have ${c_{ki}}^\top (w + \alpha x) \leq {c_{ki}}^\top (w + \alpha x + \beta y)$, whereas ${c_{ki}}^\top (w + \alpha x + \beta y) \leq d_{ki}$ by $w + \alpha x + \beta y \in S_k$. Hence ${c_{ki}}^\top (w + \alpha x) \leq d_{ki}$. In case (ii), we have ${c_{ki}}^\top (w + \alpha x) \leq {c_{ki}}^\top w \leq d_{ki}$. Therefore we have $w + \alpha x \in S_k$. Symmetrically for $w + \beta y$.

(2) By (5.1) we have $\mathrm{dom}\, f = \{z \in \mathbf{R}^n \mid {c_{ki}}^\top z \leq d_{ki} \ (\forall (k, i) \in \mathcal{I})\}$ for some $\mathcal{I}$. Then the rest is the same as in (1). □

Let us show that

$$f(w + \alpha x) + f(w + \beta y) \leq f(w) + f(w + \alpha x + \beta y) \tag{5.4}$$

holds for any $w \in \mathrm{dom}\, f$ and any $\alpha, \beta \in \mathbf{R}_+$ with $w + \alpha x + \beta y \in \mathrm{dom}\, f$. Then (5.3) is a special case with $\alpha = \beta = 1$ and $w = z$.

We say that $S_k \in \mathcal{S}_f$ *intersects with* a convex set $L \subseteq \mathrm{dom}\, f$ if $\dim (S_k \cap L) = \dim L$. For $w \in \mathrm{dom}\, f$ and $\alpha, \beta \in \mathbf{R}_+$ with $w + \alpha x + \beta y \in \mathrm{dom}\, f$, we have $P(w, \alpha, \beta) \subseteq \mathrm{dom}\, f$ by Lemma 5.2 (2). Let $p(w, \alpha, \beta)$ be the number of subsets in $\mathcal{S}_f$ intersecting with $P(w, \alpha, \beta)$, and let $q(w, \alpha, \beta)$ be the number of subsets in $\mathcal{S}_f$ intersecting with a segment connecting $w + \alpha x$ and $w + \beta y$. We show (5.4) by induction on the lexicographic ordering of the pair $(p(w, \alpha, \beta), q(w, \alpha, \beta))$, or, equivalently, induction on $|\mathcal{S}_f| \cdot p(w, \alpha, \beta) + q(w, \alpha, \beta)$.

When $p(w, \alpha, \beta) = 1$, the function f is linear in $P(w, \alpha, \beta)$. Then, (5.4) holds with equality.

Suppose that $p(w, \alpha, \beta) \geq 2$. If f is linear on the segment connecting $w + \alpha x$ and $w + \beta y$, then we have

$$\begin{aligned} f(w + \alpha x) + f(w + \beta y) &= 2f\left(w + \frac{\alpha x + \beta y}{2}\right) \\ &\leq f(w) + f(w + \alpha x + \beta y), \end{aligned} \tag{5.5}$$

which shows (5.4).

Otherwise, let γ be the maximum real number such that f is linear on the segment connecting $w + \beta y$ and $w + \gamma(\alpha x) + (1 - \gamma)(\beta y)$. Clearly, $0 < \gamma < 1$. Define $\alpha' = \gamma\alpha$, $\alpha'' = (1 - \gamma)\alpha$, $\beta' = \gamma\beta$, $\beta'' = (1 - \gamma)\beta$, and $w' = w + \alpha' x + \beta'' y$. Let $S_1 \in \mathcal{S}_f$ be a set containing w' and $w' - \varepsilon(\alpha x - \beta y)$ for some $\varepsilon > 0$, and let $S_2 \in \mathcal{S}_f$ be another set containing w' and $w' + \varepsilon(\alpha x - \beta y)$ for some $\varepsilon > 0$.

We show that (5.4) holds when (w, α, β) is replaced by

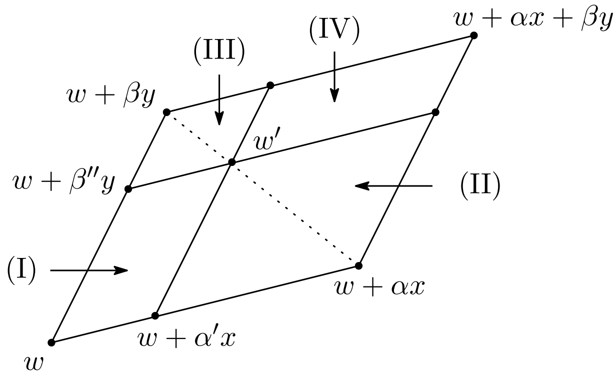

Figure 1: Four cases in the proof of Theorem 5.1

(I) (w, α', β''),

(II) $(w + \alpha' x, \alpha'', \beta'')$,

(III) $(w + \beta'' y, \alpha', \beta')$, and

(IV) (w', α'', β').

These four cases are shown in Figure 1.

<u>(I): (w, α', β'')</u> We have $p(w, \alpha', \beta'') < p(w, \alpha, \beta)$, as is shown below. Then, by induction hypothesis, (5.4) holds for (w, α', β''), that is,

$$f(w + \alpha' x) + f(w + \beta'' y) \leq f(w) + f(w'). \tag{5.6}$$

To see $p(w, \alpha', \beta'') < p(w, \alpha, \beta)$, assume that both S_1 and S_2 intersect with $P(w, \alpha', \beta'')$. Since w' is contained in both S_1 and S_2, $P(w' - \varepsilon_1 x - \varepsilon_2 y, \varepsilon_1, \varepsilon_2)$ with sufficiently small $\varepsilon_1, \varepsilon_2 > 0$ is contained in both S_1 and S_2 by Lemma 5.2 (1). This implies that f is linear on $(S_1 \cup S_2) \cap P(w, \alpha, \beta)$, which contradicts the definition of w'. Hence, at most one of S_1 and S_2 intersects with $P(w, \alpha', \beta'')$, and hence $p(w, \alpha', \beta'') < p(w, \alpha, \beta)$.

<u>(II): $(w + \alpha' x, \alpha'', \beta'')$</u> By the definition of γ, we have

$$p(w + \alpha' x, \alpha'', \beta'') \leq p(w, \alpha, \beta),$$
$$q(w + \alpha' x, \alpha'', \beta'') < q(w, \alpha, \beta).$$

Then, by the induction hypothesis, (5.4) holds for $(w + \alpha' x, \alpha'', \beta'')$, that is,

$$f(w + \alpha x) + f(w') \leq f(w + \alpha' x) + f(w' + \alpha'' x). \tag{5.7}$$

<u>(III): $(w + \beta'' y, \alpha', \beta')$</u> By the definition of γ, f is linear on the segment between $w + \beta y$ and w'. Then, in the same way as (5.5), the equation (5.4) holds for $(w + \beta'' y, \alpha', \beta')$, that is,

$$f(w') + f(w + \beta y) \leq f(w + \beta'' y) + f(w' + \beta' y). \tag{5.8}$$

(IV): (w', α'', β') In the same way as the case (I), $p(w', \alpha'', \beta') < p(w, \alpha, \beta)$. Then, by the induction hypothesis, (5.4) holds for (w', α'', β'), that is,

$$f(w' + \alpha'' x) + f(w' + \beta' y) \leq f(w') + f(w + \alpha x + \beta y). \tag{5.9}$$

By adding (5.6), (5.7), (5.8), and (5.9), we obtain

$$f(w + \alpha x) + f(w + \beta y) \leq f(w) + f(w + \alpha x + \beta y),$$

which shows (5.4). This completes the proof of Theorem 5.1.

5.2 L-convex functions

A proof of Theorem 3.3 for locally polyhedral L$^\natural$-convex functions is given here on the basis of the general result (Theorem 5.1) for locally polyhedral convex functions.

We recall two fundamental facts about L$^\natural$-convexity.

- A locally polyhedral L$^\natural$-convex function f is such that, for each $p \in \mathbf{R}^n$, $S = \arg\min(f - p)$ is an L$^\natural$-convex polyhedron.
- An L$^\natural$-convex polyhedron S can be represented as

  $$S = \{x \in \mathbf{R}^n \mid \check{\gamma}_i \leq x_i \leq \hat{\gamma}_i \ (\forall i \in V), \ x_j - x_i \leq \gamma_{ij} \ (\forall i, j \in V, \, i \neq j)\}$$

  for some $\check{\gamma}_i \in \mathbf{R} \cup \{-\infty\}$, $\hat{\gamma}_i \in \overline{\mathbf{R}}$, $\gamma_{ij} \in \overline{\mathbf{R}}$ $(i, j \in V, \, i \neq j)$; cf. (3.3).

Let f be a locally polyhedral L$^\natural$-convex function, and $\mathcal{C}_f$ be the conic partition associated with f as in Section 5.1. It follows from the above facts that $\mathcal{C}_f$ is a subfamily of the conic partition $\mathcal{C}_{\mathrm{L}}$ in (3.5). In particular, $\mathcal{C}_f$ is a finite family. Then Theorem 5.1 implies Theorem 3.3.

5.3 M-convex functions

A proof of Theorem 4.3 for locally polyhedral M$^\natural$-convex functions is given here on the basis of the general result (Theorem 5.1) for locally polyhedral convex functions.

We recall two fundamental facts about M$^\natural$-convexity.

- A locally polyhedral M$^\natural$-convex function f is such that, for each $p \in \mathbf{R}^n$, $S = \arg\min(f - p)$ is an M$^\natural$-convex polyhedron.
- An M$^\natural$-convex polyhedron S can be represented as

  $$S = \{x \in \mathbf{R}^n \mid \mu(X) \leq x(X) \leq \rho(X) \ (\forall X \subseteq V)\}$$

  for some submodular set function $\rho : 2^V \to \overline{\mathbf{R}}$ and supermodular set function $\mu : 2^V \to \mathbf{R} \cup \{-\infty\}$ satisfying some compatibility condition. This expression is the M$^\natural$-version of (4.5). See [3, Section 3.5], [13, Section 4.7] for details.

Let f be a locally polyhedral M$^\natural$-convex function, and $\mathcal{C}_f$ be the conic partition associated with f as in Section 5.1. It follows from the above facts that $\mathcal{C}_f$ is a subfamily of the conic partition $\mathcal{C}_{\mathrm{M}}$ in (4.7). In particular, $\mathcal{C}_f$ is a finite family. Then Theorem 5.1 implies Theorem 4.3.

6 Twice-Differentiable Convex Functions

6.1 General case

We first note that not every convex function is cone superadditive with respect to a finite conic partition. The following example demonstrates this.

Example 6.1. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x_1, x_2) = \exp({x_1}^2 + {x_2}^2)$. For $x = (1, 0)$, $y = (1, a)$ and $z = (at, -2t)$ we have

$$
\begin{aligned}
&f(z) + f(z + x + y) - f(z + x) - f(z + y) \\
&= \exp((4 + a^2)t^2)(1 + \exp(4 + a^2) - \exp(2at + 1) - \exp(-2at + a^2 + 1)).
\end{aligned}
$$

If this is nonnegative for all $t \in \mathbf{R}$ with a fixed, we must have $a = 0$. This implies that f is not cone superadditive with respect to any finite conic partition of $\mathbf{R}^2$. ■

Cone superadditivity with respect to a specified conic partition can be characterized by a certain form of positivity of the Hessian matrix, just as convexity can be characterized by positive semidefiniteness of the Hessian matrix. We denote by $H(z)$ the Hessian matrix of f at $z \in \mathbf{R}^n$.

Theorem 6.1. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a twice continuously differentiable function, and $C \subseteq \mathbf{R}^n$ be a convex cone. Then the following conditions are equivalent.*

(A) *For any $z \in \mathbf{R}^n$ and for any $x, y \in C$,*

$$f(z + x) + f(z + y) \leq f(z) + f(z + x + y).$$

(B) *For any $z \in \mathbf{R}^n$ and for any $x, y \in C$, $x^\top H(z)y \geq 0$.*

Proof. [(A) ⇒ (B)]: Suppose that the condition (A) holds. Fix $z \in \mathbf{R}^n$ and $x, y \in C$ in (B). For $t \in \mathbf{R}$, we define a function $g_t : \mathbf{R} \to \mathbf{R}$ by

$$g_t(s) = f(z + sx + ty) - f(z + sx).$$

Since sx and ty are contained in C for $s, t \geq 0$, it holds that $g_t(s) - g_t(0) \geq 0$ for any $s, t \geq 0$ by (A). Then, $g_t'(0) \geq 0$ for any $t \geq 0$.

Next we define a function $h : \mathbf{R} \to \mathbf{R}$ by

$$h(t) = x^\top \nabla f(z + ty).$$

Then, we have $h(t)-h(0) = g_t'(0) \geq 0$ for any $t \geq 0$, which leads to $h'(0) \geq 0$. Since $h'(0) = x^\top H(z)y$ by the definition, the condition (B) holds.

[(B) $\Rightarrow$ (A)]: Suppose that the condition (B) holds. Fix $z \in \mathbf{R}^n$ and $x, y \in C$ in (A). Assume, in order to obtain a contradiction, that

$$f(z+x) + f(z+y) > f(z) + f(z+x+y). \tag{6.1}$$

Define a function $g : \mathbf{R} \to \mathbf{R}$ by

$$g(s) = f(z+sx+y) - f(z+sx).$$

Since $g(1) - g(0) < 0$ by (6.1), there exists a real number s_0 such that $0 \leq s_0 \leq 1$ and $g'(s_0) < 0$.

Next we define a function $h : \mathbf{R} \to \mathbf{R}$ by

$$h(t) = x^\top \nabla f(z + s_0 x + ty).$$

Since $h(1) - h(0) = g'(s_0) < 0$, there exists a real number t_0 such that $0 \leq t_0 \leq 1$ and $h'(t_0) < 0$. Then, we obtain $x^\top H(z+s_0x+t_0y)y = h'(t_0) < 0$, which contradicts (B). $\square$

6.2 L-convex functions

L-convexity is defined for closed proper (nonpolyhedral) convex functions in continuous variables [18], and for a twice continuously differentiable function on $\mathbf{R}^n$ it can be characterized by its Hessian matrix as follows.

Lemma 6.2 ([19])**.** *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a twice continuously differentiable function defined on $\mathbf{R}^n$. Then, f is $L^\natural$-convex if and only if the Hessian matrix $H(z) = (H_{ij}(z) \mid i, j = 1, \ldots, n)$ satisfies the following conditions for all $z \in \mathbf{R}^n$:*

$$H_{ij}(z) \leq 0 \qquad (i \neq j; 1 \leq i, j \leq n), \tag{6.2}$$

$$\sum_{i=1}^{n} H_{ij}(z) \geq 0 \qquad (1 \leq i \leq n). \tag{6.3}$$

Then Theorem 6.1 motivates us to consider the following.

Lemma 6.3. *Let $A = (a_{ij} \mid i, j = 1, \ldots, n)$ be a symmetric matrix satisfying*

$$a_{ij} \leq 0 \qquad (i \neq j; 1 \leq i, j \leq n), \tag{6.4}$$

$$\sum_{j=1}^{n} a_{ij} \geq 0 \qquad (1 \leq i \leq n). \tag{6.5}$$

For any L-compatible vectors x and y we have $x^\top Ay \geq 0$.

Proof. Let $\mathcal{P} = (P_1, \ldots, P_m)$ be a partition of $\{0, 1, \ldots, n\}$ such that $x, y \in C_{\mathcal{P}}$. Recall that $\tilde{x} = (x_0, x_1, \ldots, x_n) = (0, x)$. For $k = 1, \ldots, m-1$, let $x'_k = \tilde{x}_i - \tilde{x}_j$ where $i \in P_k$ and $j \in P_{k+1}$. Define y'_k in the same way as x'_k. Note that x'_k and y'_k are nonnegative by the definition of $C_{\mathcal{P}}$. Suppose that $0 \in P_l$. Then we have

$$
\begin{aligned}
x &= \sum_{k=1}^{l-1} x'_k \chi_{(P_1 \cup \cdots \cup P_k)} - \sum_{k=l}^{m-1} x'_k \chi_{(P_{k+1} \cup \cdots \cup P_m)}, \\
y &= \sum_{k=1}^{l-1} y'_k \chi_{(P_1 \cup \cdots \cup P_k)} - \sum_{k=l}^{m-1} y'_k \chi_{(P_{k+1} \cup \cdots \cup P_m)}.
\end{aligned}
$$

Now we observe the following facts.

- For $X, Y \subseteq V$ with $X \cap Y = \emptyset$, it holds that $(\chi_X)^\top A(\chi_Y) \leq 0$ by (6.4).
- For $X, Y \subseteq V$ with $X \subseteq Y$, it holds that $(\chi_X)^\top A(\chi_Y) = (\chi_Y)^\top A(\chi_X) \geq 0$, because
$$(\chi_X)^\top A(\chi_Y) \geq (\chi_X)^\top A(\chi_V) \geq 0,$$
where the second inequality is by (6.5).

With these observations, we have

$$
\begin{aligned}
x^\top A y = &\sum_{k_1=1}^{l-1} \sum_{k_2=1}^{l-1} x'_{k_1} y'_{k_2} \big(\chi_{(P_1 \cup \cdots \cup P_{k_1})}\big)^\top A \big(\chi_{(P_1 \cup \cdots \cup P_{k_2})}\big) \\
&- \sum_{k_1=1}^{l-1} \sum_{k_2=l}^{m-1} x'_{k_1} y'_{k_2} \big(\chi_{(P_1 \cup \cdots \cup P_{k_1})}\big)^\top A \big(\chi_{(P_{k_2+1} \cup \cdots \cup P_m)}\big) \\
&- \sum_{k_1=l}^{m-1} \sum_{k_2=1}^{l-1} x'_{k_1} y'_{k_2} \big(\chi_{(P_{k_1+1} \cup \cdots \cup P_m)}\big)^\top A \big(\chi_{(P_1 \cup \cdots \cup P_{k_2})}\big) \\
&+ \sum_{k_1=l}^{m-1} \sum_{k_2=l}^{m-1} x'_{k_1} y'_{k_2} \big(\chi_{(P_{k_1+1} \cup \cdots \cup P_m)}\big)^\top A \big(\chi_{(P_{k_2+1} \cup \cdots \cup P_m)}\big) \\
\geq 0.&
\end{aligned}
$$

□

Theorem 6.4. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a twice continuously differentiable $L^\natural$-convex function. For any $z \in \mathbf{R}^n$ and any L-compatible vectors x and y, the inequality* (3.6) *is satisfied.*

Proof. By Lemma 6.2 the Hessian matrix $H(z)$ of f satisfies the conditions in Lemma 6.3. It then follows that f satisfies the condition (B) in Theorem 6.1. □

7 Concluding Remarks

L-convexity and M-convexity are defined also for closed proper (nonpolyhedral) convex functions in continuous variables [18]. In view of the present results it is natural to expect cone superadditivity of such general nonpolyhedral L-convex and M-convex functions.

For L-convex functions we can indeed obtain the cone superadditivity as a corollary to Theorem 3.3.

Theorem 7.1. *Let $f : \mathbf{R}^n \to \overline{\mathbf{R}}$ be a closed proper $L^\natural$-convex function. For any $z \in \operatorname{dom} f$ and any L-compatible real vectors x and y, the inequality* (3.6) *is satisfied.*

Proof. By continuity of f it suffices to prove (3.6) for rational vectors x and y. Let N be a positive integer that is a common multiple of the denominators of all the components of x and y, and put $x = p/N$ and $y = q/N$ with integer vectors p and q. Define $g : \mathbf{Z}^n \to \overline{\mathbf{R}}$ by $g(w) = f(z + w/N)$. Then g is an $\mathrm{L}^\natural$-convex function in discrete variables, which is L-cone superadditive by Theorem 3.2. In particular, we have $g(p) + g(q) \leq g(\mathbf{0}) + g(p + q)$, i.e., $f(z + x) + f(z + y) \leq f(z) + f(z + x + y)$. □

As for M-convex functions, however, the corresponding statement cannot be made, since the proof technique of Theorem 7.1 does not work. To be specific, for an $\mathrm{M}^\natural$-convex function f, the function $g(w) = f(z + w/N)$ is not guaranteed to be $\mathrm{M}^\natural$-convex.

On the other hand, cone superadditivity can be established for a subclass of $\mathrm{M}^\natural$-convex functions arising from network flows. In Example 4.1 we have seen how M-convex functions in integer variables arise from network flows. As a continuous version of (4.10) we define $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty, -\infty\}$ by

$$f(x) = \inf \left\{ \sum_{a \in A} f_a(\xi(a)) \mid \xi : A \to \mathbf{R},\ \partial\xi = x \right\} \tag{7.1}$$

with univariate convex functions $f_a : \mathbf{R} \to \overline{\mathbf{R}}$ for $a \in A$. This function f is known to be M-convex if $f > -\infty$. Furthermore, the proof given in Example 4.1 can be adapted to this case, to yield M-cone superadditivity of f in (7.1).

Acknowledgments

This work started from a conversation between the second and the third author at the workshop "Combinatorial Optimization" (Mathematisches Forschungsinstitut Oberwolfach, November 2008) organized by W. Cook, A. Frank and M. Jünger. The first author is supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

This work is supported by a Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology of Japan, and by the Global COE "The Research and Training Center for New Development in Mathematics."

Appendix: Combinatorial Proof for L-convex Functions

This section gives a purely combinatorial proof of Theorem 3.2 for L$^\natural$-convex functions in discrete variables. First recall [13, Section 7.2] that L$^\natural$-convexity of $f : \mathbf{Z}^n \to \overline{\mathbf{R}}$ can be characterized by the property:

> **(L$^\natural$-APR)** For any $x, y \in \mathbf{Z}^n$ with $\mathrm{supp}^+(x - y) \neq \emptyset$, it holds that
> $$f(x) + f(y) \geq f(x - \chi_A) + f(y + \chi_A),$$
> where $A = \arg\max(x - y)$.

The following lemma deals with the special case where x and y are $\{0, +1, -1\}$-vectors.

Lemma A.1. *Inequality* (3.6) *holds for any* $z \in \mathrm{dom}\, f$ *and any L-compatible* $\{0, +1, -1\}$*-vectors* x *and* y*.*

Proof. Put $x = \chi_A - \chi_B$ and $y = \chi_C - \chi_D$ with $A \cap B = \emptyset$ and $C \cap D = \emptyset$. By L-compatibility we have $(A \cup C) \cap (B \cup D) = \emptyset$, and by symmetry we may assume $A \subseteq C$. Then two cases are distinguished: Case 1: $B \subseteq D$ and Case 2: $B \supseteq D$.

Case 1: We have

$$x + y = 2\chi_A + \chi_{C \setminus A} - 2\chi_B - \chi_{D \setminus B}.$$

If A is nonempty, we have $\mathrm{supp}^+(x + y) \neq \emptyset$ and $A = \arg\max(x + y)$, and hence

$$f(z) + f(z + x + y) \geq f(z + \chi_A) + f(z + x + y - \chi_A) \qquad \text{(A.1)}$$

by (L$^\natural$-APR). Note that (A.1) is trivially true if $A = \emptyset$. If $B \neq \emptyset$, we have, again by (L$^\natural$-APR), that

$$\begin{aligned} & f(z + \chi_A) + f(z + x + y - \chi_A) \\ & \geq f(z + \chi_A - \chi_B) + f(z + x + y - \chi_A + \chi_B) \\ & = f(z + x) + f(z + y), \end{aligned} \qquad \text{(A.2)}$$

which is also true when $B = \emptyset$. Adding (A.1) and (A.2) yields (3.6).

Case 2: We have

$$x + y = 2\chi_A + \chi_{C \setminus A} - 2\chi_D - \chi_{B \setminus D}$$

and hence

$$\left\lfloor \frac{x+y}{2} \right\rfloor = \chi_A - \chi_D - \chi_{B\setminus D} = \chi_A - \chi_B = x,$$
$$\left\lceil \frac{x+y}{2} \right\rceil = \chi_A + \chi_{C\setminus A} - \chi_D = \chi_C - \chi_D = y.$$

Then by discrete midpoint convexity (3.2) we obtain

$$\begin{aligned} f(z) + f(z+x+y) &\geq f\left(\left\lfloor \frac{2z+x+y}{2} \right\rfloor\right) + f\left(\left\lceil \frac{2z+x+y}{2} \right\rceil\right) \\ &= f\left(z + \left\lfloor \frac{x+y}{2} \right\rfloor\right) + f\left(z + \left\lceil \frac{x+y}{2} \right\rceil\right) \\ &= f(z+x) + f(z+y). \end{aligned}$$

□

We now prove Theorem 3.2 by using Lemma A.1 as the basis of induction. For a vector $w \in \mathbf{Z}^n$ in general we define

$$\mu_+(w) = \max_{1\leq i\leq n}\{w_i \vee 0\}, \quad \mu_-(w) = \max_{1\leq i\leq n}\{(-w_i) \vee 0\},$$

and $\mu(w) = (\mu_+(w), \mu_-(w))$. We prove (3.6) by induction on the vector ordering of a 4-dimensional vector $(\mu(x), \mu(y)) \in \mathbf{Z}^4$. Lemma A.1 shows that (3.6) is true if $(\mu(x), \mu(y)) \leq (1, 1, 1, 1)$.

Suppose that $z, z + x + y \in \mathrm{dom}\, f$, and $(\mu(x), \mu(y)) \not\leq (1, 1, 1, 1)$. We have $(\mu_+(x), \mu_+(y)) \not\leq (1, 1)$ or $(\mu_-(x), \mu_-(y)) \not\leq (1, 1)$. By symmetry (or reflection) we may focus on the former case, where $\mu_+(x) \geq 2$ or $\mu_+(y) \geq 2$. By interchanging x and y, if necessary, we may assume $\arg\max(x) \subseteq \arg\max(y)$ and $\mu_+(x) \geq 1$. Put $A = \arg\max(x)$.

We divide into two cases: Case 1: $x = \chi_A$ and Case 2: $x \neq \chi_A$

Case 1: (L$^\natural$-APR) applied to $(z+x+y, z)$ yields (3.6), since $\arg\max(x+y) = A$ and $x = \chi_A$.

Case 2: If x is contained in $C_{\mathcal{P}}$ of (3.4), both $x' = x - \chi_A$ and $x'' = \chi_A$ are contained in the same $C_{\mathcal{P}}$. This implies that x' and y are L-compatible with $\mu(x') < \mu(x)$, and x'' and y are L-compatible with $\mu(x'') < \mu(x)$.

Since $A = \arg\max(x) = \arg\max(x+y)$, (L$^\natural$-APR) applied to $(z+x+y, z)$ yields $z + x + y - \chi_A = z + x' + y \in \mathrm{dom}\, f$. Then the induction hypothesis yields

$$f(z) + f(z + x' + y) \geq f(z + x') + f(z + y),$$

i.e.,

$$f(z) + f(z + x' + y) \geq f(z') + f(z + y) \tag{A.3}$$

with $z' = z + x' = z + x - \chi_A$. Since $z' \in \mathrm{dom}\, f$ by (A.3) and $z' + x'' + y = z + x + y \in \mathrm{dom}\, f$, the induction hypothesis yields

$$f(z') + f(z' + x'' + y) \geq f(z' + x'') + f(z' + y),$$

i.e.,

$$f(z') + f(z + x + y) \geq f(z + x) + f(z + x' + y). \tag{A.4}$$

Addition of (A.3) and (A.4) results in (3.6), where it is noted that all the terms in (A.3) and (A.4) are finite-valued.

The completes the proof of Theorem 3.2.

References

[1] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin: *Network Flows—Theory, Algorithms and Applications*, Prentice-Hall, Englewood Cliffs, 1993.

[2] P. Favati and F. Tardella: Convexity in nonlinear integer programming, *Ricerca Operativa*, **53** (1990), 3–44.

[3] S. Fujishige: *Submodular Functions and Optimization*, 2nd ed., Annals of Discrete Mathematics, **58**, Elsevier, 2005.

[4] S. Fujishige and K. Murota: Notes on L-/M-convex functions and the separation theorems, *Mathematical Programming*, **88** (2000), 129–146.

[5] P. Gordan: Über die Auflösung linearer Gleichungen mit reellen Coefficienten, *Mathematische Annalen*, **6** (1873), 23–28.

[6] J. E. Graver: On the foundations of linear and integer linear programming I, *Mathematical Programming*, **8** (1975), 207–226.

[7] U.-U. Haus, M. Köppe and R. Weismantel: A primal all-integer algorithm based on irreducible solutions, *Mathematical Programming*, **96** (2003), 205–246.

[8] R. Hemmecke, S. Onn and R. Weismantel: A polynomial oracle-time algorithm for convex integer minimization, *Mathematical Programming, Series A*, to appear.

[9] H. Hirai and K. Murota: M-convex functions and tree metrics, *Japan Journal of Industrial and Applied Mathematics*, **21** (2004), 391–403.

[10] J. Lee, S. Onn and R. Weismantel: On test sets for nonlinear integer maximization, *Operations Research Letters*, **36** (2008), 439–443.

[11] K. Murota: Convexity and Steinitz's exchange property, *Advances in Mathematics*, **124** (1996), 272–311.

[12] K. Murota: Discrete convex analysis, *Mathematical Programming*, **83** (1998), 313–371.

[13] K. Murota: *Discrete Convex Analysis*, SIAM Monographs on Discrete Mathematics and Applications, Vol. 10, Society for Industrial and Applied Mathematics, Philadelphia, 2003.

[14] K. Murota: Recent developments in discrete convex analysis, *Research Trends in Combinatorial Optimization, Bonn 2008* (W. Cook, L. Lovász and J. Vygen, eds.), Springer-Verlag, Berlin, 2009, Chapter 11, 219–260.

[15] K. Murota, H. Saito and R. Weismantel: Optimality criteria for a class of nonlinear integer programs, *Operations Research Letters*, **32** (2004), 468–472.

[16] K. Murota and A. Shioura: M-convex function on generalized polymatroid, *Mathematics of Operations Research*, **24** (1999), 95–105.

[17] K. Murota and A. Shioura: Extension of M-convexity and L-convexity to polyhedral convex functions, *Advances in Applied Mathematics*, **25** (2000), 352–427.

[18] K. Murota and A. Shioura: Conjugacy relationship between M-convex and L-convex functions in continuous variables, *Mathematical Programming*, **101** (2004), 415–433.

[19] K. Murota and A. Shioura: Fundamental properties of M-convex and L-convex functions in continuous variables, *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, **E87-A** (2004), 1042–1052.

[20] A. Sebő: Hilbert bases, Caratheodory's theorem and combinatorial optimization, *Integer Programming and Combinatorial Optimization* (R. Kannan and W. Pulleyblank, eds.), University of Waterloo Press, Waterloo, 1990, 431–455.

[21] J. G. van der Corput: Über Systeme von linear-homogenen Gleichungen und Ungleichungen, *Proceedings Koninklijke Akademie van Wetenschappen te Amsterdam*, **3** (1931), 368–371.

[22] R. Weismantel: Test sets of integer programs, *Mathematical Methods of Operations Research*, **47** (1998), 1–37.